MORE ZEROS OF KRAWTCHOUK POLYNOMIALS

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More zeros of Krawtchouk polynomials

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Abstract. Three theorems are given for the integral zeros of Krawtchouk polynomials. First, five new infinite families of integral zeros for the binary ($q = 2$) Krawtchouk polynomials are found. Next, a lower bound is given for the next integral zero for the degree four polynomial. Finally, three new infinite families in q are found for the degree three polynomials. The techniques used are from elementary number theory.

1. Introduction.

The Krawtchouk polynomials are of central importance in coding theory. In particular, the existence (or non-existence) of integral zeros of these polynomials is crucial for the existence (or non-existence) of combinatorial structures in the Hamming association schemes [2], [3], [5], [6], [7]. This paper studies these zeros, and is a continuation of [4]. The simultaneous integrality of all of the zeros has been studied by Hong [8].

This paper has three main results. The first, in §2, is a set of five new infinite families of integral zeros for the binary Krawtchouk polynomials, $k_n(x, 2, N)$. With these new families, the list of integral zeros for $k_n(x, 2, N)$, $N \leq 700$, in [4] contains ten zeros which do not lie in an infinite family: four each for degrees four and five, and two for degree six. It has been conjectured [7] that these four zeros for degree four are all such zeros. In §3 we show indeed that there are no more zeros if N has at most one hundred million digits. Finally, in §4 we give a three new infinite families of zeros, which depend upon the parameter q.

We now set the notation and terminology for the polynomials. The Krawtchouk polynomials are defined for $n \leq N$ by

$$k_n(x, q, N) = \sum_{j=0}^{n} (-1)^j (q-1)^{n-j} \binom{N-x}{n-j} \binom{x}{j}, \tag{1.1}$$

so that these are the eigenmatrices for the Hamming scheme $H(N, q)$ [2]. Clearly $k_n(x, q, N)$ is a polynomial of degree n in x.

For $q = 2$ it is easy to see that there is a group of order eight which acts upon the integral zeros. It is generated by two involutions: $(n, x, N) \leftrightarrow (x, n, N)$ and $(n, x, N) \leftrightarrow (n, N - x, N)$. We call two zeros equivalent if they belong to the same orbit under this group. For a representative in an orbit, we can assume that $x \leq n \leq N/2$. It is well-known [7] that $n = N/2$ and x odd is also a zero. Moreover the integral zeros for degrees one, two, and three are known [4]. This motivates the following definition.

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DEFINITION. *A non-trivial integral zero of $k_n(x,2,N)$ is an ordered triple of positive integers equivalent to some (n,x,N), with $4 \le n \le x < N/2$ and $k_n(x,2.N) = 0$.*

For $q > 2$ the group has order four and is generated by $(n,x,N) \leftrightarrow (x,n,N)$ and $(n,x,N) \leftrightarrow (N-n,N-x,N)$. The degree two case can be solved explicitly [5, Th. 4.2]: $(2,x,N)$ is an integral zero if, and only if, $x = (y^2+y)/q - y$ and $N = (y^2-(q-2)y)/(q-1)$ for some integer y. So we can assume that $3 \le n \le x \le \min(x, N-x)$.

DEFINITION. *A non-trivial integral zero of $k_n(x,q,N)$, $q \ge 3$, is an ordered triple of positive integers equivalent to some (n,x,N), with $3 \le n \le x \le min(x, N-x)$ and $k_n(x,q,N) = 0$.*

2. Five infinite families for $q = 2$.

In this section we give the five new infinite families of zeros for the Krawtchouk polynomials $k_n(x,2,N)$.

We use another explicit expression for the polynomials [4]

$$k_n(x,2,N) = \sum_{r=0}^{x/2} \binom{n}{r}\binom{N-2n}{x-2r}(-1)^r. \tag{2.1}$$

Let $N = 2n + t$ in (2.1), so that (2.1) has $\lfloor t/2 \rfloor + 1$ terms. If (2.1) is multiplied by $x!(n - x + \lfloor t/2 \rfloor)!$, we find a polynomial expression in $x/2$ (resp. $(x-1)/2$) for x even (resp. odd) of degree $\lfloor t/2 \rfloor$ (resp. $\lfloor (t-1)/2 \rfloor$). For small values of t the zeros can be explicitly found, and they are listed below. For $t > 7$ and x even, or for $t > 6, t \ne 8$, and x odd, the polynomials are cubic. The solutions are not given.

