# SINGULAR SOLUTIONS OF SOME NONLINEAR PARABOLIC EQUATIONS

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# ABSTRACT

We consider the existence and uniqueness of singular solutions for equations of the form

$$u_t = \operatorname{div}\left(|Du|^{p-2}Du\right) - \phi(u)\,,$$

with initial data $u(x,0) = 0$ for $x \neq 0$. The function $\phi$ is a nondecreasing real function such that $\phi(0) = 0$ and $p > 2$.

Under a growth condition on $\phi(u)$ as $u \to \infty$, (H1), we prove that for every $c > 0$ there exists a unique singular solution such that $u(x,t) \to c\delta(x)$ as $t \to 0$. These solutions are called fundamental solutions. Under additional conditions, (H2) and (H3), we show the existence of very singular solutions, i.e. singular solutions such that

$$\int_{|x|\le r} u(x,t)\,dx \to \infty \qquad \text{as } t \to 0\,.$$

Finally, for functions $\phi$ which behave like a power for large $u$ we prove that the very singular solution is unique. This is our main result.

In the case $\phi(u) = u^q$, $1 \le q$, there are fundamental solutions for $q < p_* = p - 1 + (p/N)$ and very singular solutions for $p - 1 < q < p_*$. These ranges are optimal.



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## 1. Introduction.

In this paper we discuss the existence and uniqueness of singular solutions for equations of the form

$$(E) = (E_\phi) \qquad u_t = \operatorname{div}(|Du|^{p-2}Du) - \phi(u)\,,$$

where $\phi$ is a nondecreasing real function such that $\phi(0) = 0$ and $p > 2$. Here $Du$ denotes the spatial gradient of $u$. The operator $\operatorname{div}(|Du|^{p-2}Du)$ is usually called the $p$-Laplacian operator and is denoted by $\Delta_p(u)$.

By a solution of $(E_\phi)$ we mean a continuous function $u(x,t)$ which is defined and non-negative in $Q$, $Du \in L^1_{loc}(0,\infty : W^{1,p-1}_{loc}(\mathbf{R}^N))$ and the equation is satisfied in the sense of distributions in $Q$. We also assume that $u$ is bounded for $t \geq \tau > 0$. By a *singular* solution of $(E_\phi)$ we will mean a solution $u$ which is continuous down to $t = 0$ for $x \neq 0$ and satisfies

$$(1.2) \qquad u(x,0) = 0 \qquad \text{for } x \neq 0\,,$$

while it is unbounded as $(x,t) \to (0,0)$, i.e. we restrict our consideration to solutions having an *isolated singularity* at $(0,0)$ .

Simpler examples of equations of diffusion-absorption type have been studied by various authors and the singular solutions have been classified into two types. One of them is the so-called *fundamental solution* (or *source-type solution*), which means that

$$(1.3) \qquad \lim_{t \to 0} u(x,t) = c\delta(x)\,,$$

where $\delta$ denotes Dirac's delta function and $c > 0$ is a constant, the initial mass. Typical diffusion equations, like the heat equation $u_t = \Delta u$, the porous medium equation $u_t = \Delta(u^m)$ and $p$-Laplacian evolution equation $u_t = \Delta_p(u)$ admit only this type of singular solution, cf. [KV].

Brezis, Peletier and Terman [BPT] found in 1986 that the semilinear heat equation $u_t = \Delta u - u^q$ admits for $1 < q < (N+2)/N$ a different type of singular solution that they called *very singular solution* (VSS for short), which has the property that

$$(1.4) \qquad \lim_{t \to 0} \int u(x,t)\,dx = \infty\,,$$

i.e. it has at (0,0) a stronger singularity than the fundamental solutions. Precisely one VSS solution exists in the above exponent range. A fundamental solution exists for every $c > 0$ in the same range and this completes the set of singular solutions of the equation according to Oswald's classification, [O].

The classification of the singular solutions has also been performed by Peletier and the authors, [KPV], in the case of the most usual model of nonlinear diffusion with absorption, i.e.

$$(1.5) \qquad u_t = \Delta u^m - u^q\,,$$

where $m, q \geq 1$. According to [KPV], equation (1.5) with $m, q > 1$ admits fundamental solutions if and only if $q < m + (2/N)$, which fits in with the results of the previous case, but a VSS exists only in the more restricted range $m < q < m + 2/N$. In the prescribed ranges the VSS is unique [KVe] and there exists a fundamental solution for every $c > 0$, see also [KP].

In this paper we continue the investigation of singular solutions of nonlinear parabolic equations in two directions. On the one hand, we are interested in understanding the influence of absorption terms of non-power form. From the point of view of the techniques involved, this will force us to abandon the ODE techniques based on selfsimilarity which go together with the scale-invariance properties of power-like nonlinearities and have been basic in many of the proofs of the cases mentioned above.

On the other hand, we want to investigate the case of $p$-Laplacian diffusion, which offers the difficulty of its degeneracy at all points where $\nabla u = 0$ and has been less considered in the literature. The main contribution of the paper is the following.

THEOREM A. *Let $\phi$ be a convex absorption function such that*

$$\lim_{u\to\infty} \frac{\phi(u)}{u^q} = a \tag{1.6}$$

*for some $a > 0$ and some $q \in (p-1, p_*)$ with $p_* = p - 1 + (p/N)$. Then there exists a unique VSS for equation $(E_\phi)$.*

Existence of VSS has been shown by Peletier and Wang [PW] in the power case

$$u_t = \Delta_p(u) - u^q \qquad \text{with } q > 1\,, \tag{1.7}$$

in the range $1 < q < p_*$. Partial uniqueness has been shown for VSS of (1.7) of selfsimilar form by Diaz and Saa [DS]. Both papers use the ODE satisfied by the solutions which can be written in this form. It can also be proved that the range $q \in (p-1, p_*)$ is optimal for the existence of a VSS. A close similarity exists in that sense between equations (1.5) and (1.7).

Let us explain our results in some more detail. In a preparatory Section 2 we construct fundamental solutions for equation $(E_\phi)$ under condition (H1) on $\phi$. This an easy task based on the techniques of [KP], [KPV] and [G]. The fundamental solution corresponding to a given mass $c$ is then shown to be unique.

Section 3 introduces additional restrictions on $\phi$, (H2) and (H3), which allow us to derive a priori estimates of the *absolute* type for singular solutions of $(E_\phi)$, i.e. the same bounds are valid uniformly for all such solutions. With these bounds we can construct (Section 4) very singular solutions. Actually, we show a more precise result : there exist both a *minimal* and a *maximal* VSS.

