SUPPORT FUNCTIONS AND ORDINAL PRODUCTS

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1. Introduction. The concept of the support function of a non-empty compact convex set was introduced by Minkowski at the end of the 19th century [Mi, pp. 106, 144, 231]. Since then it has played a vital role in many of the applications of convexity, from optimisation theory to the geometry of numbers. Support functions of non-empty compact convex subsets of a finite-dimensional Euclidean space $\mathbf{R}^d$ are characterized as positively homogeneous convex real-valued functions on $\mathbf{R}^d$, and the convex subsets are determined uniquely by their support functions.

The main aim of the current paper is to extend the concept of the support function from compact non-empty convex sets to general bounded non-empty convex sets, thus to convex sets that are not necessarily closed. The idea is to find a suitable codomain D_d, replacing the codomain $\mathbf{R}$ of Minkowski's support functions, so that non-empty bounded convex subsets of $\mathbf{R}^d$ are determined uniquely by their D_d-valued support functions defined on $\mathbf{R}^d$. Conditions must also be found, analogous to Minkowski's positive homogeneity and convexity, characterising the support functions amongst all the D_d-valued functions on $\mathbf{R}^d$.

More is demanded. Algebraic structure on $\mathbf{R}$ induces algebraic structure on the set of real-valued functions on $\mathbf{R}^d$. Ideally, the set of support functions should be closed under this algebraic structure, and should reflect comparable algebraic structure on the set of non-empty compact convex subsets of $\mathbf{R}^d$. The usual linear algebraic structure on $\mathbf{R}$ is unsuitable here; for example, the negative of a convex function is no longer convex. The authors' contention is that the correct algebraic structure on $\mathbf{R}$ for use in the context of support functions comprises convex combinations forming a barycentric algebra (see [RS1], [RS2]) and the maximum operation forming a join semilattice, the convex combinations distributing over the join so that the two structures combine to form a *modal* in the sense of [RS1]. The support functions then form a submodal of the induced modal structure on the full set of functions, and the modal structure on the support functions reflects exactly the modal structure on the compact convex sets given by convex combinations and convex hulls of unions. (See Section 3 below for an outline, and [RS1, 3.7] for details.) Given this algebraic approach to Minkowski's support functions, and the fact that non-empty bounded convex subsets of $\mathbf{R}^d$ form a modal under convex combinations and convex hulls of unions, it will then be required that the codomain D_d carry a modal structure such that the new support functions form a submodal of the induced modal of functions from $\mathbf{R}^d$ to D_d, this submodal being isomorphic to the modal of bounded convex sets.

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The codomain modals D_d are constructed from the modal $\mathbf{R}$ and a second modal by a new ordinal product method. The underlying semilattices of the two modals give ordered sets, and their set direct product may be ordered lexicographically, to give the ordinal product of the two ordered sets. The remaining algebraic structure on the product is defined componentwise. Section 2 of the paper investigates this construction generally in the context of modal theory, Theorem 2.5 giving sufficient conditions for the ordinal product of two modals again to be a modal. Section 2 may be viewed as a separate chapter in modal theory, motivated by but independent of the support function application.

The third section summarises some classical results on Minkowski's support functions, and shows how they fit nicely into the framework of modal theory. Then, in the fourth section, the support functions of general non-empty bounded convex subsets of $\mathbf{R}^d$ are introduced. The codomains D_d are defined as ordinal products (4.1) within an induction scheme on d. Conditions on functions from $\mathbf{R}^d$ to D_d, the five so-called G-*conditions* of Definition 4.11, will then characterise the support functions. Thus the G-conditions are the analogues, for the support functions of bounded convex sets, of Minkowski's positive homogeneity and convexity for support functions of compact convex sets. The fifth section verifies the validity of the induction scheme of Section 4. The verifications help to bring out the geometrical analysis of general (bounded, non-empty) convex sets achieved by their support functions.

2. Ordinal products of modals. A *mode* (A, Ω) is an algebra of type $\tau : \omega \to \mathbf{N}$ which is idempotent and entropic. *Idempotence* means that each singleton $\{a\}$ is a subalgebra $(\{a\}, \Omega)$ of (A, Ω). The *entropic* property means that each operation $\omega : A^{\omega\tau} \to A$ of Ω is a homomorphism $\omega : (A^{\omega\tau}, \Omega) \to (A, \Omega)$. A *modal* $(D, +, \Omega)$ is an algebra with (join) semilattice reduct $(D, +)$ and mode reduct (D, Ω) such that the distributive laws $x_1 \ldots (x_j + x_j') \ldots x_{\omega\tau}\omega = x_1 \ldots x_j \ldots x_{\omega\tau}\omega + x_1 \ldots x_j' \ldots x_{\omega\tau}\omega$ hold for all ω in Ω, for all $1 \leq j \leq \omega\tau$, and for all $x_1, \ldots, x_j, x_j', \ldots, x_{\omega\tau}$ in A. (See [RS1], particularly sections 1.1–1.4 and 3.1, for further details.)

For modals $(D, +, \Omega)$ and $(E, +, \Omega)$ of the same mode reduct type $\tau : \Omega \to \mathbf{N}$, the *ordinal product* $D \circ E$ is the cartesian product $D \times E$ equipped with the product mode structure $(D, \Omega) \times (E, \Omega)$ [RS1, p. 6] and with the partial order $\leq$ defined by

$$(2.1) \qquad (d, e) \leq (d', e') \quad \text{iff } d <_+ d' \quad \text{or } (d = d' \quad \text{and } e \leq_+ e').$$

Theorem 2.5 below gives sufficient conditions for the ordinal product $D \circ E$ to be a modal. The first question to be addressed is when the partial order $(D \circ E, \leq)$ is a join semilatttice. This question was examined by Slatinský [Sl], who gave necessary and sufficient conditions [Sl, 3.14] for the ordinal product of two partial orders to be a (meet) semilattice. Specialising his conditions to the case where each of these partial orders is a semilattice, and using join semilattices instead, one is led to the following

Definition 2.2. An ordered pair $((D, +), (E, +))$ of join semillatices is said *to satisfy the Slatinský condition* iff $(D, +)$ is a chain or $(E, +)$ has a least element.

