**FUNCTIONAL DIFFERENTIAL EQUATIONS
FOR THE DETERMINATION OF THE VISCOSITY
FUNCTION IN A RHEOMETER**

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FUNCTIONAL DIFFERENTIAL EQUATIONS FOR THE DETERMINATION OF THE VISCOSITY FUNCTION IN A RHEOMETER

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§1. Introduction. The helical screw rheometer is an instrument developed by Kraynik et al. [2] [3] for obtaining the viscosity function of simple non-Newtonian fluids. In a uni-directional shearing motion the viscosity function is defined as the ratio between the shear stress and the shear rate. The instrument consists of a screw rotating inside a close fitting cylinder (figure 1). The fluid whose properties are being determined fills the gap between the screw and the cylinder. The operating characteristics of this rheometer benefit measurements of the viscosity function at high temperature, high pressure, and for chemically reacting flows. As opposed to all other rheometers involving rotating flow fields, the helical screw rheometer does not require one to take torque measurements.

Figure 1

In what follows we will consider the screw to consist of an inner cylinder and a helical protuberance attached to it. The helical protuberance will be referred to as the ramp. When the screw is rotating steadily it has been observed by numerical computations and by flow visualization techniques that the velocity and pressure field are extremely simple.

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In particular, if one is not too near the ramp, the velocity field is found to be purely circumferential, and to be independent of axial and angular coordinates.

The pressure field away from the ramp can be shown to be the sum of a pressure field depending only on the radius, and a pressure field independent of the axial and radial coordinates having a constant angular gradient. This pressure field has the property that there is never any local axial pressure gradient, but there is a global pressure difference between a point A near the top of the instrument and a point B directly under it near the bottom. To get from point A to point B one can follow a helical path so that one never hits the ramp, and the pressure continually changes as one descends. One can also descend straight down from point A to point B , but in this case one will periodically hit the surface of the ramp. The pressure remains constant as one descends, but when one hits the ramp there is a discontinuity of $2\pi \frac{dp}{d\theta}$ between the pressure at the top and bottom surfaces of the ramp. In practice the pressure gradient is determined by measuring the pressure difference between a point A and a point B directly below it. The pressure gradient is then given by

$$(1.1) \quad dp/d\theta = (p(A) - p(B))/(2\pi N) ,$$

where N is the number of times one hits the ramp in descending straight down from A to B .

The distance between the inner and outer cylinder of the rheometer is much less than the radius of the inner cylinder. For this reason it is possible to ignore the curvature of the cylinders, and to approximate the flow field as though it were planar. Then the flow field looks like it is confined between two flat plates a distance H apart. The top plate is at rest, and the bottom plate moves with some velocity V . At the left edge of the lower plate there is the ramp which pushes the fluid. This is depicted in figure 2.

Figure 2

To obtain the viscosity function of a fluid using the helical screw rheometer, one must know the pressure gradient as a function of V , the velocity of the lower plate. It can be shown that this function is monotonic with respect to the velocity, so we can assume that we know V as a function of the pressure gradient. We will now derive a set of equations that determine the viscosity function once the velocity of the lower plate is known as a function of the pressure gradient.

Let y be the distance from the bottom plate. It will be convenient to use a dimensionless variable ξ defined by

$$(1.2a) \quad y = \xi H ,$$

and a pressure gradient scaled according to H

$$(1.2b) \quad \alpha = H dp/dx .$$

If we are several thicknesses H away from the ramp in figure 2, all variables are independent of x except for p which has $dp/dx = \text{constant}$. The equation balancing momentum in the x direction can be written as

$$(1.3) \quad \partial\tau/\partial\xi - \alpha = 0 ,$$

where τ is the xy component of the stress tensor. It is seen that the shear stress τ varies linearly with ξ ,

$$(1.4) \quad \tau(\xi) = \xi\alpha + \tau_0(\alpha) .$$

Let $u(\xi)$ denote the velocity in the x direction. If we assume that the no slip boundary condition holds, then

