THE UNCERTAINTY PRINCIPLE ON GROUPS

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1. Introduction. The classical uncertainty principle asserts that a function and its Fourier transform cannot both be largely concentrated on intervals of small measure. D. Donoho and P. Stark have shown recently [1] that both cannot be concentrated on any sets of small measure. They consider three separate cases: i) Fourier transforms on the line with concentrations measured in L^2; ii) Fourier transforms on the line with concentrations measured in L^1; iii) Fourier transforms on finite cyclic groups with concentrations measured in L^2. It turns out that, after simple modifications and additions, their proof in case i) applies to most locally compact abelian groups with concentrations measured in L^p, $1 \leq p \leq 2$.

In the L^2 case Donoho and Stark say that a function f is ϵ-concentrated on a set T if $\|f-\chi_T f\| \leq \epsilon\|f\|$ (χ_T denoting the characteristic function of the set T) and state their uncertainty principle as follows.

Uncertainty Principle. If $f \neq 0$ is ϵ-concentrated on T and $\hat{f}$ is δ-concentrated on W, then $|T||W| \geq (1-\epsilon-\delta)^2$, where $|T|$ denotes the measure of T.

The main inequality of Donoho and Stark, which immediately implies this uncertainty principle, is an inequality for the norm of the product of the operators

$$Pf = \chi_T f \quad \text{and} \quad Qf = (\chi_W \hat{f})^{\vee} \tag{1.1}$$

where $^\wedge$ and $^\vee$ denote the Fourier transform and its inverse. For the L^2 case their inequality is as follows.

$$\|QP\|^2 \leq |T||W| \tag{1.2}$$

Donoho and Stark provide many intriguing examples and applications of their uncertainty principle, among them the anomaly that the uncertainty principle implies that uncertain signals often can be recovered with certainty! The applications are not reproduced in this note.

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2. Standard definitions and facts. Let G be a locally compact abelian group written additively. The character group $\hat{G}$ consists of the continuous homomorphisms of G into R/Z, the reals mod the integers. $\hat{G}$ has a natural addition and a natural topology relative to which it is also a locally compact abelian group. The value of the character $\xi \in \hat{G}$ at the point $x \in G$ is written $\langle x, \xi \rangle$. See, for example, [3]. To avoid measure theoretic pathology it is assmued that non-discrete groups and character groups have σ-finite Haar measures.

The Fourier transform of a Haar integrable function f on G is the function $\hat{f}$ on $\hat{G}$ defined by

$$\hat{f}(\xi) = \int e^{-2\pi i\langle x,\xi\rangle} f(x)\, dx \tag{2.1}$$

the exponential having the obvious meaning and dx being Haar measure. When the Haar measures are normalized properly (one can be normalized arbitrarily, then the other is determined), the inverse Fourier transform is given by

$$\check{g}(x) = \int e^{2\pi i\langle x,\xi\rangle} g(\xi)\, d\xi. \tag{2.2}$$

and the following holds.

$$\|\hat{f}\|_2 = \|f\|_2. \qquad \text{(Plancherel formula)} \tag{2.3}$$

The Plancherel formula, the obvious inequality $\|\hat{f}\|_\infty \leq \|f\|_1$, and Riesz-Thorin interpolation give

$$\|\hat{f}\|_{p'} \leq \|f\|_p, \quad 1 \leq p \leq 2. \quad \text{(Hausdorff-Young inequality)} \tag{2.4}$$

(Of course, $\| \ \|_p$ denotes the norm in L^p and $1/p+1/p' = 1$.)

Convolution is defined by

$$f*g(x) = \int f(x-y)g(y)\, dy . \tag{2.5}$$

It follows immediately from the Plancherel formula that

$$(f*g)^\wedge = \hat{f}\hat{g} , \tag{2.6}$$

The inequality

$$\|f*g\|_p \leq \|f\|_1\|g\|_p \qquad \text{(Young's inequality)} \tag{2.7}$$

is proved in the classical way.

In some typical cases the Haar measures are normalized as follows. If G is discrete, each point is given measure 1. Then Plancherel applied to $\chi_{\{0\}}$ requires $|\hat{G}| = 1$. When G is finite of order N, this means that each point in $\hat{G}$ has measure $1/N$. If G is R^n, then $\hat{G} = R^n$, and on both sides the normalized Haar measure is Lebesgue measure.

Sometimes the character group is defined to be the group of continuous homomorphisms of G into $R/(2\pi)$. Then the Fourier transform involves $e^{-i\langle x,\xi\rangle}$ instead of $e^{-2\pi i\langle x,\xi\rangle}$ and the normalizations are different. In the case of R^n the Haar measures are $(2\pi)^{-n/2}$ x Lebesgue measure. (This is the convention in [4].)