This paves the way for the proof of uniqueness of the VSS. We first deal in Section 5 with the power case, i.e. equation (1.7) in the range $p - 1 < q < p_*$. For uniqueness we use first the scale invariance of the equation to conclude that both the maximal and the minimal solutions are necessarily selfsimilar. At this stage we have two options. We can use the results of [DS] to conclude that both VSS's, hence all VSS's, have to be equal. We also give another proof using purely PDE techniques (the Strong Maximum Principle and non-exact scale transformations); it borrows some ideas from [KVe] and [KV] and has moreover to tackle the degeneracy of equation (1.7) at points of maximum or minimum of the solution.

After establishing in Section 6 comparison results for the singular solutions corresponding to different absorption functions we are able to prove in Section 7 the uniqueness result in its general form. Actually, uniqueness of the VSS is obtained under weaker conditions than those stated above for simplicity (see Theorem 7.1).

It is to be noted that our results can be easily generalised to other similar equations. In particular, the reader should have little difficulty in extending most of our results to the equation

$$u_t = \Delta(u^m) - \phi(u) \tag{1.8}$$

under suitable conditions on $\phi$. Note however that the comparison arguments of Section 6 cannot be directly translated.

We devote Section 8 to an application of the uniqueness of the VSS. In the case of equation (1.7) in the range $p-1 < q < p_*$ we show that the unique VSS gives the asymptotic behaviour of all nonnegative solutions whose initial data have compact support. The asymptotic behaviour is different in the other exponent ranges.

Finally, we show in Section 9 that the range $p-1 < q < p_*$ is optimal for the existence of very singular solutions of (1.7).

**2. Existence and uniqueness of fundamental solutions.**

Throughout this paper we will work with nonnegative weak solutions of equations (E), (1.5) or (1.7), whose definition and basic properties are now standard. In particular, such solutions are continuous inside their domain of definition. We will also consider super- and subsolutions defined in a similar sense. It is well-known that the Maximum Principle holds.

To begin with, we prove the existence of singular solutions of equation (E) under a certain condition on $\phi$, which will be in all cases a nonnegative and nondecreasing function $\mathbf{R} \to \mathbf{R}$. As in [KPV] we begin with a classification result.

LEMMA 2.1. *Let $u$ be a nontrivial solution of (E) with $u(x,0) = 0$ for $x \neq 0$. Then, for every $r > 0$ there exists the limit*

$$\lim_{t\to 0} \int_{|x|<r} u(x,t)\,dx = c, \tag{2.1}$$

*where $0 < c \leq \infty$, and $c$ does not depend on $r$.*

The proof of this result is done as in [KPV] or [O].

Let

$$E_c(x,t) = t^{-1/k} f(\eta) \qquad \text{with} \quad \eta = xt^{-k/N},\ k = (p-2+(p/N))^{-1}$$

be the explicit Barenblatt fundamental solution for equation

$$u_t = \Delta_p(u) \tag{2.2}$$

(see for instance [KV]). We have

LEMMA 2.2. *Let $u$ be a solution of (E) such that (2.1) holds with a finite $c$. Then $u(x,t) \le E_c(x,t)$ in $Q$. Hence $u$ is a fundamental solution of (E).*

PROOF: Since any solution of (E) is a subsolution of (2.2) we can repeat the proof of [KV] to show that $u \le E_c$. It is then clear that $u(\cdot,t)$ has compact support for every $t > 0$ and that (1.3) holds. #

We show next the existence of fundamental solutions.

LEMMA 2.3. *If $\phi$ satisfies the condition*

$$\text{(H1)} \qquad \int_A^\infty \phi(s)\, s^{-p-\frac{p}{N}}\, ds < \infty ,$$

*then there exists a fundamental solution of (E) for every mass $c > 0$.*

We remark that in the case of a power-like nonlinearity $\phi(u) = u^q$ condition (H1) is equivalent to $q < p_*$, which is a sharp limit since for $q > p_*$ there exists no singular solutions, see [G].

PROOF: Let us define $\tilde{u}_n(x,t)$ as the solution of equation (E) for $t > 1/n$ with initial data $\tilde{u}_n(x,1/n) = E_c(x,1/n)$ at $t = 1/n$. By the Maximum Principle we know that

$$\tilde{u}_n(x,t) \le E_c(x,t) \tag{2.3}$$

for every $t \ge 1/n$ and $x \in \mathbf{R}^N$. Moreover, the sequence $\{\tilde{u}_n(x,t)\}$ is monotone decreasing as $n \to \infty$. Therefore, we can take the limit

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t) \tag{2.4}$$

and this will be a weak solution of (E) in $Q$. It is clear that $u(x,t) \le E_c(x,t)$ in $Q$, therefore $u(x,0) = 0$ for every $x \ne 0$. It remains to check that the initial mass condition (2.1) is satisfied. In order to prove that we compute the variation of mass in small time intervals for the approximate solutions $\tilde{u}_n$. Indeed, for $0 < 1/n < t$ we have with $k$ and $\eta$ as above

$$\begin{aligned} I_n(t) &= \int (\tilde{u}_n(x,t) - \tilde{u}_n(x,1/n)\, dx = \int_{1/n}^t \int_{\mathbf{R}^N} \phi(\tilde{u}_n)\, dx dt \\ &\le \int_{1/n}^t \int_{\mathbf{R}^N} \phi(E_c(x,t))\, dx dt = \int_{1/n}^t \int_{\mathbf{R}^N} \phi(t^{-k} f(\eta))\, dx dt \\ &= \int_{1/n}^t t^k (\int_{\mathbf{R}^N} \phi(t^{-k} f(\eta))\, d\eta)\, dt \end{aligned}$$

Now, the interval of $\eta$ where $f$ is nonzero is finite, and $f(\eta) \le f(0)$. Hence, if $s = t^{-k}$ we get

$$I_n(t) \le C \int_{1/n}^t t^k \phi(f(0)t^{-k})\, dt \le C \int_s^\infty s^{-2-1/k} \phi(s)\, ds$$

which is finite according to our assumption. Therefore, the integral $\int_{1/n}^t \int_{\mathbf{R}^N} \phi(\tilde{u}_n)\, dx dt$ is small for $t \approx 0$ uniformly in $n$. Since $\int u(x,1/n)\, dx = c$ by the definition of $\tilde{u}_n$, we conclude that $\int \tilde{u}_n(x,t)\, dx$ will be close to $c$ for $t \approx 0$ uniformly in $n$, and finally in the limit the same will be true for $u$. Consequently, (2.1) holds. #

LEMMA 2.4. *The fundamental solution is unique.*

PROOF: Uses above calculation plus uniqueness of $E$ and contraction principle. #

In the sequel we will denote the fundamental solution with mass $c$ by $u_c$. We remark that the fundamental solutions thus constructed form a monotone family, namely, if $c_1 > c_2$ then $u_{c_1} \geq u_{c_2}$.

