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Abstract. We consider a class of gradient systems on a suitable bounded region $\Omega \subset R^m$, where $m = 1, 2, \dots$. Let d_H and d_L denote, respectively, the Hausdorff and Lyapunov dimension of the global attractor of the gradient system. Our objective in this paper is to derive an asymptotic formula which implies that

$$\frac{d_L}{d_H} \rightarrow \left(\frac{2}{m} + 1 \right)^{m/2} \quad \text{as } d_H \rightarrow \infty.$$

Key words. gradient systems, Hausdorff dimension, Lyapunov dimension, reaction diffusion equations.

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1. Introduction. During the last few years several important advances have been made in the study of the dimension of the global attractor of certain dissipative dynamical systems. The first contribution is a paper by Mallet-Paret (1976) wherein he shows the finiteness of the Hausdorff dimension¹ d_H of the global attractor of certain dissipative dynamical systems in a Hilbert space. Later Mañé (1981) extended this result to Banach spaces.

Since that time various authors have studied specific systems with the goal of estimating the Hausdorff dimension, or the fractal dimension d_F , in terms of the physical parameters of the problem. A partial listing of such results include Constantin and Foias (1985) for the 2D Navier-Stokes equation, Foias, Manley and Temam (1986) for Bernard convection and Nicolaenko, Scheurer and Temam (1985, 1987) for the Kuramoto-Sivashinsky equation and the Cahn-Hilliard equation.

The basic approach used in these papers is to estimate the Hausdorff dimension in terms of the Lyapunov exponents on the global attractor. More precisely one calculates the Lyapunov dimension d_L by means of the Kaplan-Yorke formula, see the Appendix. Then, as shown in Constantin and Foias (1985), one has

$$d_H \leq d_L + 1, \quad d_F \leq d_L + 1.$$

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¹The three dimension concepts mentioned in this paper, Hausdorff, Lyapunov and fractal, are defined in the Appendix.

In this paper we want to study the Hausdorff and Lyapunov dimensions of the global attractor of a class of scalar reaction diffusion equations

$$(1) \quad u_t = \Delta u + \rho^2 f(x, u),$$

where $x \in \Omega \subset R^m$, $u \in R$ and ρ is a real parameter. We assume that Ω is a suitable bounded region and that (1) is given with Dirichlet boundary conditions. We assume that the nonlinearity f satisfies the following dissipative condition: There is a $R_o > 0$ such that

$$uf(x, u) < 0, \quad \text{for } x \in \bar{\Omega}, |u| > R_o.$$

We will also exclude those nonlinearities f for which (1) has a unique globally asymptotically stable stationary solution for each $\rho \in R$.

Our plan is to calculate d_H and d_L in terms of ρ and study their behavior as $\rho \rightarrow \infty$, or equivalently, as $d_H \rightarrow \infty$. Our objective in this paper is to present evidence for the following conjecture concerning the dimensions d_H and d_L :

$$(2) \quad \begin{cases} m \geq 2: & d_L = \left(\frac{2}{m} + 1\right)^{m/2} d_H \left[1 + o\left(d_H^{-1/m}\right)\right], & \text{as } d_H \rightarrow \infty, \\ m = 1: & d_L = \left(\frac{2}{m} + 1\right)^{m/2} d_H + o(1), & \text{as } d_H \rightarrow \infty. \end{cases}$$

In this paper we shall show that (2) is valid for some specific choices of f . We believe that (2) is valid, at least for a generic collection of nonlinearities (ρ, f) . Eventhough we are unable at this time to prove the validity of (2) with the generality we would like, we find the asymptotic formula (2) to be interesting and, perhaps, surprising. For this reason we are presenting our analysis of the problem here, eventhough this may not be the last word on the problem.

The fact that the coefficient in (2) satisfies

$$\left(\frac{2}{m} + 1\right)^{m/2} \rightarrow e, \quad \text{as } m \rightarrow \infty$$

is interesting. However we are unable to assign any dynamical significance to this. At a minimum it suggests that

$$D = e^{-1} d_L$$

may be a reasonable concept of dimension for gradient systems.

