**EVOLUTION OF NONPARAMETRIC SURFACES WITH
SPEED DEPENDING ON CURVATURE, II.
THE MEAN CURVATURE CASE.**

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Abstract

We consider an evolution which starts as a flow of smooth surfaces in nonparametric form propagating in space with normal speed equal to the mean curvature of the current surface. The boundaries of the surfaces are assumed to remain fixed. G. Huisken has shown that if the boundary of the domain over which this flow is considered satisfies the "mean curvature" condition of H. Jenkins and J. Serrin (that is, the boundary of the domain is convex "in the mean") then the corresponding initial boundary value problem with Dirichlet boundary data and smooth initial data admits a smooth solution for all time. In this paper we consider the case of arbitrary domains with smooth boundaries not necessarily satisfying the condition of Jenkins-Serrin. In this case, even if the flow starts with smooth initial data and homogeneous Dirichlet boundary data, singularities may develop in finite time at the boundary of the domain and the solution will not satisfy the boundary condition. However, we prove existence of solutions that are smooth inside the domain for all time and become smooth up to the boundary after elapsing of a sufficiently long period of time. From that moment on such solutions assume the boundary values in classical sense. We also give sufficient conditions that guarantee existence of smooth solutions for all time $t \geq 0$. In addition, we establish estimates of the rate at which solutions tend to zero as $t \rightarrow \infty$.

EVOLUTION OF NONPARAMETRIC SURFACES WITH SPEED DEPENDING ON CURVATURE, II. THE MEAN CURVATURE CASE.

Vladimir I. Oliker¹ and Nina N. Uraltseva²

1. Introduction

This paper is the second in the series of papers in which we investigate asymptotic properties of graphs propagating with speed controlled by curvature. In the first one [O] the case of the Gauss curvature leading to a parabolic Monge-Ampere equation was investigated. Here we consider the mean curvature case.

Let Ω be a bounded domain in R^n , $n \geq 2$, with C^∞ boundary $\partial\Omega$. Denote by Q the cylinder $\Omega \times (0, \infty)$ and let

$$H(u) := \operatorname{div} \frac{Du}{\sqrt{1 + |Du|^2}},$$

where $Du = \operatorname{grad} u$ and $|Du|^2 = \langle Du, Du \rangle$. Consider the following problem

$$u_t = (1 + |Du|^2)^{1/2} H(u) \text{ in } Q, \quad (u_t = \partial u / \partial t) \quad (1.1)$$

$$u(x, t) = 0 \text{ in } \partial\Omega \times [0, \infty), \quad (1.2)$$

$$u(x, 0) = u_0(x) \text{ in } \overline{\Omega}, \text{ where } u_0 \in C_0^\infty(\overline{\Omega}). \quad (1.3)$$

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The initial-boundary value problem (1.1)-(1.3) describes an evolution of the surface $(x, u(x,t))$, $x \in \overline{\Omega}$, propagating with normal speed equal to the mean curvature of the surface at the moment t . For the case when the boundary $\partial\Omega$ satisfies the condition of H. Jenkins and J. Serrin [JS], that is, the mean curvature of $\partial\Omega$ is nonpositive (relative to the exterior unit normal vector field on $\partial\Omega$) it was shown by G. Huisken [H-1] in 1989 that the problem (1.1)-(1.2) with (1.3) replaced by arbitrary time independent Dirichlet data admits a smooth solution. In addition, he showed that as $t \rightarrow \infty$ the solution tends to the solution of the minimal surface equation.

For arbitrary nonhomogeneous Dirichlet boundary data G. Lieberman [L] has shown that a smooth solution, in general, does not exist. Numerical examples indicate that a smooth solution may not exist even for identically vanishing Dirichlet boundary data; see the fig. on p. 56.

The main purpose of this paper is to investigate the solvability and asymptotic behavior of solutions to (1.1)-(1.3) for arbitrary domains with smooth boundaries not necessarily satisfying the condition of Jenkins and Serrin.

Here we show that for the problem (1.1)-(1.3) it is possible to construct a solution which satisfies (1.2) in some generalized sense. Such a solution may develop singularities at the boundary at some finite time, but later these singularities will disappear, the solution will become smooth up to the boundary, and it will satisfy (1.2) in classical sense. We also give sufficient conditions on the domain and initial data that guarantee solvability of (1.1)-(1.3) in classical sense for all time. In addition, we determine the rate of decay of solutions as $t \rightarrow \infty$. In particular, we show that for large time the appropriately normalized solution of (1.1)-(1.3) approaches exponentially the first eigenfunction (multiplied by a suitable constant) of the Laplace operator with Dirichlet data in Ω . This confirms the conjecture made earlier in [O-1].

Our main results are based on quite delicate local (in x and in t) estimates of the C^1 -norm of solutions to a family of regularized problems associated with (1.1)-(1.3). It is important to note that the known methods for obtaining interior gradient estimates for solutions of the mean curvature equation based on the classical maximum principle do not work when the equation (1.1) is regularized by adding the Laplace

operator multiplied by a small parameter. For the mean curvature operator alone, because of its specific form of degeneracy, this approach works (see Korevaar [K] and Evans-Spruck [ES-1]). However, when the mean curvature operator is considered in combination with the Laplace operator this degeneracy is not present anymore and the technique does not work.

For other related work on evolution of nonparametric hypersurfaces we should point out the paper by A. Lichnerowski and R. Temam [LT] who studied the equation describing evolution of graphs for which the vertical component of the velocity vector is equal to the mean curvature of the current graph. They established existence and uniqueness of pseudosolutions (see section 1 for more details) for the corresponding initial-boundary value problem. In a recent paper by K. Ecker and G. Huisken [EH] the authors study the problem (1.1), (1.3) for entire graphs over $\Omega = \mathbb{R}^n$ and show existence of a smooth solution for all time provided the initial data satisfies Lipschitz condition and grows linearly at infinity.

Compact hypersurfaces without boundaries evolving with speed depending on the mean curvature have been studied by Huisken [H] and more recently by S. Osher and J. Sethian [OS], L. C. Evans and J. Spruck [ES], [ES-1], Y-G. Chen, Y. Giga and S. Goto [CGG], and others.

This paper is organized as follows. In section 2 we formulate the regularized problem and state our main results, including some of the technical estimates. In section 3 we prove the C^0 -estimates. In section 4 we prove the interior C^1 - estimates and the theorem on existence of generalized solutions of (1.1)-(1.3). In section 5 we establish results concerning asymptotic behavior of generalized solutions. Finally, in the last section 6 we give sufficient conditions guaranteeing existence of smooth solutions to (1.1)-(1.3) for all time.

2. Main results.

2.1. It was already mentioned in the introduction that the condition (1.2) may not always be satisfied. In order to account for that we introduce the following

Definition. Let $u(x,t)$ be a function with the properties

$$u \in C^\infty(\Omega \times [0, \infty)); \quad u \in L_\infty([0, \infty); W^{1,1}(\Omega)); \quad (2.1)$$

$$u_t \in L_\infty(\Omega \times [0, \infty)); \quad (2.2)$$

$$\begin{aligned} Lu &:= u_t - \sqrt{1 + |Du|^2} H(u) \\ &\equiv u_t - \Delta u + \frac{u_i u_j}{1 + |Du|^2} u_{ij} = 0 \text{ in } Q, \end{aligned} \quad (2.3)$$

where $u_i := \partial u / \partial x_i$, $u_{ij} := \partial^2 u / \partial x_i \partial x_j$, and the summation convention over repeated indices is in effect here and everywhere in the paper;

$$-\langle Du, v \rangle \left[1 + |Du|^2 \right]^{-1/2} \in \text{Sign}(u) \text{ a.e. on } \partial\Omega \times (0, \infty), \quad (2.4)$$

where v is the exterior unit normal vector field on $\partial\Omega$;

$$u(x, 0) = u_0(x), \quad (2.5)$$

A function with such properties is called a **generalized solution** of (1.1)-(1.3).

Throughout the paper, in addition to the already made assumptions on the smoothness of $\partial\Omega$, we also assume that the initial data u_0 is a C^∞ function with compact support in Ω . This restriction can actually be relaxed by requiring a certain compatibility condition at the boundary, but we prefer the more restrictive condition in order to avoid additional nonessential technicalities.

Conditions (2.1)-(2.3) imply that $-\langle Du, v \rangle (1 + |Du|^2)^{-1/2}$ has a well defined trace on $\partial\Omega \times [0, \infty)$ which belongs to $L_\infty(\partial\Omega \times [0, \infty))$ (cf. [LT]).

The need to replace condition (1.2) by a more relaxed condition (2.4)

arises naturally when dealing with minimal surface-type equations in domains without specific assumptions on the boundary. The condition (2.4) permits occurrence of singularities at the boundary. Such singularities will actually develop for certain domains and initial data. On the figure on p. 56 consecutive stages of a flow over an annulus are shown. The gradient of the function $u(x,t)$ describing the flow becomes infinite at a certain moment along the inner circle. Later this singularity disappears. Numerical computations for this illustration were made for us by Dr. L.D. Prussner. Another example, based on a comparison of $u(x,t)$ with an expanding self-similar torus, and showing the same property was suggested to us by B. White and S. Angenent.

The form of the condition (2.4) above is the same as that in the paper by A. Lichnerowski and R. Temam [LT] where the authors study the evolution

$$u_t = H(u) \quad \text{in } Q,$$

$$u(x,t) = \psi(x,t) \text{ in } \partial\Omega \times [0,\infty), \quad u(x,0) = u_0(x). \quad \text{in } \overline{\Omega}$$

They have shown existence and uniqueness of a pseudosolution for all time and its convergence in L_1 -norm as $t \rightarrow \infty$ to the corresponding steady state solution. In contrast with the equation considered in [LT] the equation (1.1) does not have the divergence structure. However, from the geometric point of view the flow (1.1) is more natural.

Our first result is concerned with existence of generalized solutions.

Theorem A. The problem (1.1)-(1.3) admits a generalized solution.

This theorem is proved by constructing a sequence of smooth functions u^ϵ , $\epsilon > 0$, converging to a generalized solution of (1.1)-(1.3). The functions u^ϵ are solutions to the following regularized problems

$$L^\epsilon u^\epsilon := Lu^\epsilon - \epsilon(1 + |Du^\epsilon|)^{1/2} \Delta u^\epsilon = 0 \quad \text{in } \Omega \times [0,\infty), \quad (2.6)$$

$$u^\epsilon = 0 \quad \text{on } \partial\Omega \times [0, \infty) \quad (2.7)$$

$$u^\epsilon = u_0 \quad \text{on } \Omega \times \{0\}, \quad (2.8)$$

The equation (2.6) is uniformly parabolic and (2.6)-(2.8) admits for each $\epsilon > 0$ a unique C^∞ solution in $\bar{\Omega} \times [0, \infty)$. The passage to the limit as $\epsilon \rightarrow 0$ is based on a series of uniform in ϵ estimates. We begin by stating the C^0 -estimates.

Theorem B. Denote by λ_1 (> 0) the first eigenvalue of the Laplace operator in Ω with homogeneous Dirichlet boundary data. For any $\lambda < \lambda_1$ we have the estimate

$$\sup_{\Omega} |u^\epsilon(x, t)| \leq C_0(\lambda) e^{-\lambda t} \quad \text{for all } t \geq 0, \quad (2.9)$$

where $0 < C_0(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \lambda_1$.

Further, let $d(x) = \text{dist}(x, \partial\Omega)$, $x \in \bar{\Omega}$. For any $\lambda < \lambda_1$ there exist positive constants C_1 and T depending on λ , Ω , and C^0 -norm of u_0 , such that

$$|u^\epsilon(x, t)| \leq C_1(\lambda) d(x) e^{-\lambda t}, \quad (x, t) \in \bar{\Omega} \times [T, \infty). \quad (2.10)$$

Remark. Here, and everywhere below in the paper we do not indicate the dependence of constants C , C_1 , M , M_0 , c_0 , c_1 , ... on the domain Ω and some norm of the initial data u_0 . If the constants depend also on other parameters we will state this dependence. Also, speaking about eigenvalues and eigenfunctions of the Laplace operator, we always refer to the homogeneous Dirichlet boundary data.

The estimates (2.9) and (2.10) are proved by constructing appropriate

C^0 -barriers. A novel feature here is that no assumptions on the geometry of $\partial\Omega$ are made and for large time t the estimate (2.9) improves. However, exactly because no such assumptions are made, these estimates are not sufficient for establishing uniform in ϵ estimates of $|Du^\epsilon|$ in $\bar{\Omega} \times [0, \infty)$ and $|Du^\epsilon|$ may become unbounded at the boundary $\partial\Omega$ at some finite time. At the end of section 3 and in section 6 we present two types of conditions on the domain Ω and initial data sufficient for establishing uniform estimates of $|Du^\epsilon|$ in $\bar{\Omega} \times [0, \infty)$. It will be convenient to defer the exact formulations of these results till the end of this section.

The most involved part of the paper is concerned with establishing local in x and in t C^1 -estimates. These results are given, correspondingly, in Theorems C and C'. Theorem C plays a key role in the proof of existence of generalized solutions.

Theorem C. Let $\Omega' \subset\subset \Omega$ be an arbitrary subdomain of Ω such that $\bar{\Omega}' \subset \Omega$ and $T > 0$. Then there exists a constant $c = c(\text{dist}(\Omega', \partial\Omega), T)$ such that

$$\sup_{\Omega'} |Du^\epsilon(x, t)| \leq c \quad \text{for } t \in [0, T]. \quad (2.11)$$

The proof of this theorem is obtained in several steps.