For x even:

(1) $t = 2$, $x/2 = (n+1)/2$,
(2) $t = 3$, $x/2 = (n+1)/4$,
(3) $t = 4$, $x/2 = (2n+4 \pm \sqrt{2(n^2+5n+6)})/4$,
(4) $t = 5$, $x/2 = (3n+7 \pm \sqrt{5n^2+30n+41})/8$,
(5) $t = 6$, $x/2 = (n+3)/2$, or $x = (2n+6 \pm \sqrt{3n^2+21n+34})/4$.

For x odd:

(1) $t = 3$, $(x-1)/2 = (3n+3)/4$,
(2) $t = 4$, $(x-1)/2 = (n+1)/2$,
(3) $t = 5$, $(x-1)/2 = (5n+9 \pm \sqrt{5n^2+30n+41})/8$,
(4) $t = 6$, $(x-1)/2 = (2n+4 \pm \sqrt{n^2+7n+10})/4$,
(5) $t = 8$, $(x-1)/2 = (n+3)/2$, or $x = (2n+6 \pm \sqrt{2(n^2+9n+16)})/4$.

Which of these solutions represent new families of zeros? For x even, trivial zeros are given by $t = 2$ and $t = 6$ and $x/2 = (n+3)/2$; for x odd, the trivial zeros are $t = 4$ and $t = 8$ and $(x-1)/2 = (n+3)/2$. The solutions for $t = 3$ are given in [4], $(2h, 4h-1, 8h+1)$. It is easy to see, using the involution which maps x to $N - x$, that the solutions for $t = 5$ are equivalent. This leaves five families of zeros, which are our five new families.

We must find the values of n so that in the above formulas x is an integer. This can be done for each case by the explicit solution to Pell's equation [9, p. 204]. We carry out the details for $t = 4$ and x even.

In this case we must have

$$2n^2 + 10n + 12 = \delta^2,$$

where δ is an integer, so

$$(2n+5)^2 - 2\delta^2 = 1$$

is our Pell's equation. The solutions δ are given by

$$2n + 5 + \delta\sqrt{5} = \pm(3 + 2\sqrt{2})^m, m \in \mathbf{Z}.$$

Since $2n + 5 \geq 5$, we must take $m \geq 2$, and find

$$\begin{aligned} n = &\ ((3+2\sqrt{2})^m + (3-2\sqrt{2})^m - 20)/4 \\ x = &\ n + 2 - \delta/2 = n + 2 - ((3+2\sqrt{2})^m - (3-2\sqrt{2})^m - 20)/4\sqrt{2} \\ N = &\ 2n + 4 \end{aligned}$$

For $t = 6$ and x odd there are no positive integral solutions n. We collect the remaining cases to form the main result of this section.

THEOREM 1. *The Krawtchouk polynomial $k_n(x, 2, N)$ has inequivalent non-trivial integral zeros (x, n, N), at the following values:*

(1) *for $m \geq 2$ and $\rho = 3 + 2\sqrt{2}$,*

$$\begin{aligned} n =& (\rho^m + \rho^{-m} - 20)/4 \\ x =& N/2 - (\rho^m - \rho^{-m})/2\sqrt{2} \\ N =& 2n + 4, \end{aligned}$$

(2) *for $m \geq 2$ and $\rho = 9 + 4\sqrt{5}$,*

$$\begin{aligned} n =& ((\sqrt{5} \pm 1)\rho^m + (\sqrt{5} \mp 1)\rho^{-m})/2\sqrt{5} - 3 \\ x =& (3n+7)/4 \mp ((\sqrt{5} \pm 1)\rho^m - (\sqrt{5} \mp 1)\rho^{-m})/8 \\ N =& 2n + 5, \end{aligned}$$

