ON WORK AND CONSTRAINTS IN MIXTURES

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On Work and Constraints in Mixtures

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Introduction. In recent years workers in mixture theory have become aware of the central role that volume-fraction, the parameter describing the relative proportion of the volume occupied by a constituent, must play in that theory.¹ In particular, the rate of change of volume-fraction, which we here call the chority, must appear in a working term as contributing to the energy, in order to avoid various inconsistencies.² This is true both in theories in which volume-fraction appears as a parameter of microstructure and in complete mixture theories.

The presence of these working terms complicates the theory and in particular makes much more involved attempts to determine the effect of constraints on constitutive equations. In an investigation of the theory of flow of incompressible fluids

¹. The first systematic use of this quantity in what could be described as a mixture theory seems to be by Goodman and Cowin [1], in a theory of granular materials, where it entered as a variable describing microstructure. They and many others [2-9] then extended the scope of the ideas to more detailed mixture theories.

². Cf. [10] in which Müller, in order to avoid inconsistencies occurring in earlier theories, introduced an entropy flux which depended on density gradients - and hence volume-fraction - into the theory. The same effect is obtained more directly, and, we feel, more consistently, by introducing the working terms which we discuss.

through a porous medium [11], when we used computations of this sort by Sampaio and Williams [7] we discovered they were in error, due to a miscomputation. Further investigations convinced us that even the form of the working terms in [7] was not sufficiently general to take into account the diverse possibilities which may occur in a mixture. In this note we present first the modification of the working terms from [7] and then illustrate their use in computing constraints. In order to do this most conveniently we have devised what seems to be a new and rather convenient systematic way of exploiting the kinematic constraints.

Although we work in the context of the detailed thermodynamical theory of Sampaio and Williams, the ideas and methods which we discuss apply as well to more conventional mixture theories.³

³. In mixture theory, constraints on individual constituents have seldom been considered. Bedford and Drumheller [8], using a variational approach, and Nunziato and Walsh [12] have made such studies.

2. Kinematics, Constraints. We deal with a situation in which N distinct continuous bodies are pictured as occupying the same region $B \subset \mathbb{R}^3$. The relevant kinematic quantities for the i^{th} constituent are

$$\begin{aligned}\underline{v}_i &: B \times \mathbb{R} \rightarrow \mathbb{R}^3 \\ \varphi_i &: B \times \mathbb{R} \rightarrow [0,1] \\ \alpha_i &: B \times \mathbb{R} \rightarrow \mathbb{R}.\end{aligned}$$

Here $\underline{v}_i$ is the velocity field for i , φ_i is the volume-fraction for i , and α_i is the chority for i (the convected time rate of φ_i):

$$\alpha_i = \varphi_i^i = \varphi_i' + \nabla \varphi_i \cdot \underline{v}_i.$$

Here the prime denotes the partial with respect to the second argument (time), and the gradient is with respect to the first argument (space). Similarly, we may define other convected time-rates, in particular, for any j

$$\begin{aligned}\varphi_j^i &= \varphi_j' + \nabla \varphi_j \cdot \underline{v}_i \\ &= \alpha_j + \nabla \varphi_j \cdot (\underline{v}_i - \underline{v}_j).\end{aligned}$$

We assume that the constituents fill space so that

$$\sum_{i=1}^N \varphi_i = 1. \quad (1)$$

Taking the time derivative of (1) yields an a priori constraint on all motions:

$$\sum_{i=1}^N \varphi_i' = \sum_{i=1}^N [\alpha_i - \nabla \varphi_i \cdot \underline{v}_i] = 0$$

Using (1) and the equation given by taking the gradient of (1),

$$\sum_{i=1}^N \nabla \varphi_i = 0,$$

and the notation

$$2\mathbf{u}_i = \mathbf{v}_1 - \mathbf{v}_i$$

we re-express this equation as

$$\sum_{i=1}^N [\alpha_i + 2\nabla \varphi_i \cdot \mathbf{u}_i] = 0. \quad (2)$$

Next we denote the velocity gradient for i by

$$L_i = \nabla \mathbf{v}_i$$

and remark that balance of mass for i may be expressed as

$$\rho_i^i \varphi_i + \rho_i \varphi_i^i + \rho_i \varphi_i \operatorname{tr} L_i = 0$$

if ρ_i denotes the true density, that is, mass per unit volume actually occupied by i . Thus if the constituent material is incompressible it must be true that

$$\alpha_i + \varphi_i \operatorname{tr} L_i = 0. \quad (3)$$

The requirement that $\rho_i \varphi_i$ be constant would not seem to have any physical meaning. Of course, if all the constituent materials of the mixture are incompressible we may sum, use (1) and (2) and find the usual result for the mean velocity

$$\nabla \cdot \left(\sum_{i=1}^N \varphi_i \underline{v}_i \right) = 0.$$

Finally we note that if i is rigid one has

$$\underline{u}_i = \underline{0}, \quad \alpha_i = 0, \quad L_i = 0. \quad (4)$$