THEOREM 2.3. ([Sl, 3.14]) *The partial order (2.1) defines a join semilattice on the Cartesian product of the underlying sets of an ordered pair $((D,+),(E,+))$ of join semilattices iff the pair satisfies the Slatinský condition.* □

Now recall that an algebra (A,Ω) of non-empty type $\tau:\Omega\to\{n\in\mathsf{N}|n>1\}$ is said to be *cancellative* [RS1, 653] if

$$(2.4)\qquad \begin{cases} \forall\omega\in\Omega,\ \forall\, 1\le i\le\omega\tau,\ \forall x_1,\ldots,x_{\omega\tau},y,z\in A, \\ x_1\ldots x_{i-1}\,yx_{i+1}\ldots x_{\omega\tau}\omega = x_1\ldots x_{i-1}zx_{i+1}\ldots x_{\omega\tau}\omega \Rightarrow y=z. \end{cases}$$

A modal $(E,+,\omega)$ is said *to have a zero element*, usually denoted 0, if 0 is the least element of the join semilattice $(E,+)$, so that $e+0=e$ for all e in E, and if $\{0\}$ is a sink [RS1, p. 73] of (E,Ω), i.e. if $e_1\ldots e_{j-1}0e_{i+1}\ldots e_{w\tau}\omega=0$ for all ω in Ω, for all $e_1,\ldots,e_{\omega\tau}$ in E, and for all $1\le i\le\omega\tau$. With these definitions, the main result on ordinal products of modals can be formulated.

THEOREM 2.5. *Let $(D,+,\Omega)$ and $(E,+,\Omega)$ be modals whose mode reducts both have non–empty type*

$$(2.6)\qquad \tau:\Omega\longrightarrow\{n\in\mathsf{N}|n>1\}.$$

Suppose that

(2.7) *the mode (D,Ω) is cancellative, and*

(2.8) *either $(D,+)$ is a chain or $(E,+,\Omega)$ has a zero element.*

Then the ordinal product $D\circ E$ is a modal $(D\circ E,+,\Omega)$ of mode type (2.6).

Proof. The hypothesis (2.8) implies that the pair $((D,+),(E,+))$ satisfies the Slatinský condition, so by Theorem 2.3 the partial order (2.1) gives a join semillattice $(D\circ E,+)$. The product $(D\circ E,\Omega)$ is a mode of type (2.6). It remains to be proved that the operations Ω distribute over $(D\circ E,+)$, i.e. that for each ω in Ω, say with $\omega\tau=n$,

$$(2.9)\qquad \begin{cases} (d_1,e_1)\ldots(d_{i-1},e_{i-1})[(d_i,e_i)+(d'_i,e'_i)](d_{i+1},e_{i+1})\ldots(d_n,e_n)\omega \\ =(d_1\ldots d_i\ldots d_n\omega,e_1\ldots e_i\ldots e_n\omega)+(d_1\ldots d'_i\ldots d_n\omega,e_1\ldots e'_i\ldots e_n\omega) \end{cases}$$

for all $(d_1,e_1),\ldots,(d_n,e_n),(d'_i,e'_i)$ in $D\times E$. By the commutativity of $+$, there are essentially three cases to check: $d_i=d'_i$, $d_i<d'_i$, and d_i incomparable with d'_i. In the first case, (2.1) shows that the two sides of (2.9) reduce to $(d_1,e_1)\ldots(d_i,e_i+e'_i)\ldots(d_n,e_n)\omega$ and $(d_1\ldots d_i\ldots d_n\omega,\ e_1\ldots e_i\ldots e_n\omega+e_1\ldots e'_i\ldots e_nw)$ respectively. These are clearly equal by the definition of ω on $D\times E$ and the distributivity of $+$ over ω in the modal $(E,+,\omega)$.

The second case $d_i < d'_i$ is not quite so direct. Certainly $(d_i, e_i) + (d'_i, e'_i) = (d'_i, e'_i)$, so the left hand side of (2.9) reduces to

$$(d_1 \dots d'_i \dots d_n\omega, e_1 \dots e'_i \dots e_n\ \omega). \tag{2.10}$$

By the Monotonicity Lemma [RS1, 315], $d_i \le d'_i$ implies $d_1 \dots d_i \dots d_n\omega \le d_1 \dots d'_i \dots d_n\omega$. By the cancellativity hypothesis (2.7), the equality $d_1 \dots d_i \dots d_n\omega = d_1 \dots d'_i \dots d_n\omega$ would imply the equality $d_i = d'_i$, a contradiction. Thus the strict inequality $d_1 \dots d_i \dots d_n\omega < d_1 \dots d'_i \dots d_n\omega$ holds, whence by (2.1) the right hand side of (2.9) also reduces to (2.10). The third case, where d_i is incomparable with d'_i, can only arise if $(D, +)$ is not a chain. By the hypothesis (2.8), this means that $(E, +)$ has a least element 0, and $(d_i, e_i) + (d'_i, e'_i) = (d_i + d'_i, 0)$. Also $d_1 \dots d_i \dots d_n\omega$ is incomparable with $d_1 \dots d'_i \dots d_n\omega$, since $d_1 \dots d_i \dots d_n\omega \le d_1 \dots d'_i \dots d_n\omega$ say implies $d_1 \dots (d_i + d'_i) \dots d_n\omega = d_1 \dots d_i \dots d_n\omega + d_1 \dots d'_i \dots d_n\omega = d_1 \dots d'_i \dots d_n\omega$, whence the contradiction $d_i + d'_i = d'_i$ or $d_i \le d'_i$ by cancellativity. Then the right hand side of (2.9) becomes $(d_1 \dots (d_i + d'_i) \dots d_n\omega, 0) = (d_1 \dots (d_i + d'_i) \dots d_n\omega, e_1 \dots 0 \dots e_n\omega) = (d_1, e_1) \dots (d_i + d'_i, 0) \dots (d_n, e_n)\omega$, which is equal to the left hand side. □