First we note two estimates following from the equation (2.6):

$$\sup_{\Omega \times [0, \infty)} |u_t^\epsilon(x, t)| \leq M, \quad (2.12)$$

where M is a constant depending on the C^2 -norm of u_0 , and

$$\int_{\Omega} (|Du^\epsilon|^2 + \epsilon |Du^\epsilon|^4) dx \leq c_0 \quad \text{for all } t \geq 0. \quad (2.13)$$

Next we establish a result (Lemma 4.2) which shows, roughly speaking, when an estimate of the C^0 -norm of a function can be obtained

in terms of a known bound on the L_2 -norm of that function. This useful lemma is proved via an iteration procedure. We apply it in combination with above inequalities several times in different circumstances. In particular, it is applied to function $v = (1 + |Du^\epsilon|^2)^{1/2}$ and it is shown that there exists a positive constant $\epsilon_0 = \epsilon_0(T, \text{dist}(\Omega', \partial\Omega), \max_\Omega |Du_0|, M, c_0)$ such that

$$|Du^\epsilon(x, t)| \leq 2\epsilon^{-1} \text{ in } \Omega' \times [0, T] \text{ for all } \epsilon \leq \epsilon_0.$$

This estimate allows to obtain a Sobolev-type inequality on hypersurfaces $S^\epsilon(t) := (x, u^\epsilon(x, t))$, $x \in \Omega'$. Next we consider the function $w = \ln v$ and prove that there exists a constant $c = c(\zeta, T - t_1, M_0, M)$ such that for $t_1 \in [0, T]$ and any $\zeta \in C_0^\infty(\Omega)$

$$\int_{t_1}^T \int_{S^\epsilon(t)} w^2 \zeta^2 dH_n dt \leq c,$$

where $dH_n = v dx$. These two estimates and the Sobolev inequality on hypersurfaces $S^\epsilon(t)$ allow to deduce the estimate (2.11). Theorem A follows from the estimates (2.12), (2.13) and Theorems B and C.

The next result provides a local estimate in t .

Theorem C'. Suppose that for some $t_0 > 0$ the following estimate holds

$$|Du^\epsilon(x, t)| \leq c_1 \quad \text{on } \partial\Omega \times [t_0, \infty). \quad (2.14)$$

Then for any $\sigma > 0$

$$|Du^\epsilon(x, t)| \leq c_2 \quad \text{on } \Omega \times [t_0 + \sigma, \infty), \quad (2.15)$$

where $c_2 = c_2(c_1, \sigma)$.

The proof of this theorem follows the same procedure as the proof of Theorem C. In fact, the outlined above steps in the proof of Theorem C are actually carried out simultaneously in x and in t .

We use the inequality (2.10) in Theorem B and Theorem C' to show that for large time the generalized solution in Theorem A becomes smooth and satisfies the boundary condition (1.2) in the classical sense. Additionally, we determine the rate of decay to zero of the solution as $t \rightarrow \infty$. These results are summarized in the following

Theorem D. Let $u(x,t)$ be the generalized solution of (1.1)-(1.3) obtained as the $\lim_{\epsilon \rightarrow 0} u^\epsilon(x,t)$ in $\Omega \times [0, \infty)$. There exists $\bar{t} \geq 0$ such that

$$u \in C^\infty(\bar{\Omega} \times [\bar{t}, \infty)) \text{ and } u = 0 \text{ on } \partial\Omega \times [\bar{t}, \infty). \quad (2.16)$$

In addition, for any $\lambda < \lambda_1$

$$\max_{\Omega} |Du(x,t)| \leq c(\lambda)e^{-\lambda t}, \quad \text{for } t \geq \bar{t}. \quad (2.17)$$

Furthermore, there exist constants $c > 0$ and $\mu > 0$ depending on u_0 such that

$$\sup_{\Omega} |e^{\lambda_1 t} u(x,t) - c\psi_1(x)| \leq c_1 e^{-\mu t}, \quad \text{for } t \geq \bar{t}, \quad (2.18)$$

where ψ_1 is the first eigenfunction of the Laplace operator in Ω .

The estimate (2.18) shows that for sufficiently large time t the principal part of the generalized solution $u(x,t)$ is no different from the principal part of the solution to the classical heat equation. This estimate also shows that the evolution process (1.1)-(1.3) "picks up" asymptotically, the symmetries of the domain Ω . In particular, if Ω is a ball then it follows from (2.18) that asymptotically $u(x,t)$ becomes radially symmetric (cf. [O]).

Finally, we describe now sufficient conditions on the domain Ω and

initial data u_0 which guarantee that the problem (1.1)-(1.3) admits a unique smooth solution for all time.

First we note that if the C^0 -norm of u_0 is sufficiently small then it can be shown that the constant $C_0(\lambda)$ in (2.9) can be chosen small and, consequently, the inequality (2.10) is valid with $T = 0$. More details on this are given at the end of section 3.

Next, in section 6 we consider the problem (1.1)-(1.3) in starshaped domains. In this case we can establish an estimate similar to (2.10) for all time under explicit geometric hypotheses on the domain and initial data. These hypotheses tie in together the C^0 - norm of the initial data and the mean curvature of the boundary $\partial\Omega$. We state the result as an a priori estimate for smooth solutions of (1.1)-(1.3) assuming, more generally, that in (1.2) $u_0 \in C^\infty(\bar{\Omega})$. Introduce the following notation:

- ν - the (outward) unit normal vector field on $\partial\Omega$,
- $h_+(x)$ - the nonnegative part of the mean curvature of $\partial\Omega$,
- $s(x)$ - the distance from x_0 to $\partial\Omega$ along the ray in direction $(x-x_0)/|x-x_0|$ for $x \neq x_0$, where x_0 is a point in Ω relative to which Ω is starshaped,

$$\Phi(x) = \sqrt{\frac{(n-1)h_+(x)}{\langle x-x_0, \nu(x) \rangle}}, \quad x \in \partial\Omega.$$

Theorem E. Let Ω be a domain starshaped relative to some point $x_0 \in \Omega$, and let $u(x,t)$ be a solution of (1.1)-(1.3) such that u, u_i are in $C(\bar{\Omega} \times [0, \infty))$ and u_t, u_{ij} are in $C(\Omega \times (0, \infty))$. Assume also that in (1.2) $u_0 \in C^\infty(\bar{\Omega})$. Suppose, further, that the following conditions are satisfied:

$$\Phi(x)s(x) < 1, \quad x \in \partial\Omega; \tag{2.19}$$

there exists some $\delta \in (0,1]$ such that

$$\sup_{\Omega} |u(x,0)| < \frac{\delta}{2} \frac{\inf_{\partial\Omega} \sqrt{1-\Phi^2(x)s^2(x)}}{\sup_{\partial\Omega} \Phi(x)}, \quad (2.20)$$

for all $x \in \Omega_\delta = \{x \in \Omega \mid 0 < 1 - |x-x_0|/s(x) < \delta\}$ the inequality

$$|u(x,0)| < \frac{1}{2} \inf_{\partial\Omega} \frac{\sqrt{1-\Phi^2(x)s^2(x)}}{\Phi(x)} \left[1 - \frac{|x-x_0|}{s(x)}\right] \quad (2.21)$$

holds. Then for all $(x,t) \in \bar{\Omega} \times [0,\infty)$

$$|u(x,t)| \leq C \left[1 - \frac{|x-x_0|}{s(x)}\right] e^{-\sigma t}, \quad (2.22)$$

where C and σ are positive constants depending only on the domain Ω and $u(x,0)$.

Remarks. (a) The conditions (2.19)-(2.21) are trivially satisfied if $\sup_{\partial\Omega} \Phi = 0$. The latter is essentially the condition in [JS] required for solvability of the minimal surface equation with arbitrary Dirichlet boundary data. In this case the estimate (2.22) is known (in a slightly different form) [L], [H]. If $u_0(x) (= u(x,0))$ has compact support in Ω then condition (2.21) is automatically satisfied for some suitable δ .

(b) The conditions (2.19)-(2.22) have a clear geometric meaning. The quantity $\Phi(x)s(x)$ is the so called "reduced" mean curvature and it is invariant with respect to homothetic transformations of the domain Ω relative to x_0 . The function $p(x) := \langle x-x_0, \nu(x) \rangle$ for $x \in \partial\Omega$ is the support function of $\partial\Omega$ and represents the distance from x_0 to the tangent plane to $\partial\Omega$ at x . Thus if (2.19) is satisfied then (2.20) will be satisfied for a wide range of δ provided the $\inf_{\partial\Omega} p$ is sufficiently large. This is true,

for example, if Ω contains a ball of a sufficiently large radius.

Combining the estimate (2.22) with (2.10) we obtain an estimate of the gradient similar to (2.10) but for all time; see the Corollary 6.3. In section 6 we also show how the estimate (2.22) implies existence of solutions to (1.1)-(1.3) for all time and the corresponding asymptotic properties.

3. A priori C^0 -estimates.

In this section we establish uniform in ϵ estimates of the C^0 - norm of the solution to (2.6)-(2.8) in $\bar{\Omega} \times [0, \infty)$ and more precise estimates in $\Omega \times [T, \infty)$ for some sufficiently large T . In particular, we prove the Theorem B.

3.1. Lemma. Denote by λ_1 the first eigenvalue of the Laplace operator in Ω . Let $u^\epsilon(x, t)$ be a solution of (2.6)-(2.8). Then there exists $t_1 > 0$ such that for all $(x, t) \in \bar{\Omega} \times [t_1, \infty)$ and any $\lambda < \lambda_1$

$$|u^\epsilon(x, t)| \leq c(\lambda)e^{-\lambda t}, \quad (3.1)$$

where $0 < c(\lambda) < \infty$ for $\lambda < \lambda_1$ and $c(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \lambda_1$.

Proof. It will be convenient to divide the proof into two steps.

3.2. A crude estimate of $u^\epsilon(x, t)$ for all $t \geq 0$. Let $B_R \equiv B_R(x_0)$ be a ball of radius R with center at $x_0 \in \Omega$, containing $\bar{\Omega}$ strictly inside. Let $\psi(x) = m(2R^2 - (1/2)|x - x_0|^2)$ where $m = \sup_{\Omega} |u_0(x)|/R^2$. We have

$$\begin{aligned} \psi_i &= -m(x_i - x_{0i}), \quad |D\psi|^2 = m^2|x - x_0|^2, \quad \Delta\psi = -nm, \\ \sum_{ij} \psi_i \psi_j \psi_{ij} &= -m^3|x - x_0|^2, \end{aligned}$$

where $\psi_i = \partial\psi/\partial x_i$, $\psi_{ij} = \partial^2\psi/\partial x_i \partial x_j$, and Δ is the Laplace operator.

Now for the function $v(x, t) = \psi(x) e^{-\mu t}$, with $\mu = \text{const.} > 0$ to be

chosen later, we obtain

$$\begin{aligned} L^\epsilon v = & \{-\mu(2R^2 - (1/2)|x-x_0|^2) + [1 + m^2|x-x_0|^2 e^{-2\mu t}]^{-1}\} \\ & \times [n + (n-1)m^2|x-x_0|^2 e^{-2\mu t}] \\ & + \epsilon n[1 + m^2|x-x_0|^2 e^{-2\mu t}]^{1/2}\} m e^{-\mu t} > (-2\mu R^2 + n-1)m e^{-\mu t}. \end{aligned}$$

Set

$$\mu = 2(n-1)mR^{-2}. \quad (3.2)$$

Then $L^\epsilon v > 0$ in $\overline{B_R} \times [0, \infty)$. With our choice of the constant m it is clear that $|u_0(x)| \leq v(x, 0)$ in $\overline{\Omega}$. Since $\pm L^\epsilon u^\epsilon = 0 < L^\epsilon v$ in $\overline{\Omega} \times (0, \infty)$, we can apply the classical maximum principle for parabolic equations (for example, as in [LSU], Ch. I) and conclude that

$$|u^\epsilon(x, t)| \leq \psi(x) e^{-\mu t} \text{ in } \overline{\Omega} \times [0, \infty). \quad (3.3)$$

3.3. An improved estimate for large t . Let Ω_s be a tubular neighborhood of the domain Ω at a distance $s > 0$ with s small enough so that $\partial\Omega_s$ is still smooth. Let γ^s and ψ^s be, correspondingly, the first (smallest) eigenvalue and the first eigenfunction of the Laplace operator in Ω_s , that is,

$$\Delta \psi^s + \gamma^s \psi^s = 0 \text{ in } \Omega_s, \quad \psi^s = 0 \text{ on } \partial\Omega_s, \quad \psi^s > 0 \text{ in } \Omega_s.$$

Since $\overline{\Omega} \subset \Omega^s$, the function $\psi^s > 0$ in $\overline{\Omega}$ and we may assume it to be normalized so that $\inf_{\overline{\Omega}} \psi^s = 1$ in $\overline{\Omega}$. Recall also that $\|\psi^s\|^{(2)} < \infty$ where $\|\cdot\|^{(2)}$ denotes the C^2 -norm in $\overline{\Omega}$.

Fix any $\lambda \in (\mu, \gamma^s)$ and consider in $\Omega_s \times [0, \infty)$ the function $w^s(x, t, \eta) = \psi^s(x) e^{-\lambda t + \eta}$, where $\eta > 0$ is a constant to be chosen later. In order to get an estimate of $|u^\epsilon(x, t)|$ we will use again the maximum principle applying it to $w^s \pm u^\epsilon$ and starting at some large moment of time.

Namely, we show that there exists t_1 and η_1 such that $|u^\epsilon(x, t_1)| \leq w^S(x, t_1, \eta_1)$ on $\bar{\Omega}$, and $L^\epsilon w^S \geq 0$ in $\Omega \times [t_1, \infty)$. This and the inequality $|u^\epsilon(x, t)| < w^S(x, t, \eta_1)$ on $\partial\Omega \times [0, \infty)$ imply the needed estimate.

We have

$$\begin{aligned} L^\epsilon w^S = & \{[-\lambda + \gamma^S]\psi^S + [1 + |D\psi^S|^2 e^{2(-\lambda t + \eta)}]^{-1} \psi^S_i \psi^S_j \psi^S_{ij} \\ & \times e^{2(-\lambda t + \eta)} + \epsilon[1 + |D\psi^S|^2 e^{2(-\lambda t + \eta)}]^{-1} \gamma^S \psi^S\} e^{-\lambda t + \eta}, \end{aligned}$$

where $\psi^S_i = \partial\psi^S/\partial x_i$. Choose now A so that in $\bar{\Omega}$

$$-\lambda + \gamma^S - \max_{\bar{\Omega}} |\psi^S_i \psi^S_j \psi^S_{ij}| e^{-2A} \geq 0. \quad (3.4)$$

Since $\gamma^S > \lambda$, this is possible if A is sufficiently large.