We end this section with some properties of the fundamental solutions.

LEMMA 2.5. *Let $u$ be a fundamental solution of (E). Then for every $t > 0$ the function $u(\cdot, t)$ is radially symmetric and decreasing in $|x|$. Moreover, both $\int_{\mathbf{R}^N} u(x,t)dx$ and $u(0,t) = \sup\{u(x,t) : x \in \mathbf{R}^N\}$ are decreasing in time.*

PROOF: The radial symmetry comes from uniqueness combined with the rotation-invariance of the equation. The other properties are standard. #

## 3. A priori estimates.

In order to study the existence of VSS for equation (E) we have to impose on $\phi$ some additional conditions. These are

$$\text{(H2)} \qquad \int_1^\infty \frac{ds}{\phi(s)} < \infty\,,$$

$$\text{(H3)} \qquad \int_1^\infty ds\Big(\int_0^s \phi(r)\,dr\Big)^{-1/p} < \infty\,.$$

In the case $\phi(s) = s^q$ condition (H2) means $q > 1$ and condition (H3) is equivalent to $q > p - 1$.

Condition (H2) is useful because it implies the existence of a solution $V$ depending only on $t$, i.e. a *flat solution,* with infinite initial value. It is given by the implicit formula

$$t = \int_V^\infty \frac{ds}{\phi(s)}\,. \tag{3.1}$$

In this way we obtain a function $V(t)$ which is positive and decreasing and goes to 0 as $t \to T = \int_0^\infty \varphi^{-1}(s)ds$. In case this limit is finite we define $V(t) \equiv 0$ for $t \geq T$.

On the other hand, (H3) implies the existence of a stationary solution $\Psi$ which depends only on one space coordinate, say $x_1$, and is defined in $\mathbf{R}^N \setminus \{x_1 = 0\}$. It is given implicitly by

$$|x_1| = \Psi(u) \equiv C\int_u^\infty \Big(\int_0^s \phi(s)\,ds\Big)^{-1/p} ds, C = \Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \tag{3.2}$$

Obviously, similar functions can be constructed in the other variables. The function $\Psi(s)$ is positive and decreasing and goes to 0 as $s \to R = \int_0^\infty ds(\int_0^s \phi(r)dr)^{-1/p}$. Again, if $R$ is finite we define $\Psi(s) \equiv 0$ for $s \geq R$.

These solutions serve as upper bounds for our singular solutions.

LEMMA 3.1. *Suppose (H2),(H3) hold. Let $u(x,t)$ be a singular solution of (E). Then*

$$u(x,t) \le V(t)\,, \tag{3.3}$$

$$u(x,t) \le \Psi(|x|)\,. \tag{3.4}$$

PROOF: Observe that $V$ is solution of (E) with $V(x,0) = \infty$ while $F(x) = \Psi(|x| - a)$ is a supersolution of (E) in $(\mathbf{R}^N \setminus B_a(0)) \times (0,\infty)$ for very $a > 0$ such that $F(x) \ge 0$ for $|x| \ge a$ and $F(x) = \infty$ for $|x| = a$. Now apply the Maximum Principle and let $a \to 0$. #

We complete the a priori bounds by uniformly estimating the support of any singular solution.

LEMMA 3.2. *Suppose (H3) holds. There exists a continuous and strictly increasing function $\zeta(t)$ defined for $0 < t < \infty$ with $\zeta(0) = 0$ such that for any singular solution of (E) we have*

$$u(x,t) = 0 \qquad \text{if} \quad |x| \ge \zeta(t)\,. \tag{3.5}$$

*The function $\zeta$ depends only on $\phi$, and not on the particular solution considered.*

PROOF: Pick some $R > 0$. We know from (3.4) that for any singular solution $u$ we have $u(x,t) \le \Psi(R/2)$ for any $t > 0$ and $|x| = R/2$. On the other hand, we also have $u(x,0) = 0$ for $|x| \ge R/2$. The result is then a consequence of the property of finite propagation for the equation without absorption, $u_t = \Delta_p(u)$, applied to the set $\{|x| \ge R/2 : t > 0\}$. One way of proving this is to consider a travelling-wave solution of $u_t = \Delta_p(u)$, which as is well-known has the form

$$u^{\frac{p-2}{p-1}}(x,t) = b(x_1 - R - ct)^+\,, \tag{3.6}$$

with $c > 0$ large and $b$ depends on $c$, and use it as a supersolution in the aforementioned domain to control the growth of the support in the $x_1$-direction. The same happens in every other direction by symmetry. #

It is to be remarked that stronger forms of conditions (H2) and (H3) imply more powerful bounds for the solutions. Thus, if (H3) is replaced by

$$\int_0^\infty ds\Big(\int_0^s \phi(r)\,dr\Big)^{-1/p} = R < \infty\,. \tag{H3'}$$

then $\Psi(|x|)$ vanishes at $|x| = R$. Continuing the function by 0 for $|x| > R$ we obtain a supersolution of (E) which implies the following *localization* result

LEMMA 3.3. *Suppose that (H3') holds. Then the support of any singular solution of (E), (1.2) is contained in the set $\{(x,t) : |x| \le R,\ t \ge 0\}$.*

Likewise if we replace (H2) by

$$\int_0^\infty \frac{ds}{\phi(s)} = T < \infty\,, \tag{H2'}$$

we have *extinction* in a finite time which is uniformly bounded above.

LEMMA 3.4. *Suppose that (H2') holds. Then for every singular solution of (E) we have $u(x,t) = 0$ for every $x \in \mathbf{R}^N$ if $t \geq T$.*

The result is actually true for every solution which is bounded for $t \geq \tau > 0$. Results about localization and extinction are known in the literature. The interesting point here is the fact that the bounds are absolute.

## 4. Existence of a minimal and a maximal VSS.

We begin with the construction of the maximal solution.

THEOREM 4.1. *Under conditions (H1), (H2) and (H3) there exists a maximal singular solution of (E) given by*

$$V(x,t) = \sup\{u(x,t) : u \text{ is a singular solution of (E)}\} \tag{4.1}$$

PROOF: Let $V$ be defined as in (4.1). By (H1) the set of singular solutions is nonempty since it contains the fundamental solutions. By (3.3), (3.4) $V$ is finite in $Q$. Moreover, by local regularity theory, $V$ will be continuous in $Q$. We consider for every $\tau > 0$ the Cauchy Problem

$$\begin{cases} u_t = \Delta_p(u) - \phi(u) & \text{in } Q^\tau = \mathbf{R}^N \times (\tau, \infty) \\ u(x,\tau) = V(x,\tau) \end{cases} \tag{4.2}$$

and let $u^\tau$ be the corresponding solution. Since for every singular solution $u$ we have $u(\cdot,\tau) \leq u^\tau(\cdot,\tau)$ the same will happen for every $t \geq \tau$, hence taking *sup* we get $V(\cdot,t) \leq u^\tau(\cdot,t)$.