The need for the cancellativity hypothesis (2.7) is demonstrated by the

Example 2.11. Take $\tau = \{(.,2)\}$. Let $(D, +, \Omega)$ and $(E, +, \Omega)$ both be the stammered (meet-)semilattice [RS1, 327] $\{0 < 1\}$, so that $0.1 = 0 + 1 = 0$. In particular $1 <_+ 0$, which means that $(0,1) + (1,0) = (0,1)$ and $(0,1) + (0,0) = (0,0)$ in $D \circ E$. Note that (D, Ω) is not cancellative, since $0.0 = 0.1$ in D. Moreover $[(0,1) + (1,0)].(0,1) = (0,1).(0,1) = (0,1) \ne (0,0) = (0,1) + (0,0) = (0,1).(0,1) + (1,0).(0,1)$, so that the product . of $D \circ E$ does not distribute over the join $+$, i.e. $D \circ E$ is not a modal.

3. Support functions of compact convex sets. For a finite dimension d, let $\mathbf{R}^d$ denote the vector space $\mathbf{R}^d$ equipped with the Euclidean inner product $\mathbf{R}^d \times \mathbf{R}^d \longrightarrow \mathbf{R};\ \ (x, y) \longmapsto (x|y)$. Let I^0 denote the open unit interval $]0,1[$. Then $\mathbf{R}^d$ also carries an algebraic structure $(\mathbf{R}^d, I^0)$ of type $I^0 \times \{2\}$ with

$$xy\underline{p} = x(1-p) + yp \tag{3.1}$$

for p in I°. This algebraic structure is a "barycentric algebra" [RS1, 2.1] [RS2]. Convex subsets A of $\mathbf{R}^d$ are just subalgebras (A, I°) of $(\mathbf{R}^d, I^\circ)$. Let AK denote the set of all compact non–empty convex subsets of A. Then (AK, I^0) is a barycentric algebra under the complex products

$$XY\underline{p} = \{xy\underline{p} | x \in X, y \in Y\} \tag{3.2}$$

[RS1, 263]. Furthermore, $(AK, +)$ is a join semilattice with the partial order $\le_+$ being the containment relation $\subseteq$; for compact non-empty convex subsets X, Y of A, the join $X + Y$

of X and Y is just the convex hull of their set-theoretic union $X \cup Y$. (By [Gr, §2.3], this convex hull is also compact.) Together, the barycentric algebra and join semilattice structures on AK form a modal $(AK, +, I^\circ)$ [RS1, 3.7].

Another example of a modal having a barycentric algebra as its mode reduct is the modal $(\mathbf{R}, +, I^\circ)$, where $(\mathbf{R}, I^\circ)$ is the case $d = 1$ of (3.1) and where $x + y$ denotes the maximum of the real numbers x and y. If T is a topological space, the set $C(T)$ of continuous real-valued functions on T inherits the modal structures $(C(T), +, I^\circ)$ from $(\mathbf{R}, +, I^\circ)$. For a non-empty convex subset A of $\mathbf{R}^d$, the modal $(AK, +, I^0)$ embeds into the modal $(C(\mathbf{R}^d), +, I^0)$ via the *support function*

(3.3) $$H : AK \times \mathbf{R}^d \longrightarrow \mathbf{R}; (X, Y) \longmapsto \sup\{(x|y) | x \in X\} .$$

Indeed, an element X of AK is specified as

(3.4) $$X = \{x \in A |\ \forall y \in \mathbf{R}^d, \quad x \leq H(X, y)\}$$

[Bo, p. 24] [Gr, 2.2 Ex. 8(iv)]. Furthermore, for fixed X in AK, the *support function of* X

(3.5) $$\mathbf{R}^d \longrightarrow \mathbf{R}; \quad y \longmapsto H(X, y)$$

is convex [Bo, p. 24], and so continuous [Bo, p. 19]. Thus the embedding of $(AK, +, I^\circ)$ into $(C(\mathbf{R}^d), +, I^\circ)$ is given by

(3.6) $$X \longmapsto (H_X : y \longmapsto H(X, y)) .$$

This embedding is a semilattice homomorphism [Bo, p. 24] and barycentric homomorphism [Bo, p. 29], [RS1, p. 46], see also [RS1, 371]. A function $f = \mathbf{R}^d \longmapsto \mathbf{R}$ is said to be *positively homogeneous* if

(3.7) $$\forall\, p \in \mathbf{R}^+, \quad \forall\, x \textit{ in } \mathbf{R}^d, \quad f(xp) = f(x)p ,$$

where $\mathbf{R}^+$ denotes the set of positive reals. Then a function $f : \mathbf{R}^d \longmapsto \mathbf{R}$ lies in the image of the embedding (3.6) for $A = \mathbf{R}^d$, i.e. f is the support function of some non-empty compact convex set, iff f is positively homogeneous and convex [Bo, p. 26]. (Bonnesen–Fenchel require $f(0) = 0$, but this follows from (2.7) via $f(0) = f(0 + 0) = f(0) + f(0)$, i.e. taking $x = 0$ and $p = 2$.)

A point x of $\mathbf{R}^d$ is said to be a *linearity point* of a positively homogeneous function $f = \mathbf{R}^d \longrightarrow \mathbf{R}$ if $f(xp) = f(x)p$ holds additionally for negative values of p. Note that 0 is always a linearity point. The set of linearity points of a positively homogeneous convex function $f : \mathbf{R}^d \longrightarrow \mathbf{R}$ is a vector subspace $L(f)$ of $\mathbf{R}^d$ [Bo, p. 20]. The support function of a non-empty compact convex subset X of $\mathbf{R}^d$ has a non-zero linearity point x iff X lies in an affine hyperplane orthogonal to x [Bo, p. 24].