Rewrite (3.3) as $|u^\epsilon(x, t)| \leq e^{-\mu t + A'}$ for $A' = \ln \sup_{\bar{\Omega}} \psi(x)$ and select t_1 and η_1 so that simultaneously $-\lambda t_1 + \eta_1 \leq -A$ and $-\mu t_1 + A' \leq -\lambda t_1 + \eta_1$. For example, we may take $t_1 = (A + A')/\mu$ and $\eta_1 = [(\lambda - \mu)A + \lambda A']/\mu$. Then, recalling that $\psi^S(x) \geq 1$ in $\bar{\Omega}$, we conclude that for all $x \in \bar{\Omega}$

$$|u^\epsilon(x, t_1)| \leq e^{-\mu t_1 + A'} \leq w^S(x, t_1, \eta_1).$$

For $t \geq t_1$ we have $-\lambda t + \eta_1 \leq -\lambda t_1 + \eta_1 \leq -A$ and taking into account the expression for $L^\epsilon w^S$ and inequality (3.4) we see that $L^\epsilon w^S \geq 0$ in $\Omega \times [t_1, \infty)$. Obviously, $|u^\epsilon(x, t)| = 0 < w^S(x, t, \eta_1)$ on $\partial\Omega \times [t_1, \infty)$. We apply now the maximum principle and conclude that in $\bar{\Omega} \times [t_1, \infty)$

$$|u^\epsilon(x,t)| \leq w^s(x,t,\eta_1).$$

Next we note that the choice of A and, consequently of t_1 and η_1 depends on the choice of s . In particular, $\eta_1 \rightarrow \infty$ as $s \rightarrow 0$. On the other hand, since $\Omega_s \supset \bar{\Omega}$, the eigenvalue $\gamma^s < \lambda_1$ and $\gamma^s \rightarrow \lambda_1$ as $s \rightarrow 0$. Thus, for any $\lambda < \lambda_1$ we can choose $s > 0$ and correspondingly $\gamma^s \in (\lambda, \lambda_1)$ so that

$$|u^\epsilon(x,t)| \leq ce^{-\lambda t} \text{ in } \bar{\Omega} \times [t_1, \infty), \quad (3.5)$$

where $c = c(\sup_{\Omega} \psi^s, \sup_{\Omega} u_0(x), \text{diam}(\Omega))$. Note that $0 < c < \infty$ for $\gamma^s \in (\lambda, \lambda_1)$. When $\lambda \rightarrow \lambda_1$ then $s \rightarrow 0$ and $c \rightarrow \infty$. In order to indicate this dependence on λ we write $c = c(\lambda)$ as in (3.1). The lemma is proved.

3.4. Proof of Theorem B. Put

$$C_0(\lambda) = \max \{2e^{\lambda t_1} \sup_{\Omega} |u_0(x)|, c(\lambda)\}.$$

The estimate (2.9) follows from the estimates (3.3) and (3.1).

Now we prove (2.10). For that we construct, again, a suitable barrier. Let $\lambda \in (0, \lambda_1)$ and $c \equiv C_0(\lambda)$. Denote by $\Omega_\delta = \{x \in \Omega \mid d(x) < \delta\}$, where δ is small enough so that $d \in C^2(\bar{\Omega}_\delta)$. Put $f(d(x)) = (1/k)(1 - e^{-kd(x)})$, where $d \in [0, \delta]$ and $k > 0$ is a constant to be chosen later, $g(t) = [(T-t)_+]^2$, where $(\)_+$ denotes the nonnegative part of the expression. Note that

$$f(0) = 0, \quad 0 < f' \leq 1, \quad f'' < 0, \quad (3.6)$$

where $'$ and $''$ denote differentiations with respect to d .

We will show that for suitable $k > 0$, $\delta > 0$, $c_2 \geq c$, and $T \geq t_1 + 1$, where t_1 is as in Lemma 3.1, the function

$$w(x,t) = c_2[f(d(x)) + g(t)] e^{-\lambda t}$$

is the required barrier in $\Omega_\delta \times [T-1, \infty)$.

Note that $w(x,t) \geq 0 = |u^\epsilon(x,t)|$ on $\partial\Omega \times [0, \infty)$ and, because of (3.5),

$$w(x, T-1) \geq c_2 e^{-\lambda(T-1)} \geq c e^{-\lambda(T-1)} \geq |u^\epsilon(x, T-1)| \text{ in } \Omega_\delta. \quad (3.7)$$

We want now to choose k , T , δ , and c_2 so that

$$c_2 \delta e^{-k\delta} \geq c \quad (3.8)$$

and

$$L^\epsilon w \geq 0 \text{ in } \Omega_\delta \times (T-1, \infty). \quad (3.9)$$

Since $f(\delta) \geq \delta e^{-k\delta}$, the inequalities (3.8) and (3.5) will imply that $w(x,t) \geq c e^{-\lambda t} \geq |u^\epsilon(x,t)|$ for $t \geq t_1$ and $x \in \Omega$ such that $d(x) = \delta$. This together with (3.7) and (3.9) will allow to apply the maximum principle and conclude that $|u^\epsilon(x,t)| \leq w(x,t)$ in $\overline{\Omega}_\delta \times [T-1, \infty)$. Consequently,

$$u(x,t) \leq c_2 d(x) e^{-\lambda t} \text{ in } \overline{\Omega}_\delta \times [T, \infty), \quad (3.10)$$

which together with (2.9) gives the inequality (2.10).

We have

$$\begin{aligned} L^\epsilon w \geq c_2 \{ & [1 + f' 2 c_2^2 e^{-2\lambda t}]^{-1} [-f'' \\ & - f'(1 + f' 2 c_2^2 e^{-2\lambda t}) \Delta d] + g_t - \lambda(f+g) \} e^{-\lambda t}, \end{aligned}$$

where we have used the notation $g_t = \partial g / \partial t$ and the facts that

$$|Dd| = 1 \text{ and } \sum_{ij} d_i d_j d_{ij} = 0 \text{ with } d_i = \partial d / \partial x_i \text{ and } d_{ij} = \partial^2 d / \partial x_i \partial x_j.$$

Evidently, if

$$c_2 e^{-\lambda(T-1)} \leq 1 \quad (3.11)$$

then $L^\epsilon w \geq c_2 \{2^{-1}(k - 2|\Delta d|)e^{-kd} - 2 - \lambda(1+\delta)\}e^{-\lambda t}$ in $\bar{\Omega} \times [T-1, \infty)$ provided $k - 2|\Delta d| \geq 0$.

It is known that $|\Delta d(x)| < \infty$ if $d(x) < \delta_0 = \sup_{\partial\Omega} |1/\kappa_N(x)|$ where $\kappa_N(x)$ is the maximal normal curvature at the point $x \in \partial\Omega$; (see [5], p. 421). Denote by $p = \sup_{\bar{\Omega}} |\Delta d|$ and choose k , δ and c_2 so that $2^{-1}(k - 2p)e^{-1} - 2 - \lambda(1+\delta_0) \geq 0$, $\delta = \min\{\delta_0/2, 1/k\}$, and $c_2 = c\delta^{-1}e^{k\delta}$. Finally, we choose T so that (3.11) is satisfied. Then the inequalities (3.8), (3.9) hold and the estimate (3.10) follows from the maximum principle. As explained earlier this implies (2.10). The Theorem B is proved.

3.5. Remark. It can be seen from the proof of Theorem B that if the constant $C_0(\lambda)$ in the inequality (2.9) is sufficiently small then the inequality (2.10) holds with $T = 0$. On the other hand, it is clear from the proof of Lemma 3.1 that $C_0(\lambda)$ will be small if the initial data $u_0(x)$ is sufficiently small. In such circumstances (2.10) can be used to obtain a uniform bound of $|Du^\epsilon(x, t)|$ in $\bar{\Omega} \times [0, \infty)$ (cf. subsection 6.5). Then $u = \lim_{\epsilon \rightarrow 0} u^\epsilon$ is well defined and will be a smooth solution of (1.1)-(1.3) for all time. Thus, if the initial data is sufficiently small the problem (1.1)-(1.3) admits a smooth (and unique) solution for all time. In this case the proof of (2.10) with $T = 0$ amounts to rechecking the proofs of Lemma (3.1) and Theorem B. Since this result will not be used below we omit the exact formulation.

4. Local gradient estimates and proofs of Theorems C, C', and A.

4.1. The proofs of Theorems C and C' are based on several estimates which we formulate as separate lemmas. The next lemma strengthens the results in [LSU], Ch. 2, § 6.

Let $B(\rho) \equiv B(\rho, x_0)$ be a ball of radius ρ with center $x_0 \in \Omega$, $t_0 \geq 0$, $\sigma \geq 0$, and T such that $t_0 + \sigma < T$. Denote by $G(\rho, \sigma)$ a set of nonnegative functions ζ with properties:

$$\begin{aligned}\zeta(x, t) &= \omega(x)\chi(t), & (x, t) \in \bar{B}(\rho) \times [t_0, T], \\ \omega &\in \text{Lip}(B(\rho)), \chi \in \text{Lip}[t_0, T], \\ \omega &\equiv 1 \text{ on } B(\rho/2, x_0) \text{ and } \omega = 0 \text{ on } \partial B(\rho), \\ \chi &\equiv 1 \text{ on } [t_0 + \sigma, T] \text{ and if } \sigma > 0 \text{ then } \chi = 0 \text{ for } t = t_0.\end{aligned}$$

4.2. Lemma. Let z be a smooth function in $\tilde{Q} = (\bar{B}(\rho) \cap \bar{\Omega}) \times [t_0, T]$, $f \in L_\infty(\tilde{Q})$, and k^* a nonnegative constant such that

$$z(x, t) \leq k^* \text{ on } (B(\rho) \cap \partial\Omega) \times [t_0, T].$$

(If $\partial\Omega \cap B(\rho) = \emptyset$ this condition is void.) Suppose there exists $k_0 \geq \max \{ k^*, \text{ess sup}_{\tilde{Q}} |f| \}$ such that the inequalities

$$\begin{aligned}& \sup_{[t_0, T] A_k(t)} \int \psi(z, k) \zeta^2 dx + \int_{t_0}^T \int_{A_k(t)} |Dz|^2 \zeta^2 dx dt \\ & \leq \int_{t_0}^T \int_{A_k(t)} \{ \psi[(z-k)^2 (|D\zeta|^2 + \zeta^2) + \psi(z, k) |\zeta \zeta_t|] + f^2 \zeta^2 \} dx dt \quad (4.1)\end{aligned}$$

hold for any $k \in [k_0, 2k_0]$ and $\zeta \in G(\rho, \sigma)$. Here $A_k(t) = \{x \in B(\rho) \cap \Omega \mid z(x, t) > k\}$, $\zeta_t = \partial\zeta/\partial t$, ψ a Lipschitz function satisfying the conditions

$$\left. \begin{aligned} \gamma_1(z-k)^2 \leq \psi(z,k) \text{ for } z \in [k, 2k], \\ \partial\psi/\partial z \geq 0, \quad \psi(z,k) \leq \gamma_2(z-k)^2 \text{ for } z \geq k \geq 0, \end{aligned} \right\} \quad (4.2)$$

and $\gamma, \gamma_1, \gamma_2$ are positive constants.

(i) If $\sigma > 0$ and

$$\|(z-k_0)_+\|_{2,\tilde{Q}} \leq \Theta k_0 \quad (4.3)$$

with some positive $\Theta = \Theta(\gamma, \gamma_1, \gamma_2, \rho, \sigma)$ ($\|\cdot\|_{2,\tilde{Q}}$ denotes the $L_2(\tilde{Q})$ norm) then

$$z(x,t) \leq 2k_0 \text{ in } B(\rho/2) \times [t_0 + \sigma, T]. \quad (4.4)$$

(ii) If $\sigma = 0$ and (4.3) holds with some positive $\Theta = \Theta(\gamma, \gamma_1, \gamma_2, \rho)$ then the estimate (4.4) is also true with $\sigma = 0$.

Before we proceed with the proof of the lemma we point out at a special case which will be used in the proofs of Theorems C, C' and D.

4.2.1. Corollary. If in Lemma 4.2 it is assumed that inequality (4.1) holds for any $k \geq k^*$ and $\xi \in G(\rho, \sigma)$ then we have:

(i) if $\sigma > 0$ then in $B(\rho/2, x_0) \times [t_0 + \sigma, T]$

$$z(x,t) \leq 2\max\{k^*, \text{ess sup}_{\tilde{Q}} |f|, c\|(z-k_0)_+\|_{2,\tilde{Q}}\}, \quad (4.5)$$

where $c = c(\gamma, \gamma_1, \gamma_2, \rho, \sigma)$;

(ii) if $\sigma = 0$ then (4.5) holds in $B(\rho/2, x_0) \times [t_0, T]$ with $c = c(\gamma, \gamma_1, \gamma_2, \rho)$.

The corollary follows easily from Lemma 4.2 if we choose

$k_0 = \max\{k^*, \text{ess sup}_{\tilde{Q}} |f|, c\|(z-k_0)_+\|_{2,\tilde{Q}}\}$ and $c = \Theta^{-1}$ with Θ the

same as in (4.3).

Remark. When $\psi(z,k) = \chi_1(z-k)^2$ this corollary is a special case of a more general result in [LSU], Ch. 2, § 6.

Proof of Lemma 4.2. We first consider the case (i). Denote by $B_h \equiv B(\rho_h, x_0)$ a sequence of concentric balls of radii $\rho_h = \rho/2 + 2^{-h-1}\rho$, and let $t_h = t_0 + \sigma - 2^{-h}\sigma$, $k_h = 2k_0 - 2^{-h}k_0$, $h = 0, 1, 2, \dots$ (We set here $\rho_0 := \rho$.)

Let $\zeta_h(x, t) = \omega_h(x)\chi_h(t)$ where

$$\omega_h(x) = \begin{cases} 1 & \text{in } B_{h+1} \\ (\rho_h - |x - x_0|)(\rho_h - \rho_{h+1})^{-1} & \text{in } B_h \setminus B_{h+1} \\ 0 & \text{outside } B_h, \end{cases}$$

$$\chi_h(t) = \begin{cases} 0 & \text{for } t \leq t_h \\ (t - t_h)(t_{h+1} - t_h)^{-1} & \text{for } t_h < t < t_{h+1} \\ 1 & \text{for } t \geq t_{h+1}. \end{cases}$$

Evidently,

$$|D\zeta_h| \leq 2^{h+2}\rho^{-1}, \quad |(\zeta_h)_t| \leq 2^{h+1}\sigma^{-1}. \quad (4.6)$$

Put also

$$\mu_h = \sup_{[t_h, T]} \text{meas}[B_h \cap A_{k_h}(t)],$$

$$Q_h = \{(x, t) \mid x \in B_h \cap A_{k_h}(t); t \in [t_h, T]\},$$

$$J_h = \|z - k_h\|_{2, Q_h}^2.$$

Our aim is to show that $J_h \rightarrow 0$ as $h \rightarrow \infty$. The needed estimate (4.4) is obviously a consequence of this statement.