3. Norm inequalities. Let T and W be measurable sets of positive finite measure in G and $\hat{G}$, and define the operators P and Q by (1.1).

Theorem 3.1 For $1 \le p \le 2$ and $q \geq 1$,

$\|PQf\|_q \le |T|^{1/q}|W|^{1/p} \|f\|_p$, hence $\|PQ\|_p \le (|T||W|)^{1/p}$

Proof.

$$PQf(x) = \chi_T(x)\int\int e^{2\pi i\langle x,\xi\rangle}\chi_W(\xi)e^{-2\pi i\langle y,\xi\rangle} f(y)\, dy\, d\xi$$

$$= \chi_T(x)\int f(y)\int \chi_W(\xi)e^{2\pi i\langle x-y,\xi\rangle}\, d\xi\, dy.$$

Therefore,

$$PQf(x) = \langle f,k_x\rangle \quad \text{with} \quad k_x(y) = \chi_T(x)\check{\chi}_W(y-x) \tag{3.2}$$

Hölder's inequality and the Hausdorff-Young inequality give

$$|PQf(x)| \le \|f\|_p \|k_x\|_{p'} = \|f\|_p \chi_T(x) \|\check{\chi}_W\|_{p'}$$

$$\le \|f\|_p \chi_T(x) \|\chi_W\|_p = \|f\|_p \chi_T(x) |W|^{1/p}.$$

The result follows from integration of the q-th power.

Theorem 3.2 For $q \geq 2$ and $p \le q'$

$\|QPf\|_q \le |T|^{1/q}|W|^{1/p}\|f\|_p$, and $\|QPf\|_q \le (|T||W|)^{1/p} \|Pf\|_q$

Proof. As above,

$$QPf(x) = \langle f,k_x\rangle \quad \text{with} \quad k_x(y) = \chi_T(y)\check{\chi}_W(y-x)$$

By Hölder's inequality

$$\|QPf\|_q \le \|f\|_p \left(\int \|k_x\|_{p'}^q\, dx\right)^{1/q}$$

while, by Young's inequality

$$\left(\int \|k_x\|_{p'}^q\, dx\right)^{1/q} = \left(\int\left(\int \chi_T(y)|\chi_W(y-x)|^{p'}\, dy\right)^{q/p'} dx\right)^{1/q}$$

$$= \|\chi_T * |\chi_W|^{p'}\|_{q/p'}^{1/p'} \le \left(\|\chi_T\|_{q/p'} \||\chi_W|^{p'}\|_1\right)^{1/p'}$$

The first inequality in the theorem follows from

$$\|\chi_T\|_{q/p'}^{1/p'} = |T|^{1/q} \text{ and } \|\,|\chi_W|^{p'}\|_1^{1/p'} = \|\check{\chi}_W\|_{p'} \le \|\chi_W\|_p = |W|^{1/p}$$

The second inequality follows easily from the first.

4. <u>The uncertainty principle.</u> When $p = 2$, $\hat{f}$ is δ-concentrated on W if $\|f-Qf\| \le \delta\|f\|$. In this case Qf is the closest function to f with Fourier transform supported on W. For arbitrary p (actually $p = 1$), Donoho and Stark say that f is δ-bandlimited to W if there exists a function g with $\hat{g}$ supported on W, i.e. $Qg = g$, and $\|f-g\|_p \le \delta\|f\|_p$. When $p = 2$, if there is any such g, then $g = Qf$ is also one.

<u>Theorem 4.1</u> (<u>The uncertainty principle</u>) If there exists $f \ne 0$ in L^p, $p \le 2$, which is ϵ-concentrated on T and δ-bandlimited to W, then

$$(|T||W|)^{1/p} \ge \|PQ\|_p \ge (1-\epsilon-\delta)/(1+\delta) \qquad (4.2)$$

When $p = 2$, the $1+\delta$ is unnecessary.

<u>Proof.</u> $\|f\|_p - \|PQf\|_p \le \|f-PQf\|_p$

$$\le \|f-Pf\|_p + \|Pf-Pg\|_p + \|PQg-PQf\|_p \le (\epsilon + \delta + \delta\|PQ\|_p)\|f\|_p.$$

So $\|PQf\|_p \ge (1-\epsilon-\delta-\delta\|PQ\|_p)\|f\|_p$. If $p = 2$, $g = Qf$ and $PQg-PQf = 0$.

5. <u>Cases of equality when $p = 2$.</u> When $p = 2$ the operators P and Q are orthogonal projections in Hilbert space, and there are many standard characterizations of the norm of a product of two projections. In this section p is always 2 and the subscript 2 is dropped.