3. Working. A kinematic constraint in mechanics gives rise to reaction forces which are unrelated to the usual constitutive specifications. The values of such reactions are presumed to be unspecifiable a priori and only to be determined by each particular solution of the equations of motion. In continuum mechanics, where the constraints of interest are often internal, as with materials which are incompressible or inextensible in a certain direction, a principle of determinism has been stated precisely in [13] (cf.[14]). It states that forces are to be determined by constitutive relations only to within a part, the extra force, which does no work in any motion compatible with the constraint. Following [15] we choose to say, alternatively, that the extra forces are to be such that they contribute no increase in entropy in any motion compatible with the constraint.¹ In principle, then, one arrives at an equation for the extra force by finding the difference of the entropy production for processes presumed determined by the same constitutive variables and setting this equal to zero.

In [7] (cf.[17,18]) Sampaio and Williams introduce multiple energy equations, one for each constituent alone and one between each pair of constituents. The working terms which are of significance

¹ Cf. [16]. The distinction is moot for non-polar single continua.

In mixture theory several working terms, with interrelated conditions of constraint, are involved and we require only that the combination of terms to be zero.

(contribute to the entropy inequality) are, respectively,

$$T_{ii} \cdot L_i + \left(\pi_i^e + \sum_{j \neq i}^N \pi_{ij} \right) \alpha_i \quad (5)$$

$$T_{ij} \cdot \frac{1}{2}(L_i + L_j) - \underline{b}_{ij} \cdot (\underline{u}_j - \underline{u}_i) - \pi_{ij} \alpha_i - \pi_{ji} \alpha_j - \text{div } \underline{f}_{ij}.$$

Here T_{ii} is a stress tensor representing the action of i on itself, T_{ij} represents a mutual stress between i and j , $\underline{b}_{ij}$ the body force interaction between i and j , π_i^e is the volumetric tension in i due to its internal expansion from exterior sources, π_{ij} the volumetric tension on i due to j . The vector $\underline{f}_{ij}$ includes a working term and a heat term; the term involving the velocity which contributes to $\underline{f}_{ij}$ has the form

$$\int_S \underline{\tilde{g}}_{ij}^s \cdot (\underline{u}_j - \underline{u}_i) dA.$$

If we take two values of the stresses and other forces which correspond to the same constitutive values, integrate the expressions (5)₂ over a region A in each case and then subtract to set them equal to zero we find that the difference terms for $\underline{f}_{ij}$, linear in $\underline{u}_j - \underline{u}_i$, satisfy a balance equation, and standard arguments then suffice to show this difference to be expressible in terms of a tensor K_{ij} as

$$\Delta \underline{f}_{ij} = \frac{1}{2} K_{ij} (\underline{u}_j - \underline{u}_i).$$

Since (from [7]) $\underline{f}_{ij} = \underline{f}_{ji}$ it must be true that

$$K_{ij} = -K_{ji}.$$

A corresponding result from [7] for stresses is

$$T_{ij} = T_{ji}.$$

Finally we note that we wish to alter (5). To begin our argument we first justify the presence of α_i -working terms: picture a sponge saturated with water and then inclosed in a rigid container. If the water is now heated, its volume-fraction increases at the expense of the sponge. There is no deformation of either component of the mixture: $\underline{v}_1 = \underline{v}_2 = \underline{0}$, yet it is clear that there must be an increase in energy of the sponge beyond that predicted by thermal effects. This energy increase can only be accounted for by a working term in the volume-fraction rate. But the argument can be extended: consider the same sponge with both water and mercury inside. We may draw off mercury and replace it with an equal volume of water, thus maintaining the volume-fraction of the sponge. But the water can penetrate smaller (previously collapsed) cavities in the sponge which were inaccessible to the mercury due to limitations of capillarity and thus decrease its energy. Thus we might expect working to occur on the sponge even though $\alpha_1 = 0$; we contend that the working on a constituent i should also involve φ_j^i terms.¹ Thus we suggest supplementing the terms

