'COMPUTING THE BANDWIDTH OF INTERVAL GRAPHS'

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'COMPUTING THE BANDWIDTH OF INTERVAL GRAPHS'*

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Abstract. In this note we describe an $O(|V|k)$ algorithm for determining whether an interval graph on $|V|$ vertices has a bandwidth less than or equal to a given integer k . While not the first algorithm to resolve this problem, it does admit a shorter proof of its correctness than a previous algorithm of the same complexity due to D. Kratsch (1987).

I. Introduction. Let $G = (V, E)$ be a graph with vertex set V and edge set E . A labelling of G is an assignment of distinct integers from $\{1, 2, \dots, |V|\}$ to the vertices of G . The bandwidth of a labeling of G is the maximum difference between the labels of adjacent vertices. The k -bandwidth problem (BP) is the problem of finding a labeling of G with bandwidth less than or equal to some given integer k or showing that none exists. BP is NP -complete for general graphs, for a proof see Papadimitriou (1976). It has also been shown to be NP -complete for special classes of graphs – Monien (1983) and Garey *et. al.* (1978).

Recently, Johnson (1985), conjectured that BP might be NP -complete even when G is an interval graph. This has been shown not to be the case by Kratsch (1987) who describes a $O(|V|^2)$ algorithm for the problem. In this paper we describe another algorithm for BP that admits a succinct proof of its correctness. Its running time is $O(|V|k)$.

II. Preliminaries. A graph is an intersection graph if there exists a one to one correspondence between its vertices and a family V of sets such that two vertices are adjacent in the graph if and only if the corresponding sets have non-empty intersection. If V is a collection of intervals of the real line then G is called an interval graph (Golumbic (1980), Lekkerkerker (1962)). The collection V is sometimes called the interval model of G .

In what follows we shall assume that an interval graph is always represented in its interval model form. Even if this is not the case, its interval model can be obtained in $O(|E| + |V|)$ steps using the algorithm of Booth and Lueker (1976). Hence, we shall use the words vertex and interval and adjacent and overlap interchangeably. Suppose further that the intervals are ordered by increasing value of their right endpoints.

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III. The Algorithm. The left and right endpoints of interval i will be denoted by a_i and b_i respectively. The label assigned to i by the algorithm we describe is denoted by $L(i)$. This is initially set equal to zero. Each i will also be assigned another integer, $M(i)$, called its mark. This represents an upper bound on the label that can be assigned to i if we are to have a bandwidth of $\leq k$. This is initially set equal to n . We are now in a position to describe our algorithm for BP which we call k BAND.

k BAND

Step 1. Set $L(i) = 0$ and $M(i) = n \quad \forall \quad i \in V$ where $n = |V|$. Set

$$U = \{i \in V : L(i) = 0\} \quad \text{and} \quad q = 0.$$

Step 2. Select $i \in U$ with smallest b_i and set $q = q + 1$.

Step 3. Set $L(i) = q$ and $U = U \setminus i$. If $U = \phi$, stop, all vertices have been labeled.

Step 4. If $r \in U$ overlaps i and $M(r) = n$ set $M(r) = \min\{L(i) + k, n\}$.

Step 5. Let $S_j^q = \{r \in U : M(r) \leq q + j\}$. If $|S_j^q| \leq j$ for all $j \geq k - q + 1$, go to Step 7.

Step 6. There is a j such that $|S_j^q| > j$. Stop, for the bandwidth of the graph is $> k$.

Step 7. Find the smallest value j_0 , such that $|S_{j_0}^q| = j_0$.

Step 8. Select $i \in S_{j_0}^q$ with smallest b_i . Set $q = q + 1$ and go to step 3.

Suppose k BAND terminates at step 3. Then, it does so with a labeling of bandwidth $\leq k$. To see this, observe that when a vertex is labeled q by k BAND, all as yet unlabelled vertices that overlap with the vertex just labeled q are assigned a label no greater than $q + k$ by Step 4.

Suppose now that k BAND terminates at Step 6. This means that for some q (consider the first such q) there is a j' such that $|S_{j'}^q| > j'$. We claim that this implies that $|S_k^q| > k$. Suppose not. We know from the definition of q that $|S_j^{q-1}| \leq j$ for all $j \geq k - q + 2$. Hence, $|S_j^q| = |S_{j+1}^{q-1}| - 1 \leq j$ for all $j \leq k - 1$. Also, $S_j^q = S_k^q$ for $k \leq j < n - q = |S_j^q| = |S_k^q| \leq k \leq j$. Finally, $|S_{n-q}^q| = n - q$. Hence, $|S_j^q| \leq j$ for all j which is a contradiction. Thus, if k BAND terminates at Step 6, $|S_k^q| > k$. This means there are at least $k + 1$ vertices unlabeled after a vertex has been labeled q . Hence, one of the vertices in S_k^q must have a label of at least $q + k + 1$. However, each vertex in S_k^q is adjacent to at least one vertex with a label $\leq q$. Consequently, it is not possible for k BAND to produce a labeling with bandwidth $\leq k$.