A convex function $f : \mathbf{R}^d \longrightarrow \mathbf{R}$, although necessarily continuous, need not be differentiable. However, for each pair (x, y) of points of $\mathbf{R}^d$, the limit

$$\lim_{h \to 0+} \frac{f(x + yh) - f(x)}{h} = f'(x; y) \tag{3.8}$$

exists [Bo, p.19]. For each x, the function

$$y \longmapsto f'(x; y) \tag{3.9}$$

is positively homogeneous [Bo, p. 19] and convex [Bo, p. 20]. If f is the support function of the non-empty compact convex subset X of $\mathbf{R}^d$, and x is non-zero, then (3.9) is the support function of the intersection of X with its supporting hyperplane $\{z \in \mathbf{R}^d | \, (z|x) = f(x)\}$ [Bo, p. 26]. For $x = 0$, one has $f'(0; y) = f(y)$ since f is positively homogeneous.

4. Codomains for support functions. Let D denote the modal $(\mathbf{R}, +, I^\circ)$, with $x + y$ the maximum of x and y, as in the previous section. The modal D served as codomain for the support functions of non-empty compact convex subsets X of $\mathbf{R}^d$. These support functions were characterised as the positively homogeneous convex elements of the set of all functions from $\mathbf{R}^d$ to D. This set of all functions inherits the modal structure from D, and then the embedding of $\mathbf{R}^d K$ sending a non-empty compact convex subset to its support function preserves the full modal structure $(\mathbf{R}^d K, +, I^\circ)$ on $\mathbf{R}^d K$.

Let $\mathbf{R}^d B$ denote the set of all bounded non-empty convex subsets of $\mathbf{R}^d$. Then $(\mathbf{R}^d B, +, I^\circ)$ forms a modal with $X + Y$ as the convex hull of $X \cup Y$ and with I°-operations $XY\underline{p}$ defined by (3.2) as complex products. In the notation of [RS1, 3.1], $(\mathbf{R}^d B, +, I^\circ)$ is a submodal of the modal $(\mathbf{R}^d S, +, I^\circ)$ of all non-empty convex subsets of $\mathbf{R}^d$. The purpose of this section is to give an inductive definition of a modal D_d which will serve as a codomain for support functions of non-empty bounded convex subsets of $\mathbf{R}^d$, in the same way that D served as a codomain for the support functions of the non-empty compact convex subsets of $\mathbf{R}^d$. In other words, the set of all functions from $\mathbf{R}^d$ to D_d will inherit a modal structure from D_d, and there will then be an embedding of $\mathbf{R}^d B$ into this modal preserving the full modal structure $(\mathbf{R}^d B, +, I^\circ)$ on $\mathbf{R}^d B$.

Along with the series of codomain modals D_d for $d \geq 1$, there will be three other series of inductively defined modals E_d $(d \geq 0), F_d (d \geq 1)$, and $G_d (d \geq 0)$. The induction basis is the singleton G_0. The induction step begins with a modal G_{d-1} of support functions identified with $(\mathbf{R}^{d-1} B, +, I^\circ)$. Note that $\mathbf{R}^0 B$ is the singleton $\{\{0\}\}$. The modal E_{d-1} will be obtained from G_{d-1} by adding a zero element, identified with the empty subset of $\mathbf{R}^{d-1}$. The pair of modals (D, E_{d-1}) will then satisfy the conditions of Theorem 2.5. Certainly $(\mathbf{R}, I^\circ)$ is cancellative [RS1, 269 and 212], so that (2.7) is satisfied. Moreover, (2.8) is doubly satisfied: $(\mathbf{R}, +)$ is a chain *and* E_{d-1} has a zero element. The next codomain modal D_d is then defined to be the ordinal product modal

$$D_d = D \circ E_{d-1} \tag{4.1}$$

furnished by Theorem 2.5. The modal F_d is the set of functions $\mathbf{R}^d \longrightarrow D_d$, with the modal structure induced from D_d. The induction step of the definition is completed by giving five conditions on functions in F_d, the "G-conditions" of Definition 4.11 below, that define its submodal G_d. These five conditions are analogous to the two (positive homogeneity and convexity) conditions characterising the support functions of compact convex sets.

Since the underlying set of D_d is $\mathbf{R} \times E_{d-1}$, a function $f : \mathbf{R}^d \longrightarrow D_d$ may be written as

$$f : \mathbf{R}^d \longrightarrow D_d \; ; \; x \longmapsto (H_f(x), C_f(x)), \tag{4.2}$$

where $C_f(x)$ is a (possibly empty) bounded convex subset of $\mathbf{R}^{d-1}$ called the *crust shadow* (*in the x direction*), and where

$$H_f : \mathbf{R}^d \longrightarrow \mathbf{R}; \; x \longmapsto H_f(x) \tag{4.3}$$

is a real-valued function on $\mathbf{R}^d$, called the *real function part* of f. The first two conditions on f are just the positive homogeneity and convexity of H_f. If f is taken as the support function of a non-empty bounded convex subset X of $\mathbf{R}^d$, then H_f is just the traditional support function of the non-empty compact convex closure $\overline{X}$ of X in $\mathbf{R}^d$. To complete the description of X, its intersection with the supporting hyperplane of $\overline{X}$ in the direction of each non-zero vector x of $\mathbf{R}^d$ must be given. This intersection is called the *crust* of X (*in the x direction*). The crust is a (possibly empty) bounded convex subset of the supporting hyperplane. It will be described by f as the preimage of the crust shadow $C_f(x)$ in $\mathbf{R}^{d-1}$ under a certain affine isomorphism π_x from the supporting hyperplane to $\mathbf{R}^{d-1}$. Thus, before the remaining conditions on f can be given, the maps π_x must be specified. Their specification depends on the satisfaction of the first two G-conditions by f, so that H_f really is the support function of a non-empty compact convex subset of $\mathbf{R}^d$.