¹. In a two-component mixture $\varphi_1 + \varphi_2 = 1$ so the distinction is meaningless.

$$\sum_{j=1}^N \pi_{ij} \alpha_j$$

in (5)₁ with

$$\sum_{j=1}^N \lambda_{ij} \varphi_j^i$$

and the terms

$$-\pi_{ij} \alpha_i - \pi_{ji} \alpha_j$$

in (5)₂ with

$$-\lambda_{ij} \varphi_j^i - \lambda_{ji} \varphi_i^j.$$

Then the equations to be satisfied by the reaction forces are

$$T_{ii} \cdot L_i + \sum_{\substack{j=1 \\ j \neq i}}^N [\pi_{ij} \alpha_i + \lambda_{ij} \varphi_j^i] = 0 \quad (6)$$

and

$$\begin{aligned} T_{ij} \cdot \frac{1}{2}(L_i + L_j) - b_{ij} \cdot (\underline{u}_j - \underline{u}_i) - \operatorname{div} K_{ij} \cdot (\underline{u}_j - \underline{u}_i) \\ + K_{ij} \cdot \frac{1}{2}(L_i - L_j) - \pi_{ij} \alpha_i - \pi_{ji} \alpha_j - \lambda_{ij} \varphi_j^i - \lambda_{ji} \varphi_i^j = 0, \end{aligned} \quad (7)$$

for any admissible velocity fields.

Finally we note that not all of these quantities are of interest; indeed, we are generally interested only in how they contribute to the partial stress balance equation:

$$\text{div}(T_{ii} + \frac{1}{2} \sum_{j=1}^N T_{ij}) + \frac{1}{2} \sum_{j=1}^N b_{ij} = \rho_i \ddot{y}_i^i \quad (8)$$

and to the total stress

$$T = \sum_{i=1}^N \sum_{j=1}^N T_{ij}. \quad (9)$$

4. Effect of Constraints. We may regard our kinematic variables as consisting of the list

$$V = (L_1, \dots, L_N, \alpha_1, \dots, \alpha_N, u_2, \dots, u_N) \in \text{Lin}(\mathbb{R}^{3N}) \times \mathbb{R}^N \times \mathbb{R}^{3(N-1)}.$$

Then the constraint operator M is a linear mapping of such lists into $\mathbb{R}^p$, where p is the number of constraint equations, and we write the equations of constraints as the linear system $MV = 0$. The condition that the workings (6), (7) vanish for all allowable motions may be expressed as

$$B \cdot V = 0 \quad \text{for all } V \text{ for which } MV = 0. \quad (10)$$

Thus we require that each vector B be orthogonal to all vectors in the null space of M ; by the Fredholm alternative this is true if and only if each B is in the range of M^T . We proceed in each case by formulating M and requiring that each vector B be in its range, and, since the vectors B are related to one another, combining the resulting equations. Finally, we apply the results to the equations (8) and (9).

N Incompressible materials. When considering N incompressible materials we see that the equations of constraint take the form

$$\tilde{M}V = \begin{pmatrix} \alpha_1 + \varphi_1 L_1 \cdot I \\ \vdots \\ \alpha_N + \varphi_N L_N \cdot I \\ \sum_{n=1}^N \alpha_n + 2 \sum_{n=2}^N \nabla \varphi_n \cdot \tilde{u}_n \end{pmatrix} = \tilde{0}.$$

since $\text{tr } L_i$ is just the inner product of L_i with the identity tensor I .

Thus the constraint operator M has values in $\mathbb{R}^{N+1}$. To find its transpose take the inner product of $M\underline{v}$ with $\underline{q}$ (here $\underline{q} \in \mathbb{R}^{N+1}$) to obtain

$$\sum_{n=1}^N q_n \varphi_n^I \cdot L_n + \sum_{n=1}^N (q_n + q_{N+1}) \alpha_n + 2 \sum_{n=2}^N q_{N+1} \nabla \varphi_n \cdot \underline{u}_n = 0. \quad (11)$$

This is of the form $M^T \underline{q} \cdot \underline{V} = 0$ and thus

$$M^T \underline{q} = (q_1 \varphi_1^I, \dots, q_N \varphi_N^I, q_1 + q_{N+1}, \dots, q_N + q_{N+1}, 2q_{N+1} \nabla \varphi_2, \dots, 2q_{N+1} \nabla \varphi_N).$$