(3) *for* $m \geq 2$ *and* $\rho = 9 + 4\sqrt{5}$,

$$\begin{aligned} n =&((2\sqrt{5} \pm 4)\rho^m - (2\sqrt{5} \mp 4)\rho^{-m})/2\sqrt{5} - 3 \\ x =&(3n+7)/4 \mp ((2\sqrt{5} \pm 4)\rho^m + (2\sqrt{5} \mp 4)\rho^{-m})/8 \\ N =&2n+5, \end{aligned}$$

(4) *for* $m \geq 2$ *odd and* $\rho = 2 + \sqrt{3}$,

$$\begin{aligned} n =&((2\sqrt{3} \pm 1)\rho^m + (2\sqrt{3} \mp 1)\rho^{-m})/4\sqrt{3} - 7/2 \\ x =&n + 3 - (2\sqrt{3} \pm 1)\rho^m + (2\sqrt{3} \mp 1)\rho^{-m})/8 \\ N =&2n+6, \end{aligned}$$

(5) *for* $m \geq 2$ *and* $\rho = 3 + 2\sqrt{2}$,

$$\begin{aligned} n =&((5 \pm 2\sqrt{2})\rho^m + (5 \mp 2\sqrt{2})\rho^{-m})/4 - 9/2 \\ x =&n + 4 - ((5 \pm 2\sqrt{2})\rho^m + (5 \mp 2\sqrt{2})\rho^{-m})/4\sqrt{2} \\ N =&2n+8. \end{aligned}$$

3. Zeros of $k_4(x, 2, N)$.

The quartic equation $k_4(x, 2, N) = 0$ has a finite number of non-trivial solutions. This follows from a well-known theorem on hyperelliptic equations in [1 ,p. 41]. An explicit upper bound for the size of the solutions can be given from [1 , p.45]. The complete finite list of solutions is not known; however, in [7] it was conjectured that only non-trivial integral zeros are $(4, 7, 17)$, $(4, 30, 66)$, $(4, 715, 1521)$, and $(4, 7476, 15043)$. In this section we give a lower bound for the next non-trivial zero. It is unfortunately much smaller than the theoretical upper bound.

THEOREM 2. *Suppose there exists* $N > 15043$ *for which* $k_4(x, 2, N) = 0$ *has a non-trivial integral zero. Then* N *has at least one hundred million digits.*

First we rewrite the equation $k_4(x, 2, N) = 0$ as a Pell's equation. If $y = N - 2x$, it is

$$(2y^2 - 1)^2 - 6(N - 1 - y^2)^2 = -5. \tag{3.1}$$

The solutions to (3.1) are

$$2y^2 - 1 \pm (N - 1 - y^2)\sqrt{6} = (\pm\sqrt{6} \pm 1)(5 + 2\sqrt{6})^m, \text{for } m \in \mathbb{N}. \tag{3.2}$$

For an element $A + B\sqrt{6} \in \mathbb{Q}[\sqrt{6}]$, we let $Re(A + B\sqrt{6}) = A$ and $Im(A + B\sqrt{6}) = B$. By expanding $Re((\pm 1 \pm \sqrt{6})(5 + 2\sqrt{6})^m)$ and using $x \leq N/2$, we see that we must use $(\sqrt{6} \pm 1)$ on the right side of (3.2). Thus if we put

$$\begin{aligned} \alpha_m &= \frac{1}{2} Re((\sqrt{6} + 1)(5 + 2\sqrt{6})^m + 1) \\ \beta_m &= \frac{1}{2} Re((\sqrt{6} - 1)(5 + 2\sqrt{6})^m + 1), \end{aligned} \tag{3.3}$$

we find a integral zero exactly when either α_m or β_m is a square. Note that $\alpha_0 = 1$, $\alpha_1 = 9$, $\alpha_4 = 8281$, $\beta_0 = 0$, $\beta_1 = 4$, and $\beta_2 = 36$ are squares.