2. Gradient Systems. Let Ω be a minimally smooth² bounded domain in R^m , $m \geq 1$. Consider the scalar reaction diffusion equation

$$(3) \quad \begin{cases} u_t = \Delta u + \rho^2 f(x, u) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

²For the definition of minimally smooth see Edmunds and Evans (1987, p. 255)

where $u \in R$ and $\rho \in R$. We assume that $f \in \mathcal{F}$, where $\mathcal{F} = \mathcal{F}(R_o)$ is the collection of C^1 functions $f : \bar{\Omega} \times R \rightarrow R$ that satisfy

$$(4) \quad u f(x, u) < 0, \quad \text{for } x \in \bar{\Omega}, |u| > R_o,$$

and R_o is a fixed positive number.

Since (3) is a scalar equation

$$W(u) = \int_{\Omega} (|\nabla u|^2 + F(x, u)) dx$$

is a Lyapunov function for (3), where $F = \int^u f du$. Also one has

$$W'(u) = - \int_{\Omega} u_t^2 dx \leq 0.$$

Hence the omega limit set of every trajectory of (3) is some stationary solution. Because of this, equation (3) is referred to as a *gradient system*.

Let V be a stationary solution of (3). That is, V is a solution of the nonlinear elliptic equation

$$\begin{cases} \Delta V + \rho^2 f(x, V) = 0 & \text{on } \Omega \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus V satisfies $V = -\Delta^{-1}(\rho^2 f(x, V))$. Because of (4) it follows from the maximum principle that $|V(x)| \leq R_o$ for all $x \in \Omega$. This means that

$$V \in H_o^1(\Omega) \cap H^2(\Omega) \cap \{u \in L^\infty(\bar{\Omega}) : \|u\|_\infty \leq R_o\}.$$

A similar argument shows that the global attractor $\mathcal{A}$ for (3) satisfies

$$\mathcal{A} \subset H_o^1(\Omega) \cap H^2(\Omega) \cap \{u \in L^\infty(\bar{\Omega}) : \|u\|_\infty \leq R_o\}.$$

Let V be a stationary solution of (3). Then the linearization of (3) at V takes on the form

$$u_t = \Delta u + \rho^2 a(x)u$$

where $a(x) = D_u f(x, V(x))$. The stationary solution V is said to be *hyperbolic* if the linear operator

$$L = \Delta + \rho^2 a(x)$$

(with Dirichlet boundary conditions) has no eigenvalues λ with $\lambda = 0$. (Note that L is a self-adjoint operator, hence all of the eigenvalues of L are real. Since L is bounded above, it has compact resolvent, and the spectrum of L is precisely the point spectrum.)

We let $\mathcal{F}_h = \mathcal{F}_h(R_o)$ denote those functions $f \in \mathcal{F}$ for which equation (3) has, at $\rho = 1$, a finite number of stationary solutions and all of these stationary solutions are hyperbolic. For the remainder of this paper we will restrict our attention to nonlinearities $\rho^2 f \in \mathcal{F}_h$.

Let $\rho^2 f \in \mathcal{F}_h$ and let $V_0, V_1, \dots, V_N$ denote the stationary solutions of (3). Note that $N = N(\rho, f)$ depends on both ρ and f . In this case the global attractor $\mathcal{A}$ satisfies

$$\mathcal{A} = \bigcup_{i=0}^N \mathcal{W}^u(V_i),$$

where $\mathcal{W}^u(V_i)$ is the unstable manifold of V_i . Furthermore the Hausdorff dimension d_H is given by

$$d_H = d_H(\mathcal{A}) = \max\{\text{index}(V_i) : 0 \leq i \leq N\}$$

where $\text{index } V_i = \dim \mathcal{W}^u(V_i)$. The Lyapunov dimension d_L is expressable in terms of the eigenvalues of $L_i = \Delta + \rho^2 a_i(x)$, $0 \leq i \leq N$, where $a_i(x) = D_u f(x, V_i(x))$. We will examine this in the next section.

3. Calculation of Dimensions. The first step in the analysis is to calculate d_H and d_L in terms of ρ . In order to do this we shall restrict our attention to functions $f \in \mathcal{F}$ which have one additional property, *viz.*

$$(5) \quad \begin{cases} f(x, 0) \equiv 0. \\ D_u f(x, 0) \equiv \alpha = \text{constant} \\ D_u f(x, u) \leq D_u f(x, 0) \quad \text{for all } x \in \bar{\Omega} \quad \text{and } u \in R. \end{cases}$$

We assume further that³ $\alpha > 0$. For example, the Chafee-Infante nonlinearity $f = u(1-u^2)$ satisfies this additional property.