Now we prove that the sequence $u^\tau$ is monotone decreasing in $\tau$. We consider two solutions $u^\tau$ and $u^{\tau'}$ with $0 < \tau < \tau'$. Since at time $t = \tau'$ we have $u^\tau \geq V = u^{\tau'}$ we conclude that $u^\tau \geq u^{\tau'}$ for every $\tau < \tau'$.

Finally, we take the monotone limit $\lim_{\tau \to 0} u^\tau = W$. This will be a solution of (E) satisfying the bounds (3.3), (3.4) of the previous section (since they are true for every $u^\tau$). The fact that $W$ has compact support in $x$ for every $t > 0$ follows again from comparison with a travelling-wave solution. Therefore, $W$ is a singular solution. Since also $W \geq V$, by the definition of $V$ we must have $V = W$. #

REMARKS: Clearly, $V$ is necessarily a very singular solution. Notice that we could have used the word 'subsolution' instead of 'solution' in the definition of $V$, formula (4.1). #

We now show that a minimal VSS exists by the typical method of taking the set of fundamental solutions $\{u_c\}_{c>0}$ and letting $c \to \infty$.

THEOREM 4.2. *Let conditions (H1)-(H3) hold. Then the limit*

$$v(x,t) = \lim_{c \to \infty} u_c(x,t) \tag{4.3}$$

*is a very singular solution of equation (E). In fact, it is the minimal very similar solution.*

PROOF: We know that the set $\{u_c(x,t)\}$ is a monotone family of solutions of (E) which are uniformly bounded above in $Q^\tau = \mathbf{R}^N \times (\tau, \infty)$ by estimate (3.3). Therefore, the limit

$v$ exists and is finite in $Q^\tau$. Classical theory implies that $v$ will be a solution of (E) in $Q$. On the other hand, the initial data $v(x,0)=0$ for $x\neq 0$ are taken thanks to Lemma 3.2. Finally, $v$ will be a VSS since necessarily

$$\lim_{t\to 0}\int_{|x|<r} v(x,t)\,dx=\infty, \tag{4.4}$$

due to the fact that $u_c$ satisfies the formula with second member $c$ and $c\to\infty$.

The proof that it is a minimal solution proceeds as in [KPV] by showing that any VSS $\tilde v$ satisfies $\tilde v\geq u_c$ for every $c>0$. #

As in the case of the fundamental solutions we remark that since the equation is invariant under space rotations, both the maximal and the minimal VSS must be rotation-invariant, hence

$$v(x,t)=v(|x|,t) \quad \text{and} \quad V(x,t)=V(|x|,t).$$

Besides, both $v$ and $V$ are decreasing functions of $|x|$ for $t>0$ fixed. Observe also that for $t>0$ the integral $\int V(x,t)dx$ is finite and decreases as $t$ increases.

## 5. The power case. Uniqueness of the VSS.

Let us now concentrate on the case of a power-like absorption term, i.e. equation (1.7). The results of the previous section imply the existence of very singular solutions in the range $p-1<q<p_*=p-1+p/N$. This coincides with existence result of [PW]. Actually, this existence range is optimal, see Section 9.

THEOREM 5.1. *The VSS of equation (1.7) is unique .*

Once we have proved in the previous section that there exist a maximal and a minimal VSS in that exponent range the proof of uniqueness is reduced to showing that both are the same.

In the case of equation (1.7) we may exploit the extra property that our equation is invariant under the group of scaling transformations $\mathcal{T}_k$ which associates to any solution of (1.7) another solution $\tilde u=\mathcal{T}_k u$ defined for any $k>0$ by

$$(\mathcal{T}_k u)(x,t)=k^\alpha u(k^\beta x,kt), \quad \text{with} \quad \alpha=\frac{1}{q-1}, \quad \beta=\frac{q-p+1}{q(p-1)}. \tag{5.1}$$

Moreover, $\mathcal{T}_k$ transforms singular solutions into singular solutions. Since the $\mathcal{T}_k$ also preserve order, it then follows that both the minimal and the maximal solution have to be invariant under these transformations. This has as an immediate consequence that they are selfsimilar.

LEMMA 5.2. *The maximal and minimal VSS of equation (1.7) can be written in the form*

$$V(x,t)=t^{-\alpha}F(\eta), \qquad v(x,t)=t^{-\alpha}f(\eta) \tag{5.2}$$

*where* $\eta=|x|t^{-\beta}$.

PROOF: The $t$-dependence is just a consequence of the group invariance mentioned above. Just write the equation $\mathcal{T}_k u(x,t)=u(x,t)$ and set $t=1$ and $k^\beta x=y$. We thus get

$$u(y,k)=k^{-\alpha}u(k^{-\beta}y,1).$$

which implies (5.2). The fact that $F$ and $f$ depend only on $|x|$ follows from the remark at the end of the last section. #

Therefore, both $F$ and $f$ are solutions of the ODE problem

$$(|f'|^{p-2}f')' + \frac{N-1}{\eta}|f'|^{p-2}f' + \beta\eta f' + \alpha f - f^q = 0 \tag{5.3}$$

with conditions:

$$f \geq 0 \quad \text{on } [0,\infty), \quad f'(0) = 0 \quad \text{and } f \text{ has compact support.} \tag{5.4}$$

**First proof.** Uniqueness of solutions of problem (5.3),(5.4) has been proved by Díaz and Saa [DS] by ODE methods. This means that $F = f$, hence $V = v$ and all VSS coincide. #

**Second proof.** We will give below another proof which uses PDE techniques and is based on the Maximum Principle and a trick of time delay. This kind of technique has been useful in proving uniqueness of singular solutions for different nonlinear parabolic equations, see for instance [KV] and [KVe]. It is therefore of interest to show that it can also be adapted to the present situation.

We need some properties of the profiles $f$ and $F$.