$$(1.5) \quad \int_0^1 (\partial u/\partial\xi) d\xi = -V(\alpha) .$$

Conservation of mass clearly requires that

$$(1.6a) \quad \int_0^1 u(\xi) d\xi = V(\alpha) ;$$

if we integrate this equation by parts and use the no slip boundary condition, we get

$$(1.6b) \quad \int_0^1 \xi(\partial u/\partial\xi) d\xi = -V(\alpha) .$$

We assume that for any value τ of the shear stress there is a unique shear rate given by

$$(1.7) \quad \partial u/\partial y = \gamma(\tau) ;$$

also, the shear rate is an odd function of the shear stress,

$$(1.8) \quad \gamma(-\tau) = -\gamma(\tau) .$$

The function γ is of course different for different fluids. The purpose of the rheometer is to determine the viscosity function, which is equivalent to determining this function γ . If we substitute (1.3) and (1.7) into equations (1.5) and (1.6b), we arrive at some equations for determining the function $\gamma(\tau)$:

$$(1.9) \quad \int_0^1 \gamma(\alpha\xi + \tau_0(\alpha)) d\xi = k(\alpha) ,$$

$$(1.10) \quad \int_0^1 \xi \gamma(\alpha\xi + \tau_0(\alpha)) d\xi = k(\alpha) ,$$

where the function $k(\alpha)$ is defined by

$$(1.11) \quad k(\alpha) = -V(\alpha)/H .$$

The equations (1.8)–(1.10) are the final equations for the determination of the function γ . Note that these equations involve the known function $k(\alpha)$, and the unknown function $\tau_0(\alpha)$. Kraynik and Romero [4] have devised a numerical technique for determining $\gamma(\tau)$ and $\tau_0(\alpha)$ once $k(\alpha)$ is known. In this paper we prove the existence and uniqueness of a solution γ, τ_0 of (1.8)–(1.10). In §2 we transform (1.8)–(1.10) into a system of functional differential equations. In §3 we establish local existence and uniqueness. In §4 we derive a priori inequalities on the solution and then use them in order to extend the local solution into a (unique) global solution. This approach resembles somewhat the method used by Friedman and McLeod [1] to establish the existence and uniqueness of an analytic solution for a system of functional differentials arising in a refraction problem.

§2. Derivation of the functional differential equations. From the physical definition of $k(\alpha)$ one has that $k'(\alpha) > 0$.

Let $z = \alpha\xi + \tau_0(\alpha)$. Then (1.9), (1.10) take the form

$$(2.1) \quad \int_{\tau_0(\alpha)}^{\alpha + \tau_0(\alpha)} \gamma(z) dz = \alpha k(\alpha) ,$$

$$(2.2) \quad \int_{\tau_0(\alpha)}^{\alpha + \tau_0(\alpha)} (z - \tau_0(\alpha)) \gamma(z) dz = \alpha^2 k(\alpha) .$$

Differentiating (2.1) with respect to α we get

$$(2.3) \quad (1 + \dot{\tau}_0)\gamma(\alpha + \tau_0) - \dot{\tau}_0\gamma(\tau_0) = (\alpha k)_\alpha .$$

Differentiating (2.2) with respect to α we get, after some cancellations,

$$(2.4) \quad \gamma(\alpha + \tau_0)(1 + \dot{\tau}_0) - \dot{\tau}_0 k - 2k = \alpha k_\alpha .$$

Substituting $\gamma(\alpha + \tau_0)$ from (2.4) into (2.3), we find that

$$\dot{\tau}_0(\gamma(\tau_0) - k(\alpha)) = k(\alpha) .$$

and, using (1.8),

$$(2.5) \quad \dot{\tau}_0(-\gamma(-\tau_0) - k(\alpha)) = k(\alpha) .$$

In the sequel we shall study the system (2.4), (2.5) for $\alpha \geq 0$ with initial values

$$\tau_0(0) = 0 , \quad \gamma(0) = 0 .$$

From the solution to this problem we can construct immediately the solution to (1.8)–(1.10) for all α .