We let $V_0, V_1, \dots, V_N$ denote the stationary solutions of (3) and we fix $V_0(x) \equiv 0$. For each i , $0 \leq i \leq N$ we let L_i denote the linear operator

$$L_i u = \Delta u + \rho^2 a_i(x) u$$

with Dirichlet boundary conditions, where $a_i(x) = D_u f(x, V_i(x))$. Let $\lambda_1^i \geq \lambda_2^i \geq \lambda_3^i \geq \dots$ denote the eigenvalues of L_i . (Note that $\lambda_n^i \rightarrow -\infty$ as $n \rightarrow \infty$.) Recall that if $a_i(x) \geq a_j(x)$ for all x , then the eigenvalues satisfy $\lambda_n^i \geq \lambda_n^j$ for all n , see Weinberger (1974). With f satisfying (5) one then has

$$(6) \quad \lambda_n^i \leq \lambda_n^0$$

for all i and n .

³If $\alpha \leq 0$, then V_0 is a globally stable stationary solution for all ρ whenever $f \in \mathcal{F}$.

Now $\text{index}(V_i) = N_i$ is the unique integer N_i satisfying

$$\lambda_{N_i+1}^i < 0 \leq \lambda_{N_i}^i.$$

(Since V_i is hyperbolic one has $0 < \lambda_{N_i}^i$.) Because of (6) we see that

$$\dim \mathcal{A} = \max_i \text{index}(V_i)$$

is realized at $V_0 \equiv 0$. Hence the Hausdorff dimension $d_H = M$ is the unique integer satisfying

$$(7) \quad \lambda_{M+1}^0 < 0 < \lambda_M^0.$$

The Lyapunov dimension d_L is also determined by the eigenvalues at V_0 . In particular $N = \text{Int}(d_L)$ is the unique integer satisfying

$$(8) \quad \sum_{k=1}^{N+1} \lambda_k^0 < 0 \leq \sum_{k=1}^N \lambda_k^0.$$

Because of (5) the eigenvalues λ_n^0 of L_0 take on the form

$$\lambda_n^0 = -\mu_n + \alpha \rho^2,$$

where μ_n is the n^{th} eigenvalue of $-\Delta$ on the region Ω . The next step is to express μ_n as a function of n . In order to do that, it is convenient to adopt the following convention: We shall use $c_1, c_2, \dots$ to denote constants which do not depend on n , and $d_1, d_2, \dots$ will denote bounded functions of n and μ_n .

From Edmunds and Evans (1987, Chap. 11) there exists c_1, d_1 and d_2 so that

$$c_1 n = \mu_n^{\frac{m}{2}} + d_1 + d_2 \mu_n^{\frac{m-1}{2}}.$$

By solving the last equation for μ_n one obtains

$$(9) \quad \mu_n = c_2 n^{\frac{2}{m}} + d_3 n^{\frac{1}{m}}$$

for $c_2 = c_1^{\frac{2}{m}}$ and an appropriate d_3 . For $m \geq 2$ equations (7) and (8) for M and N can be written in the form

$$\lambda_M^0 + d_4 = 0 \quad , \quad \sum_{k=1}^N \lambda_k^0 + d_5 = 0,$$

or equivalently

$$(10) \quad \mu_M = \alpha \rho^2 + d_4$$

$$(11) \quad \sum_{k=1}^N \mu_k = N(\alpha \rho^2 + d_5).$$

The final step in the analysis is to combine (9) with (10) and (11). We will use the identity

$$(12) \quad \sum_{n=1}^N n^\alpha = (\alpha + 1)^{-1} N^{\alpha+1} + o(N^\alpha),$$

where $o(N^\alpha)$ is a term satisfying

$$\limsup_{N \rightarrow \infty} |N^{-\alpha} o(N^\alpha)| < \infty.$$

Similarly one has

$$(13) \quad \sum_{n=1}^N d_3 n^\alpha = d_3 N^{\alpha+1}.$$

By using (9) and (10) we obtain

$$(14) \quad \alpha \rho^2 + d_4 = c_2 M^{2/m} + d_3 M^{1/m}.$$

By combining (9), (11), (12) and (13) we obtain

$$(15) \quad \alpha \rho^2 + d_5 = c_2 \left(\frac{2}{m} + 1 \right)^{-1} N^{\frac{2}{m}} + o\left(N^{\frac{2}{m}-1}\right) + o\left(N^{\frac{1}{m}}\right).$$