5.1 If P and Q are orthogonal projections in Hilbert space, then

a) $\|PQ\|^2 = \|QP\|^2 = \|PQP\| = \|QPQ\|$ = the cosine of the angle between the ranges of P and Q.

b) When PQ is compact, $\|PQ\|^2$ is the largest eigenvalue of both QPQ and PQP, and if f and g are the corresponding eigenvectors $\|PQf\| = \|PQ\|\|f\|$ and $\|QPg\| = \|PQ\|\|g\|$.

c) If $P' \supset P$ and $Q' \supset Q$, then $\|P'Q'\| \ge \|PQ\|$.

Some facts about locally compact abelian groups are needed: a)-c) are classical results of Pontryagin [2], chapter 6; d) seems to be hard to find, so a quick proof is included for completeness.

5.2 a) $(\hat{G})^\wedge = G$.

b) G is compact if and only if $\hat{G}$ is discrete.

c) Let H be a closed subgroup of G and let $H^\perp$ be the set of characters that vanish on H. Then H, $H^\perp$, G/H, and $\hat{G}/H^\perp$ are all locally compact abelian and the following relations hold $(G/H)^\wedge = H^\perp$, $\hat{H} = \hat{G}/H^\perp$, and $(H^\perp)^\perp = H$.

d) If H is a closed subgroup with $0 < |H| < \infty$, then H and $H^\perp$ are

are compact and open, $\hat{\chi}_H = |H|\chi_{H^\perp}$, and $|H||H^\perp| = 1$.

Proof of d). Let $C \subset H$ be compact, $|C| \neq 0$, V a neighborhood of 0. Since the sets $(x+V)\cap C$, $x \in C$, cover C, some such set, hence some $(x+V)\cap H = x+(V\cap H)$ must have positive measure. Therefore, $V \cap H$ has positive measure. If H is not compact there are a neighborhood V of 0 and a sequence $x_n \in H$ such that the sets $x_n+(V\cap H)$ are disjoint. As these all have the same positive measure and are contained in H, H cannot have finite measure. It is obvious that if $\xi \in H^\perp$, then $\hat{\chi}_H(\xi) = |H|$. If $\xi \notin H^\perp$, $\xi \neq 0$ in $\hat{G}/H^\perp = \hat{H}$, so there exists $y \in H$ with $\langle y,\xi\rangle \neq 0$.

$$\hat{\chi}_H(\xi) = e^{-2\pi i\langle y,\xi\rangle} \int e^{-2\pi i\langle x-y,\xi\rangle} \chi_H(x)\, dx = e^{-2\pi i\langle y,\xi\rangle} \hat{\chi}_H(\xi).$$

by the translation invariance of the integral and of H, which implies that $\hat{\chi}_H(\xi) = 0$. $|H||H^\perp| = 1$ is obtained by taking L^2 norms. This implies that $0 < |H^\perp| < \infty$, hence that $H^\perp$ is compact. It follows that G/H is discrete, therefore that H is open.

Theorem 5.3 The equality $\|PQ\|^2 = |T||W|$ holds if and only if there exist $x_o \in T$ and $\xi_o \in W$ so that $\langle x-x_o, \xi-\xi_o\rangle = 0$ a.e. on $T \times W$. If If G or $\hat{G}$ is discrete or if either is separable metric this holds if and only if there is a compact open subgroup H such that T is almost contained in a coset of H and W is almost contained in a coset of $H^\perp$.

Proof. Let $T' = T-x_o$, $W' = W-{}_o$. It is immediate that if $\langle x,\xi\rangle = 0$ a.e. on $T' \times W'$, then $P'Q'P'\chi_{T'} = |T'||W'|\chi_{T'}$, so that $\|P'Q'P'\| \geq |T||W| = |T'||W'|$. Moreover, $\|P'Q'\| = \|P'Q\| = \|QP'\| = \|QP\| = \|PQ\|$. This proves part of the theorem and shows in passing that for any measurable set T, $|T||T^\perp| \leq 1$. Indeed, if either measure is 0, the left side is interpreted as 0 and there is nothing to prove. If $T' \subset T$ and $W' \subset T$ have finite measure, the above gives $|T'||W'| = \|P'Q'\|^2 \leq 1$. Since this holds for all T' and W' it follows that $|T||T^\perp| \leq 1$.

