**SOME GLOBAL DYNAMICAL PROPERTIES
OF A CLASS OF PATTERN FORMATION EQUATIONS**

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SOME GLOBAL DYNAMICAL PROPERTIES OF A CLASS OF PATTERN FORMATION EQUATIONS

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INTRODUCTION

The present paper is concerned with the mathematical study of two nonlinear evolution equations: the so called Kuramoto-Sivashinsky and Cahn-Hilliard equations. They are respectively given by:

$$(0.1) \quad \frac{\partial u}{\partial t} + \Delta^2 u + \Delta u + \frac{1}{2} |\nabla u|^2 = 0$$

$$(0.2) \quad \frac{\partial u}{\partial t} + \Delta^2 u + \Delta u - \beta \Delta(u^3) = 0$$

As we will see below, these equations occur in a wide range of different physical contexts and have been advocated as “models” for transition to turbulence and pattern formation. Mathematical study of (0.1), (0.2) is of interest for several reasons. The linear parts are both 4th order, non-dissipative, operators: use of any maximum principle is precluded and the linear parts do not control the nonlinear term, in contrast with, for instance, with the Navier-Stokes equation. Thus, we have two unusual examples of nonlinear parabolic equations. Surprisingly, they appear in a variety of physical contexts to describe the time evolution, near the threshold of instability, of any perturbation of some known solution of the system of interest. In this context, both equations are obtained from a bifurcation analysis assuming that a typical parameter is small. The Kuramoto-Sivashinsky equation occurs in hydrodynamics, to describe film layer flow on an inclined plane [6], [32], in chemistry to describe Belousov-Zabotinskii reactions [26a], [26b], [27], in combustion to describe thermodiffusive flame fronts [48a], [48b], [49], in fluid mechanics to describe various types of interfacial instabilities [3], [22], in plasma physics to describe Alfvén drift waves [11], [29]. The Cahn-Hilliard equation occurs principally in phase transition to describe spinodal decomposition in binary alloys [7], [8], [28], [39b]. Our mathematical study of (0.1) and (0.2) and their natural generalization (see (2.1) part I, (1.1) part II) will be achieved by conceiving them as infinite dimensional dynamical systems. In this setting we are mainly concerned with the description of the asymptotic behavior of the solutions of (0.1), (0.2). Precisely, this amounts to perform a nonlinear stability analysis, to describe the structure of

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the associated global attractor, to compute an upper bound for its dimension, to construct an inertial manifold. For the first step of this program, we address the question of bounded absorbing sets. In fact, we will make precise an uniform dissipativity property resulting from the nonlinear term in (0.1), (0.2). For (0.2) this is a consequence of the existence (already noticed in [7], [38], [39]) of a Lyapunov functional. For (0.1), the situation seems to be much more difficult and we restricted the analysis to one dimensional even solutions. In this case, we give a much shorter and general proof of the result stated in [9]. Existence of absorbing sets allows us to define universal attractors and to compute upper bounds for their dimension. We follow the theoretical framework developed by C.F.T. [12], [13a] (see also [19a], [4a,b,c]), based on construction of uniform Lyapunov exponents. Moreover, concerning equation (0.2), we construct an inertial manifold, that is a Lipschitz invariant manifold attracting all the trajectories. Here we follow the recent work of [16b] (see also [17b] for the K-S equation) although new difficulties occur due to the strong nonlinearity and invariance of the spatial average of the solutions. The existence of inertial manifolds for the Cahn-Hilliard equation was also obtained in [13b] by a totally different method. Finally, we would like to mention several numerical works related to the equations studied here [42a,b,c], [37], [35], [2], [23].

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PART I: GENERALIZED KURAMOTO-SIVASHINSKY EQUATIONS

1. Lower bound for the first eigenvalue of a Schrödinger operator.

This section has for purpose to study the first eigenvalue of a Schrödinger operator $\mathcal{H}$ of the form:

$$(1.1) \quad \mathcal{H}w = \nu P \left(\frac{\partial}{\partial x} \right) w - q(x)w.$$

In the sequel, x will always belongs to the one dimensional torus $\mathbb{T} = [-\frac{L}{2}, \frac{L}{2}]$, w will be a *real* function defined on $\mathbb{T}$ and ν is a positive parameter. The linear operator P in (1.1) is defined in terms of the Fourier series of w . If we set:

$$(1.2) \quad w(x) = \sum_{k \in \mathbb{Z}} w_k \exp i \frac{2\pi}{L} kx$$

then, by definition

$$P \left(\frac{\partial}{\partial x} \right) w = \sum_{k \in \mathbb{Z}} \left| \frac{2\pi}{L} k \right|^m w_k \exp i \frac{2\pi}{L} kx$$

where the strictly positive real number m is the order of $P \left(\frac{\partial}{\partial x} \right)$.