Since $\underline{B}$ is in the range of M^T , each force vector $\underline{B}$ must have this form. The work equations are now¹

$$-T_{jj} \cdot L_j - \sum_{\substack{n=1 \\ n \neq j}}^N (\pi_{jn} \alpha_j + \lambda_{jn} (\alpha_n + 2 \nabla \varphi_n \cdot (\underline{u}_n - \underline{u}_j))) = 0, \quad (12)$$

which is the j^{th} material equation and

$$\begin{aligned} & -\frac{1}{2} T_{ij} \cdot L_i - \frac{1}{2} T_{ij} \cdot L_j + \tilde{b}_{ij} \cdot (\underline{u}_j - \underline{u}_i) + \text{div}(K_{ij}^T) \cdot (\underline{u}_j - \underline{u}_i) \\ & + \frac{1}{2} K_{ij}^T \cdot L_i - \frac{1}{2} K_{ij}^T \cdot L_j + \pi_{ij} \alpha_i + \pi_{ji} \alpha_j + \lambda_{ij} (\alpha_j + 2 \nabla \varphi_j \cdot (\underline{u}_j - \underline{u}_i)) \\ & + \lambda_{ji} (\alpha_i + 2 \nabla \varphi_i \cdot (\underline{u}_i - \underline{u}_j)) = 0 \end{aligned} \quad (13)$$

which is the $i-j^{\text{th}}$ interaction.

We would like to determine $T_{k\ell}, K_{k\ell}^T$ and $b_{k\ell}$ for all k and ℓ . Let us begin by considering the j^{th} fluid interacting with itself. Setting the coefficients of L_n from (11) and (12)

¹. Recall that $\underline{u}_1 = 0$.

equal we see that

$$q_n \varphi_n^I = \begin{cases} 0 & n \neq j, N+1 \\ -T_{jj} & n = j \end{cases}$$

so that for every $n \neq j, N+1$, $q_n = 0$. If we now compare the coefficients of α_n we find that $q_{N+1} = -\lambda_{jn}$ for all $n \neq j$ and $q_j = -\sum_{\substack{n=1 \\ n \neq j}}^N \pi_{jn} - q_{N+1}$. Hence

$$T_{jj} = -[\lambda_j - \sum_{\substack{n=1 \\ n \neq j}}^N \pi_{jn}] \varphi_j^I \quad (14)$$

where $\lambda_{jn} = \lambda_j$ for all $n \neq j$. Notice that with the above choices of q_n the coefficients of $\underline{u}_n$ in (11) and (12) are equal. In fact, to show that this is true it is necessary to recognize that $\sum_{n=1}^N \nabla \varphi_n = 0$.

Examining the i^{th} fluid interaction with the j^{th} fluid we see that, by equating the coefficients of L_n in (11) and (13),

$$q_n \varphi_n^I = \begin{cases} 0 & n \neq i, j, N+1 \\ -\frac{1}{2} T_{ij} + \frac{1}{2} K_{ij}^T & n = i \\ -\frac{1}{2} T_{ij} - \frac{1}{2} K_{ij}^T & n = j \end{cases} \quad (15)$$

Therefore $q_n = 0$ for all $n \neq i, j, N+1$. If we now consider the coefficients of α_n in (11) and (13) we find that $q_{N+1} = 0$, $q_i = \pi_{ij} + \lambda_j$ and $q_j = \pi_{ji} + \lambda_i$. Substituting this into (15)

we have that

$$\begin{aligned} T_{ij} &= -[(\pi_{ij} + \lambda_j)\varphi_i + (\pi_{ji} + \lambda_i)\varphi_j]I, \\ K_{ij}^T &= [(\pi_{ij} + \lambda_j)\varphi_i - (\pi_{ji} + \lambda_i)\varphi_j]I \end{aligned} \quad (16)$$

for all i and j not equal.

Finally, equating the coefficients of $\underline{u}_n$ we arrive at

$$\underline{\mathfrak{L}}_{ij} = -\text{div } K_{ij}^T - 2\lambda_i \nabla \varphi_j + 2\lambda_j \nabla \varphi_i.$$

Using (16)₂ we see that

$$\underline{\mathfrak{L}}_{ij} = -\nabla [(\pi_{ij} + \lambda_j)\varphi_i - (\pi_{ji} + \lambda_i)\varphi_j] - 2\lambda_i \nabla \varphi_j + 2\lambda_j \nabla \varphi_i. \quad (17)$$

Now the balance equation for the i^{th} fluid involves

$$T_i = T_{ii} + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N T_{in}.$$

Substituting (14) and (16) into this we arrive at

$$T_i = [-\lambda_i \varphi_i + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N ((\pi_{in} - \lambda_n)\varphi_i - (\pi_{ni} + \lambda_i)\varphi_n)]I. \quad (18)$$