It is easy to find the following recurrences, generating functions, and explicit formulas

$$\begin{aligned} \alpha_{m+1} &= 12\alpha_m - 5\beta_m - 3 \\ \beta_{m+1} &= 5\alpha_m - 2\beta_m - 1 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \sum_{m=0}^{\infty} \alpha_m t^m &= \frac{(1 - 3t)(1 + t)}{(1 - t)(1 - 10t + t^2)} \\ \sum_{m=0}^{\infty} \beta_m t^m &= \frac{4t(1 - 2t)}{(1 - t)(1 - 10t + t^2)}. \end{aligned} \tag{3.5}$$

$$\begin{aligned} \alpha_m &= ((5 + 2\sqrt{6})^m(1 + \sqrt{6}) + (5 - 2\sqrt{6})^m(1 - \sqrt{6}) + 2)/4 \\ \beta_m &= ((5 + 2\sqrt{6})^m(-1 + \sqrt{6}) + (5 - 2\sqrt{6})^m(-1 - \sqrt{6}) + 2)/4 \end{aligned} \tag{3.6}$$

Theorem 2 will follow from the next proposition.

PROPOSITION 1. *If α_m and β_m are not squares for $4 < m < M$, then N for the next non-trivial zero must have at least $.99M$ digits.*

Proof. From (3.6) it is clear that both α_m and β_m grow exponentially in m, $c(5 + 2\sqrt{6})^m$. An explicit computation shows that $Im(\sqrt{6} \pm 1)(5 + 2\sqrt{6})^m)$ also grows exponentially. From (3.2) we find that N grows exponentially in m, a lower bound is $.066(5 + 2\sqrt{6})^m)$. Since $log(5 + 2\sqrt{6}) \approx .995$, the result follows.

Our goal is to prove that for many m, α_m and β_m are not squares. We do this by eliminating m in certain residue classes. Fix a positive integer k and consider $\alpha_m \mod k$. This sequence is periodic, because $\alpha_m \mod k$ satisfies a three-term recurrence relation with a finite number of possible initial conditions. Let $P_A(k)$ be the period of this sequence. If we happen to know that $\alpha_i \mod k$ is not a square $\mod k$, then no α_m, with $m \equiv i \mod P_A(k)$ could be a square. This is our basic technique.

For example, let $k = 5$ for which $P_A(5) = 4$. Since $\alpha_3 = 837 \equiv 2 \mod 5$ which is not a square $\mod 5$, α_m is not a square when $m \equiv 3 \mod 4$. In the Appendix many choices of k are given, which eliminate all but the following classes for m.

PROPOSITION 2. *If* $m \not\equiv 0, 1, 4, 58198140$ *or* $89008924 \mod 116396280$, *then* α_m *is not a square.*

For β_m, the Appendix gives a similar result.

PROPOSITION 3. *If* $m \not\equiv 0, 1, 2, 98017921$ *or* $116396280 \mod 232792560$, *then* β_m *is not a square.*

4. Infinite families in q.

The numerical evidence indicates that there are fewer non-trivial zeros for $q > 2$ than $q = 2$. In this section we give the first infinite families in q. Each zero occurs for a cubic polynomial. It has previously been shown [5, Th. 4.14] that for fixed q, $k_3(x, q, N) = 0$ has a finite number of solutions.

THEOREM 3. *The following values of* x *and* N *give non-trivial integral zeros for* $k_3(x, q, N)$*:*

$$(1) \qquad \begin{aligned} x &= (q-1)^2(2q+3)(2q^2-5q+3)/27 \\ N &= (2q+3)(2q^4-7q^3+8q^2-12q+18)/27, \quad \text{if} \quad q \equiv 3, 4, 6 \quad \text{or} \quad 7 \mod 9 \end{aligned}$$

$$(2) \qquad \begin{aligned} x &= 2(2q+1)(q^2-q-3)(4q^2-10q+3)/27 \\ N &= 2q(2q+1)(4q^3-10q^2-9q+27)/27, \quad \text{if} \quad q \equiv 3, 4, 6 \quad \text{or} \quad 7 \mod 9 \end{aligned}$$

$$(3) \qquad \begin{aligned} x &= (q-3)(q+2)(2q-5)(2q^2+q+3)/108 \\ N &= (2q^2+q+3)(2q^3-5q^2-12q+36)/108, \\ &\quad \text{if } q \equiv 3, 4, 6 \quad \text{or} \quad 7 \mod 9 \quad \text{and} \quad q \equiv 2 \quad \text{or} \quad 3 \mod 4. \end{aligned}$$