It will be shown that

$$\tau_0(\alpha) < 0, \quad \gamma(\alpha) > 0 \quad \text{for all } \alpha > 0 .$$

For simplicity of notation we set $\tau = \tau_0$; we also take $\alpha = y$, so that the system becomes

$$(2.6) \quad \tau'(y)(\gamma(-\tau(y)) + k(y)) = -k(y),$$

$$(2.7) \quad \gamma(y + \tau(y))(1 + \tau'(y)) - \tau'(y)k(y) = yk'(y) + 2k(y) ,$$

$$(2.8) \quad \tau(0) = \gamma(0) = 0 .$$

We assume that the given function k is in $C^1[0, \infty)$ and satisfies

$$(2.9) \quad k(y) > 0, k'(y) > 0 \quad \text{if } y \geq 0 , \quad \text{and} \quad k(0) = 0 .$$

Since $(k, \tau, \gamma) \rightarrow (\lambda k, \tau, \lambda \gamma)$ leaves the system invariant, we may normalized k by

$$(2.10) \quad k'(0) = 1 .$$

We also assume, for technical reasons, that

$$(2.11) \quad k \in C^2[0, \infty) .$$

In §3 we prove that (2.6)–(2.8) has a unique solution for $0 \leq y \leq \rho$, ρ small, with

$$(2.12) \quad \begin{aligned} \tau &\in C^2[0, \rho] , \quad \gamma \in C^1[0, \rho] , \\ \tau(y) &= -\frac{y}{3} + O(y^2) , \quad \gamma(y) = 6y + O(y^2) . \end{aligned}$$

In §4 we shall extend the solution to all $y > 0$.

§3. Local existence and uniqueness. We wish to establish existence of a solution for (2.6)–(2.8) by a fixed point theorem. Given $\tilde{\tau}(y)$, $\tilde{\gamma}(y)$, we wish to solve

$$(3.1) \quad \tau'(y)(\tilde{\gamma}(-\tilde{\tau}(y)) + k(y)) = -k(y) ,$$

$$(3.2) \quad \gamma(y + \tau(y))(1 + \tau'(y)) - \tau'(y)k(y) = yk'(y) + 2k(y) ,$$

$$(3.3) \quad \tau(0) = 0, \quad \gamma(0) = 0 ,$$

and show that the mapping S defined by

$$S(\tilde{\tau}, \tilde{\gamma}) = (\tau, \gamma)$$

has a fixed point. By analyzing the lowest order terms in the equations we anticipate a solution of the form

$$(3.4) \quad \begin{aligned} \tau(y) &= -\frac{y}{3} + T(y) , \quad 0 \leq y \leq \rho, \\ \gamma(y) &= 6y + \Gamma(y), \quad 0 \leq y \leq \rho/2, \end{aligned}$$

with T, Γ “small.” Denote by X the space of functions (T, Γ) where $T \in L^\infty(0, \rho)$, $\Gamma \in L^\infty(0, \frac{\rho}{2})$, and denote by $Y = X_{A,B,C,D}$ the subset of X defined by

$$\begin{aligned} |T(y)| &\leq A y^2, |T'(y)| \leq 2Ay, |T''(y)| \leq B \quad \text{for } 0 \leq y \leq \rho, \\ |\Gamma(y)| &\leq C y^2, |\Gamma'(y)| \leq Dy \quad \text{for } 0 \leq y \leq \frac{\rho}{2}, \end{aligned}$$

where A, B, C, D are positive constants. The derivatives T'', Γ' are taken in the weak sense; thus $T \in W^{2,\infty}[0, \rho]$ and $\Gamma \in W^{1,\infty}[0, \frac{\rho}{2}]$.

For any $(\tilde{T}, \tilde{\Gamma})$ in Y we substitute

$$(3.5) \quad \begin{aligned} \tilde{\tau}(y) &= -\frac{y}{3} + \tilde{T}(y), \\ \tilde{\gamma}(y) &= 6y + \tilde{\Gamma}(y) \end{aligned}$$

into (3.1) and write the solution of (3.1)–(3.3) in the form (3.4).