Setting $k = k_h$, $\zeta = \zeta_h$ in (4.1), and using (4.2) and (4.6), we obtain

$$\begin{aligned} & \sup_{[t_h, T]} \int_{A_{k_h}(t)} \psi(z, k_h) \zeta_h^2 dx + \iint_{Q_h} |D[(z-k_h)\zeta_h]|^2 dx dt \\ & \leq \sup_{[t_h, T]} \int_{A_{k_h}(t)} \psi(z, k_h) \zeta_h^2 dx + 2 \iint_{Q_h} [|D(z-k_h)|^2 \zeta_h^2 + 2(z-k_h)^2 |D\zeta_h|^2] dx dt \\ & \leq c_1 2^{2h} J_h + 2\bar{f}^2 |Q_h|, \end{aligned} \quad (4.7)$$

where $c_1 = 4 \max\{\vartheta, 1\}(16\rho^{-2} + \vartheta_2 \sigma^{-1} + 1)$, $\bar{f} = \text{ess sup}_{\tilde{Q}} |f|$, and $|Q_h|$ is the $n+1$ -dimensional Lebesgue measure of Q_h .

The second integral on the left in the first line of (4.7) is estimated from below with the use of a Sobolev-type inequality

$$\int_{\Omega} g^2 dx \leq \beta [\text{meas supp } g]^{2/n} \int_{\Omega} |Dg|^2 dx. \quad (4.8)$$

It holds for any $g \in H_0^1(\Omega)$ with $\beta = \beta(n)$.

Taking $g = (z-k_h)_+ \zeta_h$ and integrating (4.8) from t_h to T we obtain after combining the result with (4.7)

$$\|(z-k_h)_+ \zeta_h\|_{2, Q_h}^2 \leq \beta \mu_h^{2/n} (2^{2h} c_1 J_h + 2\bar{f}^2 |Q_h|).$$

Since

$$J_{h+1} \leq \|z - k_h\|_{2, Q_{h+1}}^2 \leq \|(z - k_h)\zeta_h\|_{2, Q_h}^2,$$

we get

$$J_{h+1} \leq \beta \mu_h^{2/n} (2^{2h} c_1 J_h + 2\bar{r}^2 |Q_h|). \quad (4.9)$$

We need to estimate μ_h and $|Q_h|$ from above. We have

$$\bar{r}^2 |Q_{h+1}| \leq k_0^2 (k_{h+1} - k_h)^{-2} J_h = 2^{2h+2} J_h$$

and by (4.2)

$$\psi(k_{h+1}, k_h) \mu_{h+1} \leq \sup_{[t_{h+1}, T]} \int_{A_{k_h}(t)} \psi(z, k_h) \zeta_h^2 dx,$$

$$\psi(k_{h+1}, k_h) \geq \gamma_1 (k_{h+1} - k_h)^2 \geq \gamma_1 2^{-(2h+2)} k_0^2.$$

Combining these estimates with (4.7) and (4.9) we conclude that

$$\mu_{h+1} \leq 2^{4h+2} \gamma_1^{-1} k_0^{-2} (c_1 + 2) J_{h-1} \quad (4.10)$$

and

$$J_{h+1} \leq c_2 b^h J_{h-2} k_0^{1+2/n-4/n}, \quad h = 2, 3, \dots \quad (4.11)$$

where $c_2 = \beta \gamma_1^{-2/n} (c_1 + 2)^{1+2/n}$, $b = 2^{2+8/n}$.

Put $\Theta = [c_2 b^{2+3n/2}]^{-n/4}$. We check by induction in h that if the condition (4.3) holds with such Θ then $k_0^{-2} J_h \leq \Theta^2 b^{-hn/2} \rightarrow 0$ when $h \rightarrow \infty$ (cf. [LSU], Ch. II, Lemma 4.7). The estimate (4.4) is now proved for $\sigma > 0$.

Consider now the case (i). In the preceding arguments set $t_h = 0$, $\zeta_h(x, t) = \omega_h(x)$ for $h = 0, 1, 2, \dots$, $c_1 = 4(16\rho^{-2} + 1)\max\{\gamma, 1\}$,

$c_2 = \beta \gamma_1^{-2/n} (c_1 + 2)^{1+2/n}$, $b = 2^{2+8/n}$, and $\Theta = [c_2 b^{2+3n/2}]^{-n/4}$.

Consequently, we again obtain (4.4). The lemma is proved.

Next we consider solutions u^ϵ of the problem (2.6)-(2.8). Note that the equations (2.6) are uniformly parabolic and $u^\epsilon \in C^\infty(\bar{\Omega} \times [0, \infty))$.

4.3. Proposition. Let $M_0 = \max_{\Omega} |u_0|$. Then

$$\sup_{\Omega \times [0, \infty)} |u^\epsilon(x, t)| \leq M_0, \quad (4.12)$$

$$\sup_{\Omega \times [0, \infty)} |u^\epsilon_t(x, t)| \leq M, \quad (4.13)$$

where M is a constant depending on the C^2 - norm of u_0 . Also,

$$\int_{\Omega} (|Du^\epsilon| + \epsilon |Du^\epsilon|^2) dx \leq c_0 \quad \text{for all } t \geq 0, \quad (4.14)$$

Proof. The inequality (4.12) follows from the classical maximum principle applied to the problem (2.6)-(2.8).

The inequality (4.13) is also obtained with the use of the maximum principle applied to the differentiated in t equation (2.6) after we note that $u^\epsilon_t = 0$ on $\partial\Omega \times [0, \infty)$ and

$$u^\epsilon_t = (1 + |Du_0|^2)^{1/2} \{ H[u_0] + \epsilon \Delta u_0 \} \quad \text{in } \Omega \times \{0\}.$$

Next we observe that

$$\int_{\Omega} \left[\frac{u^\epsilon_t \eta + u^\epsilon_i \eta_i}{\sqrt{1 + |Du^\epsilon|^2}} + \epsilon u^\epsilon_i \eta_i \right] dx = 0 \quad \text{for all } \eta \in H^1_0(\Omega) \text{ and } t \geq 0, \quad (4.15)$$

where $u^\epsilon_i = \partial u^\epsilon / \partial x_i$ and $\eta_i = \partial \eta / \partial x_i$. Letting $\eta = u^\epsilon$ in (4.15) and taking

into account (4.12), (4.13), we obtain (4.14). The proposition is proved.

4.4. It will be convenient to introduce the following notation.

For $p = (p_1, p_2, \dots, p_n)$ we put

$$F(p) = (1 + |p|^2)^{1/2}, \quad F^\epsilon(p) = F(p) + (\epsilon/2)|p|^2,$$

$$F_i(p) = \partial F(p)/\partial p_i = p_i/F(p), \quad F_{ij} = \partial^2 F/\partial p_i \partial p_j = (1/F)(\delta_{ij} - F_i F_j),$$

$$F^\epsilon_i = F_i + \epsilon p_i, \quad F^\epsilon_{ij} = F_{ij} + \epsilon \delta_{ij}.$$

Put $v^\epsilon(x, t) = F(Du^\epsilon(x, t))$. Rewrite the equation (2.6) as

$$u^\epsilon_t - v^\epsilon \frac{dF^\epsilon_i}{dx_i} = 0 \quad \text{in } \Omega \times [0, \infty). \quad (4.16)$$

Here and below in this section we omit the arguments at F , F_i , etc.

assuming always that $p = Du^\epsilon$; the notation d/dx_i is used for the total derivative and summation over i is assumed.

Since in this section we deal exclusively with the solution $u^\epsilon(x, t)$ of the problem (2.6)-(2.8), we omit for brevity the superscript ϵ at $v^\epsilon(x, t)$ and its derivatives.

We will need to use here the tangential operator δ defined as follows. Let $g \in C^1(\Omega)$. Put

$$\delta g = \nabla g - \langle \nabla g, N \rangle N,$$

where $\nabla g = (g_1, g_2, \dots, g_n, 0)$, $g_i = \partial/\partial x_i$, and $N = (-Du^\epsilon, 1)/v$ is the unit normal vector field on the hypersurface

$$S^\epsilon(t) = \{ (x, x_{n+1}) \mid x \in \overline{\Omega}, x_{n+1} = u^\epsilon(x, t) \}. \quad (4.17)$$

Note that

$$|\delta g|^2 = |Dg|^2 - \langle \nabla g, N \rangle^2 \geq |Dg|^2 (1 - |Du^\epsilon|^2/v^2) = |Dg|^2/v^2, \quad (4.18)$$

$$v^{-1} |\delta g|^2 = F_{ij} g_i g_j \leq v^{-1} |Dg|^2. \quad (4.19)$$

Applying the operator $F_k d/dx_k$ to (4.16) we obtain an equation for v

$$v_t - v \frac{d}{dx_i} (F_{ij}^\epsilon v_j) + v \Delta = (F_k v_k) \frac{dF_i^\epsilon}{dx_i}, \quad (4.20)$$

where $v_t = \partial v / \partial t$, $v_i = \partial v / \partial x_i$ and

$$\Delta = F_{ij}^\epsilon F_{kl} u_{ik}^\epsilon u_{jl}^\epsilon \geq 0. \quad (4.21)$$

Multiply (4.20) by ηv^{-1} , $\eta \in H_0^1(\Omega)$, and integrate over Ω . Then, after integration by parts, using equation (4.16), inequality (4.18) and $|F_k v| \leq |Dv_k|$, we obtain

$$\int_{\Omega} \left(\frac{v_t}{v} \eta + F_{ij}^\epsilon v_i \eta_j + \Delta \eta \right) dx \leq M \int_{\Omega} \frac{|\delta v|}{v} |\eta| dx \quad \text{for all } t \geq 0, \quad (4.22)$$

where M is the constant from the inequality (4.13).

The next two lemmas are needed for the proofs of Theorems C and C'. The estimates contained in these theorems are obtained in two steps. First we obtain estimates depending on ϵ and then we use them to get the required uniform estimates. The scheme in which one first obtains an estimate depending on ϵ and then improves on it in order to get a uniform estimate was used previously by R. Temam in [T] where he studied regularization of the (time independent) mean curvature equation. For the capillary problem the same scheme was used later by C. Gerhardt [G]. The techniques for proving such estimates on both stages of these procedure are based on the earlier work of O.A. Ladyzhenskaya and N.N. Ural'tseva [LU], [LSU].

In the following the constants M_0 , M , and c_0 always denote constants in the inequalities (4.12), (4.13), and (4.14).

4.5. Lemma. Let $\Omega' \subset\subset \Omega$ and $T > 0$, $0 \leq \sigma < T$. Then the following estimate holds

$$[1 + |Du^\epsilon(x,t)|^2]^{1/2} \leq 2\epsilon^{-1} \text{ in } \Omega' \times [\sigma, T] \text{ for all } \epsilon \leq \epsilon_0, \quad (4.23)$$

where $\epsilon_0 = \epsilon_0(T, \sigma, \text{dist}(\Omega', \partial\Omega), M, c_0)$ if $\sigma > 0$ and $\epsilon_0 = \epsilon_0(T, \text{dist}(\Omega', \partial\Omega), \max_{\Omega} |Du_0|, M, c_0)$ if $\sigma = 0$.

4.6. Lemma. If for some $t_0 \geq 0$ the estimate

$$(1 + |Du^\epsilon(x,t)|^2)^{1/2} \leq k^* \text{ on } \partial\Omega \times [t_0, T] \quad (4.24)$$

holds then for each $\sigma > 0$ there exists a positive $\epsilon_0 = \epsilon_0(T - t_0, \sigma, M, c_0, k^*)$ such that for all $\epsilon \leq \epsilon_0$

$$(1 + |Du^\epsilon(x,t)|^2)^{1/2} \leq \epsilon^{-1} \text{ on } \Omega \times [t_0 + \sigma, T]. \quad (4.25)$$

The **proofs** of both lemmas are almost identical. For that reason we prove only Lemma 4.5 and then point out the changes that need to be made for proving Lemma 4.6.

Proof of Lemma 4.5. We will show that function $z = v$ satisfies the hypotheses of Lemma 4.2 with $f \equiv 0$, $k_0 = \epsilon^{-1}$, and some suitable function ψ .

Fix a ball $B(\rho, x_0)$ ($\equiv B(\rho)$), $x_0 \in \Omega'$, $\rho = \text{dist}(\Omega', \partial\Omega)$. Put $t_0 = 0$ and take $k^* = \max_{\Omega} (1 + |Du_0|^2)^{1/2}$ if $\sigma = 0$ and $k^* = 1$ if $\sigma > 0$. We continue writing t_0 instead of 0 so that our notation match those in Lemma 4.2 and because in the proof of Lemma 4.6 t_0 is not necessarily

equal to zero. Set in (4.22) $\eta = (v-k)_+ \zeta^2$, $k \geq k^*$, and let $\zeta \in G(p, \sigma)$. Note that $\eta = 0$ on $\partial\Omega \times [t_0, T]$. Put

$$A_k(t) = \{x \in \Omega \cap B(\rho) \mid v(x, t) > k\}.$$

Then we obtain from (4.22)

$$\begin{aligned} & \int_{A_k(t)} \left[\frac{v_t(v-k)}{v} \zeta^2 + F_{ij}^\epsilon v_i v_j \zeta^2 + \Delta(v-k) \zeta^2 \right] dx \\ & \leq \int_{A_k(t)} \left[M \frac{|\delta v|}{v} (v-k) \zeta^2 - 2\zeta(v-k) F_{ij}^\epsilon v_i \zeta_j \right] dx \quad (4.26) \end{aligned}$$

The terms on the right are estimated as follows:

$$M |\delta v| (v-k) \leq (1/4) |\delta v|^2 + M^2 (v-k)^2;$$

$$2 |\zeta(v-k) F_{ij}^\epsilon v_j \zeta_i| \leq (1/4) \zeta^2 F_{ij}^\epsilon v_i v_j + 4 |D\zeta|^2 (v^{-1} + \epsilon) (v-k)^2. \quad (4.27)$$

Put

$$\Phi(v, k) = \int_k^v \frac{(y-k)_+}{y} dy \quad (4.28)$$

For the second term on the left of (4.26) we have because of (4.19)

$$F_{ij}^\epsilon v_i v_j = |\delta v|^2 v^{-1} + \epsilon |Dv|^2. \quad (4.29)$$

