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WITH SOURCE**

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ON SOME MONOTONICITY IN TIME PROPERTIES FOR A QUASILINEAR PARABOLIC EQUATION WITH SOURCE

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Abstract. We consider the Cauchy problem for a one-dimensional quasilinear degenerate heat equation with source. The property of monotone in time behavior of the weak solution at a fixed spatial point $x_0 \in R$ is studied. It is shown that the conditions of a such a behavior depend on “a nonlinear interaction” between the nonlinear heat operator and the source of energy considered. There are two different cases: (i) the solution is monotone in time for any x_0 which is far enough from the initial support and (ii) the solution is monotone in time if it becomes large enough. In a general situation the monotone in time behavior at a given point $x = x_0$ is proved to depend on the shape of the initial function in some neighborhood of $x = x_0$.

Proofs are based on the method of intersection comparison of the solution and the continuous set of stationary solutions of the same equation.

Key words. quasilinear heat equation with source, stationary solutions, intersection comparison

AMS(MOS) subject classifications. 35K55, 35K60

1. Introduction and the main results. We consider the Cauchy problem for a one-dimensional quasilinear heat equation with source

$$(1.1) \quad u_t = (u^\sigma u_x)_x + u^\beta \text{ in } \omega_T = R \times (0, T),$$

$$(1.2) \quad u(x, 0) = u_0(x) \geq 0 \text{ in } R = (-\infty, \infty),$$

where $\sigma > 0$ and $\beta > 1$ are fixed constants and $T \in (0, +\infty]$ is the maximal existence time of the solution $u = u(x, t) \geq 0$. The initial function is assumed to satisfy

$$(1.3) \quad \begin{aligned} \sup u_0 &= M_1 < +\infty; \sup |(u_0^\sigma)'| = M_2 < +\infty; \\ \text{supp } u_0(x) &\equiv \{x \in R | u_0(x) > 0\} = (h_-(0), h_+(0)) \\ &\text{is a bounded connected interval.} \end{aligned}$$

Under above hypotheses there exists the weak local (in time) solution $u(x, t)$ which is a nonnegative continuous function, see survey [10]. If $T < +\infty$, then the solution blows up in a finite time and (see references in [2], [13])

$$(1.4) \quad \lim_{t \rightarrow T} \sup_{x \in R} u(x, t) = +\infty.$$

T is called a finite blow-up time.

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This paper is devoted to the investigation of some monotonicity (in time) properties of the solution $u(x, t)$ at a point $x = x_0$.

Denote for a given $x_0 \in R$

$$d(x_0, \text{supp } u_0) \equiv \inf_{y \in \text{supp } u_0} |x_0 - y|$$

and

$$0_\delta(x_0) \equiv \{x \mid |x - x_0| < \delta\},$$

where $\delta > 0$ is a fixed constant. Set $m = [\beta - (\sigma + 1)]/2$,

$$c_1 = [2(\beta + \sigma + 1)]^{-1/2} B\left(\frac{\sigma + 1}{\beta + \sigma + 1}, \frac{1}{2}\right),$$

where $B(p, q)$ is Euler's beta function,

$$(1.5) \quad c_2 = \left(\frac{(mc_1)^2 + m}{(1 + mc_1^2)^2}\right)^{-1/2} > 0 \quad \text{for } \beta > \sigma + 1,$$

$$(1.6) \quad M_k = \left\{ M_1^{\beta + \sigma + 1} + \frac{\beta + \sigma + 1}{2\sigma^2} (M_1 M_2)^2 \right\}^{1/(\beta + \sigma + 1)}.$$

Denote for a fixed $\lambda > 0$

$$(1.7) \quad \bar{x}(\lambda) \equiv c_1 \lambda^{-m}$$

and $l_k = 2\bar{x}(M_k) > 0$.

We now state the main results. We consider two different cases.

The point x_0 is outside $\text{supp } u_0$. We now assume that $x_0 \notin \text{supp } u_0$. Then the hypotheses on u_0 for monotonicity of $u(x_0, t)$ are absolutely different for $1 < \beta \leq \sigma + 1$ and $\beta > \sigma + 1$.

THEOREM 1. *Let $\beta \in (1, \sigma + 1]$. If*

$$(1.8) \quad d(x_0, \text{supp } u_0) > \ell_k \equiv 2\bar{x}(M_k),$$

then

$$(1.9) \quad u(x_0, t) \text{ does not decrease in time in } [0, T).$$

Remark. For $\beta = \sigma + 1$ the corresponding length $\ell_k = \pi/(\sigma + 1)^{1/2}$ is independent of the initial function. This result has been recently proved in [4].

THEOREM 2. Let $\beta > \sigma + 1$. If for some $t_0 \in (0, T)$

$$(1.10) \quad u(x_0, t_0) > \left\{ \frac{1}{c_2} d(x_0, \text{supp } u_0) \right\}^{-\frac{2}{\beta - (\sigma + 1)}},$$

then

$$(1.11) \quad u_t(x_0, t) \geq 0 \text{ for all } t \in [t_0, T).$$

There is much difference between Theorems 1 and 2 which describe the monotone behavior of $u(x_0, t)$ outside the initial support. In comparison with Theorem 1 the second one shows that for $\beta > \sigma + 1$ the solution $u(x_0, t)$ is monotone in time if it becomes large enough. We shall show later by using a self-similar solution that Theorem 2 is optimal and cannot be improved. Another result on monotone behavior of large solutions without using particular properties of u_0 given in (1.3) has been proved in [7] (see also [6] and [8] where a general heat equation with arbitrary nonlinearities was considered). It can be stated as follows:

$$(1.12) \quad \begin{array}{l} \text{if for arbitrary fixed point } x = x_0 \in R \text{ there exists} \\ \text{some } t_0 \in (0, T) \text{ such that } u(x_0, t_0) \geq M_k, \text{ then (1.11) holds.} \end{array}$$

Thus, Theorems 1 and 2 describe monotone in time behavior depending on more detailed structure of the initial function.

To this end, we notice that there are some common features of monotonicity in time of $u(x_0, t)$ for $\beta \in (1, \sigma + 1]$ and $\beta > \sigma + 1$. In particular, we shall show that for $\beta > \sigma + 1$ the following result on monotonicity of a large solution is valid.

THEOREM 3. Let $\beta \in (1, \sigma + 1]$. Assume that x_0 satisfies the inequality

$$\rho \equiv \{d(x_0, \text{supp } u_0) - \ell_k/2\} \in (0, \ell_k/2),$$

and there exists $t_0 \in (0, T)$ such that:

i) for $\beta \in (1, \sigma + 1)$

$$(1.13) \quad u(x_0, t_0) \geq \left\{ M_k^{\sigma + 1 - \beta} - \frac{\sigma + 1 - \beta}{2} \rho^2 \right\}^{1/(\sigma + 1 - \beta)},$$

ii) for $\beta = \sigma + 1$

$$(1.13') \quad u(x_0, t_0) \geq M_k \cos((\sigma + 1)^{1/2} \rho).$$

Then (1.11) is valid.