We will restrict ourselves to *odd* functions of $\dot{H}^m(\mathbb{T})$, the Sobolev spaces of order m of L -periodic functions ([33]) with zero spatial average, and we will take for domain of $\mathcal{H}$:

$$(1.4) \quad D(\mathcal{H}) = \{w \in \dot{H}^m(\mathbb{T}), w(-x) = -w(x), x \in \mathbb{T}\}$$

Now, the first eigenvalue of $\mathcal{H}$ is¹:

$$(1.5) \quad \lambda_1 = \min_{w \in D(\mathcal{H})} \frac{(\mathcal{H}w, w)}{\|w\|^2}.$$

The question we address here is to study the following Max Min problem:

$$(1.6) \quad \max_{q \in \dot{C}^\infty} \min_{w \in D(\mathcal{H})} \frac{(\mathcal{H}w, w)}{\|w\|^2},$$

where $\dot{C}^\infty$ is the space of smooth real functions on $\mathbb{T}$ with *zero mean*. The main result is:

¹Hereafter, $(\cdot, \cdot)$ and $\|\cdot\|$ are the scalar product and the norm in $L^2(\mathbb{T})$.

THEOREM 1.1. *Let $\mathcal{H}$ and $D(\mathcal{H})$ be defined by (1.1), (1.3), (1.4). Suppose $m > 1$. Then, for any $\tau \geq 2$, there exists an even function q of $\dot{C}^\infty$ such that:*

$$(1.7) \quad \lambda_1 \geq \frac{\nu}{2}\tau,$$

where λ_1 is defined by (1.5).

Proof of Theorem 1.1. We first introduce the standard notation $[w]_s = (L \sum_k (\frac{2\pi}{L}k)^{2s} |w_k|^2)^{\frac{1}{2}}$ for $s > 0$. Now from (1.3) and Parseval identity, we obtain:

$$(1.8) \quad \begin{aligned} (\mathcal{H}w, w) &= \nu[w]_{\frac{m}{2}}^2 - \int_{-\frac{L}{2}}^{\frac{L}{2}} q(x)|w(x)|^2 dx \\ &= \nu[w]_{\frac{m}{2}}^2 + \nu\tau\|w\|^2 - \int_{-\frac{L}{2}}^{\frac{L}{2}} [\nu\tau + q(x)]|w(x)|^2 dx. \end{aligned}$$

The proof consists in choosing q such that the last integral of (1.8) is arbitrary small. Since $w(0) = 0$, for M chosen below, we define q by:

$$(1.9) \quad +q(x) = \nu\tau \sum_{0 < |k| \leq M} \exp i \frac{2\pi}{L} kx = 2\nu\tau \sum_{k=1}^M \cos \frac{2\pi}{L} kx$$

Of course, q belongs to $\dot{C}^\infty$ and :

$$\begin{aligned} \int_{-\frac{L}{2}}^{\frac{L}{2}} [\nu\tau + q(x)]|w(x)|^2 dx &= \nu\tau \sum_{|k| \leq M} \int_{-\frac{L}{2}}^{\frac{L}{2}} |w(x)|^2 \exp i \frac{2\pi}{L} kx dx \\ &= \nu\tau L \sum_{|k| \leq M} f_k \end{aligned}$$

where f_k is the k^{th} Fourier coefficient of the even function $f(x) \equiv |w(x)|^2$. As $f(0) = \sum_{k \in \mathbb{Z}} f_k = |w(0)|^2 = 0$, we have for any $s > \frac{1}{2}$:

$$(1.11) \quad \begin{aligned} \left| \sum_{|k| \leq M} f_k \right| &= \left| \sum_{|k| \geq M} f_k \right| \\ &\leq \left(\sum_{|k| \geq M} \left(\frac{2\pi}{L} k \right)^{2s} |f_k|^2 \right)^{\frac{1}{2}} \left(\sum_{|k| \geq M} \left(\frac{2\pi}{L} k \right)^{-2s} \right)^{\frac{1}{2}} \\ &\leq c_1 [f]_s \cdot M^{-(s-\frac{1}{2})} L^{s-\frac{1}{2}}, \end{aligned}$$

where we used Cauchy-Schwarz inequality. For $s > \frac{1}{2}$ the Sobolev space H^s is an algebra; therefore, since $w \in \dot{H}^m(\mathbf{T})$, $[f]_s \equiv [|w|^2]_s \leq C_2 L^{s-\frac{1}{2}} [w]_s^2$ and (1.10), (1.11) gives:

$$(1.12) \quad \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} [\nu\tau + q(x)] |w(x)|^2 dx \right| \leq c_3 \nu\tau M^{-(s-\frac{1}{2})} L^{2s} [w]_s^2$$