Now collecting together (4.26)-(4.29) and taking into account (4.21), when estimating the third term on the left, we obtain

$$\begin{aligned}
& \int_{\Omega} \frac{d}{dt} [\Phi(v,k)\zeta^2] dx + \frac{1}{2} \int_{A_k(t)} [|\delta v|^2 v^{-1} + \epsilon |Dv|^2] \zeta^2 dx \\
& \leq \int_{A_k(t)} \left[M^2 (v-k)^2 v^{-1} \zeta^2 + 2\Phi(v,k)\zeta\zeta_t + 4(v^{-1} + \epsilon) |D\zeta|^2 (v-k)^2 \right] dx \quad (4.30)
\end{aligned}$$

Consider now such ϵ that $\epsilon^{-1} \geq k^*$ and set

$$k_0 = \epsilon^{-1}. \quad (4.31)$$

Note that $(\epsilon v)^{-1} \leq (\epsilon k)^{-1} \leq (\epsilon k_0)^{-1} = 1$ on $A_k(t)$ for $k \geq k_0$. Integrating (4.30) over the interval $[t_0, T]$ and estimating each term on the left by the entire right hand side we obtain for any $k \geq k_0$

$$\begin{aligned}
& k_0 \sup_{[t_0, T]} \int_{A_k(t)} \Phi(v,k)\zeta^2 + \frac{1}{2} \int_{t_0}^T \int_{A_k(t)} |Dv|^2 \zeta^2 dx dt \\
& \leq 2 \int_{t_0}^T \int_{A_k(t)} \left[\left(M^2 \zeta^2 + 8 |D\zeta|^2 \right) (v-k)^2 + 2k_0 \Phi(v,k) |\zeta\zeta_t| \right] dx dt. \quad (4.32)
\end{aligned}$$

Set $\psi(v,k) = (1/2)k\Phi(v,k)$. This function satisfies the conditions (4.2) with $\gamma_1 = 1/16$, $\gamma_2 = 1/4$. Moreover, when $k \in [k_0, 2k_0]$ the inequalities (4.1) with $z = v$, $\gamma = 4(M^2 + 8)$ and $f \equiv 0$ follow from (4.32). Also, because of (4.14), we have (recalling (4.31))

$$\|(v-k_0)_+\|_{2,\tilde{Q}} \leq [(T-t_0)c_0\epsilon]^{1/2} k_0.$$

Hence, the inequality (4.3) will be satisfied for $\epsilon \leq \Theta^2 (T-t_0)^{-1} c_0^{-1}$ where Θ is determined as in Lemma 4.2. Thus, all conditions of Lemma 4.2 will be satisfied for any $\epsilon \leq \epsilon_0$ if we set $\epsilon_0 = \min \{(k^*)^{-1}, \Theta^2 (T-t_0)^{-1} c_0^{-1}\}$. This completes the proof of the Lemma 4.5.

Proof of Lemma 4.6. It is similar to the proof of Lemma 4.5. The difference is that in the beginning of the proof of Lemma 4.5 one should take k^* and t_0 from (4.24) and $\rho = \max\{1, 2\text{diam}(\Omega)\}$.

We will also need a variant of Sobolev's inequality on hypersurfaces.

Lemma 4.7. Suppose $u^\epsilon(x, t)$ is such that $\max_{\Omega'} |Du^\epsilon| \leq 2\epsilon^{-1}$ for some $\Omega' \subseteq \Omega$ and some fixed $t \geq 0$. Then there exists a constant $\beta = \beta(M, M_0)$ such that the inequality

$$\int_{S^\epsilon(t)} g^2 dH_n \leq \beta H^{2/n}(\hat{S}^\epsilon(t)) \int_{S^\epsilon(t)} |\delta g|^2 dH_n \quad (4.33)$$

holds for any $g \in C^1(\Omega')$ and vanishing on $\partial\Omega'$. Here $dH_n = v dx$ and

$$\hat{S}^\epsilon(t) = \{(x, x_{n+1}) \in S^\epsilon(t) \mid x \in \text{supp } g\}$$

with $S^\epsilon(t)$ as in (4.17).

Proof. An inequality of this type for elliptic equations was proved in [LU], Lemma 2, part 2. Since inequality (4.33) is stated for a fixed t , we will need only to reformulate the conditions on $u^\epsilon(x, t)$ so that Lemma 2 in [LU] applies.

We rewrite (4.16) in the form

$$\frac{u_t^\epsilon}{v} - \frac{dF_i^\epsilon}{dx_i} = 0$$

and note that

$$F_i^\epsilon u_i^\epsilon \geq v - v^{-1} \geq v - 1,$$

$$\sum_i |F_i^\epsilon| + |u_t^\epsilon/v| \leq c(M);$$

the last inequality is obtained with the use of (4.13).

Under these circumstances the inequality (4.33) follows from Lemma

2 in [LU], part 2.

4.8. Lemma. Let $T > 0$ and $t_1 \in [0, T]$. Put $w = \ln v$. For any $\zeta \in C_0^\infty(\Omega)$ there exists a constant $c = c(\zeta, T - t_1, M_0, M, c_0)$ such that

$$\int_{t_1}^T \int_{S^\epsilon(t)} w^2 \zeta^2 dH_n dt \leq c. \quad (4.34)$$

If, in addition, it is known that $w(x, t) \leq \bar{k}$ on $\partial\Omega \times [t_0, T]$ for some $\bar{k}$ then (4.34) holds with $\zeta \equiv 1$ and any $t_1 \geq t_0$. The constant c in this case depends also on $\bar{k}$.

Proof. Set $\eta = u^\epsilon w^2 \zeta^2$, $w_i = \partial w / \partial x_i$, $\zeta_i = \partial \zeta / \partial x_i$. After substitution into (4.15) and taking into account (4.12), (4.13) we obtain

$$\begin{aligned} \int_{\Omega} F_i^\epsilon (u_i^\epsilon w^2 \zeta^2 + 2u^\epsilon w w_i \zeta^2 + 2u^\epsilon w^2 \zeta \zeta_i) dx &= - \int_{\Omega} \frac{u^\epsilon}{v} u^\epsilon w^2 \zeta^2 dx \\ &\leq M_0 M \int_{\Omega} \frac{w^2}{v} \zeta^2 dx \end{aligned} \quad (4.35)$$

Next we estimate individual integrands in the first integral in (4.35). We have

$$\begin{aligned} F_i^\epsilon u_i^\epsilon &= v - v^{-1} + \epsilon |Du^\epsilon|^2, \\ 2|u^\epsilon w F_i^\epsilon w_i| &= 2|u^\epsilon w (v^{-1} + \epsilon) u_i^\epsilon w_i| \\ &\leq (1/2) w^2 (v + \epsilon |Du^\epsilon|^2) + 2M_0^2 (v^{-1} + \epsilon) |Dw|^2, \\ |2u^\epsilon w^2 \zeta F_i^\epsilon \zeta_i| &\leq 2M_0 (1 + \epsilon v) w^2 |\zeta| |D\zeta|. \end{aligned}$$

We now substitute these estimates in (4.35). But first we note that because $w(x, t) = 0$ when $v(x, t) = 1$ it suffices to integrate in (4.35) over

$A_1(t)$. Thus, after rearranging the terms we obtain

$$\int_{A_1(t)} v w^2 \zeta^2 dx \leq I + 4M_0^2 \int_{A_1(t)} (v^{-1} + \epsilon) |Dw|^2 \zeta^2 dx, \quad (4.36)$$

where

$$I = \int_{\Omega} [2(1 + M_0 M) v^{-1} \zeta^2 + 4M_0 (1 + \epsilon v) |\zeta| |D\zeta|] w^2 dx.$$

Using (4.14) we estimate I by a constant $c_1 = c_1(M_0, M, c_0, \|\zeta\|^{(1)})$

where $\|\cdot\|^{(1)}$ denotes the C^1 -norm.

Integrating (4.36) in t over the interval $[t_1, T]$ we obtain

$$\int_{t_1}^T \int_{A_1(t)} v w^2 \zeta^2 dx dt \leq c_1 (T - t_1) + 4M_0^2 \int_{t_1}^T \int_{A_1(t)} (v^{-1} + \epsilon) |Dw|^2 \zeta^2 dx dt. \quad (4.37)$$

We now need to estimate the second term on the right in (4.37). We proceed as follows. In (4.22) we set $\eta = (v-1)_+ \zeta^2$ with the same ζ as before. Repeating the arguments in the beginning of the proof of Lemma 4.5 for $k = 1$ we obtain the inequality (4.30). Integrating this inequality in t we get

$$\begin{aligned} & \int_{\Omega} \Phi(v, 1) \zeta^2 dx \Big|_{t_1}^T + \frac{1}{2} \int_{t_1}^T \int_{A_1(t)} [|\delta v|^2 v^{-1} + \epsilon |Dv|^2] \zeta^2 dx \\ & \leq \int_{t_1}^T \int_{A_1(t)} \left[M^2 (v-1)^2 v^{-1} \zeta^2 + 4(v^{-1} + \epsilon) |D\zeta|^2 (v-1)^2 \right] dx dt. \end{aligned}$$

Noting that $0 \leq \Phi(v, 1) \leq v$ and using (4.14) we estimate the first integral on the left and the integral on the right in terms of c_0 , ζ and

$T - t_1$. Collecting all these estimates and noting that $|Dv|^2 = |\delta v|^2 v^2$, we obtain

$$\int_{t_1}^T \int_{A_1(t)} [|\delta v|^2 v^{-1} + \epsilon |Dv|^2] \zeta^2 dx dt \leq c_2 = c_2(c_0, M, \|\zeta\|^{(1)}, T-t_1).$$

Observe that $|\delta v|^2 \geq |Dv|^2 v^{-2} = |Dw|^2$. Combining the last inequality with (4.37) and noting that $v dx = dH_n$ we complete the proof of the first part of the lemma.

Suppose now that an upper bound $\bar{k}$ for w on $\partial\Omega \times [t_0, T]$ is known. In this case we set the function $\zeta \equiv 1$ and replace w in (4.35) by $\bar{w} = (w - \bar{k})_+$. Then in (4.36)-(4.37) the integration should be performed over $A_{k'}(t)$ with $k' = e^{\bar{k}}$. The inequality (4.30) in this case after integration in t gives

$$\begin{aligned} & \int_{\Omega} \Phi(v, k') \zeta^2 dx \Big|_{t_1}^T + \frac{1}{2} \int_{t_1}^T \int_{A_{k'}(t)} [|\delta v|^2 v^{-1} + \epsilon |Dv|^2] \zeta^2 dx \\ & \leq \int_{t_1}^T \int_{A_1(t)} [|\delta v|^2 v^{-1} + \epsilon |Dv|^2] \zeta^2 dx dt \leq c_2 = c_2(c_0, M, \|\zeta\|^{(1)}, T-t_1). \end{aligned}$$

Here we have also used the inequality (4.14).

Now, arguing as before, we obtain the estimate

$$\int_{t_1}^T \int_{A_{k'}(t)} v w^2 dx dt \leq c_3 = c_3(c_0, M_0, M, T-t_1).$$

Finally, note that, since $w \leq \bar{k}$ on $\Omega \setminus A_{k'}(t)$, we obtain (4.34) with the bound depending this time also on $\bar{k}$. The lemma is proved.

4.9. Lemma. Let $\Omega' \subset\subset \Omega$ and ϵ_0 be such that (4.23) holds with $\sigma = 0$ for all $\epsilon \leq \epsilon_0$. Let, as in previous lemma, $w = \ln v$. Then for $\epsilon \leq \epsilon_0$,

any $\zeta \in H^1_0(\Omega')$, and $k > \max_{\Omega} w(x,0)$ the following inequalities hold

$$\begin{aligned} & \sup_{[0,T]} \int_{S_k^\epsilon(t)} (w-k)^2 \zeta^2 dH_n + \int_0^T \int_{S_k^\epsilon(t)} |\delta w|^2 \zeta^2 dH_n dt \\ & \leq c(M) \int_0^T \int_{S_k^\epsilon(t)} (w-k)^2 (\zeta^2 + |D\zeta|^2) dH_n dt, \end{aligned} \quad (4.38)$$

where

$$S_k^\epsilon(t) = \{(x, x_{n+1}) \in S^\epsilon(t) \mid w(x,t) > k\}.$$

Proof. Put $\eta = v[(w-k)_+^2 + 2(w-k)_+]\zeta^2$. Note that

$$\frac{v_t}{v} \eta = \frac{d}{dt} [v(w-k)_+^2 \zeta^2].$$

Substituting η in (4.22) we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} v(w-k)_+^2 \zeta^2 dx + \int_{\Omega \cap \{w>k\}} \left\{ F_{ij}^\epsilon v_i v_j [(w-k)_+^2 + 4(w-k)_+ + 2]\zeta^2 + \Delta \eta \right\} dx dt \\ & \leq \int_{\Omega} (M |\delta v| \zeta^2 - 2F_{ij}^\epsilon v_i \zeta \zeta_j v) [(w-k)_+^2 + 2(w-k)_+] dx. \end{aligned} \quad (4.39)$$

The integrand on the right is estimated with the use of (4.23):

$$|M |\delta v| \zeta^2 - 2F_{ij}^\epsilon v_i \zeta \zeta_j v| [(w-k)_+^2 + 2(w-k)_+] \leq \frac{1}{4} \frac{|\delta v|^2}{v} [(w-k)_+^2 + 2]\zeta^2$$

$$\begin{aligned}
& + 3M^2(w-k)_+^2 \zeta^2 v + \frac{1}{4} F_{ij}^\epsilon v_i v_j [(w-k)_+^2 + 2] \zeta^2 + 12 |D\zeta|^2 (v + \epsilon v^2) (w-k)_+^2 \\
& \leq \frac{1}{2} F_{ij}^\epsilon v_i v_j [(w-k)_+^2 + 2] \zeta^2 + 3(w-k)_+^2 v [M^2 \zeta^2 + 12 |D\zeta|^2].
\end{aligned}$$

Integrating (4.39) from 0 to an arbitrary $t \in [0, T]$ and using these estimates, we get

$$\begin{aligned}
& \sup_{[0, T]} \int_{\Omega} v(w-k)_+^2 \zeta^2 dx \Big|_0^t + \frac{1}{2} \int_0^T \int_{\Omega \cap \{w > k\}} (F_{ij}^\epsilon v_i v_j \zeta^2 + \Delta \eta) dx dt \\
& \leq 6 \int_0^T \int_{\Omega} (w-k)_+^2 v (M^2 \zeta^2 + 12 |D\zeta|^2) dx dt.
\end{aligned}$$

Now, noting that $\Delta \geq 0$, $F_{ij}^\epsilon v_i v_j \geq v^{-1} |\delta v|^2 = v |\delta w|^2$, and $v dx = dH_{n-1}$ on $S^\epsilon(t)$, we obtain (4.38).