The point x_0 is inside $\text{supp } u_0$. Assume now that $x_0 \in \text{supp } u_0$. It is well-known that for any fixed point $x_0 \in R$ inequality (1.9) is valid if, e.g., $u_0^{\sigma+1} \in C^2(R)$ and if the initial function satisfies

$$(1.14) \quad u_t(x, 0) \equiv (u_0^\sigma(u_0)_x)_x + u_0^\beta \geq 0 \text{ for all } x \in R.$$

The proof is a straightforward consequence of the analysis of the following linear parabolic equation

$$(1.15) \quad w_t = (u^\sigma w_x)_x + \beta u^{\beta-1} w$$

satisfied by the function $w \equiv u_t$. Indeed, $w(x, 0) \geq 0$ by (1.14) and hence the Maximum Principle [3, Chapter II] yields that $w(x, t) \geq 0$ in $R \times (0, T)$. (This proof for the weak solution $u(x, t)$ can be made rigorous by the approximation).

We now show that for $\beta \in (1, \sigma + 1]$ the monotonicity in time of $u(x_0, t)$ for $x = x_0 \in \text{supp } u_0$ depends on above mentioned inequality (1.14) in some neighborhood of this point.

THEOREM 4. *Let $\beta \in (1, \sigma + 1]$ and $x_0 \in \text{supp } u_0$. Denote $G \equiv \text{supp } u_0 \cap 0_\delta(x_0)$, where $\delta = \ell_k$. Assume that $u_0^{\sigma+1} \in C^2(G)$ and*

$$(1.16) \quad (u_0^\sigma(u_0)_x)_x + u_0^\beta > 0 \text{ in } G.$$

Then (1.9) holds.

For the case $\beta > \sigma + 1$ we can prove a weakened form of above Theorem 4.

THEOREM 5. *Let $\beta > \sigma + 1$ and $x_0 \in \text{supp } u_0$ be the maximum point of the function $u_0(x)$ in the domain $0_\delta(x_0)$, where $\delta = 2\bar{x}(u_0(x_0))$, i.e., $u_0(x_0) \geq u_0(x)$ in $0_\delta(x_0)$. Assume that $u_0^{\sigma+1} \in C^2(G)$, $G \equiv \text{supp } u_0 \cap 0_\delta(x_0)$, and (1.16) holds. Then (1.9) is valid.*

Finally, we state some general result for arbitrary $\sigma > 0$, $\beta > 1$ under more restrictive hypotheses on the initial function u_0 .

THEOREM 6. *Let $\sigma > 0$ and $\beta > 1$. Set $\delta = \ell_k$ for $\beta \in (1, \sigma + 1]$ and $\delta = 2\bar{x}(u_0(x_0))$ for $\beta > \sigma + 1$ and assume that $u_0^{\sigma+1}(x) \in C^2(0_\delta(x_0))$. Denote*

$$(1.17) \quad \lambda_0 = \inf_{x \in 0_\delta(x_0)} u_0(x)$$

(we set $0_\delta(x_0) = R$ if $\delta = \infty$). If (cf. (1.16))

$$(1.18) \quad (u_0^\sigma(u_0)_x)_x + \lambda_0^\beta \geq 0 \text{ in } 0_\delta(x_0),$$

then (1.9) is valid.

The proofs are based on the method of intersection comparison which has been applied to many problems for nonlinear parabolic equations and systems, see references in [13, Chapter IV]. The plan of the paper is as follows. The set of stationary solutions of the equation (1.1) is studied in Section 2. Section 3 is devoted to intersection comparison techniques. The proofs of Theorems 1–6 are given in Section 4. The final remarks are discussed in Section 5.

2. Stationary solutions. The proofs of the main results are based on the intersection comparison with stationary solutions of equation (1.1). Fix arbitrary $\lambda > 0$ and $a \in R$. Let $U = U(x; \lambda, a)$ be the solution of two Cauchy problems (for $x > a$ and $x < a$ respectively) for the following ordinary differential equation:

$$(2.1) \quad \begin{aligned} (U^\sigma U_x)_x + U^\beta &= 0 \text{ for } x \in R \setminus \{a\}; \\ U(a; \lambda, a) &= \lambda, U_x(a; \lambda, a) = 0. \end{aligned}$$

It is easy to verify that for any given $a \in R, \lambda > 0$ the solution $U(x; \lambda, a)$ exists and it is strictly positive on the interval $\omega(\lambda, a) \equiv \{|x - a| < \bar{x}(\lambda)\}$, $U(a \pm \bar{x}(\lambda); \lambda, a) = 0$, where $\bar{x}(\lambda)$ is given in (1.7). It follows from (1.7) that for any given constants $\lambda_1 > \lambda_2 > 0$

$$(2.2) \quad \begin{aligned} \text{if } \beta \in (1, \sigma + 1) \text{ then } \omega(\lambda_2, a) &\subset \omega(\lambda_1, a), \\ \text{if } \beta = \sigma + 1 \text{ then } \omega(\lambda_2, a) &\equiv \omega(\lambda_1, a), \\ \text{if } \beta > \sigma + 1 \text{ then } \omega(\lambda_1, a) &\subset \omega(\lambda_2, a). \end{aligned}$$

In particular, for $\beta \in (1, \sigma + 1]$ we have

$$(2.2') \quad U(x; \lambda_2, a) \leq U(x; \lambda_1, a) \text{ for all } x \in \omega(\lambda_2, a) \subseteq \omega(\lambda_1, a).$$

The function $U(x; \lambda, a)$ is symmetric with respect to $x = a$: $U(x; \lambda, a) = U(2a - x; \lambda, a)$, $U_x(x; \lambda, a) < 0$ for $x \in (a, a + \bar{x}(\lambda))$ and

$$(2.3) \quad U(x; \lambda, a) = U(x - a; \lambda, 0) \text{ for } x \in \omega(\lambda, a).$$

We shall use some other simple properties of stationary solutions which are consequence of the invariance of equation (2.1) under the rescaling transformation:

$$(2.4) \quad \begin{aligned} U(x; \lambda, a) &\equiv \lambda U(\lambda^m(x - a); 1, 0), \\ U_x(x; \lambda, a) &\equiv \lambda^{(\beta+1-\sigma)/2} U_s(s; 1, 0), s = \lambda^m(x - a). \end{aligned}$$

By using (2.4) we conclude that

$$(2.5) \quad \begin{aligned} U(x; \lambda, a) &\rightarrow \infty \text{ in } \omega_1(\lambda, a) \equiv \{|x - a| \leq \bar{x}(\lambda)/2\} \text{ as } \lambda \rightarrow \infty, \\ |(U^\sigma)_x(x; \lambda, a)| &\rightarrow \infty \text{ in } \omega(\lambda, a) \setminus \omega_1(\lambda, a) \text{ as } \lambda \rightarrow \infty. \end{aligned}$$

Multiplying (2.1) by $(U^{\sigma+1})_x$ and integrating over (a, x) yield that

$$(2.6) \quad \begin{aligned} & (U^{\sigma+1})_x^2(x; \lambda, a) = \\ & \frac{2(\sigma+1)^2}{\beta+\sigma+1} [\lambda^{\beta+\sigma+1} - U^{\beta+\sigma+1}(x; \lambda, a)] \text{ for } x \in \omega(\lambda, a). \end{aligned}$$

We need to derive some upper estimate of $U(x; \lambda, a)$ for arbitrary fixed $\lambda > 0$ and $a \in R$. Suppose $m \equiv [\beta - (\sigma + 1)]/2 > 0$. By integrating (2.1) over $[a, x]$, $x \in (a, a + \bar{x}(\lambda))$, and by using the monotonicity in x of $U(x; \lambda, a)$ there, we obtain that

$$U^\sigma(x)U_x(x) = - \int_a^x U^\beta dz < -U^\beta(x)(x - a),$$

and hence $(U^{-2m})_x > 2m(x - a)$. Then integrating yields

$$(2.7) \quad \begin{aligned} & U(x; \lambda, a) \leq U_+(x; \lambda, a) \equiv \\ & \equiv (\lambda^{-2m} + m(x - a)^2)^{-1/2m} \text{ for } x \in \omega(\lambda, a). \end{aligned}$$

It is easy to verify that for $\beta \in (1, \sigma + 1)$ (i.e., $m < 0$) the estimate (2.7) is also valid. Notice that for $\beta \in (1, \sigma + 1)$ the function $U_+(x; \cdot)$ exists on a finite interval.