LEMMA 3.1. *There exists positive constants A, B, C, D , and ρ such that there exists a unique solution (T, Γ) of (3.1)–(3.3) with $(T, \Gamma) \in Y$.*

Proof. We begin by computing

$$(3.6) \quad \begin{aligned} \tilde{\gamma}(-\tilde{\tau}(y)) &= -6\tilde{\tau}(y) + \tilde{\Gamma}(-\tilde{\tau}(y)) \\ &= 6\left(\frac{y}{3} - \tilde{T}(y)\right) + \tilde{\Gamma}\left(\frac{y}{3} - \tilde{T}(y)\right) = 2y + \gamma_* , \\ |\gamma_*| &\leq 6A y^2 + C\left(\frac{y}{3} + Ay^2\right)^2 \leq \left(6A + \frac{C}{8}\right)y^2 \quad \text{if } \rho < \rho_0 \end{aligned}$$

where ρ_0 depends on A ; notice that since

$$-\tau(y) = \frac{y}{3} + O(y^2), \quad 0 \leq y \leq \rho,$$

the function $\tilde{\gamma}(-\tau(y))$ is well defined for $0 \leq y \leq \frac{\rho}{2}$ if ρ_0 is sufficiently small. Also, for $0 \leq y \leq \frac{\rho}{2}$,

$$(3.7a) \quad \begin{aligned} |\gamma'_*(y)| &\leq 6|\tilde{T}'| + |\tilde{\Gamma}'| \cdot \left| \frac{1}{3} - \tilde{T}' \right| + O(y^2) \\ &\leq \frac{101}{300} Dy \quad \text{if } D \gg A, \quad \rho_0 \text{ small;} \end{aligned}$$

hence

$$(3.7b) \quad |\gamma_*(y)| \leq \frac{1}{2} \frac{101}{300} D y^2.$$

It will be convenient to write $k(y) = y + K(y)$; then

$$|K(y)| \leq cy^2, |K'(y)| \leq cy, |K''(y)| \leq c \quad (c \text{ constant}).$$

We can solve (3.1) as follows:

$$(3.8) \quad \begin{aligned} \tau'(y) &= -\frac{y + K(y)}{3y + \gamma_*(y) + K(y)} = -\frac{1}{3} \frac{1 + \frac{K(y)}{y}}{1 + \frac{1}{3} \frac{\gamma_*(y)}{y} + \frac{K(y)}{3y}} \\ &= -\frac{1}{3} \left(1 + \frac{K(y)}{y} - \frac{1}{3} \frac{\gamma_*}{y} - \frac{K}{3y} \right) + O(y^2). \end{aligned}$$

Hence

$$T'(y) = \frac{\gamma_*}{9y} - \frac{2}{9} \frac{K}{y} + O(y^2).$$

Recalling (3.6) we get

$$(3.9) \quad |T'(y)| \leq \left(\frac{6}{9} A + \frac{C}{71} \right) y, \quad 0 \leq y < \rho$$

provided C is large enough depending on k , and ρ_0 is small enough depending on A, C .

Using (3.7a), (3.7b), we also get

$$(3.10) \quad |T''(y)| \leq \frac{3}{2} \frac{1}{9} \frac{102}{300} D < \frac{D}{17}, \quad 0 < y < \rho$$

provided ρ_0 is small enough as before and D is large enough depending on k . We now turn to solving (3.2). Since $y + \tau(y)$ varies in an interval containing $\{0 \leq y \leq \frac{\rho}{2}\}$, we can solve for $\gamma(x)$ directly:

$$(3.11) \quad \gamma(x) = \frac{yk'(y) + 2k(y) + \tau'(y)k(y)}{1 + \tau'(y)}$$

where $y = y(x)$ is defined by

$$x = y + \tau(y) = \frac{2}{3} y + T(y) ,$$

or

$$(3.12) \quad y = \frac{3}{2} x - T(y) .$$

We get

$$\begin{aligned} \gamma(x) &= \frac{\frac{8}{3} y + T'(y)k(y) + yK'(y) + \frac{5}{3}K(y)}{\frac{2}{3} + T'(y)} \\ &= 4y \left\{ 1 - \frac{3T'}{2} + \frac{3}{8y} (kT' + yK' + \frac{5}{3}K) \right\} + O(y^3) . \end{aligned}$$