Finally, we choose $s = \frac{m}{2}$ (with $m > 1$) and:

$$(1.13) \quad M = (c_3 \tau)^{\frac{2}{m-1}} L^\gamma, \gamma \equiv \frac{4s}{2s-1} = \frac{2m}{m-1}.$$

This achieves the definition (1.9) of q . By inserting (1.12), (1.13) into (1.8) we find, for $\tau \geq 2$, $(\mathcal{H}w, w) \geq \nu \frac{\tau}{2} \|w\|^2$. The proof is complete. $\square$

Remark 1.3. A counter example. Here we show that the result (1.7) is not true without the hypothesis of oddness for functions of $D(\mathcal{H})$. Just compute $\frac{(\mathcal{H}w, w)}{\|w\|^2}$ for the particular $w = \exp i \frac{2\pi}{L} kx$, $k \in \mathbf{Z}$. One finds $\frac{(\mathcal{H}w, w)}{\|w\|^2} = \nu (\frac{2\pi}{L} |k|)^m$, since q is of zero mean and w of modulus one; so the above ratio is 0 for $k = 0$, which contradicts (1.7). $\square$

2. Non Linear stability analysis for the generalized K-S equation.

In this section, we study the generalized K-S equation. Precisely, we consider the following non linear evolution equation in space dimension ≤ 3 , for real u :

$$(2.1) \quad \frac{\partial u}{\partial t} + \nu P \left(\frac{\partial}{\partial x} \right) u + R \left(\frac{\partial}{\partial x} \right) u + \frac{1}{2} |\nabla u|^2 = 0 \text{ in } \mathbf{R}^n \times \mathbf{R}_+$$

$$(2.2) \quad u(x, 0) = u_0(x) \text{ in } \mathbf{R}^n,$$

where ν is strictly positive. We use the L -periodic boundary conditions, for a given $L > 0$:

$$(2.3) \quad u(x + Le_i, t) = u(x, t) \quad 1 \leq i \leq n.$$

It is thus natural to introduce the n -dimensional torus $\mathbf{T}^n = [-\frac{L}{2}, \frac{L}{2}]^n$ and to expand u in the Fourier series

$$(2.4) \quad u(x, t) = \sum_{k \in \mathbf{Z}^n} u_k \exp i \frac{2\pi}{L} k \cdot x.$$

In (2.1), $P(\frac{\partial}{\partial x})$ is a linear operator defined by:

$$(2.5) \quad P \left(\frac{\partial}{\partial x} \right) u = \sum_{k \in \mathbf{Z}^n} \left(\frac{2\pi}{L} \right)^m |k|^m u_k \exp i \frac{2\pi}{L} k \cdot x,$$

where $m > 0$ and real is the order of P . The natural domain for $P(\frac{\partial}{\partial x})$ is thus the Sobolev space $H^m(\mathbf{T}^n)$. The operator $R(\frac{\partial}{\partial x})$ is an arbitrary linear operator and we just assume a boundedness property:

$$(2.6) \quad |(R \left(\frac{\partial}{\partial x} \right) u, u)| \leq \rho \|u\|_{H^{\frac{m'}{2}}}^2, \quad 0 < m' < m,$$

where ρ is an absolute constant.