If we recall that $\sum_{n=1}^N (\nabla \varphi_n) = 0$ we see that

$$\text{div } T_i + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N b_{in} = -\varphi_i \sum_{n=1}^N (\nabla \lambda_n).$$

Hence the "arbitrary pressure" term p is represented by $\nabla(\sum_{n=1}^N \lambda_n)$. The total stress is

$$\operatorname{div} T = \sum_{n=1}^N \nabla p_n = \sum_{n=1}^N \nabla \lambda_n = \nabla(\sum_{n=1}^N \lambda_n) = \nabla p$$

so that what appears in the partial stress equation is $\varphi_i \nabla p$.

Some compressible constituents. In the case of k incompressible and $N - k$ compressible materials we have

$$\underline{M}\underline{V} = \begin{pmatrix} \alpha_1 + \varphi_1 L_1 \cdot I \\ \vdots \\ \alpha_k + \varphi_k L_k \cdot I \\ \sum_{n=1}^N \alpha_n + 2 \sum_{n=2}^N \nabla \varphi_n \cdot \underline{u}_n \end{pmatrix} = 0.$$

Here $1 \leq n \leq k$ represents the incompressible constituents and for $k < n \leq N$ we are considering the compressible constituents. Notice that $M:V \rightarrow \mathbb{R}^{k+1}$. Taking the inner product of this equation with $\underline{q} \in \mathbb{R}^{k+1}$ we arrive at

$$\begin{aligned} \sum_{n=1}^k q_n \varphi_n I \cdot L_n + \sum_{n=1}^k (q_n + q_{k+1}) \alpha_n \\ + \sum_{n=k+1}^N q_{k+1} \alpha_n + 2 \sum_{n=2}^N q_{k+1} \nabla \varphi_n \cdot \underline{u}_n = 0. \end{aligned} \tag{19}$$

Suppose $1 \leq j \leq k < \ell \leq N$. Following the method of the previous section we see that the j^{th} interaction of the fluid with itself is

$$T_{jj} = -[\lambda_j - \sum_{\substack{n=1 \\ n \neq j}}^N \pi_{jn}] \varphi_j I \quad 1 \leq j \leq k, \quad (20)$$

where $\lambda_j = \lambda_{jn}$ for all $n \neq j$. The ℓ^{th} fluid interaction with itself is

$$T_{\ell\ell} = 0 \quad k < \ell \leq N \quad (21)$$

and $\lambda_\ell = \sum_{\substack{n=1 \\ n \neq \ell}}^N \pi_{\ell n}$ where $\lambda_\ell = \lambda_{\ell n}$ for all $n \neq \ell$ and $\ell > k$.

Remark. We arrived at the above conclusions by setting the coefficients of L_n, α_n and $\underline{u}_n$ equal in equations (19) and (12) and considered first the incompressible fluid and then the compressible material.

Remark. We can write (20) and (21) as $T_{nn} = -[\lambda_n - \sum_{\substack{i=1 \\ i \neq n}}^N \pi_{ni}] \varphi_n I$

and note that $\lambda_n = \sum_{\substack{i=1 \\ i \neq n}}^N \pi_{ni}$ for all $n > k$.

We must now consider the interactions between incompressible-incompressible, incompressible-compressible and compressible-compressible materials. Let $1 \leq i, j \leq k < \ell, m \leq N, i \neq j$ and $\ell \neq m$.

As in the previous section we see that, for the i^{th} incompressible constituent interacting with the j^{th} incompressible

constituent T_{ij} and K_{ij}^T are given by (16) and $\tilde{\Sigma}_{ij}$ is represented by (17). Here $1 \leq i, j \leq k$ and $i \neq j$.

Let us now examine the j^{th} incompressible material interacting with the ℓ^{th} compressible constituent. Equating the coefficients of L_n in (13) and (19) and recalling that $\ell > k$ we arrive at

$$q_n \varphi_n I = \begin{cases} 0 & n \neq j, k+1 \\ -\frac{1}{2} T_{j\ell} + \frac{1}{2} K_{j\ell}^T & n = j, \end{cases}$$

so that $q_n = 0$ for all $n \neq j, k+1$. Since there is no coefficient of L_ℓ in (9) we see that $-\frac{1}{2} T_{j\ell} - \frac{1}{2} K_{j\ell}^T = 0$.