Now suppose that $\|PQ\|^2 = |T||W|$, let g be an eigenfunction in 5.1b, and choose $x_o \in T$ so that $PQPg(x_o) = |T||W|g(x_o) \neq 0$. Since $\|QPg\| = \|PQ\|\|g\|$, equality, $|\langle g,k_x\rangle| = \|g\|\|k_x\|$, holds for almost all x in the application of the Hölder (Cauchy-Schwarz) inequality in the proof of Theorem 3.2, and since both sides are continuous, it holds for all x. For $x = x_o$ this implies that (up to a constant multiple) $g(y) = k_{x_o}(y) = \chi_T(y)\check{\chi}_W(y-x_o)$. Therefore,

$$|T||W|g(x_o) = PQPg(x_o) = \iint e^{-2\pi i\langle y-x_o,\xi\rangle} \chi_W(\xi)\chi_T(y)\check{\chi}_W(y-x_o)\,dy\,d\xi.$$

Since the maximum of the absolute value of the integrand is $|W| = g(x_o)$ it follows that $e^{-2\pi i\langle x,\xi\rangle}\check{\chi}_W(x) = |W|$ a.e. on $T-x_o \times W$. Fix $\xi_o \in W$

so that $\check{\chi}_{W-\xi_0}(x) = e^{-2\pi i\langle x,\xi_0\rangle}\check{\chi}_W(x) = |W|$ a.e. on $T-x_0$. Then

$$e^{-2\pi i\langle x,\xi\rangle}\chi_{W-\xi_0}(\xi)d\xi \;=\; |W| = |W-\xi_0| \text{ a.e. on } T-x_0$$

requires $e^{-2\pi i\langle x,\xi\rangle} = 1$ a.e. on $T-x_0 \times W-\xi_0$.

Replace T by $T-x_0$, W by $W-{}_0$. It will be shown that with the extra conditions there exist $T' \subset T$ and $W' \subset W$ with $|T-T'| = |W-W'| = 0$ and $\langle x,\xi\rangle = 0$ everywhere on $T' \times W'$. Then $H = W'^{\perp}$ is the compact open subgroup. H has positive measure because it contains T' and has finite measure because $|W'||W'^{\perp}| \leq 1$, with $|W'| \neq 0$. Because of the symmetry, the conditions "discrete" or "separable metric" can be imposed on G or $\hat{G}$ at will.

The fact that $\langle x,\xi\rangle = 0$ a.e. on $T \times W$ means that the set W' of $\xi \in W$ such that $\langle x,\xi\rangle = 0$ a.e. on T satisfies $|W-W'| = 0$. If G is discrete, each point has positive measure, so $T' = T$ will do. If $\hat{G}$ is separable metric, let ξ_n be a dense sequence in W', let T_n be the set of x in T such that $\langle x,\xi_n\rangle = 0$, and let T' be the intersection of the the T_n. Then $|T-T'| = 0$ and for each $x \in T'$, $\langle x,\xi_n\rangle = 0$ for all n. By continuity, $\langle x,\xi\rangle = 0$ for all $\xi \in W'$,

The theorem shows that the equality $\|PQ\|^2 = |T||W|$ is pretty rare. For example, if G or $\hat{G}$ is discrete or separable metric and $\hat{G}$ is connected, it holds only when G is discrete and T is a single point. Indeed, the only possibility is $H^{\perp} = \hat{G}$, therefore $H = \{0\}$. If G is discrete and contains no element of finite order, then (see [3]) $\hat{G}$ is connected, and the above holds. If G is discrete and does contain elements of finite order, such elements provide compact open subgroups, therefore nontrivial sets T and W with equality.

Since $\|PQ\| \leq 1$, if $|T||W| > 1$, the relevant question is when $\|PQ\| = 1$. (The applications of Donoho and Stark to signal recovery depend on the condition $\|PQ\| < 1$ which allows inversion of 1-PQ by a Neumann series. $|T||W| < 1$ is one means of ensuring $\|PQ\| < 1$.)

From 5.1c, 5.2d, and Theorem 5.3 it follows that $\|PQ\| = 1$ if there is a compact open subgroup H such that T contains a coset of H and W contains a coset of $H^{\perp}$. In some groups this is the only possibility for equality, in others it is not, and in general we do not know how to decide. A start is as follows. If equality holds and g is the eigenfunction, then $Pg = g$ and $Qg = g$, i.e. g has support in T and $\hat{g}$ in W so it can be assumed without loss of generality that T is the support of g and W is the support of $\hat{g}$ and that both contain 0. In this case, g and $\hat{g}$ are constant on cosets of $W^{\perp}$ and $T^{\perp}$, so $W^{\perp}$ and $T^{\perp}$ have finite measure. If $|W^{\perp}| \neq 0$, then $H = W^{\perp}$ is compact and open and T is a finite union of cosets of H, so

$$g = \sum c_j \chi_{H+x_j}, \qquad \hat{g} = |H|\,\chi_H \sum c_j e^{-2\pi i\langle x_j,\xi\rangle}.$$

If the trigonometric polynomial

$$q(\xi) = \sum c_j e^{-2\pi i\langle x_j,\xi\rangle}$$

$\neq 0$ a.e. on $H^{\perp}$, then the support of $\hat{g}$ is $H^{\perp}$, and the situation is as above. If $q = 0$ on a subset A of positive measure in H equality holds with $W = H^{\perp} - A$, and the situation is not as above.