For each 1-dimensional vector subspace V of $\mathbf{R}^d$, pick a linear isomorphism

$$\theta_V : \mathbf{R}^d/V \longrightarrow \mathbf{R}^{d-1} . \tag{4.4}$$

For each non-zero point x of $\mathbf{R}^d$, define

$$\pi_x : \{z \in \mathbf{R}^d | \; (z|x) = H_f(x)\} \longrightarrow \mathbf{R}^{d-1} \tag{4.5}$$

to be the composite of the restriction of the projection

$$\mathbf{R}^d \longrightarrow \mathbf{R}^d/x\mathbf{R}; \; z \longmapsto z + x\mathbf{R} \tag{4.6}$$

with the isomorphism $\theta_{x\mathbf{R}} : \mathbf{R}^d/x\mathbf{R} \longrightarrow \mathbf{R}^{d-1}$. Then π_x is an affine isomorphism, since satisfaction of the first two conditions on f guarantees that the domain of π_x, as the supporting hyperplane of the non-empty compact convex subset of $\mathbf{R}^d$ with support function H_f, is an affine subspace of $\mathbf{R}^d$ of dimension $d-1$, orthogonal to the axis $x\mathbf{R}$ of the projection (4.6). By convention, π_0 is defined to be the zero map

(4.7) $$\pi_0 : \{z \in \mathbf{R}^d | (z|0) = 0 = H_f(0)\} = \mathbf{R}^d \longrightarrow \mathbf{R}^{d-1}; z \longmapsto 0.$$

Of course, this is not an affine isomorphism.

One more concept is required for the formulation of the five conditions determining G_d. For a function f in F_d with positively homogeneous convex real function part H_f, and for x in $\mathbf{R}^d$, define the ***supercrust*** *(at x)*

(4.8) $$K_f(x) = \{z \in \mathbf{R}^d | \ \forall y \in \mathbf{R}^d, \quad (z|y) \leq H_f'(x; y)\}.$$

As observed in connection with (3.9), for non-zero x the set $K_f(x)$ is the intersection of the convex set having support function H_f with its supporting hyperplane $\{z \in \mathbf{R}^d| \ (z|x) = H_f(x)\}$.

Thus

(4.9) $$K_f(x) \subseteq \{z \in \mathbf{R}^d| \ (z|x) = H_f(x)\}$$

and

(4.10) $$\forall \ y \ \in \ \mathbf{R}^d, \quad K_f(x) \subseteq \{z \in \mathbf{R}^d| \ (z|y) \leq H_f(y)\}.$$

The third condition on f will be that its crust shadow in the x direction has to be contained in the image of the supercrust at x under π_x (whence the name "supercrust"). Since the positively homogeneous convex real function part H_f satisfies $H_f'(0; y) = H_f(y)$, the supercrust $K_f(0)$ is just the non-empty compact convex set with support function H_f. The containments (4.9) and (4.10) still hold if x is zero. The fourth condition on f is formulated in terms of the relative interior of the supercrust. Recall that the ***affine hull*** of a non-empty convex subset A of $\mathbf{R}^d$ is the smallest affine subspace of $\mathbf{R}^d$ containing A. The affine hull is topologised by the restriction of the Euclidean topology on $\mathbf{R}^d$. The ***relative interior*** A^0 of A is then defined to be the interior of A in this restriction topology on its affine hull. (An alternative, purely algebraic definition of A^0 is that (A^0, I^0) is the smallest non-empty sink of the barycentric algebra (A, I^0) – see [RS1, 386], [RS2].)

The inductive definition may now be completed.

Definition 4.11. The subset G_d of F_d, for $d > 0$, is defined to be the set of functions

(4.2) $$f : \mathbf{R}^d \longrightarrow D_d \ ; \ x \longmapsto (H_f(x), C_f(x))$$

satisfying the following *G-conditions*:

(GC1) H_f is positively homogeneous;

(GC2) H_f is convex;

(GC3) $\forall\, x \in \mathbf{R}^d, \quad C_f(x) \subseteq \pi_x(K_f(x))$;

(GC4) $\forall\, x \in L(H_f), \quad C_f(x) \supseteq \pi_x(K_f(x)^0)$;

(GC5) for all non-zero x, y in $\mathbf{R}^d$, $\pi_x^{-1}(C_f(x)) \cap \{z \in \mathbf{R}^d |\ (z|y) = H_f(y)\}$
$= \pi_y^{-1}(C_f(y)) \cap \{z \in \mathbf{R}^d |\ (z|x) = H_f(x)\}$

It remains to be shown that G_d is a submodal of F_d isomorphic with the modal $(\mathbf{R}^d B, +, I^0)$ of non-empty bounded convex subsets of $\mathbf{R}^d$. This will be done in the next section.

5. Characterisation of support functions. The singleton modals G_0 and $(\mathbf{R}^0 B, +, I^\circ)$ are clearly isomorphic, and may be identified. On the inductive assumption that G_{d-1} is a modal (of support functions) that has been identified with $(\mathbf{R}^{d-1} B, +, I^\circ)$, for $d > 0$, Definition 4.11 of the previous section defined G_d as the set of functions from $\mathbf{R}^d$ to the ordinal product modal D_d of (4.1) satisfying the G-conditions. To show that the G-conditions characterise the support functions of non-empty bounded convex subsets of $\mathbf{R}^d$, this section will show that G_d is a submodal of the induced modal F_d isomorphic to $(\mathbf{R}^d B, +, I^\circ)$.