LEMMA 5.3. *Let $f$ be a solution of problem (5.3), (5.4). Then $f$ is strictly decreasing and $C^1$-smooth inside its support $[0, R]$. Moreover, we have the interface condition*

$$\lim_{\eta \to R-} \frac{|f'(\eta)|^{p-1}}{f(\eta)} = R\beta\,.$$

PROOF: These properties follow rather easily from (5.3) under the assumed conditions. See [DS] or [PW]. See also a similar result in [KV] for equation (1.5). #

We will also use the following transformation. For a solution $u$ of (1.7) and a $\lambda > 0$ we define $\mathcal{S}_\lambda u = u_\lambda$ by

$$u_\lambda(x,t) = \lambda u(\lambda^{-\delta}x, t) \qquad \text{with } \delta = \frac{p-2}{p} \tag{5.5}$$

LEMMA 5.4. *If $u$ is a nonnegative solution of (1.7) in $Q$ then for every $\lambda > 1$ $u_\lambda$ defined by (5.5) is a supersolution of the equation in the same domain.*

PROOF: We have

$$\partial_t u_\lambda - \Delta_p(u_\lambda) + (u_\lambda)^q = \lambda(\lambda^{q-1} - 1)u^q \geq 0\,. \tag{5.6}$$

Let us now proceed with the proof that $F = f$. Suppose they are not equal, i.e. $v(x,t) \neq V(x,t)$. Then the function $v_\lambda$ defined according to the transformation (5.5) will be supersolution of (1.7) and moreover for $\lambda$ large enough it will be larger than $V$. In terms of the profiles $f$ and $F$ this means that

$$f_\lambda(\eta) = \lambda f(\lambda^{-\delta}\eta) \geq F(\eta) \tag{5.7}$$

for every $\eta \in \mathbf{R}^N$. Now we define

$$l = \min\{\lambda > 1 : \text{ (5.7) holds }\} \tag{5.8}$$

Clearly $l \geq 1$. The uniqueness proof is reduced to showing that $l$ is not greater than 1. This will be a consequence of the following result

LEMMA 5.5. *Suppose that (5.7) holds for some $l > 1$. Then there exist $\varepsilon > 0$, $t_1 > 1$ and $\tau \geq 0$ such that for $t \geq t_1$*

$$V(x,t) \leq v_{l-\varepsilon}(x, t+\tau)\,. \tag{5.9}$$

PROOF: We first observe that $v_l$ is a supersolution of (1.7) such that $v_l \geq V$ at $t = 1$. By the Maximum Principle we will have $v_l(x,t) \geq V(x,t)$ for every $x \in \mathbf{R}^N$ and $t > 1$. Let $B_R(0)$ be the support of $F$, i.e. the support of $V$ at $t = 1$.

Take now some $t_1 > 1$. We would like to make sure that not only $v_l(\cdot, t_1) \geq V(\cdot, t_1)$, but also that both functions are *strictly separated* in the sense that they have strictly different supports and that inside the support of $V$ they do not touch. In order to obtain this situation we will need to introduce a time delay. Once the functions are so separated we can slightly reduce the factor $l$ and still have the same inequality, thus proving (5.9). The analysis is done in three steps.

(i) We begin by studying the situation inside the support of $V$. Let $R$ be the radius of the support of $F$. The interface of $V$ is then given by $|x| = R(t) \equiv Rt_1^{\beta}$. In the region $\{0 < |x| < R(t)\}$ both equations (for $v_\ell$ and $V$) are not degenerate by virtue of the Lemma 5.3. In fact, both $v_l$ and $V$ are $C^1$ functions inside the support of $V$ unless possibly at $x = 0$ and both are strictly decreasing in $r = |x|$. The Strong Maximum Principle implies then that in case they would touch at a point $(x,t)$ with $0 < |x| < Rt^{\beta}$, then they should be identical for all $t \leq t_1$. Now, $v_l$ is a strict supersolution for $|x| < R(t)$ while $V$ is an exact solution, hence this statement cannot hold.

(ii) We check now the possibility of contact at $x = 0$, $t > 1$. To eliminate this possibility we take a small $a > 0$ and consider for $t > 1$ the function

$$W(x,t) = v_l(x,t) - a(t-1)\,. \tag{5.10}$$

It is easy to check that there exists $\mu > 0$ such that $W$ is still a supersolution of (1.7) in a neighbourhood of $x = 0$ for $0 < t - 1 < \mu$. Besides, it follows from (i) above that $W(x,t) \geq V(x,t)$ if $|x| = R/2$ and $0 < t - 1 < \mu$ provided again that $\mu$ is small enough. Finally $W \geq V$ at $t = 1$. Using again the Maximum Principle we conclude that $W(x,t) \geq V(x,t)$ for $(x,t) \in B_{R/2}(0) \times (1, 1+\mu)$. Therefore, for $1 < t < 1 + \mu$

$$v_l(0,t) \geq a(t-1) + V(0,t) > V(0,t) \tag{5.11}$$

Take $t_1 = 1 + \mu$.

(iii) The last step is the separation of the boundaries of the supports. It could happen that the graphs of $v_l$ and $V$ do not meet at $|x| = R$ for $t = t_1$. Then we would be allowed to replace $v_l(x,t_1)$ by $v_{l-\varepsilon}(x,t_1)$ and the inequality will be preserved. i.e. $v_{l-\varepsilon}(x,t_1) \geq V(x,t_1)$, which proves (5.9) with $\tau = 0$.

Assume now that we are in the less fortunate situation where $v_l$ and $V$ touch at the boundary for $t = t_1$. Then we replace $v_l$ by

$$\tilde{W}(x,t) = v_{l-\varepsilon}(x, t+\tau)$$

and compare $\tilde{W}$ with $V$ at $t = t_1$. Since $v_l$ and $V$ are strictly separated inside $B_R(0)$ at $t = t_1$ the same will happen with $\tilde{W}$ and $V$ in a smaller ball $B_{R'}$, $R' < R$, if $\tau$ is small enough. In order to assert the separation of $\tilde{W}$ and $V$ near $|x| = R$ we use the behaviour of the solutions near the boundary given by Lemma 5.3 .

LEMMA 5.6. *For $t \approx 1$ and $x \approx R$ we have*

$$v(x,t) < v(x,t+\tau)$$

PROOF: We have from (5.2)

$$v_t = t^{-(\alpha+1)}\{\beta\eta|f'(\eta)| - \alpha f(\eta)\} \tag{5.12}$$

where $\eta = xt^{-\beta}$, and this quantity will be positive for $R - \varepsilon < \eta < R$ thanks to Lemma 5.3. #

END OF SECOND PROOF OF THEOREM 5.1: Uniqueness is now an immediate consequence of the form of the solutions and Lemma 5.5. Indeed, formula (5.9) means that for every $x \in \mathbf{R}^N$ and $t \geq t_1$

$$(t+\tau)^{-\alpha}(l-\varepsilon)f(\frac{x}{(t+\tau)^{\beta}(l-\varepsilon)^{\delta}}) \geq t^{-\alpha}F(\frac{x}{t^{\beta}}) \tag{5.13}$$

Put now $y = xt^{-\beta}$. Then (5.13) can be written as

$$(l-\varepsilon)f((1+\frac{\tau}{t})^{-\beta}\frac{y}{(l-\varepsilon)^{\delta}}) \geq (1+\frac{\tau}{t})^{\alpha}F(y)$$

if $t \geq t_1$. We may let $t \to \infty$ to get

$$(l-\varepsilon)f(\frac{y}{(l-\varepsilon)^{\delta}}) \geq F(y)\,. \tag{5.14}$$

This means that $l$ was not the minimum, contrary to the definition (5.8), hence the assumption that $l > 1$ cannot hold and $V = v$. #

## 6. Comparison of singular solutions.

We want to relate the various singular solutions described above for equation (E) when we consider different absorption functions $\phi$. We are particularly interested in considering the relation between singular solutions when the absorption functions are related by an inequality of the form

$$\phi_2(u) \leq a\phi_1(u) + b\,, \tag{6.1}$$

valid for every $u > 0$ and some $a > 0$ and $b \geq 0$.