One can see that the set of stationary solutions can be defined by the following slightly different way. Fix arbitrary $x_0 \in R$, $\mu > 0, \nu \in R$ and consider the following Cauchy problems (for $x > x_0$ and $x < x_0$) for the function $V = V(x; x_0, \mu, \nu)$ (cf. (2.1)):

$$(2.8) \quad \begin{aligned} & (V^\sigma V_x)_x + V^\beta = 0 \text{ for } x \in R \setminus \{x_0\}; \\ & V(x_0; x_0, \mu, \nu) = \mu, (V^\sigma)_x(x_0; x_0, \mu, \nu) = \nu. \end{aligned}$$

The solutions of problems (2.8) exist and the function $V(x; x_0, \mu, \nu)$ is strictly positive on some maximal interval $\omega(x_0, \mu, \nu) \subset R$. The function V satisfies the identify (cf. (2.6))

$$(2.9) \quad \begin{aligned} & (V^{\sigma+1})_x^2(x; x_0, \mu, \nu) = \left(\frac{\sigma+1}{\sigma} \mu \nu \right)^2 + \\ & + \frac{2(\sigma+1)^2}{\beta+\sigma+1} [\mu^{\beta+\sigma+1} - V^{\beta+\sigma+1}(x; x_0, \mu, \nu)] \text{ for } x \in \omega(x_0, \mu, \nu). \end{aligned}$$

It is easy to calculate that for any fixed x_0, μ, ν there exists a unique pair of constants $\lambda \geq \mu$ and $a \in \omega(x_0, \mu, \nu)$ such that

$$(2.10) \quad V(x; x_0, \mu, \nu) \equiv U(x; \lambda, a) \text{ in } \omega(x_0, \mu, \nu) \equiv \omega(\lambda, a).$$

3. Main intersection comparison techniques. For fixed x_0, μ, ν and $t_0 \in [0, T)$ denote by $N(t_0; x_0, \mu, \nu)$ the number of intersections in $\omega(x_0, \mu, \nu)$ of the functions $u(x, t_0)$ and $V(x; x_0, \mu, \nu)$ or, which is the same, the number of sign changes in $\omega(x_0, \mu, \nu)$ of the difference $u(x, t_0) - V(x; x_0, \mu, \nu)$. Let $N(t_0; \lambda, a)$ be the number of intersections in $\omega(\lambda, a)$ of the functions $u(x, t_0)$ and $U(x; \lambda, a)$.

Under some hypotheses on solutions considered one can obtain an upper bound of the number of intersections, see, e.g., [1,14,6,11,12] and references in [13, Chapter IV]. The following Lemma 1 will be used in the proof of the main results. The detailed proof of Lemma 1 is based on the Maximum Principle and can be found in [7] (see also [4]). We shall assume that (1.3) holds.

LEMMA 1. Let for fixed $x_0, \nu \in R, \mu \geq u_0(x_0)$

$$(3.1) \quad \begin{aligned} & \text{either } N(0; x_0, \mu, \nu) < 2, \\ & \text{or } N(0; x_0, \mu, \nu) = 2 \text{ and } \bar{\omega}(x_0, \mu, \nu) \subset \text{supp } u_0. \end{aligned}$$

Then

$$N(t_0; x_0, \mu, \nu) \leq 2 \text{ for all } t_0 \in (0, T).$$

The main result of the intersection comparison with the set of stationary solutions is given below, see [7] and also [6,8].

LEMMA 2. Fix arbitrary $x_0 \in R$. Assume that there exists a constant $\mu_* \geq u_0(x_0)$ such that (3.1) holds for arbitrary $\mu \geq \mu_*$ and $\nu \in R$. Suppose that there exists $t_0 \in [0, T)$ such that $u(x_0, t_0) = \mu_*$. Then

- (i) if $\mu_* = u_0(x_0)$, then (1.9) holds;
- (ii) if $\mu_* > u_0(x_0)$, then (1.11) holds.

The proof is based on the analysis of the number of intersections $N(t_0; x_0, \mu, \nu)$ of the functions $u(x, t_0)$ and $V(x; x_0, \mu, \nu)$ for all $\mu \geq \mu_*$. The sketch of the proof is given below. We now introduce some basic notations. Assume that the conditions of Lemma 2 are valid and, e.g., $\mu_* > u_0(x_0) \geq 0$. Then for a fixed $t = t_0$ we construct at the point $x = x_0$ the *tangent stationary solution* $V(x; x_0, \mu_*, \nu)$ to the profile $u(x, t_0)$, $u(x_0, t_0) > 0$, where

$$\mu_* = u(x_0, t_0), \nu = (u^\sigma)_x(x_0, t_0).$$

We have to show that in some small neighborhood of the point $x = x_0$ the inequality $u(x, t_0) \geq V(x; x_0, \mu_*, \nu)$ is valid, i.e., $x = x_0$ is the point of *tangency*. Then one can see by comparison that $u_t(x_0, t_0) \geq V_t(x_0; x_0, \mu_*, \nu) = 0$. Suppose that it is not valid and hence the point $x = x_0$ is a point of *inflection*. In this case we can consider some other stationary solutions $V = V(x; x_0, \mu_*, \nu_1)$ such that $V(x_0; x_0, \mu_*, \nu_1) = \mu_*$ and $|\nu - \nu_1| > 0$ is sufficiently small. It is easily seen that there exists some constant ν_1 such that $N(t_0; x_0, \mu_*, \nu_1) \geq 3$ and hence we arrive at the contradiction of Lemma 1. Since the condition (3.1) is valid for any $\mu \geq \mu_*$, we can conclude that $u_t(x_0, t) \geq V_t(x_0; x_0, \mu, \nu) \equiv 0$ for any $t \in [t_0, T)$ such that $u(x_0, t) = \mu$ (see a similar detailed analysis in [7] and similar techniques in [6,8]).

4. Proofs of Theorems 1–6.

Proof of Theorem 1. First we consider the stationary solutions $U(x; \lambda, a)$ with $\lambda \geq M_k$ and $a \in R$, where the constant M_k is given in (1.6). It follows from (1.3), (1.6) and (2.6) that (see [7])

$$(4.1) \quad \begin{aligned} N(0; \lambda, a) &\leq 2 \text{ for any } \lambda \geq M_k \text{ and } a \in R \\ \text{and if } N(0; \lambda, a) &= 2 \text{ then } \bar{\omega}(\lambda, a) \subset \text{supp } u_0(x). \end{aligned}$$

Thus we now assume that (1.8) holds for some $x_0 \in R$. Fix arbitrary constants $\mu > 0$ and $\nu \in R$. By using (2.10) we can uniquely choose the constants $\lambda > 0$, $a \in R$, so that $V(x; x_0, \mu, \nu) \equiv U(x; \lambda, a)$ and $\omega(x_0, \mu, \nu) = \omega(\lambda, a)$. We need to consider two cases.