2.A Even one-dimensional solutions. In view of (2.5), the operator $P(\frac{\partial}{\partial x})$ is invariant by the reflection $x \rightarrow -x$ or $k \rightarrow -k$. In this section, we assume $m > 1$, (2.6) and moreover that $R(\frac{\partial}{\partial x})$ is also invariant by reflection. Finally, we restrict ourselves to space dimension one. Thus (2.1)-(2.3) becomes:

$$(2.7) \quad \frac{\partial u}{\partial t} + \nu P\left(\frac{\partial}{\partial x}\right)u + R\left(\frac{\partial}{\partial x}\right)u + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 = 0, \quad \text{in } \mathbf{R} \times \mathbf{R}_+$$

$$(2.8) \quad u(x, 0) = u_0(x) \quad \text{in } \mathbf{R}$$

$$(2.9) \quad u(x + L, t) = u(x, t) \quad \text{in } \mathbf{R} \times \mathbf{R}_+$$

Due to the invariance property, if $u_0(x)$ is *even* i.e. $u_0(-x) = u_0(x)$, then the solution $u = u(x, t)$ of (2.7)–(2.9) *remains even for all times*. Introducing $\varphi = \varphi(x)$ a smooth *even* function independent of time, L -periodic and $v = u - \varphi$ we notice that (2.7)–(2.9) amounts to finding $v = v(x, t)$ such that:

$$(2.10) \quad \begin{aligned} \frac{\partial v}{\partial t} + \nu P\left(\frac{\partial}{\partial x}\right)v + R\left(\frac{\partial}{\partial x}\right)v + \varphi' \frac{\partial v}{\partial x} + \frac{1}{2}\left|\frac{\partial v}{\partial x}\right|^2 = \\ - \nu P\left(\frac{\partial}{\partial x}\right)\varphi - R\left(\frac{\partial}{\partial x}\right)\varphi - \frac{1}{2}|\varphi'|^2 \end{aligned}$$

$$(2.11) \quad v(x, 0) = u_0(x) - \varphi(x)$$

$$(2.12) \quad v(x + L, t) = v(x, t)$$

Assuming u_0 even, bounds uniform in time for v , hence for u , will result from an appropriate choice of φ by using Theorem 1.1.

THEOREM 2.1. *Let u_0 be given in $H^1(\mathbf{T})$. Choose φ as:*

$$(2.13) \quad \varphi(x) = 2\nu\tau \sum_{k=1}^M \left(\frac{2\pi}{L}k\right)^{-2} \cos \frac{2\pi}{L}kx$$

where $\tau \geq \tau_0$ and M is defined by (1.13). Then, any solution v of (2.10)–(2.12) remains uniformly bounded in time by an expression of the form:

$$(2.14) \quad \left\| \frac{\partial v}{\partial x}(t) \right\| \leq \left\| \frac{\partial v}{\partial x}(0) \right\| \exp\left(-\frac{\lambda}{2}t\right) + \left[\frac{S(\varphi)}{\lambda}(1 - \exp - \lambda t) \right]^{\frac{1}{2}},$$

where λ and $S(\varphi)$ are defined by :

$$(2.15) \quad \lambda = \left(c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} + c_9 \right)$$

$$(2.16) \quad S(\varphi) = 2 \left(c_4 \nu [\varphi]_{\frac{m}{2}+1}^2 + \left(1 + \frac{c_6}{\nu} \right) \|\varphi\|_{m'+1-\frac{m}{2}}^2 + \|\varphi'\|_{L^\infty}^2 \|\varphi''\|^2 \right)$$

$c_0, \dots, c_9$ denoting absolute constants (independent of ν, L).

Proof. We use energy estimates and the following standard notations $\|w\|_s$ for $\|w\|_{H^s}$, $\|w\|$ for $\|w\|_{L^2}$ and $[w]_s \equiv (L \sum_k |\frac{2\pi}{L}k|^{2s} |w_k|^2)^{\frac{1}{2}}$. We multiply (2.10) by $-\frac{\partial^2 v}{\partial x^2}$ and integrate by parts; setting $w = \frac{\partial v}{\partial x}$, we obtain:

$$(2.17) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu(Pw, w) + \frac{1}{2}(\varphi''w, w) &= -(Rw, w) \\ &+ \nu \left(P\varphi, \frac{\partial w}{\partial x} \right) + \left(R\varphi, \frac{\partial w}{\partial x} \right) + \left(\frac{1}{2}|\varphi'|^2, \frac{\partial w}{\partial x} \right) \end{aligned}$$

We first derive a lower bound (coercivity type inequality) for the left-hand side of (2.17); due to the choice of φ we can apply Theorem 1.1 with $q = -\varphi''$ and, for any $\tau \geq 2$, after using (2.5) we obtain :

$$(2.18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu(Pw, w) + \frac{1}{2}(\varphi''w, w) &\geq \frac{\nu}{2}(Pw, w) + \frac{\nu}{4}\tau \|w\|^2 \\ &\geq \frac{\nu}{2}[w]_{\frac{m}{2}}^2 + \frac{\nu}{4}\tau \|w\|^2 \end{aligned}$$