We treat first the case where $a = 1$.

LEMMA 6.1. *Let (H1) hold for $\phi_i$, $i = 1,2$ and assume that $\phi_2(u) \leq \phi_1(u) + b$ for some $b > 0$ and all $u > 0$. If $u_1$ and $u_2$ are the respective fundamental solutions with a certain mass $c > 0$, then we have*

$$u_1(x,t) \leq u_2(x,t) + bt\,. \tag{6.2}$$

*A similar comparison holds for maximal and minimal VSS under the additional hypotheses (H2), (H3).*

PROOF: **(i)** Let us begin with the fundamental solutions. We observe that, if $u_2$ is a fundamental solution for $E_{\varphi_2}$, the function

$$W(x,t) = u_2(x,t) + bt$$

is a supersolution for $(E_{\phi_1})$ since

$$\begin{aligned} &W_t - \Delta_p(W) + \phi_1(W) \geq u_{2,t} + b - \Delta_p(u_2) + \phi_1(u_2) \geq \\ &u_{2,t} - \Delta_p(u_2) + \phi_2(u_2) = 0 \end{aligned} \tag{6.3}$$

(note that $\phi(u)$ is nondecreasing). Therefore, we may replace $E_c$ by $W$ in the construction of the fundamental solution for the equation with $\phi_1$ done in Lemma 2.3 and thus obtain a fundamental solution $u_1 \leq W$ with the same initial mass as $W$ and $u_2$.

**(ii)** The result for minimal VSS is now a consequence of formula (4.3), which gives the VSS as the limit of fundamental solutions.

**(iii)** Finally, for maximal VSS we argue as follows. In case $u_1$ is the maximal VSS for $\phi_1$, then the function

$$Z(x,t) = \max\{0, u_1(x,t) - bt\} \tag{6.4}$$

is a subsolution of the equation with $\phi_2$ (use a computation similar to (6.3) and the general fact that the maximum of two subsolutions is again a subsolution). We have observed in the remark after Lemma 4.1 that the maximal VSS can be obtained as the sup over all the singular subsolutions of the equation, therefore we get $u_2(x,t) \geq Z(x,t)$. This ends the proof. #

In order to treat absorption functions related by $\phi_2(u) = a\phi_1(u)$, $a > 0$, we simply have to observe that the transformation $u \to \mathcal{R}_a u$ given by

$$(\mathcal{R}_a u)(x,t) = u(a^{1/p}x, at) \tag{6.5}$$

establishes a one-to-one correspondence between solutions of equation $(E_\phi)$ and solutions with $E_{a\phi}$. Moreover, this correspondence maps a fundamental solution of the first equation with mass $c$ into a fundamental solution of the second with mass $a^{-N/p}c$. Finally, the correspondence maps a maximal (or minimal or arbitrary) VSS into the same type of VSS for the other equation.

Combining both results we obtain comparison for solutions corresponding to absorption functions related by (6.1). In particular, the following result will be needed in the next section

COROLLARY 6.2. *Let $\phi_i$, $i = 1,2$ be two absorption functions satisfying (H1)-(H3) and related by* (6.1) *and let $V_{\phi_i}$ be the corresponding maximal VSS. Then*

$$V_{\phi_1}(a^{1/p}x, at) \leq V_{\phi_2}(x,t) + bt\,. \tag{6.6}$$

*The same holds for minimal VSS.*

## 7. Uniqueness for more general $\phi$'s.

We are now ready to prove our more general uniqueness result.

THEOREM 7.1. *Let $\phi_0$ be an absorption function which satisfies (H1)-(H3) and for which the VSS is unique and let $\phi$ be another absorption function such that*

$$\lim_{u\to\infty} \frac{\phi(u)}{\phi_0(u)} = a \tag{7.1}$$

*for some $a > 0$. Let us also assume that*

$$\phi(\lambda u) \geq \lambda\phi(u) \tag{7.2}$$

*for every $u > 0$ and $\lambda > 1$. Then equation $(E_\phi)$ has exactly one VSS.*

PROOF: We begin by observing that the case $a \neq 1$ can be reduced to the case $a = 1$ by means of the transformation (6.5), replacing $\phi_0$ by $a\phi_0$. Henceforth, we take $a = 1$. It then follows from (7.1) that there for every $\varepsilon > 0$ there exists $b > 0$ such that

$$(1+\varepsilon)^{-1}\phi_0(u) - b \leq \phi(u) \leq (1+\varepsilon)\,\phi_0(u) + b \tag{7.3}$$

It is clear that $\phi$ also satisfies conditions (H1)-(H3), hence there exist a minimal and a maximal VSS which we denote by $V_1$ and $V_2$. Let $V$ be the unique VSS of $(E_{\phi_0})$. If we apply the comparison results of Section 6 for maximal VSS we obtain, thanks to the left-hand side of (7.3),

$$V_2((1+\varepsilon)^{1/p}x, (1+\varepsilon)t) \leq V(x,t) + b(1+\varepsilon)t\,, \tag{7.4.a}$$

while for minimal VSS we get from the right-hand side of (7.3)

$$V_1(x,t) \geq V((1+\varepsilon)^{1/p}\,x, (1+\varepsilon)\,t) - bt\,. \tag{7.4.b}$$

Therefore,

$$V_1(x,t) \leq V_2(x,t) \leq V_1(\frac{x}{(1+\varepsilon)^{2/p}}, \frac{t}{(1+\varepsilon)^2}) + 2bt\,. \tag{7.5}$$

Now we observe that for $\lambda > 1$ and $c > 0$ the function

$$W(x,t) = \mathcal{S}_\lambda V_1(x,t) + c$$

where $\mathcal{S}_\lambda$ is defined as in (5.5), is a supersolution of $(E_\phi)$. This fact is verified with a computation like (5.6) using property (7.2). Let us take some $t_1 > 0$ and put $c = 2bt_1$ and $\lambda = (1+\varepsilon)^{2/(p-2)}$. We may deduce from (7.5) that

$$V_2(x,t_1) \leq \lambda V_1(\frac{x}{\lambda^\delta}, \frac{t_1}{(1+\varepsilon)^2}) + c = \mathcal{S}_\lambda V_1(x, t_1 - \tau) + c \tag{7.6}$$

with $\tau = t_1 - t_1/(1+\varepsilon)^2 \in (0, t_1)$. By the Maximum Principle we conclude that the inequality holds with $t_1$ replaced by $t$ if $t \geq t_1$. Keeping $\varepsilon$ fixed and letting $t_1 \to 0$ we get in the limit

$$V_2(x,t) \leq \mathcal{S}_\lambda V_1(x,t)\,. \tag{7.7}$$

since $\tau \to 0$. This estimate being valid for every $\varepsilon > 0$, i.e. for every $\lambda > 1$ we get in the limit $\lambda \to 1$, $V_2(x,t) \leq V_1(x,t)$ which ends the proof. #

In view of the uniqueness result of Section 5 for power-like absorption functions, Theorem A of the Introduction is a consequence of Theorem 7.1.