Putting (14) and (15) together and noting that $m \geq 2$ we obtain

$$\left(\frac{2}{m} + 1 \right) M^{\frac{2}{m}} + o\left(M^{\frac{1}{m}}\right) = N^{\frac{2}{m}} + o\left(N^{\frac{1}{m}}\right).$$

Raising the last equality to the $\frac{m}{2}$ power we obtain

$$\left(\frac{2}{m} + 1 \right)^{\frac{m}{2}} M + o\left(M^{1-\frac{1}{m}}\right) = N + o\left(N^{1-\frac{1}{m}}\right),$$

which implies that

$$N = \left(\frac{2}{m} + 1 \right)^{m/2} M \left[1 + o\left(M^{-1/m}\right) \right]$$

for $m \geq 2$.

The argument for $m = 1$ is somewhat simpler since $\mu_n = c_0 n^2$ for some $c_0 > 0$. In this case (10) and (11) take the form

$$\mu_M = \alpha \rho^2 + d_4 M, \quad \sum_{k=1}^N \mu_k = N(\alpha \rho^2 + d_5 N).$$

Also (14) becomes

$$(16) \quad \alpha \rho^2 + d_4 M = c_0 M^2$$

while (15) becomes

$$(17) \quad \alpha \rho^2 + d_5 N = \frac{1}{3} c_0 N^2 + O(N).$$

Putting (16) and (17) together we obtain

$$3^{1/2} M = N + O(1).$$

4. Conclusions. We believe that with Ω fixed, the collection of nonlinearities $\rho^2 f$, which are in $\mathcal{F}_h(R_o)$ and which satisfy (5) for some $\alpha > 0$, forms an open dense set in the space of smooth functions on $\bar{\Omega} \times R$. As we have shown, the conjecture is valid for all such nonlinearities.

An alternate point of view is to fix (ρ, f) so that f is independent of x and $f \in \mathcal{F}(R_o)$ and such that (5) is satisfied for some $\alpha > 0$. For example, one might fix $f(u) = u(1 - u^2)$. Then a result of Henry (1986, 1988) implies that for *most* regions $\Omega \subset R^m$ one has $\rho^2 f \in \mathcal{F}_h(R_o)$. The conjecture is valid for all such nonlinearities.

Finally, it is interesting to speculate on whether some variation of this conjecture may be valid for other classes of (nongradient) nonlinear dissipative dynamical systems.

Appendix A. Dimension Concepts. We recall here three definitions of the dimension for a compact set $X \subset H$, where H is a given Hilbert space.

First, the *Hausdorff dimension* is defined by

$$d_H = d_H(X) \stackrel{\text{def}}{=} \inf\{d > 0 : \mu_H^d(X) = 0\}$$

where

$$\mu_H^d(X) = \lim_{r \rightarrow 0^+} \mu_{H,r}^d(X),$$

$$\mu_{H,r}^d(X) = \inf \left\{ \sum_{i=1}^k r_i^d : X \subset \bigcup_{i=1}^k B_i, \quad r_i = \text{radius}(B_i) \leq r \right\},$$

and B_i denotes an open ball in H .

Secondly the fractal dimension, see Mandelbrot (1977), is defined by

$$d_F = d_F(X) \stackrel{\text{def}}{=} \inf\{d > 0 : \mu_F^d(X) = 0\}$$

where

$$\mu_F^d(X) = \limsup_{r \rightarrow 0^+} r^d n_X(r)$$

and $n_X(r)$ is the minimum number of balls of H of radius equal to r required to cover X .

Finally the Lyapunov dimension is given by the Kaplan-Yorke formula

$$d_L = d_L(X) \stackrel{\text{def}}{=} j_0 + \frac{\mu_1 + \cdots + \mu_{j_0}}{|\mu_{j_0+1}|}$$

where μ_i are the Lyapunov exponents and the integer j_0 is defined by

$$j_0 = \max\{j : \mu_1 + \cdots + \mu_j \geq 0\},$$

see Kaplan and Yorke (1979).

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