Lemma 5.4 If G_1 and G_2 have the property that no non trivial trigonometric polynomial vanishes on a set of positive measure, then so does $G = G_1 \oplus G_2$.

Proof. $\hat{G} = \hat{G}_1 \oplus \hat{G}_2$, and $\langle x,\xi\rangle = \langle x^1,\xi^1\rangle + \langle x^2,\xi^2\rangle$, so

$$q(\xi) = \sum c_j e^{-2\pi i\langle x_j^1,\xi^1\rangle}\, e^{-2\pi i\langle x_j^2,\xi^2\rangle}$$

If $q = 0$ on a set A of positive measure, then for some ξ^1, $q(\xi^1,\xi^2) = 0$ on a set A_2 of positive measure in G_2. Since the exponentials are $\neq 0$, this requires $c_j \neq 0$.

Example 5.5 If $G = Z^m$, hence $\hat{G} = (R/Z)^m$, then $\|PQ\| = 1$ if and only if $W = \hat{G}$.

Proof. In this case all subgroups of G have positive measure, but the only one with finite measure is $H = \{0\}$, so the proof reduces to showing that no trigonometric polynomial vanishes on a set of positive measure, and by the Lemma it is sufficient to show this for $G = Z$. If the trigonometric polynomial $q(\xi) = \sum c_j e^{-2\pi i n_j \xi}$ vanishes on a set of on a set of positive measure, then the polynomial $p(z) = \sum c_j z^{n_j}$ vanishes on an infinite set, and all c_j are 0.

Example 5.6 Let G be finite of order N. If T has n elements and W has m, then $\|PQ\| = 1$ if $n+m > N$. If G is cyclic and T is an interval the condition $n+m > N$ is also necessary.

Proof. Let $T = \{x_1,\ldots,x_n\}$. Since $\hat{G}-W$ has $N-m < n$ elements, there exist $c_1,\ldots,c_n$ not all 0 such that $\sum c_j e^{-2\pi i\langle x_j,\xi\rangle}$ vanishes on $\hat{G}-W$. If $g = \sum c_j \chi_{\{x_j\}}$, then g has support in T, $\hat{g}$ in W. If G is cyclic and T is an interval, then by the translation invariance of $\|PQ\|$ it can be assumed that $T = \{0, \ldots, n-1\}$, in which case

$$\hat{g}(\xi) = p(e^{-2\pi i\xi/N}), \qquad p(z) = \sum_{j=0}^{n-1} c_j z^j .$$

The polynomial p cannot vanish at $N-m \geq n$ points.

Example 5.7 If $G = R^n$, then $\|PQ\| = 1$ never holds if either T or W is bounded because the Fourier transform of a function with bounded support is analytic. (We do not know if there exist unbounded sets T, W of finite measure that support $f, \hat{f}$ in L^2. One would expect the answer to be well known, but we have not found it.)

For the case where T and W are intervals in R^1, Landau, Pollak, Slepian, and Sonnenblick have made a detailed analysis of the operator QPQ, and, in particular, have computed the eigenvalues. They show that $\|PQ\|$ depends only on the product $|T||W|$, and that, while $\|PQ\|$ is never 1, it is close to 1 as soon as $|T||W|$ is a little larger than 1. E.g. they show in [4] that if $|T||W| = 1,2,3,4,5$, then $\|PQ\| = .757, .939, .988, .998, .999$.

Example 5.6 shows that when G is finite cyclic $\|PQ\|$ depends on the sum m+n, not just on the product mn, even when T and W are intervals. For example, let $m = m_1 d$, $n = n_1 d$, $m' = m_1 n_1 d$, $n' = d$. Then $mn = m'n'$, while if $m_1, n_1 \geq 2$, $d \geq 1$, $m+n < m'+n'$, so if G has order m+n, then $\|PQ\| < 1$, while $\|P'Q'\| = 1$. On the other hand, after examination of some 30000 cases, Donoho and Stark have found $\|PQ\| \geq .999$ if $|T||W| \geq 3$.

References

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Additional references and many interesting facts and applications are given in [1] and [4].

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