1. If $\lambda \geq M_k$, then $N(0; \lambda, a) \equiv N(0; x_0, \mu, \nu) \leq 2$ by the property (4.1). Since $x_0 \in \omega(x_0, \mu, \nu)$ and $x_0 \notin \text{supp } u_0$, we obtain that $N(0; x_0, \mu, \nu) \leq 1$.

2. If $\lambda \in (0, M_k)$, then $\text{mes } \omega(x_0, \mu, \nu) \equiv \text{mes } \omega(\lambda, a) < \text{mes } \omega(M_k, a) \equiv \ell_k$, see (2.2). Since $x_0 \in \omega(x_0, \mu, \nu)$ and (1.8) holds, we deduce that $\text{supp } u_0 \cap \omega(x_0, \mu, \nu) = \emptyset$. Hence $N(0; x_0, \mu, \nu) = 0$.

In both cases we arrive at the necessary estimate $N(0; x_0, \mu, \nu) \leq 1$. Therefore Theorem 1 is a straightforward consequence of Lemma 2. $\square$

Proof of Theorem 2. Without loss of generality we assume that $x_0 > h_+(0)$. Let us introduce the following set of stationary solutions:

$$P_U = \{U_* | U_*(x; \lambda) = U(x; \lambda, h_+(0) + \bar{x}(\lambda)), \lambda > 0\}.$$

Any function $U_*(x; \lambda) \in P_U$ is strictly positive on $\omega_h(\lambda) = (h_+(0), h_+(0) + 2\bar{x}(\lambda))$ and $U_*(h_+(0); \lambda) \equiv 0$. Set formally $U_*(x; \lambda) \equiv 0$ in $R \setminus \omega_h(\lambda)$.

Consider for $\beta > \sigma + 1$ the following tangent curves to the set P_U and to the set (2.7) of functions $\{U_+\}$, “shifted” in x , respectively:

$$(4.2) \quad \begin{aligned} L_*(x) &= \sup_{\lambda > 0} U_*(x; \lambda), \\ L_+(x) &= \sup_{\lambda > 0} U_+(x; \lambda, h_+(0) + \bar{x}(\lambda)). \end{aligned}$$

The existence of $L_*(x)$ follows from (2.2) and (2.4). One can see that $L_*(x) > 0$ for $x > h_+(0)$ and $L_*(h_+(0)) = +\infty$. On the other hand, the function $L_+(x)$ can be calculated explicitly:

$$L_+(x) = [(x - h_+(0))/c_2]^{-2/[\beta - (\sigma + 1)]} \text{ for } x > h_+(0),$$

where the constant c_2 is given in (1.5). Using (2.7) yields

$$(4.3) \quad \begin{aligned} L_*(x) &\leq L_+(x) \\ &= [(x - h_+(0))/c_2]^{-2/[\beta - (\sigma + 1)]} \text{ for } x > h_+(0). \end{aligned}$$

Fix arbitrary $x_0 > h_+(0)$, $\mu \geq L_+(x_0) > L_*(x_0)$ and $\nu \in R$. It follows from (2.10) that there exist the constants $a \in R$, $\lambda > 0$ so that $V(x; x_0, \mu, \nu) \equiv U(x; \lambda, a)$ and $U(x_0; \lambda, a) = \mu > L_*(x_0)$. We shall show that $\text{supp } u_0 \cap \omega(\lambda, a) = \emptyset$ and $N(0; \lambda, a) = 0$.

Suppose now for contradiction that $\text{supp } u_0 \cap \omega(\lambda, a) \neq \emptyset$ and therefore $h_+(0) > a - \bar{x}(\lambda)$. Set $\delta = h_+(0) - a + \bar{x}(\lambda) > 0$ and consider the stationary solution $U_\delta = U(x; \lambda, a + \delta)$. One can see that $U_\delta \in P_U$. Then, (2.3) yields that for $x_1 = x_0 + \delta$ there holds $U(x_1; \lambda, a + \delta) > L_*(x_1)$. Hence, we arrive at the contradiction of (4.2). Thus, $\text{supp } u_0 \cap \omega(\lambda, a) = \emptyset$ and $N(0; \lambda, a) = 0$. Using Lemma 2 completes the proof. $\square$

Proof of Theorem 3. Without loss of generality we assume that $x_0 \in \omega_+ \equiv (h_+(0) + \ell_k/2, h_+(0) + \ell_k)$. Set $a_+ \equiv h_+(0) + \ell_k/2$.

Consider the case $\beta \in (1, \sigma + 1)$. We now prove the stronger condition than (1.13). Notice that the term given in the right-hand side of the inequality (1.13) is precisely the function $U_+(x; M_k, a_+)$ for $x = x_0$, which is the upper bound of $U(x; M_k, a_+)$ so that $U(x; M_k, a_+) \leq U_+(x; M_k, a_+)$ for $x \in \omega_+$, see (2.7). Thus, if we prove that the property (1.11) is the consequence of the inequality $u(x_0, t_0) \geq U(x_0; M_k, a_+)$ for some $t_0 \in (0, T)$, then Theorem 3 for $\beta \in (1, \sigma + 1)$ will be valid.

Fix arbitrary constants $\mu \geq U(x_0; M_k, a_+)$ and $\nu \in R$. Using (2.10), choose the constants $\lambda > 0, a \in R$ so that $V(x; x_0, \mu, \nu) \equiv U(x; \lambda, a)$. Consider two cases.

1. If $\lambda \geq M_k$, then (4.1) yields (3.1). (Notice that since $x_0 \in \omega(x_0, \mu, \nu)$ and $x_0 \notin \text{supp } u_0$ we obtain that $N(0; x_0, \mu, \nu) \leq 1$).

2. If $\lambda \in (0, M_k)$, we now prove that $\text{supp } u_0 \cap \omega(\lambda, a) = \emptyset$ and hence $N(0; \lambda, a) \equiv N(0; x_0, \mu, \nu) = 0$. Show that $a_+ = h_+(0) + \ell_k/2 < a$. Indeed, this estimate follows from (2.2') and the condition $\mu \geq U(x_0; M_k, a_+)$. Since $\text{supp } u_0 \cap \omega(M_k, a_+) = \emptyset$, $a_+ < a$ and $\lambda < M_k$, we deduce from (2.2) that $\text{supp } u_0 \cap \omega(\lambda, a) = \emptyset$.

Thus, (3.1) holds and Lemma 2 completes the proof for $\beta \in (1, \sigma + 1)$.

Consider the case $\beta = \sigma + 1$. Notice that $\text{mes } \omega(\lambda, a) \equiv \ell_k = \pi/(\sigma + 1)^{1/2}$ for any $\lambda > 0, a \in R$ and $U(x_0; M_k, a_+) \equiv M_k \cos((\sigma + 1)^{1/2}(x - a_+))$ for $x \in \omega_+$. Thus, one can see in this case that the proof is quite the same as for $\beta \in (1, \sigma + 1)$.