## 8. VSS and asymptotic behaviour.

In this section we exhibit an interesting application of the uniqueness of VSS, namely to describe the asymptotic behaviour of the class of solutions of equation (1.7) whose initial data are compactly supported. In fact, in the parameter range $p-1<q<p_* \equiv p-1+p/N$ all such solutions will be approximately described by the unique VSS.

THEOREM 8.1. *Let $V(x,t) = t^{-\alpha}F(xt^{-\beta})$ be the VSS of equation (1.7) in the range $p-1<q<p_*$ and let $u(x,t)$ be any solution of (1.7) whose initial data have compact support. Then, as $t \to \infty$ we have*

$$\lim t^{\alpha}|u(x,t)-V(x,t)| = \lim |t^{\alpha}u(x,t)-F(xt^{-\beta})| = 0 \tag{8.1}$$

*uniformly in $x \in \mathbf{R}^N$.*

PROOF: (i) Let us assume moreover that $u_0$ is continuous with $u_0(x) \geq V(x,\tau)$ for some $\tau > 0$. We apply to $u$ the transformation $\mathcal{T}_k$ of (5.1) and consider for $k=1,2,\cdots$ the sequence

$$u_k(x,t) = k^{\alpha}u(k^{\beta}x, kt) \tag{8.2}$$

Certain estimates apply uniformly to the $u_k$. Thus, we have from (3.3)

$$u_k(x,t) \leq Ct^{-\alpha}\,. \tag{8.3}$$

On the other hand, if the support of $u_0$ is contained in the ball of radius $R$ around 0, then the support of $u_k(x,0)$ is contained in the ball of radius $Rk^{-\beta}$. A uniform estimate of the support of $u_k(\cdot,t)$ for $t>0$ is provided by the following Lemma.

LEMMA 8.2. *Let $q > p-1$. If $u$ is solution of (1.7) with initial support in the ball of radius $R$ then there exists $C = C(p,q,N)$ such that the support of $u(\cdot,t)$ is contained in the ball of radius $R = Ct^{\beta}$.*

PROOF: For $t=1$ it is a consequence of the property of finite propagation as in Lemma 3.2. For $t \neq 1$ we use the transformation $\mathcal{T}_k$ to reduce ourselves to the above case. #

We continue with the proof of the Theorem. By known regularity theory the sequence $u_k$ is relatively compact in $C(S)$ for every compact subdomain of $Q$, hence we may extract a subsequence and pass to the limit $k \to \infty$ to obtain a funtion $\tilde{u}$ which is a solution of (1.7) in $Q$. By Lemma (8.2) the support of such limit solution is contained in the set $\{(x,t) \in Q : |x| \leq Ct^{\beta}\}$. Therefore, $\tilde{u}$ is continuous down to $t=0$ for $x \neq 0$ and $\tilde{u}(x,0)=0$ in that case.

Finally, we verify that $\tilde{u}$ is a VSS by observing that since $V$ is invariant under the transformation $\mathcal{T}_k$ we have

$$u_k(x,t) = (T_k u)(x,t) \geq k^{\alpha}V(k^{\beta}x, kt+\tau) = V(x, t+k^{-1}\tau)\,, \tag{8.4}$$

hence in the limit $\tilde{u} \geq V$. This implies that $\tilde{u}$ is a VSS and by uniqueness it must coincide with $V$.

(ii) The assumption of continuity on $u_0$ can be realised without loss of generality for any solution $u$ by simply translating the origin of time to some $t_0 > 0$. By known regularity theory all solutions are continuous in $Q$.

The other assumption, that $u_0(x) \geq V(x,\tau)$ for some $\tau > 0$ is not always satisfied. However, we may choose a $\lambda < 1$ such that, after suitably displacing the $x$-axis so that $u_0(0) > 0$, we have

$$u_0(x) \geq \mathcal{S}_\lambda V(x,\tau) . \tag{8.5}$$

Next we observe that for $\lambda < 1$ the function $\mathcal{S}_\lambda V$ is a subsolution of (1.7) in $Q$. Replacing $V$ by $\mathcal{S}_\lambda V$ in the above argument we conclude as in (8.4) that $\tilde{u} \geq \mathcal{S}_\lambda V$ in $Q$. From this we now conclude that

$$\int \tilde{u}(x,t)\, dx \to \infty \qquad \text{as } t \to 0\,,$$

so that $\tilde{u}$ is a VSS and the argument ends as before with $\tilde{u} = V$. #

For the sake of completeness and comparison we will also review the asymptotic behaviour of the solutions of (1.7) with compactly supported initial data for the whole exponent range $p > 2$, $q > 1$. Such a study has been performed for equation (1.5): $u_t = \Delta u^m - u^q$ with $m, q > 1$; we can say that a main conclusion is that the large-time behaviour of the whole class of solutions considered is intimately connected with the existence and uniqueness of singular solutions and can be described in terms of them. As we shall see, this is also the case for equation (1.7).

The large-time behaviour of solutions with compact support for equation (1.7) is described in the exponent range $q > p_* = p - 1 + (p/N)$ by the fundamental solutions of the purely diffusive equation $u_t = \Delta_p u$. The proof of this result in no way differs from the case $q > p_* = m + (2/N)$ for equation (1.5), cf. [GmV] for $m = 1$, and we will leave it to the reader. The situation behind the result is however worth a comment. In fact, what happens is that the total mass $\int u(x,t)\, dx$ decreases as $t \to \infty$ towards a positive quantity $M_\infty$, and $u(x,t)$ looks for $t \gg 0$ very much like the Barenblatt fundamental solution with that mass, the influence of the absorption term being negligible for large times.