If we now consider the coefficients of α_n we have that $q_{k+1} = 0$ and $q_j = \pi_{j\ell} + \lambda_\ell$. We now notice that the coefficient of α_ℓ is zero in (19) and hence

$$\pi_{\ell j} = -\lambda_j \quad \text{for all } \ell > k \text{ and } j \leq k.$$

By noting this identity the stress tensors can again be written as in (16). This is also equivalent to

$$\left. \begin{aligned} T_{j\ell} &= -[(\pi_{j\ell} + \lambda_\ell) \varphi_j] I = T_{\ell j}, \\ K_{j\ell}^T &= [(\pi_{j\ell} + \lambda_\ell) \varphi_j] I = -K_{\ell j}^T. \end{aligned} \right\} \begin{array}{l} \text{for all } j \leq k, \\ \ell > k \end{array} \quad (22)$$

We also have that

$$\tilde{\Sigma}_{j\ell} = -\nabla[(\pi_{j\ell} + \lambda_\ell) \varphi_j] - 2\lambda_j \nabla \varphi_\ell + 2\lambda_\ell \nabla \varphi_j = -\tilde{\Sigma}_{\ell j}. \quad (23)$$

Remark. For $1 \leq j \leq k < l \leq N$ notice that $T_{jl} = -K_{jl}^T$. It does not follow that $T_{lj} = -K_{lj}^T$. In fact it is an easy exercise to show that $T_{lj} = K_{lj}^T$.

Finally, let us consider the l^{th} compressible constituent interacting with the m^{th} compressible constituent. Equating coefficients of L_i and α_i in (13) and (19) we see that $q_n = 0$ for $1 \leq n \leq k+1$. Since the coefficients of L_l , L_n , α_l and α_n are zero in (19) we have that

$$\begin{aligned} T_{lm} &= K_{lm} = 0, \\ \pi_{lm} &= -\lambda_m, \\ \pi_{ml} &= -\lambda_l \end{aligned} \tag{24}$$

for all $l, m > k$, $l \neq m$. $\underline{\underline{b}}_{lm}$ must now satisfy

$$\underline{\underline{b}}_{lm} = -2\lambda_l \nabla \varphi_m + 2\lambda_m \nabla \varphi_l. \tag{25}$$

Let us now compute the balance equation for the i^{th} fluid. By considering (20), (21), (16), (22)₁, (24)₁ we see that

$$T_i = \begin{cases} [-\lambda_i \varphi_i - \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^k (\pi_{ni} + \lambda_i) \varphi_n + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N (\pi_{in} - \lambda_n) \varphi_i] I & i \leq k \\ -\frac{1}{2} \sum_{n=1}^k (\pi_{ni} + \lambda_i) \varphi_n I & i > k. \end{cases}$$

Finally,

$$\operatorname{div} T_i + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N b_{in} = \begin{cases} -\varphi_i \sum_{n=1}^N \nabla \lambda_n & i \leq k \\ (\nabla \varphi_i) \sum_{n=1}^N \lambda_n & i > k. \end{cases}$$

But $\sum_{n=1}^N \lambda_n = 0$. Hence

$$\operatorname{div} T_i + \frac{1}{2} \sum_{\substack{n=1 \\ n \neq i}}^N b_{in} = 0.$$

Remark. To arrive at this, we recall that $\lambda_\ell = \sum_{\substack{n=1 \\ n \neq \ell}}^N \pi_{\ell n}$ for all $\ell > k$. Also, we have that $-\lambda_n = \pi_{\ell n}$ for all $n \leq k$ and $\ell > k$ and that $\pi_{\ell n} = -\lambda_n$ for all $\ell, n > k$. Hence

$$\lambda_\ell = \sum_{n=1}^k (-\lambda_n) + \sum_{\substack{n=k+1 \\ n \neq \ell}}^N (-\lambda_n)$$

and we see that $\sum_{n=1}^N \lambda_n = 0$.

5. A particular example. A specific application of the above theory is to fluid flowing through a rigid porous medium (see Williams [19] for a derivation of a single fluid flowing through a solid and Pericak-Spector [20] for a discussion of the properties of these equations). Here there are two different situations. The first is when all the pores are filled with fluid, so we have two incompressible materials under consideration and the pressure term is given by $p = p(\varphi)$. In the second case, the solid is assumed to be filled with fluid and air, so we consider two incompressible materials and a compressible material, and find that the pressure term is zero. Physically one can argue that a uniform pressure is maintained by the third constituent air that occupies the remainder of the volume.

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