Recalling (3.12) we get

$$(3.13) \quad \Gamma(x) = -6 T(y) - 3y \frac{3T'}{2} + \frac{3}{2} KT' + \frac{3}{2} yK' + \frac{5}{2} K + O(y^3) .$$

Using (3.9) we find that

$$|\Gamma(x)| \leq \frac{13}{2} \left(\frac{6}{9} A + \frac{C}{70} \right) y^2$$

if C is large enough depending on k and ρ_0 is small enough, or, by (3.12),

$$(3.14) \quad |\Gamma(x)| \leq \frac{9}{4} \frac{13}{2} \left(\frac{6}{9} A + \frac{C}{69} \right) x^2 .$$

Similarly, by (3.13), (3.9), (3.10),

$$\begin{aligned} |\Gamma'(x)| &\leq \frac{3}{2} \left\{ \frac{19}{2} |T'(y)| + \frac{9}{2} y |T''(y)| + 4K'(y) + \frac{3}{2} y |K''(y)| \right\} + O(y^2) \\ &\leq \frac{57}{4} \left(\frac{6}{9} A + \frac{C}{70} \right) y + \frac{27}{4} \frac{D}{17} y \end{aligned}$$

if C is large enough (depending on k), or

$$(3.15) \quad |\Gamma'(x)| \leq \frac{171}{8} \left(\frac{6}{9} A + \frac{C}{69} \right) x + \frac{81}{8} \frac{D}{17}.$$

We now have to show that (T, Γ) belongs to Y ; for this it suffices to prove that

$$(3.16) \quad \frac{6}{9} A + \frac{C}{71} \leq 2 A, \quad \frac{9}{4} \frac{11}{2} \left(\frac{6}{9} A + \frac{C}{69} \right) \leq C$$

and

$$(3.17) \quad \frac{D}{17} \leq B, \quad \frac{171}{8} \left(\frac{6}{9} A + \frac{C}{69} \right) + \frac{81}{8 \cdot 17} D \leq D.$$

Taking $A = \theta C$, (3.16) holds if

$$\frac{2}{3} \theta + \frac{1}{71} \leq 2\theta, \quad \frac{99}{8} \left(\frac{2\theta}{3} + \frac{1}{69} \right) \leq 1$$

which indeed is valid if, say, $\theta = \frac{3}{4} \cdot \frac{1}{71}$. As for (3.17) we simply choose $B = D/17$ and choose D large enough (depending on A, C).

THEOREM 3.2. *If ρ is positive and sufficiently small then there exists a unique solution (τ, γ) of (2.6)–(2.8) in an interval including $0 \leq y \leq \rho$, with $\tau \in C^2[0, \rho]$, $\gamma \in C^1[0, \rho]$. The solution has the form (3.4) with $T = O(y^2)$, $\Gamma = O(y^2)$.*

Proof. By Lemma 3.1 the mapping S maps Y into Y . Further, Y is a compact convex subset of X and, as easily checked, S is continuous on Y (with respect to the X -topology). Applying the Schauder fixed point theorem we conclude that S has a fixed point; this establishes existence.

To prove uniqueness suppose $(\tilde{\tau}, \tilde{\gamma})$ is another solution of (2.6)–(2.8). Then we can compute $\tilde{\tau}'(0)$ from (2.6) and $\tilde{\gamma}'(0)$ by differentiating (2.7), and thus deduce that

$$(3.18) \quad \tilde{\tau}(y) = -\frac{y}{3} + O(y^2), \quad \tilde{\gamma}(y) = 6y + O(y^2).$$

Set $T = \tilde{\tau} - \tau$, $\Gamma = \tilde{\gamma} - \gamma$ and define

$$\epsilon = \sup_{0 < y < \rho} \frac{|T'(y)|}{y}, \quad \delta = \sup_{0 < y < \frac{\rho}{2}} \frac{|\Gamma(y)|}{y^2}.$$