In order to prove Theorems 4 and 5 we need some preliminaries. First, we state for convenience the following result.

LEMMA 3. *Let $\sigma > 0, \beta > 1$ and $x_* \in \text{supp } u_0$. Assume that there exists a neighborhood $0_\delta(x_*)$, $\delta > 0$, of the point $x = x_*$ such that $u_0(x) > 0$ for $x \in 0_\delta(x_*)$, $u_0^{\sigma+1} \in C^2(0_\delta(x_*))$ and*

$$(u_0^\sigma(u_0)_x)_x + u_0^\beta > 0 \text{ for all } x \in 0_\delta(x_*).$$

Denote by $V = V(x; x_, u_0(x_*), (u_0^\sigma(x_*))_x)$ the tangent stationary solution to $u_0(x)$ at the point $x = x_*$. Then there exists some small neighborhood $0_{\delta_1}(x_*) \subset 0_\delta(x_*)$ such that*

$$V(x; x_*, u_0(x_*), (u_0^\sigma(x_*))_x) < u_0(x) \text{ for all } x \in 0_{\delta_1}(x_*) \setminus \{x = x_*\}.$$

Proof. It is based on the simple analysis of the ordinary differential inequality for the difference $w \equiv u_0(x) - V(x; \cdot)$ near the point $x = x_*$. $\square$

LEMMA 4. *Let $1 < \beta \leq \sigma + 1$. Fix arbitrary $x_0 \in \text{supp } u_0, \mu \geq u_0(x_0), \nu \in R$, and denote $G_1 \equiv \omega(x_0, \mu, \nu) \cap \text{supp } u_0$. Assume that $u_0^{\sigma+1} \in C^2(G_1)$ and $(u_0^\sigma(u_0)_x)_x + u_0^\beta > 0$ on G_1 . Then (3.1) holds.*

Proof. We denote by $a \in R$ and $\lambda \geq \mu$ the constants given in (2.10) such that $U(x; \lambda, a) \equiv V(x; x_0, \mu, \nu)$ and $\omega(\lambda, a) \equiv \omega(x_0, \mu, \nu)$.

1. Suppose now for a contradiction that $N(0; x_0, \mu, \nu) \equiv N(0; \lambda, a) \geq 3$. Then there exist at least two points $x_1, x_2 \in G_1, x_1 < x_2$, such that

$$(4.4) \quad \begin{aligned} u_0(x_1) = U(x_1; \lambda, a) &> 0, u_0(x_2) = U(x_2; \lambda, a) > 0, \\ u_0(x) &> U(x; \lambda, a) \quad \text{for all } x \in (x_1, x_2). \end{aligned}$$

Then since $\beta \in (1, \sigma + 1]$ from (2.2') we have that for any $\bar{\lambda} > \lambda$

$$(4.5) \quad \begin{aligned} U(x_1; \bar{\lambda}, a) &> u_0(x_1), \quad U(x_2; \bar{\lambda}, a) > u_0(x_2), \\ U(x; \bar{\lambda}, a) &> U(x; \lambda, a) \quad \text{for all } x \in [x_1, x_2]. \end{aligned}$$

Notice that $U(x; \bar{\lambda}, a) \rightarrow +\infty$ as $\bar{\lambda} \rightarrow +\infty$ uniformly in $[x_1, x_2]$, see (2.4). Then by continuity of the function $U(x; \lambda, a)$ with respect to λ we can conclude that there exist some constant $\lambda_* > \lambda$ and at least one point $x_* \in (x_1, x_2)$ such that $U(x_*; \lambda_*, a) = u_0(x_*)$ and $U(x; \lambda_*, a) \geq u_0(x)$ for all $x \in (x_1, x_2)$. Since the functions $U(x; \lambda, a)$ and $u_0(x)$ are smooth in G_1 , the function $U(x; \lambda_*, a)$ is the tangent stationary solution to the initial function $u_0(x)$. Setting $0_{\delta_1}(x_*) \equiv 0_\delta(x_*) \subset (x_1, x_2)$ for some small $\delta > 0$, we arrive at the contradiction of Lemma 3.

2. Assume now that $N(0; x_0, \mu, \nu) = 2$ and $\bar{\omega}(x_0, \mu, \nu) \cap \text{supp } u_0 \neq \bar{\omega}(x_0, \mu, \nu)$. Then it can be easily seen that $\text{supp } u_0 \subseteq \omega(x_0, \mu, \nu)$. In this case there exist two points $x_1, x_2 \in G_1 \equiv \text{supp } u_0, x_1 < x_2$, such that (4.4) hold. As before it leads to a contradiction of Lemma 3. $\square$

Proof of Theorem 4. Fix arbitrary $\mu \geq u_0(x_0), \nu \in R$. Choose by (2.10) the constants $\lambda > 0$ and $a \in R$ such that $U(x; \lambda, a) \equiv V(x; x_0, \mu, \nu)$ and $\omega(\lambda, a) = \omega(x_0, \mu, \nu)$. If $\lambda \geq M_k$, then (4.1) and (3.1) hold. Notice that for $\lambda = M_k$ we have $a \in (x_0 - \ell_k/2, x_0 + \ell_k/2)$, where $\ell_k/2 \equiv \bar{x}(M_k)$. Since $\delta = \ell_k$, we conclude that $\omega(M_k, a) \subset 0_\delta(x_0)$.

Assume now that $\lambda \in (0, M_k)$. It follows from (1.7) that $\bar{x}(\lambda) < \ell_k/2 \equiv \bar{x}(M_k)$. Since $a \in (x_0 - \bar{x}(\lambda), x_0 + \bar{x}(\lambda))$ we have that $\omega(\lambda, a) \subset \omega(M_k, a) \subset 0_\delta(x_0)$. Then $G_1 \equiv \omega(\lambda, a) \cap \text{supp } u_0 \subset G$, and hence by Lemma 4 the property (3.1) holds. Using Lemma 2 completes the proof of Theorem 4. $\square$

Proof of Theorem 5 is based on the following result.

LEMMA 5. Let $\beta > \sigma + 1$ and $x_0 \in \text{supp } u_0$. Fix arbitrary $\mu \geq u_0(x_0)$ and $\nu \in R$ and denote $G_1 = \omega(x_0, \mu, \nu) \cap \text{supp } u_0$. Let $x = x_0$ be a local maximum point of $u_0(x)$ in the following sense: $u_0(x_0) \geq u_0(x)$ for all $x \in G_1$. Assume that $u_0^{\sigma+1} \in C^2(G_1)$ and $(u_0^\sigma(u_0)_x)_x + u_0^\beta > 0$ on G_1 . Then (3.1) holds.

Proof. For given $\mu \geq u_0(x_0), \nu \in R$ choose the constants $a \in R$ and $\lambda > 0$ such that (2.10) holds. Notice that $\lambda \geq \mu \geq u_0(x_0)$. Assume that (3.1) is not valid.

1. Suppose that $N(0; x_0, \mu, \nu) \equiv N(0; \lambda, a) \geq 3$. Then there exist at least two points $x_1, x_2 \in G_1$ such that (4.4) holds. Without loss of generality we assume that $x_2 > x_1 > a$ and hence $U(x_1; \lambda, a) > U(x_2; \lambda, a)$. Let $x = x_3 \in [x_1, x_2]$ be a point of maximum of $u_0(x)$ on $[x_1, x_2]$.