It remains to find an upper bound for the right-hand side of (2.17). For the first term, we use (2.6) and standard interpolation argument. ([33]) For $\theta \equiv 1 - \frac{m'}{m}$, we find:

$$\begin{aligned} -(Rw, w) &\leq \rho \|w\|_{\frac{m}{m'}}^2 \\ &\leq \rho \|w\|_{\frac{m}{2}}^{2(1-\theta)} \|w\|^{2\theta} \\ &\leq \rho c_2 \left(\|w\|^{2(1-\theta)} + [w]_{\frac{m}{2}}^{2(1-\theta)} \right) \|w\|^{2\theta} \\ &\leq \rho c_2 \|w\|^2 + \frac{\nu}{6} [w]_{\frac{m}{2}}^2 + c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} \|w\|^2 \end{aligned}$$

For the second (resp. third) term, we first integrate by parts and then use Parseval identity and (2.5) (resp. (2.6)). We obtain:

$$(2.20) \quad \begin{aligned} \left(P\varphi, \frac{\partial w}{\partial x} \right) &= -(P\varphi', w) \\ &\leq \nu [\varphi]_{\frac{m}{2}+1} [w]_{\frac{m}{2}} \\ &\leq \frac{\nu}{6} [w]_{\frac{m}{2}}^2 + c_4 \nu [\varphi]_{\frac{m}{2}+1}^2 \end{aligned}$$

$$(2.21) \quad \begin{aligned} \left(R\varphi, \frac{\partial w}{\partial x} \right) &= -(R\varphi', w) \\ &\leq \rho \|\varphi\|_{m'+1-\frac{m}{2}} \|w\|_{\frac{m}{2}} \\ &\leq \rho c_5 \|\varphi\|_{m'+1-\frac{m}{2}} (\|w\| + [w]_{\frac{m}{2}}) \\ &\leq \left(1 + \frac{c_6}{\nu} \right) \|\varphi\|_{m'+1-\frac{m}{2}}^2 + c_7 \|w\|^2 + \frac{\nu}{6} [w]_{\frac{m}{2}}^2 \end{aligned}$$

For the last term, we integrate by parts and apply Hölder inequality:

$$\begin{aligned}
(2.22) \quad \left(\frac{1}{2} |\varphi'|^2, \frac{\partial w}{\partial x} \right) &= (\varphi' \varphi'', w) \\
&\leq \|\varphi'\|_{L^\infty} \|\varphi''\| \|w\| \\
&\leq \|\varphi'\|_{L^\infty}^2 \|\varphi''\|^2 + c_8 \|w\|^2
\end{aligned}$$

We collect (2.18)–(2.22) into (2.17). The terms $[w]_{\frac{m}{2}}^2$ cancel and we obtain:

$$\begin{aligned}
(2.23) \quad \frac{1}{2} \frac{d}{dt} \|w\|^2 + \left(\frac{\nu}{4} \tau - c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} + c_9 \right) \|w\|^2 \\
\leq c_4 \nu [\varphi]_{\frac{m}{2}+1}^2 + \left(1 + \frac{c_6}{\nu} \right) \|\varphi\|_{m'+1-\frac{m}{2}}^2 + \|\varphi'\|_{L^\infty}^2 \|\varphi''\|^2
\end{aligned}$$

For $c_{10}(\nu) \equiv \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} c_3 + c_9$, the parameter τ (≥ 2 in view of Theorem 1.1) is now chosen as $\tau \geq \text{Max}\{2, \frac{6}{\nu} c_{10}(\nu)\} \equiv \tau_0$. Thus this condition implies $\frac{\nu}{4} \tau \geq \frac{3}{2} c_{10}(\nu)$; Therefore (2.23) becomes finally:

$$\begin{aligned}
(2.24) \quad \frac{d}{dt} \|w\|^2 + c_{10}(\nu) \|w\|^2 \\
\leq 2 \left(c_4 \nu [\varphi]_{\frac{m}{2}+1}^2 + \left(1 + \frac{c_6}{\nu} \right) \|\varphi\|_{m'+1-\frac{m}{2}}^2 + \|\varphi'\|_{L^\infty}^2 \|\varphi''\|^2 \right)
\end{aligned}$$

We finish by applying Gronwall's Lemma to (2.24). $\square$

We now state an easy, but important, Corollary of Theorem 2.1.