Subtracting the equations (2.6) for (τ, γ) and $(\tilde{\tau}, \tilde{\gamma})$ from one another we easily get, upon using (3.4), (3.18),

$$3y|T'(y)| \leq \frac{1 + O(y)}{3} \left(\frac{y}{3} + O(y^2) \right)^2 \delta + 2(1 + O(y))|T(y)|.$$

Hence

$$\frac{|T'(y)|}{y} \leq \left(\frac{\delta}{81} + \frac{\epsilon}{3} \right) (1 + O(\rho)) .$$

Taking “sup” on y we get

$$\epsilon \leq \left(\frac{\delta}{81} + \frac{\epsilon}{3} \right) (1 + O(\rho)) ,$$

so that

$$(3.19) \quad \epsilon \leq \frac{\delta}{54} (1 + O(\rho)) .$$

Next, subtracting the equations (2.7) for (τ, γ) and $(\tilde{\tau}, \tilde{\gamma})$ from one another, we find that

$$\frac{2}{3} |\Gamma(x)| \leq \left[\frac{2}{3} 6|T(y)| + 3y|T'(y)| \right] (1 + O(\rho))$$

where $x = y + \tau(y) \sim \frac{2}{3} y$. Hence

$$\frac{|\Gamma(x)|}{x^2} \leq \left(\frac{3}{2} \right)^3 5\epsilon(1 + O(\rho)) ;$$

taking the “sup” we get, after using (3.19),

$$\delta \leq \left(\frac{3}{2} \right)^3 5 \frac{\delta}{54} (1 + O(\rho)) = \frac{5}{16} \delta (1 + O(\rho)) .$$

Hence $\delta = 0$ if ρ is small enough, and therefore also $\epsilon = 0$ (by (3.19)). It follows that $\tau = \tilde{\tau}, \gamma = \tilde{\gamma}$.

REMARK 3.1. One can alternately prove uniqueness by showing that the mapping S defined above is a contraction.

§4. Existence of a global solution. In this section we prove:

THEOREM 4.1. *There exists a unique solution (τ, γ) of (2.6)–(2.8) for all $y > 0$, with $\tau \in C^2[0, \infty)$, $\gamma \in C^1[0, \infty)$; further,*

$$(4.1) \quad -1 < \tau'(y) < 0 ,$$

$$(4.2) \quad y + \tau(y) > -\tau(y) ,$$

$$(4.3) \quad \gamma(y) > 0$$

for all $y > 0$.

Notice that (4.1) ensures that the coefficient of τ' in (2.6) is nonzero and thus τ can be solved from (2.6). The inequality (4.2) ensures that the argument $y + \tau(y)$ of γ in (2.7) is larger than the argument $\tau(y)$ of γ in (2.6), and this should enable us to extend γ , from (2.7), once τ has been extended by solving (2.6).

The proof of Theorem 4.1 relies upon a priori estimates summarized in the following lemma.

LEMMA 4.2. Suppose there exist functions $\tau(y)$, $\gamma(x)$ for $x = y + \tau(y)$, $0 < y < y_0$ satisfying (2.6)–(2.8) and (4.1)–(4.3). Then there exists a positive constant η such that

$$(4.4) \quad \tau'(y) \leq -\eta ,$$

$$(4.5) \quad 1 + \tau'(y) \geq \eta ,$$

$$(4.6) \quad y + \tau(y) \geq -\tau(y) + \eta$$

for $0 < y < y_0$, and

$$(4.7) \quad yk'(y) + k(y) < \gamma(y) < \frac{1}{\eta}$$

for all $0 < y < y_0 + \tau(y_0)$.

Notice that since $|\tau'| \leq 1$ (by (4.4), (4.5)), $\tau(y_0) \equiv \tau(y_0 - 0)$ exists.