THEOREM. Algorithm k BAND correctly determines if the bandwidth of an interval graph is $\leq k$ in $O(|V|k)$ steps.

Proof. Clearly, it is sufficient to show that when k BAND terminates at Step 6 after labeling $q < n$ vertices of an interval graph G , there is no labeling L' of G with bandwidth

$\leq k$. Suppose not and let G be the smallest graph which admits a labeling of bandwidth $\leq k$ but k BAND terminates at step 6 when applied to G .

As k BAND terminates at step 6, the set of vertices in S_k^q are unlabelled and $|S_k^q| \geq k+1$. Since $M(\alpha) \leq q+k \forall \alpha \in S_k^q$, $a_\alpha \leq b_\beta \forall \alpha \in S_k^q$ where β is such that $L(\beta) = q$. If $b_\alpha \geq b_\beta \forall \alpha \in S_k^q$, then, the set $S_k^q \cup \{\beta\}$ forms a clique of size $\geq k+2$. Hence, G can admit no labeling of bandwidth $\leq k$. Thus, $\exists \alpha \gamma \in S_k^q$ such that $b_\gamma < b_\beta$.

Observe now that as γ is unlabelled but $b_\gamma < b_\beta$ it must mean that γ was not selected during an iteration of step 8 of k BAND to be labeled. Hence, $\exists$ an integer j such that $\gamma \in S_j^{q-1}$ and $|S_j^{q-1}| = j$. Let j be the smallest such integer. Note that $j \leq k-1$.

The vertices of S_j^{q-1} all overlap the vertex δ where $L(\delta) = q+j-1-k$ or some vertex with a smaller right hand end point. Since $b_\beta \leq b_\alpha \forall \alpha \in S_j^{q-1}$ it follows that all vertices in S_j^{q-1} overlap δ as well all vertices θ that overlap β such that $b_\theta \geq b_\gamma$.

Let $L'_{\max} = \max_{\alpha \in S_j^{q-1}} L'(\alpha)$ and $L'_{\min} = \min_{\alpha \in S_j^{q-1}} L'(\alpha)$. Let θ be some vertex that overlaps all elements of S_j^{q-1} . Then $L'(\theta) \geq L'_{\max} - k$ and $L'(\theta) \leq L'_{\min} + k$. This means that the range of feasible values for $L'(\theta)$ is $\leq 2k + L'_{\min} - L'_{\max} + 1 \leq 2k - j + 2$.

Consider now the graph G' formed by removing all vertices in $S_k^q \setminus S_j^{q-1}$ except β , as well as all vertices as yet unlabeled by k BAND and extend β leftwards until it just overlaps δ . Label the vertices of G' with consecutive integers in the same order that those vertices are labeled by L' . Call this labeling $\bar{L}$. A total of $|S_k^q \setminus S_j^{q-1}| + 1 = k - j + 1$ were removed from S_k^q to form G' . Hence, the range of distinct labels assigned by $\bar{L}$ to the vertices in G' that overlap all members of S_j^{q-1} in G' is reduced to $k+1$. Thus, $\bar{L}$ has a bandwidth of $\leq k$, because β , and any vertices it might newly overlap in G' all overlap all members of S_j^{q-1} in G and hence all have labels in this range; and must therefore have label difference along any edge in G' of at most k .

There are $q+k+1-k-1+j = q+j$ vertices in G' all of whom receive a mark of $\leq q+j-1$ when k BAND is applied to G' ; which means that k BAND cannot produce a labeling of G that is of bandwidth $\leq k$. This contradicts the definition of G and so the required result is obtained.

Finally, note that one pass through steps 3,4,5,7 and 8 takes $O(k)$ steps. This will be repeated at most $|V|$ times. Hence k BAND is $O(|V|k)$. $\square$

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REFERENCES

BOOTH, K.S. AND G.S. LUEKER, *Testing for the consecutive ones property, interval graphs, and graph*

- planarity using PQ-tree algorithms, *J. Comput. Systems Sci.* 13, 335-379, 1976.
- GOLUMBIC, M.C., *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
- JOHNSON, D.S., *The NP-completeness column: an ongoing guide* (16th ed, *Journal of Algorithms*, 6, 434-451, 1985.
- KRATSCH, DIETER, *Finding the Minimum Bandwidth of an Interval Graph*, *Information and Computation*, 74, 140-158, 1987.
- LEKKERKERKER, C. G. AND BOLAN, J.C., *Representations of a finite graph by a set of intervals on the real line*, *Fund. Math.*, 51, 45-64, 1962.
- MONIEN, B., *The bandwidth minimization problem for caterpillars with hair length 3 is NP-complete*, Paderbon, Reibe Theoretische Informatik, Bericht Nr. 17, Bibliographisches Inst., Mannheim, 1983.
- PAPADIMITRIOU, C.H., *The NP-completeness of the bandwidth minimization problem*, *Computing*, 16, 263-270, 1976.