The main task is to obtain a set isomorphism between G_d and $\mathbf{R}^d B$ from two mutually inverse mappings

$$(5.1) \qquad \phi : \mathbf{R}^d B \longrightarrow G_d; X \longmapsto (f_X : x \longmapsto (H_X(x), C_X(x)))$$

and

$$(5.2) \qquad \xi : G_d \longrightarrow \mathbf{R}^d B; f \longrightarrow X_f\ .$$

The mapping ϕ is defined initially as $\phi : \mathbf{R}^d B \longrightarrow F_d$ by taking the real function part H_X of f_X to be the (traditional) support function $H_{\overline{X}}$ of the closure $\overline{X}$ of X in $\mathbf{R}^d$; this closure is clearly a non-empty, compact, convex (cf. [Br, Th. 3.4(a)]) subset of $\mathbf{R}^d$. Using H_X in place of H_f, (4.5) then defines affine isomorphisms π_x for each non-zero vector x of $\mathbf{R}^d$. The crust shawdow $C_X(x)$ is defined to be the image under π_x of the crust $X \cap \{z \in \mathbf{R}^d | (z|x) = H_X(x)\}$ of X in the x direction. The crust shadow $C_X(0)$ is defined to be the zero subspace $\{0\}$ of $\mathbf{R}^{d-1}$.

LEMMA 5.3. *The function $X\phi = f_X$ satisfies the G-conditions.*

Proof. Since H_X is just the support function $H_{\overline{X}}$, the first two G-conditions are satisfied. Write $K_X(x)$ for the supercrust $K_{X\phi}(x)$. Then $C_X(x) = \pi_x(X \cap \{z \in \mathbf{R}^d | (z|x) = H_X(x)\}) \supseteq \pi_x(\overline{X} \cap \{z \in \mathbf{R}^d | (z|x) = H_X(x)\}) = \pi_x(K_X(x))$, so that (GC3) is satisfied. If x is zero, then (GC4) is immediate. Suppose that x is a non-zero linearity point for

H_X, so that $\overline{X} = K_X(x)$. Also $X \subseteq \{z \in \mathbf{R}^d | \ (z|x) = H_X(x)\}$, so $C_X(x) = \pi_x(X)$. Now $X \supseteq X^0 = (\overline{X})^0$ [Br, Th. 3.4(d)]. Applying π_x gives $C_X(x) = \pi_x(X) \supseteq \pi_x((\overline{X})^0) = \pi_x(K_X(x)^0)$, which is (GC4) for the non-zero x. Finally, the equality (GC5) holds for all non-zero x and y, since each side of the equality is $X \cap \{z \in \mathbf{R}^d|(z|x) = H_f(x)\} \cap \{z \in \mathbf{R}^d|(z|y) = H_f(y)\}$. □

The mapping ξ of (5.2) is defined by

$$X_f = \bigcap_{x \in \mathbf{R}^d} P_f(x) \tag{5.4}$$

for f in G_d, where

$$P_f(x) = \{z|(z|x) < H_f(x)\} \cup \pi_x^{-1}(C_f(x)). \tag{5.5}$$

LEMMA 5.6. *If f satisfies the G-conditions, then X_f is a non-empty bounded convex subset of $\mathbf{R}^d$.*

Proof. If f satisfies the G-conditions, its real function part H_f is the support function of a non-empty compact convex subset $X = K_f(0)$ of $\mathbf{R}^d$. Since the closed convex set X is non-empty, and $X = \overline{X} = \overline{(X^0)}$ (cf. [Br, Th. 3.4(c)]), the relative interior X^0 of X is also non-empty. By (GC4) for the linearity point $x = 0$, combined with (GC3), $\{0\} = \pi_0(X^0) \subseteq C_f(0) \subseteq \pi_0(X) = \{0\} \subseteq \mathbf{R}^{d-1}$. Then $P_f(0) = \emptyset \cup \pi_0^{-1}(\{0\}) = \mathbf{R}^d$, which certainly contains X^0. For non-zero x, the set $P_f(x)$ also contains X^0: in $\{z|(z|x) < H_f(x)\}$ if x is not a linearity point of H_f, and in $\pi_x^{-1}(C_f(x))$ by (GC4) if x is a linearity point of H_f. Since the non-empty set X^0 is contained in each $P_f(x)$, it is contained in their intersection (5.4), so that the set X_f is non-empty.

Now by (GC3) and (4.9), $P_f(x) \subseteq \{z|(z|x) \leq H_f(x)\}$. Taking the intersection of these containments over all x in $\mathbf{R}^d$ shows that X_f is contained in the bounded set X, and so is itself bounded. Finally, note that each $P_f(x)$ is convex, so that their intersection X_f is also. □

Lemmas 5.3 and 5.6 show that the codomains of the mappings ϕ and ξ are as claimed in (5.1) and (5.2). It must now be shown that the mappings are mutually inverse. The first result, Lemma 5.7 below, shows that $\phi\xi = 1 : \mathbf{R}^d B \longrightarrow \mathbf{R}^d B$. A non-empty bounded convex subset X of $\mathbf{R}^d$ furnishes a function $X\phi = f$ in G_d, to which corresponds a subset X_f of $\mathbf{R}^d$.

LEMMA 5.7. *The sets X and X_f are equal.*

Proof. Since the real function part H_f is the support function of the closure of X, one has that $X \subseteq \{z \in \mathbf{R}^d|(z|x) \leq H_f(x)\} \cap X \subseteq \{z|(z|x) < H_f(x)\} \cup [\{z|(z|x) = H_f(x)\} \cap X] = \{z|(z|x) < H_f(x)\} \cup \pi_x^{-1}(C_f(x)) = P_f(x)$ for each x in $\mathbf{R}^d$. The intersection of these containments $X \subseteq P_f(x)$ gives $X \subseteq X_f$.