4.10. Proof of Theorem C. Once the estimates in Lemmas 4.5, 4.8, 4.9, and Sobolev inequality (4.33) are available, the proof of the theorem is obtained by arguments similar to the ones in the proof of Lemma 4.2 where the inequalities (4.1) for z should be replaced by (4.38) for w . Instead of using the Sobolev inequality (4.8) for Euclidian domains we use the inequality (4.33) for hypersurfaces. The function J_h is taken in the form

$$J_h = \iint_{Q_h} (w - k_h) dH_n(t) dt,$$

$$Q_h = \{(x, x_{n+1}, t) \mid (x, x_{n+1}) \in S_{k_h}^\epsilon(t), x \in B_{\rho_h}, t \in [0, T]\},$$

where k_h and ρ_h are as in Lemma 4.2. We use the same arguments as there to prove (4.11). Consequently, choosing a sufficiently large k_0 so

that $k_0^{-2} J_0 \leq \Theta^2$, we show that $J_h \rightarrow 0$ as $h \rightarrow \infty$. Such k_0 can be chosen because of the estimate (4.34). This completes the proof of the Theorem B.

Remark. For a minimal surface-type equation (stationary case) a similar scheme for obtaining local gradient estimates was given in [BDGM] and [LU].

In order to prove Theorem C' we need one more auxiliary result.

4.11. Lemma. Suppose that for some $t_0 > 0$ the inequality (4.24) holds and ϵ_0 is the constant in Lemma 4.6 corresponding to some $\sigma > 0$. Then for any $k \geq k^*$ and any $\zeta \in \text{Lip } [t_0 + \sigma, T]$ vanishing at $t = t_0 + \sigma$ the following inequalities hold

$$\begin{aligned} & \sup_{[t_0 + \sigma, T]} \int_{S_k(t)} (w-k)^2 \zeta^2 dH_n + \int_{t_0 + \sigma}^T \int_{S_k^\epsilon(t)} |\delta w|^2 \zeta^2 dH_n dt \\ & \leq c(M) \int_{t_0 + \sigma}^T \int_{S_k^\epsilon(t)} (w-k)^2 (\zeta^2 + |\zeta \zeta_t|) dH_n dt. \end{aligned} \quad (4.40)$$

The proof of this lemma is similar to the proof of Lemma 4.9.

4.12. Proof of Theorem C'. Using Lemmas 4.11, 4.8, 4.6, and 4.7 in the same way as in the proof of Theorem C is shown that $|Du^\epsilon(x, t)| \leq \tilde{c}_2$ on $\bar{\Omega} \times [t_0 + \sigma, t_0 + \sigma + 1]$ where $\tilde{c}_2 = \tilde{c}_2(c_1, \sigma)$. On the other hand, by the maximum principle applied to the differentiated in x_j , $j = 1, 2, \dots, n$, equation (2.6) (cf. the proof of Lemma 5.1) we obtain the inequality (2.15) with $c_2 = \max \{\tilde{c}_2, c_1\}$.

4.13. Proof of Theorem A. It follows from the estimates (2.11), (2.12) and standard results on uniformly parabolic equations that we can select a subsequence of $\{u^\epsilon\}$ converging in $C^k(\bar{\Omega}' \times [0, T])$ for any $k \geq 0$, any $T > 0$, and any $\Omega' \subset\subset \Omega$ to some function $u \in C^\infty(\Omega \times [0, \infty))$. The

function u satisfies (1.1) in $\Omega \times [0, \infty)$ and (1.3). Additionally, it follows from (4.12)-(4.14) that $u_t \in L_\infty(\Omega \times [0, \infty))$ and $u \in W^{1,1}(\Omega)$ for all $t \geq 0$.

It remains to show that the boundary condition (2.4) is satisfied. This is done using the same arguments as in Lichniewsky-Temam [LT], section 3.2. One only needs to observe that u satisfies the following variational inequality

$$\int_0^T \int_\Omega \frac{u_t}{\sqrt{1 + |Du|^2}} \eta \, dx \, dt + E(u + \eta) - E(u) \geq 0, \quad (4.41)$$

for any $\eta(\cdot, t) \in L_\infty((0, \infty); W^{1,1}(\Omega))$,

where

$$E(u) = \int_0^T \int_\Omega \sqrt{1 + |Du|^2} \, dx \, dt + \int_0^T \int_{\partial\Omega} |u| \, ds \, dt, \quad ds - \text{volume element on } \partial\Omega.$$

Except for the factor $(1 + |Du|^2)^{-1/2}$ in the first term of (4.41) the latter is the same as in Lichniewsky-Temam [LT]. However, because $u_t(1 + |Du|^2)^{-1/2} \in L_\infty(\Omega)$ for each $t \in (0, \infty)$, one shows, following [LT], that (2.4) is satisfied.

5. Proof of Theorem D.

We will need several auxiliary results before we can proceed with the proof of Theorem D.

5.1. Lemma. Let $u(x, t)$ be a solution of (1.1)-(1.3) such that $u \in C^\infty(\bar{\Omega} \times [\bar{t}, \infty))$ for some $\bar{t} \geq 0$. Suppose that for some positive constants c and λ

$$\sup_{\partial\Omega} |Du(x, t)| \leq c e^{-\lambda(t-\bar{t})} \quad \text{for any } t \geq \bar{t}, \quad (5.1)$$

$$\sup_\Omega |Du(x, t)| \leq c \quad \text{for any } t \geq \bar{t}, \quad (5.2)$$

Then there exist positive constants c_1 and β depending on c , λ , and Ω such that

$$\sup_{\Omega} |Du(x,t)| \leq c_1 e^{-\beta(t-\bar{t})} \quad \text{for any } t \geq \bar{t}. \quad (5.3)$$

Proof. The estimate (5.3) will be proved by constructing a suitable barrier. But first we note that because of (5.2) the equation (2.3) (which is a more convenient form of (1.1)) is uniformly parabolic for $t \geq \bar{t}$. Then for any fixed $\bar{t}' > \bar{t}$

$$\|u(\cdot, t)\|^{(2)} \leq c_2 \quad \text{for any } t \geq \bar{t}', \quad (5.4)$$

where $\|u\|^{(2)}$ denotes the C^2 -norm in Ω and c_2 a positive constant that does not depend on t (see [LSU], Ch. IV).

Differentiate (2.3) with respect to x_k and set $z = u_k$. Then we obtain

$$\bar{L}z := z_t - g_{ij}z_{ij} + b_i z_i = 0, \quad (5.5)$$

where $z_t = \partial z / \partial t$, $z_i = \partial z / \partial x_i$, $z_{ij} = \partial^2 z / \partial x_i \partial x_j$, $g_{ij} = \delta_{ij} - u_i u_j v^{-2}$. By (5.2) and (5.4) the coefficients b_i , $i = 1, 2, \dots, n$, satisfy the inequality $|b_i| \leq \text{const.} \equiv \bar{b}$ in $\bar{\Omega} \times [\bar{t}', \infty)$ where $\bar{b} = \bar{b}(c, c_2)$.

Since the equation (2.3) is invariant relative to parallel translations of the origin of coordinate system in the hyperplane $x_{n+1} = 0$, we may assume that the domain Ω is positioned so that $x_1 \geq 0$ in Ω . Consider the function

$$\omega(x, t) = A e^{-\beta t} (2 - e^{-\mu x_1}),$$

where the constants A , β , and μ are to be chosen later. We have

$$A^{-1} e^{\beta t + \mu x_1} \bar{\omega} = (1 - 2e^{\mu x_1})\beta + (1 - \frac{u_1^2}{v^2})\mu^2 + b_1\mu.$$

Choose μ so that

$$(1 - \frac{u_1^2}{v^2})\mu > |b_1|$$

and then choose β small enough so that $\beta \leq \lambda$ and

$$\bar{\omega} > 0, \quad (5.6)$$

where λ is as in (5.1). Put also $A = ce^{\beta \bar{t}}$. Because of our choice of β and A we get from (5.1) and (5.2)

$$|z| \leq c e^{-\beta(t-\bar{t})} = A e^{-\beta t} \leq \omega \text{ on } \partial\Omega \times [\bar{t}, \infty) \quad (5.7)$$

$$|z(x, \bar{t})| \leq A e^{-\beta \bar{t}} \leq \omega(x, \bar{t}) \quad \text{on } \Omega. \quad (5.8)$$

It follows from (5.6), (5.7), (5.8) and the maximum principle that

$$|z(x, t)| \leq \omega(x, t) \leq 2A e^{-\beta t} \leq c_1 e^{-\beta(t-\bar{t})} \quad \text{on } \Omega \times [\bar{t}, \infty)$$

where $c_1 = 2c$. The lemma is proved.

It will be convenient to rewrite the equation (2.3) in the form

$$u_t(x, t) - \Delta u(x, t) = f(x, t) \quad \text{in } Q_{\bar{t}} = \Omega \times [\bar{t}, \infty), \quad (5.9)$$

where

$$f(x, t) = - \frac{u_i u_j u_{ij}}{1 + |Du|^2}.$$

Let $0 < \lambda_1 < \lambda_2 \leq \dots$ be the spectrum of the Laplace operator in Ω

with Dirichlet data and $\psi_1, \psi_2, \dots$, the corresponding orthonormal set of eigenfunctions. Denote by H_m a subspace of $L_2(\Omega)$ orthogonal to $\psi_1, \dots, \psi_m$. Put $H_0 := L_2(\Omega)$. The next lemma is known, possibly in a different form. We present it here with a proof in order to make our presentation reasonably self-contained.

5.2. Lemma. Let $u \in C^\infty(\bar{\Omega} \times [\bar{t}, \infty))$, satisfy (5.9) and

$$u = 0 \text{ on } \partial\Omega \times [\bar{t}, \infty), u = \psi \text{ on } \Omega \times \{\bar{t}\}.$$

Suppose that for some integer $s \geq 1$ and any $t \geq \bar{t}$ $\psi \in H_{s-1}$, $f(\cdot, t) \in H_{s-1}$ and

$$\|f(x, t)\|_{2, \Omega} \leq ce^{-2\beta t} \quad \text{for } t \geq \bar{t} \quad (5.10)$$

with some positive constants c and $\beta \neq (1/2)\lambda_s$. Then for any $t \in [\bar{t}, \infty)$

$$\|u(x, t)\|_{2, \Omega} \leq c_1 e^{-\gamma t}, \quad (5.11)$$

$$\int_{\bar{t}}^{t+1} \|Du(\cdot, \tau)\|_{2, \Omega}^2 d\tau \leq c_2 e^{-2\gamma t}, \quad (5.12)$$

where $\gamma = \min\{2\beta, \lambda_s\}$,

$$c_1 = e^{\lambda_s \bar{t}} \|\psi\|_{2, \Omega} + \frac{c}{|\lambda_s - 2\beta|}, \quad c_2 = \frac{c^2}{2} + \frac{cc_1}{2\gamma}.$$

Proof. Note first that $u(x, t) \in H_{s-1}$ for any $t \geq \bar{t}$ and therefore

$$\lambda_s \|u\|_{2, \Omega}^2 \leq \|Du\|_{2, \Omega}^2 \quad \text{for any } t \geq \bar{t}. \quad (5.13)$$

Put $\phi(t) = \|u(\cdot, t)\|_{2,\Omega}$. Multiplying (5.9) by $u(x,t)$, integrating over Ω and using (5.10), we obtain

$$\frac{1}{2} \frac{d(\phi^2)}{dt} + \|Du\|_{2,\Omega}^2 \leq c\phi e^{-2\beta t}, \quad t \geq \bar{t}. \quad (5.14)$$

This and (5.13) give

$$\frac{d\phi}{dt} + \lambda_s \phi \leq ce^{-2\beta t}, \quad t \geq \bar{t}.$$

Then

$$\phi(t) \leq e^{-\lambda_s t} \left[e^{\lambda_s \bar{t}} \phi(\bar{t}) - \frac{c}{\lambda_s - 2\beta} e^{(\lambda_s - 2\beta)\bar{t}} \right] + \frac{c}{\lambda_s - 2\beta} e^{-2\beta t}.$$

This implies (5.11).

The inequality (5.12) is obtained by integrating (5.14) from t to $t+1$ and using (5.11). The lemma is proved.

Lemma 5.3. Suppose the conditions of Lemma 5.2 are satisfied except for (5.10) which is replaced by

$$\sup_{\Omega} |f(\cdot, t)| \leq ce^{-2\beta t}, \quad t \geq \bar{t}. \quad (5.15)$$

Then

$$\sup_{\Omega} |u(\cdot, t)| \leq c_3 e^{-\delta t}, \quad t \geq \bar{t} + 1/2. \quad (5.16)$$

Moreover, if for some $\lambda \in (0, \lambda_1)$ the estimate

$$\sup_{\partial\Omega} |Du(\cdot, t)| \leq ce^{-\lambda t}, \quad t \geq \bar{t} \quad (5.17)$$

holds then

$$\sup_{\Omega} |Du(\cdot, t)| \leq c_4 e^{-\delta t}, \quad t \geq \bar{t} + 1/2, \quad (5.18)$$

where $\tilde{\delta} = \min \{\lambda, \delta\}$, δ is as in Lemma 5.2, $c_3 = c_3(c_1, c)$, and $c_4 = c_4(c_2, c)$. Both constants c_3 and c_4 depend on $\bar{t}$.

Remark. The proof of Lemma 5.3 is obtained by combining estimates in Lemma 5.2 with local estimates of C^0 - and C^1 -norms of solutions to uniformly parabolic equations in terms of L_2 -norms of these solutions. Such estimates are given in [LSU], Ch. II. However, they are presented there in a setting in which it is difficult to trace the dependence of the estimates on the data. Since in the following we will need to know this dependence in order to determine the rates of decay of solutions to (1.1), we prefer to give here the proofs of the needed estimates. The proofs given below are based on the Corollary 4.2.1.