Proof. A calculation shows that the results are correct, but we show in fact how to derive these formulas. Again we change variables, putting $t = qx$ and $y = N(q-1) - qx$. The equation $k_3(x, q, N) = 0$ becomes

$$(4.1) \qquad y(y-(q-1))(y-2(q-1)) = (3y-2(q-2))t.$$

If we put $3y - 2(q-2) = i$, then (4.1) is equivalent to

$$27t = (i+2q-4)(i-q-1)(i-4q+2)/i.$$

Because t is an integer, we must have

$$(4.2) \qquad 3y - 2(q-2) \mid 4(q+1)(q-2)(2q-1).$$

Thus the divisibility condition (4.2) is our key necessary condition for integral solutions to (4.1). This also shows the number of solutions for a fixed q is finite.

It remains to put $3y - 2(q-2) = d$, for a divisor d of the right side of (4.2). The possible choices for d are multiples $\pm 1, \pm 2, \pm 4$, of 1, $(q+1)$, $(q-2)$, $(2q-1)$, $(q+1)(q-2)$, $(q-2)(2q-1)$, and $(q+1)(q-2)(2q-1)$. We may also use the fact that $(q-2)(q+1)$ is even to choose d to be $\pm(q-2)(q+1)/2$ or $\pm(q-2)(q+1)(2q-1)/2$. This gives 52 cases to check the necessary congruences $t \equiv 0 \mod q$ and $y + t \equiv 0 \mod q-1$.

We find 52 more cases in the following way. If $q \equiv 2 \mod 3$, $q = 3\theta + 2$, then $3y - 2(q-2) = 3(y - 2\theta)$ contains the factor 3. We find 52 cases as in the previous paragraph for $y - 2\theta | 4\theta(\theta+1)(2\theta+1)$.

We will explicitly do two of these 104 cases, and then list the results.

First take $3y - 2(q-2) = 1$, so that $q \equiv 0 \mod 3$ and (4.1) becomes

$$27t = q(2q-3)(4q-3),$$

which implies

$$27(y+t) = (2q-3)(4q^2-3q+9) = (2(q-1)-1)(4(q-1)^2+5(q-1)+10). \tag{4.3}$$

Since $y + t \equiv 0 \mod q-1$, (4.3) implies that $10 \equiv 0 \mod q-1$. The solutions are $q = 3$, $y = 1$, $t = 3$, $x = 1$, $N = 2$, and $q = 6$, $y = 3$, $t = 42$, $x = 7$, $N = 9$, which are both trivial.

Secondly, take $3y - 2(q-2) = -(q-2)(q+1)(2q-1)$, or $3y = -(q-1)(q-2)(2q+3)$. We find that

$$27t = q(q-1)^2(2q+3)(2q^2-5q+3)$$

and

$$27(y+t) = (q-1)(2q+3)(2q^4-7q^3+8q^2-12q+18).$$

Clearly the modular conditions for t and $y + t$ hold, so the solutions $x = t/q$ and $N = (y+t)/(q-1)$ give the first infinite family of Theorem 3.

The other two infinite families correspond to the choices $3y-2(q-2) = -(q-2)(q+1)/2$ and $3y - 2(q-2) = 2(q-2)(q+1)(2q-1)$. There are also five sporadic non-trivial zeros which occur. They are $(3, 3212, 3432)$ for $q = 14$, $(3, 1326, 1379)$ and $(3, 5526, 5833)$ for $q = 21$, (3,86736,89377) for $q = 35$, and $(3, 46102, 46992)$ for $q = 56$. □

Note that for a given value of q, there are very likely many more than the 104 cases in the proof of Theorem 3. We checked by computer all values of $q \leq 100$. Only one more non-trivial zero was found: $(3, 162, 170)$ for $q = 13$.

5. Remarks.

With the infinite families given in Theorem 1, there remain exactly six non-trivial zeros for $N \leq 700$ which do not lie in infinite families: two for degree four: $(4, 715, 1521)$ and

$(4, 7476, 15043)$, three for degree five: $(5, 22, 67)$, $(5, 28, 67)$, and $(5, 133, 289)$, and one for degree six: $(6, 155, 345)$.