The critical case $q = p_*$ has been recently investigated by Galaktionov and one of the authors [GV] for both equations (1.5) and (1.7). In this case the decay rates of the solutions of the purely diffusive and the purely absorptive equations are the same. It happens that for large times the mass of the solutions to the complete equations (1.5) or (1.7) goes to 0 exactly like $c(\log t)^{-N/2}$ and the solution itself behaves like the corresponding Barenblatt solution subject to a contraction in both $u$ and $x$ which expresses the influence of the absorption term ("contracted Barenblatt profiles").

In the exponent range $p - 1 < q < p_*$ considered above a simple computation shows that the decay rate corresponding to the absorption term becomes greater than the one corresponding to pure diffusion. The asymptotic behaviour of the solutions is therefore strongly influenced by the absorption. This explains why the unique VSS found above expresses the behaviour of all solutions with compact spatial support.

## 9. Nonexistence of VSS.

The existence of VSS of equation (E) has been obtained in the preceding sections under the hypotheses (H1)-(H3) on the absorption function $\phi$. In the case of a power-like nonlinearity, $\phi(u) = u^q$, these conditions are equivalent to $p-1 < q < p_*$. It is interesting to remark that this range is actually optimal, since no VSS exists for equation (1.7) in the cases $q \geq p_*$ or $1 < q \leq p-1$. A similar situation has been described in [KPV] for equation (1.5). In fact, the similarity extends to the proofs, which can be adapted without major changes. The nonexistence of singular solutions when $q \geq p_*$ has been shown in [G].

Nevertheless, we will give here a proof of nonexistence of VSS in the range $1 < q \leq p-1$, since we want to present a direct method of proof based on the analysis of the ODE satisfied by the space profile. This considerably simplifies the method used in [KPV] for equation (1.5) by avoiding all use of the asymptotic behaviour of the solutions. We already know by Theorem 4.2 that in case there exists at least one VSS, then the limit as $c \to \infty$ of the fundamental solutions $u_c(x,t)$ gives exactly the minimal VSS. Therefore, the nonexistence of a VSS for equation (1.7) in the range $q \in (1, p-1]$ is a direct consequence of the following result

THEOREM 9.1. *Let $u_c(x,t)$ be the fundamental solution of equation (1.7) with initial data $c\delta(x)$. If $1 < q \leq p-1$ we have*

$$\lim_{c\to\infty} u_c(x,t) = c_* t^{-\frac{1}{q-1}} \quad \text{with } c_* = (q-1)^{-1/(q-1)}. \tag{9.1}$$

PROOF: This result is the counterpart of Proposition 2.2 of [KPV]. The function $U(t) = c_* t^{-\alpha}$, $\alpha = 1/(q-1)$, is an exact solution of equation (E), with no space dependence, the so-called *flat solution*, and takes infinite initial data. The phenomenon that the fundamental solutions tend to $U$ can thus be interpreted as a controlled explosion at $t = 0$.

The first stages of the proof follow [KPV]. Since $\{u_c\}_{c>0}$ forms an increasing family of solutions of (1.7), each of them being radially symmetric and decreasing as a function of $|x|$, and since $U(t)$ is an upper bound for all of them, we can take the limit

$$V(x,t) = \lim_{c\to\infty} u_c(x,t), \tag{9.2}$$

and $0 \leq V(x,t) \leq U(t)$. Moreover, $V(x,t) \geq u_c(x,t)$ for every $c > 0$. We now repeat the scaling analysis of [KPV] to conclude that $V$ is necessarily self-similar of the form

$$V(x,t) = t^{-\alpha} f(\eta) \quad \text{with} \quad \eta = x t^{-\beta}, \tag{9.3}$$

where $\alpha = 1/(q-1)$ and $\beta = (q-p+1)/(q(p-1))$ as before. Since $q \leq p-1$ we have $\beta \leq 0$. This is important in the sequel.

We have to prove that $f(\eta) \equiv c_*$. We know that $0 \leq f(\eta) \leq c_*$. It is clear that $f$ is also a radially symmetric function, $f = f(r)$, if $r = |\eta|$, and nonincreasing in $r$. We now depart from [KPV] and analyze $f$ by studying the ODE it satisfies, namely (5.3), that we write as

$$\frac{1}{r^{N-1}}(r^{N-1}|f'|^{p-2}f')' = -\beta r f' - f(c_*^{q-1} - f^{q-1}), \tag{9.4}$$

where $'$ denotes differentiation with respect to $r \in (0, \infty)$. Observe that the term $-\beta r f'$ is always nonpositive.

By monotonicity there exists the limit $k = \lim_{r\to\infty} f(r)$ and $0 \le k \le c_*$. Therefore, if we exclude the possibility $0 \le k < c_*$ we will be done. We have to introduce the function

$$g = \frac{p-1}{p-2} |f|^{\frac{-1}{p-1}} f \,,$$

so that $|f'|^{p-2} f'/f = |g'|^{p-2} g'$. Then (9.4) gives

$$r^{1-N}(r^{N-1}|g'|^{p-2}g'f)' \le -f(c_*^{q-1} - f^{q-1}) \,,$$

which can be transformed, after differentiation of the first member and dividing throughout by $f$, into

$$(9.5) \qquad (r^{N-1}|g'|^{p-2}g')' + r^{N-1}|g'|^{p-2}g'\frac{f'}{f} \le -r^{N-1}(c_*^{q-1} - f^{q-1}) \,.$$

We now observe that the second term is positive and can be dropped. Since $k < c_*$ the second member can be estimated from above by $-l\, r^{N-1}$ with $0 < l < c_*^{q-1}$. Integration of the resulting inequality gives at once

$$(9.6) \qquad |g'|^{p-2}g' \le -\frac{l}{N} r \,,$$

for all large $r$, hence $g$ necessarily becomes negative. This contradicts the fact that $f$ is nonnegative. The theorem is proved. #

In fact, the nonexistence of a VSS can be easily generalized to equation (E) with a more general $\phi$. Only the behaviour of $\phi(s)$ for large $s$ matters.

THEOREM 9.2. *Assume that $\phi$ satisfies for $s > 0$*

$$(9.7) \qquad \phi(s) \le cs^{p-1} + d$$

*for some constants $c$ and $d > 0$. Then equation (E) does not admit any VSS.*

PROOF: It is a simple consequence of the comparison result, Lemma 6.1, applied to the fundamental solutions of (E), taking $\phi_1(s) = s^{p-1}$ and $\phi_2$ equal to the present $\phi$. Note that (9.7) implies (H1), so that fundamental solutions exist. On the contrary, (9.7) is incompatible with (H3). We remark that, by Theorem 9.1, the limit of the fundamental solutions for $\phi_1$ when the mass goes to infinity is the flat solution, which takes on infinite initial value everywhere. By comparison the same happens for the limit of the fundamental solutions with $\phi$. Now, this limit is the minimal VSS when there exist very singular solutions. We conclude that they cannot exist. #

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