Proof of Lemma 5.3. We begin with the estimate of $\sup_{\Omega} |u(x, t)|$. For that we use the part (i) of the Corollary 4.2.1. In the notation of the corollary we set $z = u$ ($z = -u$), $T = t_0 + 1$, with an arbitrary but fixed $t_0 \geq \bar{t}$, $\rho = \max \{1, 2 \text{diam } \Omega\}$, $k^* = 0$, and $\sigma = 1/2$. Multiply the inequality $z_t - \Delta z \leq |f|$ by $(z-k)_+ \zeta^2(t)$ for $k \geq 0$, and integrate the result. After integration by parts in the term containing the Laplace operator and obvious manipulations we obtain the inequalities (4.1) with $\psi = (1/2)(z-k)_+^2$ and any $k \geq 0$. Consequently, (5.15), (5.11), and (4.5) for $z = \pm u$ imply the estimate $\sup_{\Omega} |u(\cdot, t)| \leq c_3 e^{-\tilde{\delta} t}$, for $t \in [t_0 + 1/2, t_0 + 1]$, where c_3 depends on the constants in the inequalities (5.15), (5.11) and does not depend on t_0 . Since $t_0 \geq \bar{t}$ but otherwise arbitrary, we obtain (5.16).

Now we prove (5.18). Fix some $j \in \{1, 2, \dots, n\}$ and set $z = u_j$ ($z = -u_j$). Again, we check that z satisfies the conditions of part (i) in the Corollary 4.2.1.

As before we consider an arbitrary fixed $t_0 \geq \bar{t}$ and set $T = t_0 + 1$, $\rho = \max \{1, 2 \text{diam}(\Omega)\}$, $\sigma = 1/2$, and $k^* = k_0 = ce^{-\lambda t_0}$. Note that the ball $B_{\rho/2} := B(\rho/2, x_0)$ contains Ω for any $x_0 \in \Omega$. Put $\tilde{Q} = \bar{\Omega} \times [t_0, T]$. Let $\eta(x, t)$ be such that $\eta(\cdot, t) \in \text{Lip}(\Omega)$ and has compact support in Ω for all

t. Multiply (5.9) by $\eta_j = \partial\eta/\partial x_j$, integrate over Ω and then integrate by parts the terms containing u_t and the Laplace operator. Then we obtain

$$\int_{\Omega} (u_{jt} \eta + u_{ji} \eta_i + f \eta_j) dx = 0 \quad \forall t \geq \bar{t}. \quad (5.19)$$

Take $\eta = (z-k)_+ \zeta^2(t)$, $k \geq k^*$, and $\zeta \in G(p, 1/2)$ where $G(p, 1/2)$ is the same as in subsection (4.1) and where we set $\omega \equiv 1$ in $\bar{\Omega}$. Substituting in (5.19) we obtain

$$\int_{A_k(t)} [z_t (z-k) \zeta^2 + |Dz|^2 \zeta^2 + f z_j \zeta^2] dx = 0.$$

(When considering $z = -u_j$ we obtain this relation with f replaced by $-f$.)

Then for any $k \geq k^*$ and $t \in [t_0, T]$

$$\begin{aligned} & \int_{\Omega} \left[\frac{\partial}{\partial t} [(z-k)_+^2 \zeta^2] \right] dx + \int_{A_k(t)} |Dz|^2 \zeta^2 dx \\ & \leq 4 \int_{A_k(t)} \left[(z-k)^2 |\zeta \zeta_t| + f^2 \zeta^2 \right] dx. \end{aligned}$$

Integrating over the interval $[t_0, t]$, with $t \in [t_0, T]$, setting in Lemma 4.2

$\psi(z, k) = (z-k)_+^2$, $\gamma = 4$, we obtain, after estimating straight forwardly (in t),

$$\sup_{[t_0, T]} \int_{A_k(t)} \psi(z, k) \zeta^2 dx + \int_{t_0}^T \int_{A_k(t)} |Dz|^2 \zeta^2 dx dt$$

$$\leq 4 \int_{t_0}^T \int_{A_k(t)} \left[\psi(z, k) |\zeta \zeta_t| + r^2 \zeta^2 \right] dx dt \quad \text{for any } k \geq k^*.$$

This inequalities imply the inequalities (4.1) of Lemma 4.2.

Accordingly, the inequalities (5.17), (4.5), (5.15), (5.11) and (5.12) imply

$$\sup_{\Omega} |Du(x, t)| \leq c_5 e^{-\tilde{\delta} t_0} \quad \text{for } t \in [t_0 + 1/2, T]$$

where the constant c_5 depends on the domain Ω , the constants in above listed inequalities and it does not depend on t_0 . Again, since $t_0 \geq \bar{t}$ but otherwise arbitrary, we obtain (5.18) if we set $c_4 = c_5 e^{\tilde{\delta}}$.

5.4. Proof of Theorem D. It follows from inequality (2.10) in Theorem B that for any $\lambda \in (0, \lambda_1)$ there exists $T(\lambda) \geq 0$ such that

$$|Du^\epsilon(x, t)| \leq C_1(\lambda) e^{-\lambda t}, \quad (x, t) \in \partial\Omega \times [T, \infty). \quad (5.20)$$

We fix one such λ so that $2\lambda > \lambda_1$ and in the following do not indicate the dependence of constants on λ . By Theorem C'

$$|Du^\epsilon(x, t)| \leq c_5, \quad (x, t) \in \bar{\Omega} \times [T+1, \infty), \quad (5.21)$$

It follows from estimate (5.21) and standard results on quasilinear uniformly parabolic equations [LSU], Ch. IV, that u^ϵ and all of its derivatives are uniformly bounded in $\bar{\Omega} \times [\bar{t}, \infty]$ with $\bar{t} = T+2$. The appropriate bounds do not depend on ϵ . The family $\{u^\epsilon\}$ is compact in $C^1(\bar{\Omega} \times [\bar{t}, T'])$ for any $T' > \bar{t}$. Therefore, the generalized solution $u(x, t)$ of (1.1)-(1.3) is in $C^1(\bar{\Omega} \times [\bar{t}, T'])$. Therefore, $u \in C^\infty(\bar{\Omega} \times [\bar{t}, T'])$ and $u = 0$ on $\partial\Omega \times [\bar{t}, T']$. Thus we have obtained (2.16).

Obviously, the estimates (5.20)-(5.21) hold also for $u(x, t)$, possibly, with some larger $\bar{t}$. Applying Lemmas 5.1 and 5.3, we have for $t \geq \bar{t}$

$$\sup_{\Omega} |Du(\cdot, t)| \leq c_6 e^{-\tilde{\beta}t}, \quad \tilde{\beta} = \min \{2\beta, \lambda\}, \quad (5.22)$$

with some $\beta > 0$ as in Lemma 5.1. If $2\beta \geq \lambda$ then (5.22) implies (2.17).

Suppose $2\beta < \lambda$. We want to show now that the exponent $\tilde{\beta}$ can be improved. Consider again the equation (5.9). It follows from (5.22) that (5.15) is satisfied with β replaced by $\tilde{\beta}$. Then by Lemma 5.3 we obtain (5.22) with $\tilde{\beta}$ replaced by $\min \{4\beta, \lambda\}$ and with some larger $\bar{t}$.

Let $q = [\lambda/2\beta] + 1$. Repeating the above arguments q times we obtain (5.22) with $\tilde{\beta}$ replaced by $\min \{2q\beta, \lambda\} = \lambda$ and some appropriately adjusted $\bar{t}$. This completes the proof of (2.17).

It remains to prove (2.18). Write $u(x, t)$ as a sum of the projections in $L_2(\Omega)$ on ψ_1 and its orthogonal complement H_1 , that is,

$$u = \bar{u} + \tilde{u}, \quad \bar{u} = (u, \psi_1) \psi_1.$$

Then $\tilde{u}$ is a solution of the problem

$$\tilde{u}_t - \Delta \tilde{u} = \tilde{f} \quad \text{in } \Omega \times [\bar{t}, \infty),$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega \times [\bar{t}, \infty),$$

$$\tilde{u} = \psi := u - \bar{u} \quad \text{in } \Omega \times \{\bar{t}\}$$

where $\tilde{f} = f - (f, \psi_1) \psi_1$. It follows from (2.17) that $|\tilde{f}| \leq c_7 e^{-2\lambda t}$. Also, $(\tilde{f}, \psi_1) = 0$, and $(\psi, \psi_1) = 0$. By Lemma 5.3 we have

$$\sup_{\Omega} |\tilde{u}(x, t)| \leq c_8 e^{-\tilde{\alpha}t}, \quad \tilde{\alpha} = \min \{2\lambda, \lambda_2\} > \lambda_1. \quad (5.23)$$

On the other hand,

$$\bar{u}(x, t) = e^{-\lambda_1(t-\bar{t})} \left[(u(\cdot, \bar{t}), \psi_1) + \int_{\bar{t}}^{\infty} e^{-\lambda_1(\bar{t}-\tau)} f_1(\tau) d\tau \right] \psi_1(x) + r(x, t), \quad (5.24)$$

where

$$f_1(\tau) = (f(\cdot, \tau), \psi_1),$$

$$r(x, t) = - \left[\int_t^\infty e^{-\lambda_1(t-\tau)} f_1(\tau) d\tau \right] \psi_1(x).$$

By (2.17) $|f_1(\tau)| \leq c_9 e^{-2\lambda\tau}$ and, since $\lambda_1 < 2\lambda$, we get

$$|r(x, t)| \leq \frac{c_{10} e^{-2\lambda t}}{2\lambda - \lambda_1}.$$

From this inequality, (5.23) and (5.24) we obtain (2.18) if we set

$$c = e^{\lambda_1 \bar{t}} \left[(u(\cdot, \bar{t}), \psi_1) + \int_{\bar{t}}^\infty e^{-\lambda_1(\bar{t}-\tau)} f_1(\tau) d\tau \right],$$

$$c_1 = c_8 + c_{10}/(2\lambda - \lambda_1), \quad \mu = \min \{ (2\lambda - \lambda_1), (\lambda_2 - \lambda_1) \}.$$

This completes the proof of Theorem D.

6. A priori C^0 - and C^1 -estimates for starshaped domains. Existence of a smooth solution to (1.1)-(1.3) for all time.

In this section we prove Theorem E and indicate how the conclusions in Theorem D change with the additional assumptions on the domain Ω and initial data. Before proceeding with the proof of Theorem E we want to make the following observations.

6.1. Here we demonstrate that the "reduced" mean curvature $\Phi(x)s(x)$ is invariant with respect to homothetic transformations of the domain Ω relative to x_0 . This can be seen as follows. Let $S^{n-1} (\equiv S)$ be a unit sphere centered at x_0 . We represent $\partial\Omega$ as a graph over S by setting $v = (x - x_0)/|x - x_0|$ for $x \neq x_0$ and $r(v) = s(v)v$, where v is considered as a point on S and also as a unit vector in R^n . Thus $r(v)$ is the position

vector determining $\partial\Omega$. Assume that a set of smooth local coordinates $v^1, \dots, v^{n-1}$ is chosen on S . Since we assume that $\partial\Omega$ is smooth, the function $s: S \rightarrow (0, \infty)$ is smooth. In terms of this function the mean curvature of $\partial\Omega$ is given by a well known expression

$$h = - \frac{1}{\sqrt{s^2 + |\nabla s|^2}} + \frac{1}{(n-1)s} \operatorname{div}_S \frac{\nabla s}{\sqrt{s^2 + |\nabla s|^2}}$$

where ∇ and div_S denote respectively the gradient and divergence on S .

Similarly, letting $p(x) = \langle x - x_0, \nu(x) \rangle$ for $x \in \partial\Omega$, we see that in our parametrization it becomes

$$p(v) = \langle r(v), \nu(v) \rangle = s^2(v)(s^2(v) + |\nabla s(v)|^2)^{-1/2}.$$

The function $p(v)$ is the support function of $\partial\Omega$.

Now, if $\tilde{r} = cr$ for some $c = \text{const.} > 0$ then $\tilde{s} = cs$, $\tilde{h} = c^{-1}h$, and $\tilde{p} = cp$. Consequently, for the function Φ defined above we have $\tilde{\Phi} = c^{-1}\Phi$ and $\tilde{\Phi}\tilde{s} = \Phi s$ and our claim is proved.

6.1.1. Previous remark shows that if (2.19) is satisfied then (2.20) will be satisfied for a wide range of δ provided the $\inf_{\partial\Omega} p$ is sufficiently large. This is true if Ω contains a ball of a sufficiently large radius. Note also that (2.21) is satisfied for any function whose support is in $\overline{\Omega} \setminus \Omega_\delta$.

6.2. Proof of Theorem E. The problem (1.1)-(1.3) is invariant relative to parallel translations of the origin of the Cartesian coordinate system in the hyperplane $x_{n+1} = 0$. Therefore we may assume that the origin $O = \{0\}$ coincides with the point x_0 .

We construct now a special family of hypersurfaces and compute their mean curvatures. From the computational viewpoint it is more convenient here to use directly the definition of the mean curvature rather than compute it using the corresponding expression for H in (1.1).

We continue to use the notation introduced in subsection 6.1.

Let $f(\tau)$ and $g(\tau)$ be two smooth real valued functions defined on the

interval $[0,1]$ with the properties

$$f(0) = 1, f(1) = 0, f' < 0, f'' < 0, \quad g(0) = 1, g' > 0, g'' \leq 0.$$

Consider the hypersurface F defined by the map $R: S \times [0,1] \rightarrow \mathbb{R}^{n+1}$,
 $R(v, \tau) = (f(\tau)r(v), g(\tau))$.

Put $R_i = \partial R / \partial v^i$, $R_{ij} = \partial^2 R / \partial v^i \partial v^j$, and similarly r_i, r_{ij} . Also, let $R_\tau = \partial R / \partial \tau$ and $R_{\tau\tau} = \partial^2 R / \partial \tau^2$. Denote by $e_{ij} = \langle r_i, r_j \rangle$ the metric of $\partial\Omega$. Put $\gamma_i = \langle r, r_i \rangle$, $[e^{ij}] = [e_{ij}]^{-1}$ and $\gamma^i = e^{ij}\gamma_j$, where, as before, the summation convention over repeated indices is in effect.