It can be shown that $k_4(x, q, N) = 0$ has finitely many solutions for a fixed $q \geq 3$. We conjecture that the same statement holds for any degree $n > 4$.

Appendix.

In this Appendix we list the residue classes eliminated in §3 for choices of the modulus k. Given a period P, integers k such that $P(k) = P$ can be found in the following way. If α has period $P \mod k$, then $k|(\alpha_P - \alpha_0)$ and $k|(\alpha_{P+1} - \alpha_1)$. Thus k divides the greatest common divisor of $\alpha_P - \alpha_0$ and $\alpha_{P+1} - \alpha_1$, and any factor of the greatest common divisor will have a period dividing P. For example, if $P = 7$, we find that

$$gcd(\alpha_7 - \alpha_0, \alpha_8 - \alpha_1) = 4316 = 2^2 \times 13 \times 83$$

gives the values of $k = 13$ and $k = 83$ for $P = 7$ below. (In fact it appears that the greatest common divisors are the same for α and β.)

The computations were completed using MAPLE.

Residue classes for α_m

k	P(k)	Residue classes mod P(k) eliminated
5	4	3
7	8	5
9	6	5
8	4	2,3
11	3	2
97	24	8,9,20,21
9601	24	16

Note that the residue classes mod 24 which remain thus far are 0,1,4, and 12.

k	P(k)	Residue classes mod P(k) eliminated
17	18	5,6,12,14,15,17
19	18	5,6,15,16
73	36	3,5,6,7,10,11,15,20,21,22,23,27,28,29,30,31,33
81	18	3,5,11,15,17
971	9	2,3,5,7
91009	72	49

The residue classes mod 72 which remain thus far are 0,1,4, and 36.

k	P(k)	Residue classes mod P(k) eliminated
109	5	2
89	10	8,9
59	30	3,5,6,9,10,11,17,19,20,21,23,24,25,26,28,29
179	30	5,9,11,19,23,24,25,27,29
8641	15	2,3,5,8,13
1901	60	16

The residue classes mod 360 which remain are 0,1,4, and 360.

k	P(k)	Residue classes mod P(k) eliminated
13	7	2,3
29	28	2,5,6,8,9,15,16,18,19,22,25,27
83	7	2,6
881	14	2,11,12,13
32117	28	21 (and more)
41	42	2,3,6,7,8,9,18,27,28,29,30,33,34,37,38,40,41
251	63	22,43
71	35	11

The residue classes mod 2520 which remain are 0,1,4, and 1260.

By continuing in this way, k may be chosen so that the period $P(k)$ is divisible by 11, 13, 17, and 19. These four more cases, and the above result give Proposition 2 because $2520 \times 11 \times 13 \times 17 \times 19 = 116396280$. The values 58198160 and 89008924 are eliminated by $k = 11593, P(k) = 46$, residue class 44, and $k = 7006537, P(k) = 46$, residue class 28.

Residue classes for β_m

k	P(k)	Residue classes mod P(k) eliminated
5	4	3
49	8	6,7
8	16	4,5,6,7,9,10,12,13,14,15

The residue classes mod 16 which remain are 0,1,2, and 8.

k	P(k)	Residue classes mod P(k) eliminated
27	6	4
97	12	9,10
9601	24	4,6,7,9,15,17,18,19,20
17	18	3,4,10,12,13,15
19	18	3,4,11,12
81	18	4,6,10,12,16
73	36	3,7,8,11,12,13,15,21,23,24,25,26,27,31,32,33,34

12889	36	5,16
91009	72	14,50,56
9727489	72	37,53

The residue classes mod 144 which remain are 0,1,2, and 72.

k	P(k)	Residue classes mod P(k) eliminated
109	5	4
89	10	6,7
25	20	3,4,7,8,11,12,15,16,19
1901	20	13,18
79	80	10,21,22,24,30,60,61
884376377281	45	20
92188801	40	25

The residue classes mod 720 which remain are 0,1,2, and 360.

Again by continuing to insert the primes 7, 11, 13, 17, and 19, we find Proposition 3. The values 98017922 and 116396280 are eliminated by $k = 11593, P(k) = 46$, residue class 17, and $k = 47, P(k) = 23$, and residue class 18.

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