Denote $k_- = u_0(x_2)$ and $k_+ = u_0(x_3) \geq u_0(x_1) > k_-$. By hypotheses of Lemma 5 we have that $k_+ \leq u_0(x_0)$. Fix arbitrary $k \in [k_-, k_+]$. One can see that there exists

$$(4.6) \quad x_{k,U} = \sup\{x \in \omega(\lambda, a) | U(x; \lambda, a) = k\}.$$

Since $x = x_0$ is the maximum point of $u_0(x)$ in G_1 , we conclude that for arbitrary fixed $k \in [k_-, k_+]$ there exists

$$(4.7) \quad x_{k,u} = \sup\{x \in (x_1, x_2) | u_0(x) = k\}.$$

Then by assumption we have that

$$(4.8) \quad \delta = \sup_{k \in [k_1, k_2]} \{x_{k,u} - x_{k,U}\} > 0.$$

One can see by definition (4.8) that there exists $k_\delta \in [k_-, k_+]$ such that

$$(4.9) \quad \delta = x_{k_\delta,u} - x_{k_\delta,U}.$$

Denote $x_* = x_{k_\delta,u}$. Consider the difference $w_\delta(x) = u_0(x) - U(x; \lambda, a + \delta)$. Then, by the construction $w_\delta(x_*) = 0$. It follows from (4.6) – (4.9) that

$$w_\delta(x) \leq 0 \text{ in } [x_{k_+,U} + \delta, x_{k_-,U}] \equiv I_\delta$$

and $x_* \in I_\delta$. It is easily seen that

$$w'_\delta(x_*) = u'_0(x_*) - U_x(x_*; \lambda, a + \delta) = 0.$$

Indeed, if such a tangency condition is not valid then this contradicts the definition of δ , see (4.8). Hence $U(x; \lambda, a + \delta) \equiv V(x; x_0 + \delta, \mu, \nu)$ is the tangent stationary solution to

the function $u_0(x)$ at the point $x = x_*$ and this fact leads to a contradiction of Lemma 3. Therefore $N(0; x_0, \mu, \nu) \equiv N(0; \lambda, a) \leq 2$.

2. Consider the case $N(0; x_0, \mu, \nu) = 2$ and suppose that $\bar{\omega}(x_0, \mu, \nu) \cap \text{supp } u_0 \neq \bar{\omega}(x_0, \mu, \nu)$, i.e., $\text{supp } u_0 \subseteq \omega(\lambda, a)$. Then the end of the proof is quite similar to the proof of Lemma 4. This completes the proof of Lemma 5. $\square$

Proof of Theorem 5. Consider arbitrary fixed constants $\mu \geq u_0(x_0)$ and $\nu \in R$. We first show that $\omega(x_0, \mu, \nu) \subset 0_\delta(x_0)$.

Let a and λ be such constants that (2.10) holds. Notice that $\lambda \geq \mu$ and $x_0 \in \omega(\lambda, a)$. Since $\lambda \geq u_0(x_0)$ and $m = [\beta - (\sigma + 1)]/2 > 0$, it follows from (1.7) that $\bar{x}(\lambda) \leq \bar{x}(u_0(x_0))$. By using the fact that $x_0 \in \omega(\lambda, a)$ and $\delta = 2\bar{x}(u_0(x_0))$ we obtain that $\omega(x_0, \mu, \nu) \equiv \omega(\lambda, a) \subset 0_\delta(x_0)$. Hence $G_1 = \omega(x_0, \mu, \nu) \cap \text{supp } u_0 \subset G$ and we can use Lemma 5 and Lemma 2 after. $\square$

Proof of Theorem 6. Fix arbitrary constants $\mu \geq u_0(x_0)$ ($\mu > 0$ if $u_0(x_0) = 0$) and $\nu \in R$ and consider the corresponding stationary solution $V(x; x_0, \mu, \nu)$. Let $\lambda > 0$ and $a \in R$ be the constants such that $V(x; x_0, \mu, \nu) \equiv U(x; \lambda, a)$, see (2.10). If $\lambda \geq M_k$, then (4.1) and (3.1) hold.

Assume that $\lambda \in (0, M_k)$. Then we can show by the same way as in proofs of Theorems 4 and 5 that $\omega(\lambda, a) \equiv \omega(x_0, \mu, \nu) \subset 0_\delta(x_0)$. Denote $z(x) = u_0^{\sigma+1}(x) - V^{\sigma+1}(x; x_0, \mu, \nu)$ and notice that the number of sign changes in $\omega(x_0, \mu, \nu)$ of the function $z(x)$ is equal to $N(0; x_0, \mu, \nu)$. Consider the first case, where $\lambda_0 = 0$ in (1.17) and (1.18). Then $u_0^{\sigma+1}(x)$ is a monotone convex function in $\omega(x_0, \mu, \nu)$. Since $V^{\sigma+1}(x; x_0, \mu, \nu)$ is strictly concave in $\omega(x_0, \mu, \nu)$, $\text{supp } u_0$ is the bounded connected interval and $\lambda_0 = 0$, we conclude that $N(0; x_0, \mu, \nu) \leq 1$ and hence (3.1) holds.

Suppose now that $\lambda_0 > 0$. Then by hypotheses we have $\omega(x_0, \mu, \nu) \subset 0_\delta(x_0) \subset \text{supp } u_0$. By (1.18) the function $z(x) \in C^2(0_\delta(x_0))$ satisfies

$$(4.10) \quad z_{xx} = (u_0^{\sigma+1})_{xx} - (V^{\sigma+1})_{xx} \geq (\sigma + 1)[V^\beta - \lambda_0^\beta] \text{ in } \omega(x_0, \mu, \nu).$$

Denote $\omega_1 \equiv \{x \in \omega(x_0, \mu, \nu) | V(x; x_0, \mu, \nu) \geq \lambda_0\} \subset \omega(x_0, \mu, \nu)$. Then $z(x) > 0$ for all $x \in \omega(x_0, \mu, \nu) \setminus \omega_1$. If $x \in \omega_1$, then $z_{xx} \geq 0$ by (4.10) and hence $z(x)$ is a convex function in ω_1 . Evidently this implies that $N(0; x_0, \mu, \nu) \leq 2$ and for such constants μ, ν condition (3.1) is valid. Finally, Lemma 2 completes the proof of Theorem 6. $\square$

5. Final Remarks. 1. The hypotheses of Theorems 1 and 2 can be weakened if we use some additional properties of $u_0(x)$ inside the support near its end points. In particular, the simplest generalization is as follows. Assume in addition that $u_0(x)$ is monotone on some $\mathcal{H}$ -neighborhoods, $\mathcal{H} > 0$ is a constant, of the end points of the support $x = h_\pm(0)$. Then by a similar way we prove that Theorem 1 remains be valid if (1.8) is replaced by the following condition

$$(5.1) \quad d(x_0, \text{supp } u_0) > \max\{\ell_k/2, \ell_k - \mathcal{H}\}.$$

One can see that (5.1) for $\mathcal{H} = 0$ coincides with the initial hypothesis (1.8).