The metric E of the hypersurface F has the following structure:

$$E_{ij} = \langle R_i, R_j \rangle = f^2 \langle r_i, r_j \rangle = f^2 e_{ij}, \quad i, j = 1, \dots, n-1,$$

$$E_{i\tau} = \langle R_i, R_\tau \rangle = f f' \gamma_i, \quad E_{\tau\tau} = f'^2 r^2 + g'^2.$$

The unit normal vector field on F is given by

$$N = (g'/q)(v, -(f'/g')\langle r, v \rangle) \text{ with } q = (g'^2 + f'^2 \langle r, v \rangle^2)^{1/2}.$$

Note that since Ω is starshaped relative to O , there exists a constant c_0 , depending on the $\sup_S |\text{grad } r|$ and $\sup_S |r|$ such that

$$\langle r, v \rangle \geq c_0 = \text{const.} > 0 \text{ on } S.$$

We will also need the expression for the matrix inverse to E ; namely,

$$E^{ij} = (1/f^2)[e^{ij} + (f'/q)^2 \gamma^i \gamma^j], \quad E^{i\tau} = -f' \gamma^i / (fq)^2, \quad E^{\tau\tau} = 1/q^2.$$

The coefficients of the second fundamental form of F are given by

$$B_{ij} = \langle R_{ij}, N \rangle = (g'f/q)b_{ij}, \text{ where } b_{ij} = \langle r_{ij}, v \rangle;$$

$$B_{i\tau} = \langle R_{i\tau}, N \rangle = 0; \quad B_{\tau\tau} = \langle R_{\tau\tau}, N \rangle = (g'f'/q)\langle r, v \rangle [f''/f' - g''/g'].$$

Finally, the mean curvature H of F is given by

$$\begin{aligned} nH &= E^{ij}B_{ij} + E^{\tau\tau}B_{\tau\tau} \\ &= \frac{g'\langle r, v \rangle}{q^3 f} \left[\frac{q^2 e^{ij} + f'^2 \gamma^i \gamma^j}{\langle r, v \rangle} b_{ij} + f f' \left(\frac{f''}{f'} - \frac{g''}{g'} \right) \right]. \end{aligned} \quad (6.1)$$

We want to estimate from above the first term in square brackets in (6.1). Recall that $q^2 = g'^2 + f'^2 \langle r, v \rangle^2$ and consider the form $(\langle r, v \rangle^2 e^{ij} + \gamma^i \gamma^j) \eta_i \eta_j$, $\eta = (\eta_1, \dots, \eta_{n-1}) \in \mathbb{R}^{n-1}$. Since $s^2 = \langle r, v \rangle^2 + \gamma^2$, where $\gamma^2 = \gamma_i \gamma_j e^{ij}$, we have

$$(\langle r, v \rangle^2 e^{ij} + \gamma^i \gamma^j) \eta_i \eta_j = [(s^2 - \gamma^2) e^{ij} + \gamma^i \gamma^j] \eta_i \eta_j \leq s^2 \eta^2,$$

where $\eta^2 = \eta_i \eta_j e^{ij}$. Thus, the eigenvalues of $\langle r, v \rangle^2 e^{ij} + \gamma^i \gamma^j$ relative to the metric $[e_{ij}]$ do not exceed s^2 . Then, noting that the mean curvature of $\partial\Omega$ is $h = (n-1)^{-1} e^{ij} b_{ij}$ we obtain

$$(\langle r, v \rangle^2 e^{ij} + \gamma^i \gamma^j) b_{ij} \leq (n-1) s^2 h_+. \quad (6.2)$$

We now choose the functions f and g so that $H < 0$ on F . It will be convenient to consider **two cases**:

(a) $\sup_{\partial\Omega} \Phi(x) \neq 0$ and (b) $\sup_{\partial\Omega} \Phi(x) = 0$.

Case (a). It follows from (2.19) that there exists $\xi > 0$ such that $\Phi^2(x) s^2(x) < 1 - \xi$. Noting that, we set $f(\tau) = 1 - \tau$ and $g(\tau) = B(\tau - \tau^2/2)$

where

$$B^2 = \inf_{\partial\Omega} \{\Phi^{-2}(x)[1-\Phi^2(x)s^2(x)-\xi]\}.$$

Then, we obtain for $\tau < 1$ after substituting f and g in (6.1) and taking into account (6.2)

$$\begin{aligned} nH(g) &\leq B(1-\tau)\langle r, \nu \rangle q^{-3} [(1-\tau)^2 B^2 + s^2] \Phi^2(x) - 1] \\ &\leq B\langle r, \nu \rangle q^{-3} [(B^2 + s^2) \Phi^2(x) - 1] \leq -\xi B\langle r, \nu \rangle q^{-3} < 0. \end{aligned} \quad (6.3)$$

We now relate our preceding construction to the construction of a C^0 -barrier for a solution $u(x,t)$ of the problem (1.1)-(1.3). First of all note that any point x in domain $\overline{\Omega}$ can be represented as $x = (1-\tau)r(v)$ and, therefore, $\tau = 1 - |x|/s(x)$. Then

$$g(\tau) \equiv g(x) = B \left[1 - \frac{|x|}{s(x)} - \frac{1}{2} \left(1 - \frac{|x|}{s(x)} \right)^2 \right].$$

Let $\tilde{g}(x,t) = e^{-\sigma t} g(x)$ where $\sigma > 0$ and will be specified later. It is not difficult to see by examining the inequalities in (6.3) that $H(\tilde{g}) \leq 0$ in $\Omega \setminus \{0\}$. Evaluating the operator L in (1.1), we obtain for all for $(x,t) \in (\Omega \setminus \{0\}) \times [0, \infty)$

$$L\tilde{g} = -\sigma\tilde{g} - (1 + e^{-2\sigma t} |Dg|^2)^{1/2} H(\tilde{g}) \geq -\sigma\tilde{g} - H(\tilde{g}).$$

Since the function g is bounded in $\overline{\Omega}$ and the inequality $H(\tilde{g}) < 0$ is strict, we can select σ so that

$$L\tilde{g} > 0 \text{ in } (\Omega \setminus \{0\}) \times [0, \infty). \quad (6.4)$$

Let now $u(x,t)$ be a solution of the problem (1.1)-(1.3) and consider it in the subdomain

$$\overline{\Omega}_\delta = \{x \in \overline{\Omega} \mid s(x) - |x| \leq \delta s(x)\}$$

where $\delta \in (0,1)$ is as in the statement of the theorem. The boundary $\partial\Omega_\delta$ consists of two disjoint components; one is $\partial\Omega$ and the other one is $Y = \{x \in \Omega \mid s(x) - |x| = \delta s(x)\}$. We want to show that

$$|u(x,t)| \leq \tilde{g}(x,t) \text{ on the hypersurface } Y \text{ for all } t \geq 0. \quad (6.5)$$

Because of the estimate (3.3) (valid with $\epsilon = 0$ and regardless of whether u_0 has compact support or not) it suffices to show that

$2\sup_\Omega |u(x,0)| e^{-\mu t} \leq \tilde{g}(x,t)$ on $Y \times [0,\infty)$. We assume here that $\sigma \leq \mu$. If this is not the case then we decrease σ ; obviously, (6.4) will continue to hold. Thus, one needs to show only that $2\sup_\Omega |u(x,0)| \leq B(\delta - \delta^2 2^{-1})$ and, since $(1/2)\delta \leq \delta - \delta^2 2^{-1}$ for any $\delta \in [0,1]$, it suffices to show that $2\sup_\Omega |u(x,0)| \leq 2^{-1}B\delta$. We have

$$\begin{aligned} B &= \{\inf_{\partial\Omega} [\Phi^{-2}(x)(1-\Phi^2(x)s^2(x)-\xi)]^{1/2} \\ &\geq \inf_{\partial\Omega} [1-\Phi^2(x)s^2(x)-\xi]^{1/2} / \sup_{\partial\Omega} \Phi(x) \\ &\geq \inf_{\partial\Omega} [1-\Phi^2(x)s^2(x)]^{1/2} / \sup_{\partial\Omega} \Phi(x) + O(\xi), \end{aligned}$$

where $O(\xi)$ denotes a term of order ξ . Then

$$\begin{aligned} 2^{-1}B\delta - 2\sup_\Omega |u(x,0)| &\geq 2^{-1}\delta \inf_{\partial\Omega} [1-\Phi^2(x)s^2(x)]^{1/2} / \sup_{\partial\Omega} \Phi(x) \\ &\quad - 2\sup_\Omega |u(x,0)| + O(\xi). \end{aligned}$$

Since the inequality in (2.20) is strict, one can choose $\xi > 0$ so small that

$$2^{-1}B\delta - 2\sup_\Omega |u(x,0)| \geq 0$$

and (6.5) is proved.

In order to complete the case (a) we need to show that

$$|u(x,0)| \leq \tilde{g}(x,0) \text{ in } \overline{\Omega}_\delta. \quad (6.6)$$

We have for $\tau \in [0, \delta]$

$$\begin{aligned} B(\tau - \tau^2 2^{-1}) - |u((1-\tau)r(v), 0)| \\ = \{\inf_{\partial\Omega} \Phi^{-2}(x)[1-\Phi^2(x)s^2(x)-\xi]\}^{1/2}(\tau - \tau^2 2^{-1}) - |u((1-\tau)r(v), 0)| \\ \geq \inf_{\partial\Omega} \{\Phi^{-1}(x)[1-\Phi^2(x)s^2(x)]\}^{1/2} 2^{-1} \tau - |u((1-\tau)r(v), 0)| + O(\xi). \end{aligned}$$

Consequently, decreasing ξ , if necessary, and taking into account (2.21) we obtain (6.6).

According to the maximum principle the inequalities (6.4)-(6.6) imply that

$$|u(x,t)| \leq \tilde{g}(x,t) \text{ in } \overline{\Omega}_\delta \times [0, \infty).$$

Evidently,

$$\tilde{g}(x,t) \leq e^{-\sigma t} \{\inf_{\partial\Omega} \Phi^{-2}(x)[1-\Phi^2(x)s^2(x)]\}^{1/2} [1 - |x|/s(x)].$$

Choose now a constant B' so that $|u(x,t)| \leq B'\tilde{g}(x,t)$ in $\Omega \setminus \Omega_\delta$. This is possible, since (3.3) holds. Setting $C_2 = \max \{B', 1\}$ we obtain (2.22).

It remains to consider the

Case (b). In this case $\sup_{\partial\Omega} \Phi(x) = 0$. Then we take $f(\tau) = 1-\tau$ and $g(\tau) = B(\tau - \tau^2/2)$ with $B = 8\sup_{\Omega} |u(x,0)|$. It follows from (6.1) that $nH(g) \leq -B\langle r, \nu \rangle q^{-3} < 0$ and this implies (6.4) for the modified functions g and $\tilde{g}$.

Next we take $\delta = 1/2$ and verify that on the hypersurface Y $2\sup_{\Omega} |u(x,0)| \leq g(x)$. This implies (6.5).

In order to satisfy (6.6) we increase the constant B (if necessary). The estimate (2.22) follows now from the maximum principle (possibly after adjusting the constant B once again to make it valid in Ω).

The theorem is proved.

6.3. Corollary. Let Ω be a domain starshaped relative to some point $x_0 \in \Omega$, and let $u(x,t)$ be a solution of (1.1)-(1.3) such that u, u_i are in $C(\bar{\Omega} \times [0, \infty))$ and u_t, u_{ij} are in $C(\Omega \times (0, \infty))$. Suppose, also, that conditions (2.21) and (2.20) in Theorem E are satisfied with $\delta = \text{dist}(\partial\Omega, \text{supp } u_0)$. Then there exist positive constants C_1

$$|u(x,t)| \leq C_1 \left[1 - \frac{|x-x_0|}{s(x)}\right] e^{-\sigma t} \quad \text{in } \bar{\Omega} \times [0, \infty). \quad (6.7)$$

Proof. Since $\text{supp } u_0 \subset\subset \Omega$ has compact support the condition (2.21) is also satisfied and the Corollary follows from Theorem E.

6.4. Proposition. Suppose the conditions of the Corollary 6.3 are satisfied. Then

$$|Du(x,t)| \leq C_2 e^{-\sigma t} \text{ on } \bar{\Omega} \times [0, \infty), \quad (6.8)$$

where C_2 is a positive constant.

Proof. Put $\omega^\pm(x,t) = \pm C_1 W(x) e^{-\sigma t}$, where C_1 and σ are as in (6.7) and $W(x) = 1 - |x-x_0|/s(x)$. The normal derivative of W on $\partial\Omega$ is given by

$$\frac{\partial W(x)}{\partial \nu} = \frac{1}{s(x)} [\langle \text{grad } s(x), \nu(x) \rangle - \frac{x-x_0}{|x-x_0|}, \nu(x) \rangle.$$

Since $W(x) = 0$ on $\partial\Omega$ and $s(x) > 0$ on $\partial\Omega$, $\sup_{\partial\Omega} |D\omega^\pm| \leq C_3 e^{-\sigma t}$.

Because $u(x,t) = 0$ on $\partial\Omega \times [0, \infty)$ and, since (6.7) holds, we get

$$|Du(x,t)| \leq C_3 e^{-\sigma t} \text{ on } \partial\Omega \times [0, \infty).$$

Let $Q = \Omega \times (0, \infty)$ and ∂Q be the boundary of Q . It is well known (see, for example, [LSU], remark 3.1 in §3, Ch. VI) that

$$\sup_Q |Du(x,t)| = \sup_{\partial Q} |Du(x,t)|.$$

Consequently, in $\bar{Q}$

$$|Du(x,t)| \leq \max \{ \sup_{\Omega} |Du_0(x)|, C_3 e^{-\sigma t} \}.$$

Obviously, this implies (6.8).

6.5. Corollary. Suppose the conditions of the proposition (6.3) are satisfied. Then the problem (1.1)-(1.3) admits a unique solution $u(x,t) \in C^\infty(\bar{\Omega} \times [0,\infty))$.

Proof. The a priori estimate (6.8) implies that equation (1.1) is uniformly parabolic in $\bar{\Omega} \times [0,\infty)$. then, according to standard results on quasilinear uniformly parabolic equations (see [LSU], Ch. VI), the problem (1.1)-(1.3) admits a solution $u(x,t) \in C^\infty(\bar{\Omega} \times [0,\infty))$. By the maximum principle this solution is unique.

Remark. The exponent σ in (6.8) can be improved; namely, it can be shown that (6.8) holds with σ replaced by any $\lambda \in [\sigma, \lambda_1)$. In any case, since a smooth solution of (1.1)-(1.3) is also a generalized solution the asymptotic properties (2.17) and (2.18) hold in this case as well.

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Mean curvature flow on an annulus

FIGURE

Inner radius = .1, outer radius = 2.0; $t = .1$ corresponds to 100 iterates with time step $\Delta t = .001$; initial surface is a graph of a function with compact support with height = 1.9.

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