Proof. From (2.7)

$$(4.8) \quad \gamma(y + \tau(y)) = \frac{yk' + 2k + \tau'k}{1 + \tau'} \geq \frac{yk' + k}{1 + \tau'} > yk' + k > 0 .$$

In order to complete the proof of (4.7) we have to show that $\gamma(x)$ is bounded from above for $0 < x < y_0 + \tau(y_0)$. Suppose this is not true. Then

$$\gamma(y_n + \tau(y_n)) \rightarrow \infty \quad \text{for a sequence } y_n \rightarrow y_0 .$$

From (2.7) we then deduce that

$$\tau'(y_n) \rightarrow -1$$

and (2.6) yields

$$\tau'(y_n)\gamma(-\tau(y_n)) \rightarrow 0 .$$

Hence $\gamma(-\tau(y_n)) \rightarrow 0$, a contradiction to (4.8).

Next we prove (4.5). If this is not true than there exists a sequence $y_n \rightarrow y_0$ such that

$$1 + \tau'(y_n) \rightarrow 0 .$$

Taking $y = y_n$ in (2.7) we get

$$k(y_0) = y_0 k'(y_0) + 2k(y_0)$$

which is a contradiction.

To prove (4.4) suppose it is not true. Then there exists a sequence $y_n \rightarrow y_0$ such that $\tau'(y_n) \rightarrow 0$. Using this in (2.6) we get $k(y_0) = 0$ which is a contradiction.

It remains to prove (4.6). If this is not true then

$$(4.9) \quad y + 2\tau(y) > 0 \quad \text{if} \quad 0 < y < y_0$$

but

$$(4.10) \quad y_0 + 2\tau(y_0) = 0 .$$

By the intermediate value theorem, for any $\tilde{y} \in (0, y_0)$ there is a y in $(0, \tilde{y})$ such that

$$y + \tau(y) = -\tau(\tilde{y})$$

since $\tau' < 0$, $\tilde{y}$ is uniquely determined, and we shall write $\tilde{y} = \tilde{y}(y)$ or $\tilde{y} = \tilde{y}(y)$. By (4.10),

$$(4.11) \quad \tilde{y} \rightarrow y_0 \quad \text{if} \quad y \rightarrow y_0 .$$

From (2.6) at $\tilde{y}$,

$$(4.12) \quad \tau'(\tilde{y})\gamma + \tau'(\tilde{y})k(\tilde{y}) = -k(\tilde{y})$$

where $\gamma = \gamma(-\tau(\tilde{y}))$. From (2.7),

$$(4.13) \quad (1 + \tau'(y))\gamma - \tau'(y)k(y) = yk'(y) + 2k(y) .$$

Multiplying (4.13) by $\tau'(\tilde{y})/(1 + \tau'(y))$ and subtracting from (4.10) we obtain

$$\begin{aligned} \tau'(\tilde{y})k(\tilde{y}) + \frac{\tau'(y)\tau'(\tilde{y})}{1 + \tau'(y)} k(y) &= -k(y) \\ - \frac{[y k'(y) + 2k(y)]\tau'(\tilde{y})}{1 + \tau'(y)} &, \end{aligned}$$

or

$$(4.14) \quad \begin{aligned} & (1 + \tau'(\tilde{y}))^2 k(\tilde{y}) + \tau'(y) \tau'(\tilde{y}) k(y) \\ & + \tau'(\tilde{y}) [y k'(y) + 2k(y)] = O(|y - y_0|). \end{aligned}$$

From (4.9), (4.10) it follows that there is a sequence $y_n \rightarrow y_0$ such that

$$\frac{d}{dy} (y + 2\tau(y)) < 0 \quad \text{at} \quad y = y_n.$$

We take $\tilde{y} = y_n$, so that

$$(4.15) \quad \tau'(\tilde{y}) < -\frac{1}{2}.$$

We shall use this inequality to derive a contradiction to (4.14). Since $\tau'(y) \tau'(\tilde{y}) < -\tau'(\tilde{y})$, (4.14) implies that the function

$$f(x) \equiv (1 + x)^2 k(y_0) + x k(y_0) + x [y_0 k'(y_0) + 2k(y_0)] + O(|y - y_0|)$$

is ≥ 0 for $x = \lambda \equiv \tau'(\tilde{y})$. Also, by (4.15), $\lambda < -\frac{1}{2}$ and, by (4.1), $\lambda > -1$. Setting

$$m \equiv \frac{k'(y_0)}{k(y_0)} > 0$$

we conclude that

$$(4.16) \quad g(x) \equiv (1 + x)^2 + x + x(m + 2) \geq O(|y - y_0|) \quad \text{for} \quad x = \lambda;$$

the right-hand side can be made arbitrarily small (by taking $\tilde{y} = y_n \rightarrow y_0$).