Conversely, it will be shown that each element u of

$$(5.8) \qquad X_f = \bigcap_{x\in\mathbf{R}^d} [\{z|(z|x) < H_f(x)\} \cup \pi_x^{-1}(C_f(x))]$$

is an element of X. If u lies in $\pi_x^{-1}(C_f(x))$ for some non-zero x in $\mathbf{R}^d$, then $u \in X \cap \{z|(z|x) = H_f(x)\} \subseteq X$. So suppose

$$(5.9) \qquad u \in \bigcap_{0\neq x\in\mathbf{R}^d} \{z|(z|x) < H_f(x)\} .$$

Since $X^0 = (\overline{X})^0$ (cf. [Br, Th. 3.4(d)]) and $X^0 \subseteq X$, it suffices to show that $u \in (\overline{X})^0$. Apply a translation so that $u = 0$. Let $2\delta = \inf\{H_f(x)|(x|x) = 1\}$. If δ is positive, then the ball of radius δ centered on 0 lies in $\overline{X}$, so that $u = 0 \in (\overline{X})^0$. Otherwise, $\delta = 0$. A contradiction will be derived from this assumption. The assumption implies the existence of a sequence of points x_n on the unit sphere with $\lim_{n\to\infty} H_f(x_n) = 0$. For each n, define $y_n = x_n\, H_f(x_n)$. Then $\lim_{n\to\infty} y_n = 0$, and each y_n lies on the boundary of $\overline{X}$. But this boundary is closed (cf. [Bu, (1.4)]), and so contains 0, whence (cf. [Bu, (1.8)]) 0 lies on a supporting hyperplane of $\overline{X}$. Translating back to the original position of u, this gives a contradiction to (5.9). □

Lemma 5.10 below shows that $\xi\pi = 1 : G_d \longrightarrow G_d$. A function f in G_d specifies a convex set $f\xi = X$ according to (5.4). The set X then determines a function $X\phi = f_X$ in G_d.

LEMMA 5.10. *The functions f and f_X agree.*

Proof. The real function part of f is the support function of the supercrust $Y = K_f(0)$. The real function part of f_X is the support function of the closure $\overline{X}$ of $X = \bigcap_{x\in\mathbf{R}^d} P_f(x)$. Thus the equality of the real function parts of the two functions will follows from the equality $Y = \overline{X}$. Now $\overline{X} = \overline{\bigcap_{x\in\mathbf{R}^d} P_f(x)} \subseteq \bigcap_{x\in\mathbf{R}^d} \overline{P_f(x)} \subseteq \bigcap_{x\in\mathbf{R}^d} \{z|(z|x) \leq H_f(x)\} = Y$. Conversely, consider $Y = \overline{Y} = \overline{(Y^0)}$ (cf. [Br, Th. 3.4(c)]). Each element y of Y is the limit of a sequence of points y_n of Y^0. If x is not a linearity point of H_f, then $(y_n|x) < H_f(x)$. If x is a linearity point of H_f, so that $Y \subseteq K_f(x)$, then $y_n \in Y^0 \subseteq K_f(x)^0 \subseteq \pi_x^{-1}(C_f(x))$, the latter containment holding by (GC4). Thus y_n lies in $P_f(x)$ for all x in $\mathbf{R}^d$, whence $y_n \in X$. Taking the limit, one obtains $y \in \overline{X}$, as required.

With the equality of the real function parts demonstrated, it remains to show that the crust shadows $C_f(x)$ of f and

$$(5.11) \qquad \pi_x\left(\left[\bigcap_{0\neq y\in\mathbf{R}^d} P_f(y)\right] \cap \{z|(z|x) = H_f(x)\}\right)$$

of f_X agree for all non-zero x in $\mathbf{R}^d$ (for $x = 0$, both equal the zero subspace of $\mathbf{R}^{d-1}$). Now for non-zero y, one has $\pi_x^{-1}(C_f(x)) = [\pi_x^{-1}(C_f(x)) \cap \{z|(z|y) < H_f(y)\}] \cup [\pi_x^{-1}(C_f(x)) \cap \{z|(z|y) = H_f(y)\}] = [\pi_x^{-1}(C_f(x)) \cap \{z|(z|y) < H_f(y)\}] \cup [\pi_y^{-1}(C_f(y)) \cap \{z|(z|x) = H_f(x)\}] \subseteq \{z|(z|y) < H_f(y)\} \cup \pi_y^{-1}(C_f(y)) = P_f(y)$, the second equality holding by (GC5). Thus $\pi_x^{-1}(C_f(x)) \subseteq \left[\bigcap_{0\neq y\in\mathbf{R}^d} P_f(y)\right] \cap \{z|(z|x) = H_f(x)\}$. Applying the affine isomorphism π_x gives the containment of $C_f(x)$ in (5.11). Conversely, $\pi_x\left(\left[\bigcap_{0\neq y\in\mathbf{R}^d} P_f(y)\right] \cap \{z|(z|x) = H_f(x)\}\right) \subseteq \pi_x(P_f(x) \cap \{z|(z|x) = H_f(x)\}) = \pi_x(\pi_x^{-1}(C_f(x)) \cap \{z|(z|x) = H_f(x)\}) = \pi_x(\pi_x^{-1}(C_f(x))) = C_f(x)$, giving the containment of (5.11) in $C_f(x)$ and completing the proof of the lemma. □

Now that $\phi : \mathbf{R}^d B \longrightarrow G_d$ and $\xi = G_d \longrightarrow \mathbf{R}^d B$ have been proved to be mutually inverse set isomorphisms, it remains to show that ϕ is a modal homomorphism, for then G_d becomes a modal isomorphic with $(\mathbf{R}^d B, +, I^0)$, as required to complete the inductive definition of Section 4. A preliminary result is needed.