The same idea is easily used for Theorem 2. One can calculate that Theorem 2 holds if under above assumptions on $u_0(x)$ we replace $d(x_0, \text{supp } u_0)$ in (1.10) by $d(x_0, \Omega_0^{\mathcal{H}})$, where $\Omega_0^{\mathcal{H}}$ denotes the set

$$(5.2) \quad \Omega_0^{\mathcal{H}} = \{x \in \text{supp } u_0 \mid |x - h_{\pm}(0)| > \min\{\mathcal{H}, \ell_k/2\}\}.$$

The proofs are quite similar to those given in Section 4. We use the fact that under hypotheses given above the initial function $u_0(x)$ and any stationary solutions $V(x; x_0, \mu, \nu)$ are such that $\text{supp } u_0 \cap \omega(x_0, \mu, \nu) \neq \emptyset$ for $\lambda \in (0, M_k)$ but $\text{supp } u_0 \cap \omega(x_0, \mu, \nu) \subseteq \text{supp } u_0 \setminus \Omega_0^{\mathcal{H}}$. This implies that the functions $u_0(x)$ and $V(x; x_0, \mu, \nu)$ have different monotonicity in $\text{supp } u_0 \setminus \Omega_0^{\mathcal{H}}$. Therefore for any given $\lambda \in (0, M_k)$ we deduce that $N(0; \lambda, a) \equiv N(0; x_0, \mu, \nu) \leq 1$. The ends of proofs are the same.

As for Theorem 3, by using the hypothesis on monotone behavior of $u_0(x)$ near the end points of $\text{supp } u_0$, we can replace ρ by

$$(5.3) \quad \rho = d(x_0, \text{supp } u_0) - \max\{0, \ell_k/2 - \mathcal{H}\} \in (0, \ell_k/2).$$

2. We now show that Theorem 2 cannot be improved for arbitrary exponents $\beta > \sigma + 1$. Equation (1.1) admits a global in time self-similar solution of the form

$$(5.4) \quad u_A(x, t) = (T + t)^{-1/(\beta-1)} f(\xi),$$

$$(5.5) \quad \xi = x/(T + t)^n, n = [\beta - (\sigma + 1)]/2(\beta - 1),$$

where $T > 0$ is a fixed constant and the function $f(\xi) \geq 0$ solves the following ordinary differential equation:

$$(5.6) \quad (f^\sigma f')' + n f' \xi + \frac{1}{\beta - 1} f + f^\beta = 0 \text{ for } \xi \in R.$$

The existence of a nontrivial compactly supported solution $f = f(|\xi|) \not\equiv 0$ to equation (5.6) for any $\beta > \sigma + 3$ has been proved in [13, Chapter IV], a similar result for $\sigma = 0, \beta > 3$, has been obtained in [9].

This self-similar solution satisfies

$$(5.7) \quad \frac{d}{dt} u_A(0, t) \equiv -\frac{1}{\beta - 1} (T + t)^{-\beta/(\beta-1)} f(0) < 0$$

for any $t > 0$. Moreover, one can see that for a given $x_0 \in R$ there exists $t_0 = t_0(x_0) \geq 0$ such that

$$(5.8) \quad \frac{d}{dt} u_A(x_0, t) < 0 \text{ for any } t > t_0.$$

Thus, for $\beta > \sigma + 1$ there exist some “small enough” self-similar solutions $u_A(x, t)$ which are monotone decreasing in time as $t \rightarrow \infty$ for any fixed point $x \in R$.

Consider now more detailed spatial structure of the set $\{u_A(x, t), t > 0\}$. Let $T = 0$ in (5.4), (5.5) and denote $f_0 = f(0)$, $a_0 = \sup\{\xi > 0 | f(\xi) > 0\}$. Then for any $t > 0$

$$(5.9) \quad u_A(x, t) \leq W(x, t) \equiv f_0 t^{-1/(\beta-1)} \text{ for } |x| \leq a_0 t^n$$

and $W(x, t) \equiv 0$ for $|x| > a_0 t^n$. Hence we may conclude that for any given $x_0 \in R$

$$(5.10) \quad \begin{aligned} u_A(x_0, t) &\leq \sup_{t>0} W(x_0, t) \equiv W(x_0, (|x_0|/a_0)^{1/n}) \\ &\equiv f_0 (a_0)^{\frac{2}{\beta-(\sigma+1)}} |x_0|^{-\frac{2}{\beta-(\sigma+1)}}. \end{aligned}$$

Thus, by using (5.9), (5.10) we conclude that the tangent curve to the set $\{u_A(x, t), t > 0\}$ has the behavior for large $|x_0| > 0$ which is similar to the function given in the right hand side of (1.10). The exponents $-2/[\beta - (\sigma + 1)]$ of the power decay at infinity given in (1.10) and (5.9) coincide.

3. By using the same method of proofs given in Section 3 one can easily derive some new conditions of monotone in time behavior of $u(x_0, t_0)$ if the value of derivative $(u^\sigma)_x(x_0, t_0)$ is known.

In particular, the following result holds for the case $\beta \in (1, \sigma + 1)$ (cf. Theorem 1).

THEOREM 5.1. *Let $\beta \in (1, \sigma + 1)$ and (cf. (1.8))*

$$(5.11) \quad 0 < x_0 - h_+(0) < \ell_k.$$

Then for any given $\mu_ \in (0, M_k)$ there exists a constant $\nu_* < 0$, $|\nu_*|$ is small enough, such that if for some $t_0 \in (0, T)$*

$$(5.12) \quad u(x_0, t_0) = \mu_*, (u^\sigma)_x(x_0, t_0) \geq \nu_*,$$

then

$$(5.13) \quad u_t(x_0, t_0) \geq 0.$$

Proof is quite similar. By using properties of stationary solutions for $\beta \in (1, \sigma + 1)$ we conclude that under hypotheses (5.11) and (5.12) the tangent stationary solution $V(x; x_0, \mu_*, \nu)$ satisfies

$$(5.14) \quad \text{supp } u_0 \cap \omega(x_0, \mu_*, \nu) = \emptyset$$

for any $\nu \geq \nu_*$ and arbitrary $\nu < \nu_*$ if $\nu_* - \nu > 0$ is small enough. Thus, we have $N(0; x_0, \mu_*, \nu) = 0$, and hence the proof of Theorem 1 can be used.

Remark. One can see that (5.13) is valid if $|(u^\sigma)_x(x_0, t_0)|$ is large enough since in this case $\lambda \geq M_k$, see (4.1).

As for the case $\beta > \sigma + 1$, the following result is true (cf. Theorem 2).

THEOREM 5.2. Let $\beta > \sigma + 1$ and for some $x_0 > h_+(0)$ and $t_0 \in (0, T)$ (cf. (1.10))

$$(5.15) \quad u(x_0, t_0) \leq \left\{ \frac{1}{c_2} (x_0 - h_+(0)) \right\}^{-\frac{2}{\beta - (\sigma + 1)}}.$$

There exists $\nu_* < 0$ such that if

$$(5.16) \quad (u^\sigma)_x(x_0, t_0) \leq \nu_*,$$

then (5.13) holds

Proof is the same, and it is based on the fact that (5.14) is valid for any $\nu \leq \nu_*$, i.e., if $|\nu|$ is large enough. This is a straightforward consequence of properties of stationary solutions for $\beta > \sigma + 1$ given in Section 2. Thus, it follows from (5.15) and (5.16) that $N(0; x_0, \mu_*, \nu) = 0$, $\mu_* = u(x_0, t_0)$, $\nu \leq \nu_*$, and the end of the proof is similar.