But

$$g'(x) = 2(1 + x) + 1 + m + 2 > 0 \quad \text{if} \quad x > -1$$

and $g(-\frac{1}{2}) < 0$; hence $g(x) < 0$ if $-1 \leq x \leq -\frac{1}{2}$ which is a contradiction to (4.16) since $-1 < \lambda < -\frac{1}{2}$.

Proof of Theorem 4.1. Suppose a solution $\tau(x), \gamma(y)$ exists for $0 < x < x_0$, $0 < y < -\tau(x_0)$. In view of (4.6), $\gamma(x)$ is defined in the interval $0 < x < x_1$ where $x_1 = x_0 + \tau(x_0) \geq -\tau(x_0) + \eta$, $\eta > 0$. Hence, in particular, $\gamma(x)$ is defined in a neighborhood of $(-\tau(x_0))$. Thus (2.6) can be written in the form

$$(4.17) \quad \frac{d\tau}{dx} = F(\tau, x)$$

where $F \in C^1$ in a neighborhood of $(\tau(x_0), x_0)$. It follows that, for some $\delta > 0$, there exists a unique solution $\tau = \tau(x)$ for $|x - x_0| \leq \delta$, with $\tau = \tau(x_0)$ at $x = x_0$. Thus $\tau(x)$ has been uniquely extended to $0 \leq x \leq x_0 + \delta$ as a solution of (2.6).

We now turn to (2.7) and write it in the form

$$(4.18) \quad \gamma(\hat{x}) = \frac{\tau'k + xk' + 2k}{1 + \tau'} \Big|_{x=x(\hat{x})}$$

where $\hat{x} = x + \tau(x)$. Recalling (4.5) and the inequalities $x + \tau(x) > -\tau(x)$, $1 + \tau'(x) > 0$, which imply that $x \rightarrow \hat{x}$ is one-to-one C^2 mapping with non-vanishing first derivative, we deduce that $\gamma(\hat{x})$ can be extended, uniquely, from (4.18) to $0 < \hat{x} < -\tau(x_0 + \delta)$, as a C^1 function satisfying (2.7).

Having thus extended the solution of (2.6)–(2.8) to a larger interval, we note that η which entered into the above arguments is independent of x_0 as long as x_0 remains in a bounded interval (this follows from the proof of Lemma 4.2); the same is true of δ . Thus the extension of the solution $\tau(x), \gamma(y)$ can be carried out for $0 < x < x_0, 0 < y < -\tau(x_0)$ for any $x_0 \in (0, \infty)$. By uniqueness, there is thus a unique solution $\tau(x), \gamma(x)$ defined for all $x > 0$.

REMARK 4.1. If $k(y)$ is real analytic for $0 \leq y < \infty$ then the same is true of $\gamma(y), \tau(y)$. Indeed, for small y this follows by repeating the proof of Theorem 3.2 with y complex in a disc $|y| < \rho$ and $\tau(y), \gamma(y)$ holomorphic. Next we observe that the extension proof shows that if $k(y)$ is real analytic then the extension of the real analytic functions $\tau(x), \gamma(y)$ ($0 < x < x_0, 0 < y < -\tau(x_0)$) are again real analytic (Notice that F in (4.17) is real analytic.)

REMARK 4.2. If $k(y)$ is in $C^m[0, \infty)$ for some integer $m > 2$ then $\tau \in C^m[0, \infty)$ and $\gamma \in C^{m-1}[0, \infty)$. Indeed, for y small we repeat the proof of Theorem 3.2 with a set Y of functions (T, Γ) with $T \in W^{m, \infty}(0, \rho)$, $\Gamma \in W^{m-1, \infty}(0, \rho)$; the argument for establishing smooth extension of the solution is as before, noting that F is in C^{m-1} .

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