LEMMA 5.12. *For non-empty bounded convex subsets A, B of $\mathbf{R}^d$, and for p in I°, non-zero x in $\mathbf{R}^d$, $[A \cap \{z|(z|x) = H_{\overline{A}}(x)\}][B \cap \{z|(z|x) = H_{\overline{B}}(x)\}]\underline{p} = AB\underline{p} \cap \{z|(z|x) = H_{\overline{A}\,\overline{B}\underline{p}}(x)\}$.*

Proof. If $(z|x) = H_{\overline{A}}(x)$ and $(z'|x) = H_{\overline{B}}(x)$, then $(zz'\underline{p}|x) = (z|x)(z'|x)\underline{p} = H_{\overline{A}}(x)H_{\overline{B}}(x)\underline{p} = H_{\overline{A}\,\overline{B}\underline{p}}(x)$. Thus $\{z|(z|x) = H_{\overline{A}}(x)\}\{z|(z|x) = H_{\overline{B}}(x)\}\underline{p} = \{z|(z|x) = H_{\overline{A}\,\overline{B}\underline{p}}(x)\}$. The containment of the left hand side of the equation of the Lemma in its right hand side follows. Conversely, suppose $a \in A$, $b \in B$, and $(ab\underline{p}|x) = H_{\overline{A}\,\overline{B}\underline{p}}(x)$, so that $ab\underline{p}$ is a typical element of the right hand side. It will be shown that a and b lie in the respective arguments of the left hand side. If a is not in $\{z|(z|x) = H_{\overline{A}}(x)\}$, then $(a|x) < H_{\overline{A}}(x)$. But $(b|x) \leq H_{\overline{B}}(x)$, so $(ab\underline{p}|x) = (a|x)(b|x)\underline{p} < H_{\overline{A}}(x)H_{\overline{B}}(x)\underline{p} = H_{\overline{A}\,\overline{B}\underline{p}}(x)$, a contradiction. Thus a lies in the first argument. Similarly, b lies in the second. □

The concluding result may now be given

LEMMA 5.13. *The mapping $\phi : (\mathbf{R}^d B, +, I^\circ) \longrightarrow G_d$ is a modal homorphism.*

Proof. For non-empty bounded convex subsets A, B of $\mathbf{R}^d$, and for x in $\mathbf{R}^d$, it must be shown that

$$(5.14) \qquad (H_A(x), C_A(x))(H_B(x), C_B(x))\underline{p} = (H_{AB\underline{p}}(x), C_{AB\underline{p}}(x))$$

for p in I° and

$$(5.15) \qquad (H_A(x), C_A(x)) + (H_B(x), C_B(x)) = (H_{A+B}(x), C_{A+B}(x)).$$

The first component of (5.14) restates the fact that (2.6) is a barycentric homomorphism. The second component is trivial if x is zero. To verify it for non-zero x, take $\pi_x : \mathbf{R}^d \longrightarrow \mathbf{R}^{d-1}$ to be the full composite of (4.6) with $\theta_{x\mathbf{R}}$, rather than just its restriction to a supporting hyperplane. Note that π_x, being linear, is a barycentric homomorphism. Then $C_A(x)C_B(x)\underline{p} = \pi_x(X \cap \{z|(z|x) = H_A(x)\})\pi_x(Y \cap \{z|(z|x) = H_B(x)\})\underline{p} = \pi_x([A \cap \{z|(z|x) = H_A(x)\}][B \cap \{z|(z|x) = H_B(x)\}]\underline{p}) = \pi_x(AB\underline{p} \cap \{z|(z|x) = H_{AB\underline{p}}(x)\}) = C_{AB\underline{p}}(x)$, the penultimate equality holding by Lemma 5.12.

In the verification of (5.15), the expression

$$A + B = \bigcup_{p \in I} AB\underline{p} \tag{5.16}$$

is used, where (2.1) and (2.2) are extended to all of the closed unit interval I. (Thus $ab\underline{0} = a$ and $ab\underline{1} = b$, etc.) Given the commutativity of the semilattice operation $+$, and the fact that $(D, +)$ is a chain, (5.15) breaks up into just two cases. In the first, $H_A(x) > H_B(x)$, so the left hand side of (5.15) becomes $(H_A(x), C_A(x))$. Then $H_{A+B}(x)$ is the maximum of $\{H_A(x), H_B(x)\}$, namely $H_A(x)$, since (2.6) is a semilattice homorphism. Equality of the second component of (5.15) follows on applying π_x to the equality

$$A \cap \{z|(z|x) = H_A(x)\} = (A + B) \cap \{z|(z|x) = H_A(x)\} \ . \tag{5.17}$$

Here the right hand side clearly contains the left. Conversely, an element $ab\underline{p}$ of the right hand side ($a \in A$, $b \in B$) not contained in the left would have p positive. But then $(a|x) \leq H_A(x)$ and $(b|x) \leq H_B(x) < H_A(x)$ imply $(ab\underline{p}|x) = (a|x)(b|x)\underline{p} < H_A(x)H_A(x)\underline{p} = H_A(x)$, a contradiction.

The second case of (5.15) to be considered is where $H_A(x) = H_B(x)$, so that $H_{AB\underline{p}}(x) = H_A(x)H_B(x)\underline{p} = H_A(x)$. The left hand side of (5.15) becomes $(H_A(x), C_A(x) + C_B(x))$, and the first component of (5.15) certainly holds. The second component is trivial for zero x. For non-zero x, set $S = \{z|(z|x) = H_A(x)\}$. Since π_x is an affine isomorphism, $C_A(x) + C_B(x) = \pi_x(A \cap S) + \pi_x(B \cap S) = \pi_x((A \cap S) + (B \cap S))$, and $C_{A+B}(x) = \pi_x((A + B) \cap S)$. It will be shown that $(A \cap S) + (B \cap S) = (A + B) \cap S$, whence the second component of (5.15) follows on applying π_x. Indeed, $(A \cap S) + (B \cap S) = \bigcup_{p \in I}(A \cap S)(B \cap S)\underline{p} = \bigcup_{p \in I}(AB\underline{p} \cap S) = (\bigcup_{p \in I} AB\underline{p}) \cap S = (A + B) \cap S$. The first and last equalities here are cases of (5.16), while the second follows by Lemma 5.12.

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