4. It is easily verified that all techniques work for the solution $u(x, t)$ of the initial-boundary value problem in $(-L, L) \times (0, T)$ with Dirichlet boundary conditions

$$(5.17) \quad u(-L, t) = u(L, t) = 0 \text{ for } t \geq 0.$$

5. The main results remain be valid for more general equation of the form

$$(5.18) \quad u_t = (\varphi(u))_{xx} + Q(u) \text{ in } \omega_T,$$

where $\varphi(u)$ and $Q(u)$ are given nonnegative functions smooth enough; $\varphi'(u) > 0$, $Q(u) > 0$ for $u > 0$, $Q \in C^1([0, \infty))$. In particular, the property (1.12) of monotonicity in time of any solution large enough has been proved in [8] (by using the same technique as in [6]) for uniformly parabolic equation (5.18) under the single addition hypothesis on the coefficients $\varphi(u)$ and $Q(u)$:

$$(5.19) \quad \int_1^\infty Q(z)\varphi'(z)dz = +\infty.$$

This assumption implies that any stationary solution $U(x; \lambda, a)$, satisfying (cf. (2.1))

$$(5.20) \quad (\varphi(U))_{xx} + Q(U) = 0 \text{ for } x \in R \setminus \{a\}$$

with boundary conditions given in (2.1), intersects the function $\varphi(u_0(x))$ smooth enough at two points at most. Indeed, one can see that $U(x) \equiv U(x; \cdot)$ satisfies the identity (cf. (2.6))

$$(5.21) \quad (\varphi(U(x)))_x^2 = 2 \int_{U(x)}^\lambda Q(z)\varphi'(z)dz.$$

Hense, under hypothesis (5.19) there holds

$$(5.22) \quad |(\varphi(U))_x| \rightarrow +\infty \text{ as } \lambda \rightarrow \infty \text{ uniformly in the set } \{x \in R | 0 < U(x) \leq M_1\}.$$

Therefore, for arbitrary $a \in R$ and any $\lambda > 0$ large enough the stationary profile $U(x; \lambda, a)$ intersects $u_0(x)$ at two points at most if

$$(5.23) \quad \varphi(u_0(x)) \text{ is a Lipschitz continuous function in } R.$$

This implies that there exists $M_k > 0$ such that (1.12) is valid.

We now state some more detailed properties which are similar to those given in Theorems 1–3. We assume that equation (5.18) describes processes with a finite speed of propagation of disturbances, see [10]. Since $Q \in C^1([0, \infty))$, this implies that

$$(5.24) \quad \int_0^1 \frac{\varphi'(z)}{z} dz < \infty.$$

We consider the Cauchy problem for (5.18) with the initial function u_0 satisfying (1.3) where the second assumption is replaced by (5.23).

Using identity (5.21) yields that for any given $a \in R$ and $\lambda > 0$ the function $U(x; \lambda, a)$ is positive iff $|x - a| < x_0(\lambda)$, where

$$(5.25) \quad x_0(\lambda) = \int_0^\lambda \varphi'(s) \left(2 \int_s^\lambda Q(z) \varphi'(z) dz \right)^{-\frac{1}{2}} ds < \infty.$$

The properties, which are similar to those given in Theorems 1 and 2, depend on the behavior of $x_0(\lambda)$ as $\lambda \rightarrow 0$. More exactly, the following result is true (cf. Theorem 1).

THEOREM 5.3. *Assume that under hypothesis given above there holds*

$$(5.26) \quad x_0(\lambda) \text{ is uniformly bounded in } (0, 1].$$

If (cf. (1.8))

$$(5.27) \quad d(x_0, \text{supp } u_0) > \ell_k \equiv \sup_{\lambda \in (0, M_k)} (2x_0(\lambda)),$$

then (1.9) is valid.

Proof is absolutely the same as the proof of Theorem 1 in Section 4.

On the other hand, if (5.26) is not valid and

$$(5.28) \quad x_0(\lambda) \rightarrow \infty \text{ as } \lambda \rightarrow 0,$$

then we have the case which is similar to Theorem 2. More exactly, there exists a function $P(s) > 0$, $P(s) \rightarrow 0$ as $s \rightarrow \infty$, such that if (cf. (1.10))

$$(5.29) \quad u(x_0, t_0) > P(d(x_0, \text{supp } u_0)),$$

then (1.11) is valid. The function $P(s)$ depending on φ and Q can be constructed as a tangent curve (envelope) to a specific family of stationary solutions $U(x; \lambda, a)$ where $a = a(\lambda) > 0$ is chosen so that $U(0; \lambda, a(\lambda)) \equiv 0$ for $\lambda > 0$ (see a similar construction in the proof of Theorem 2, Section 4). Then $P(s)$ for large $s > 0$ has the form

$$(5.30) \quad P(s) = \sup_{\lambda \in (0, M_k)} U(s; \lambda, a(\lambda)).$$

Theorems 3-6 can be also extended to general equation (5.18) under some hypothesis on coefficients $\varphi(u)$ and $Q(u)$.

Remark. Notice that more complicated equation

$$(5.31) \quad u_t = \phi(u)((\varphi(u))_{xx} + Q(u)),$$

where $\phi(u) > 0$ for $u > 0$ is a given smooth function, has the same stationary solutions as equation (5.18). Therefore all results discussed above are valid for equation (5.31) with Dirichlet boundary conditions (5.17). Property (1.12) for (5.18) with $\varphi(u) = u^2$, $\varphi(u) = Q(u) = u$, has been recently proved in [15] by the same intersection comparison with a set of stationary solutions.

6. Finally, we notice that these results can be stated for a general fully nonlinear equation

$$(5.32) \quad u_t = a(u, u_x, u_{xx}) \text{ in } \omega_T,$$

where $a(p, q, r)$ is a given smooth function, $a'_r > 0$. We now assume for a moment that there exists the unique classical solution $u(x, t)$ to the Cauchy problem with the initial function $u_0(x) \geq 0$ satisfying

$$(5.33) \quad \sup u_0 = M_1 < \infty, \sup |u'_0| = M_2 < \infty.$$

Consider first the property (1.12) on monotonicity in time of large solutions. By using the same technique we conclude that (1.12) is valid, i.e., such a constant $M_k > 0$ exists, if stationary solutions $U(x; \lambda, a)$

$$(5.34) \quad a(U, U_x, U_{xx}) = 0 \text{ for } x \in R \setminus \{a\}$$

with boundary conditions given by (2.1) exist for any $\lambda > 0$ and (cf. (5.22))

$$(5.35) \quad |U_x(x; \cdot)| \rightarrow +\infty \text{ as } \lambda \rightarrow \infty \text{ uniformly in } \{x \in R | 0 < U(x; \cdot) \leq M_1\}.$$

Properties given in Theorem 1 and 2 depend on the following function (cf. (5.25))

$$2x_0(\lambda) = \text{mes}\{x \in R | U(x; \lambda, a) > 0\}.$$

More exactly, if (5.26) holds, then we expect the property given in Theorem 1 (or 5.3). On the other hand, if (5.28) is valid, then the result similar to Theorem 2 is expected to be true.

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