COROLLARY 2.2. *Under the assumptions of Theorem 2.1 for any solution u of (2.7)–(2.9), the following holds:*

$$(2.25) \quad \limsup_{t \rightarrow \infty} \left\| \frac{\partial u}{\partial x}(t) \right\| \leq \|\varphi'\| + \left(\frac{S(\varphi)}{\lambda} \right)^{\frac{1}{2}} \equiv \mathcal{R}_0$$

$$(2.26) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t [u(s)]_{\frac{m}{2}+1}^2 ds \leq c_{11} \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} \mathcal{R}_0^2 \equiv \mathcal{Y}_0,$$

where c_{11} is an absolute constant and $\lambda, S(\varphi)$ are defined by (2.15), (2.16).

Proof. The first bound is immediate from (2.14) and $u = v + \varphi$. For the second, we come back to (2.7), multiply by $-\frac{\partial^2 u}{\partial x^2}$ and integrate by parts to find:

$$(2.27) \quad \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial x} u(t) \right\|^2 + \nu \left(P \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right) \leq \left| \left(R \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right) \right|$$

From (2.5) we deduce $\nu(P \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) \geq \nu[u]_{\frac{m}{2}+1}^2$ and we bound the right handside of (2.27) as in (2.29). We obtain:

$$(2.28) \quad \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial x} u(t) \right\|^2 + \frac{\nu}{2} [u]_{\frac{m}{2}+1}^2 \leq C_{11} \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} \left\| \frac{\partial u}{\partial x}(t) \right\|^2$$

Since $\left\| \frac{\partial u}{\partial x}(t) \right\| \leq \|u'\| + \left\| \frac{\partial v}{\partial x}(t) \right\|$, (2.26) is obtained after using (2.14) and taking time average. $\square$

Remark 2.3. As it will be seen in section 2B, the uniform bound for $\left\| \frac{\partial v}{\partial x}(t) \right\|$ or $\left\| \frac{\partial u}{\partial x}(t) \right\|$ imply indeed uniform bound in any Sobolev norms. Using standard argument ([33]), it is then an easy matter to show that, for u_0 even and smooth, there exists a unique smooth (C^∞) solution of (2.7)–(2.8).

Remark 2.4. It follows from Corollary 2.2 that any trajectory $u = u(t)$ enters for t sufficiently large in the ball of $H^1(\mathbb{T})$ centered at 0 of radius $\mathcal{R}'_0 > \mathcal{R}_0$, say $\mathcal{R}'_0 = \mathcal{R}_0 + 1$. A careful analysis of the proof of Corollary 2.2 shows that, for any bounded set $\mathcal{B}$ of $H^1(\mathbb{T})$, there exists a time $t = t(\mathcal{B})$ such that $S(t)\mathcal{B}$ is included in this ball, hereafter denoted $\mathcal{A}$. A set $\mathcal{A}$ which enjoys this property is called an *absorbing set* (cf. [13a], [51], [19a]).

2.B C^∞ -regularity of gradient-bounded solutions. Coming back to the problem (2.1)–(2.3), in space dimension $n \leq 3$, we now ask the question of the smoothness of the solution u (for u_0 smooth) assuming however, in view of the result (2.25), that:

$$(2.29) \quad \|\nabla u(t)\| \leq \mathcal{R}, \quad t > 0.$$

We need the following standard notation:

$$\begin{aligned} \|D^k u\|_{L^p} &= \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p} \\ \|D^k u\|_s &= \sum_{|\alpha|=k} \|D^\alpha u\|_s \\ \|(u_1, \dots, u_J)\|_{L^p} &= \max_{1 \leq j \leq J} \|u_j\|_{L^p}, \end{aligned}$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = \frac{\partial}{\partial x_j}$.

As before, P is defined by (2.5) and we assume (2.6) for R . The result is:

THEOREM 2.4. *Let $k \in \mathbb{N}$, $k \geq 1$ and u_0 given in $H^k(\mathbb{T}^n)$, $n \leq 3$. Assume (2.29), for u solution of (2.1)–(2.3), and the order m of P satisfying $m \geq 2 + \frac{n}{2}$. Then:*

$$(2.30) \quad \|D^j u(t)\| \leq K_j \quad 1 \leq j \leq k, \quad t > 0,$$

where the K_j are constants independent of t .

Proof. We proceed by induction, using classical arguments. For $k = 1$, (2.30) reduce to (2.29); we will assume $\|D^j u(t)\| \leq K_j$, $1 \leq j \leq k-1$ ($k \geq 2$) and prove $\|D^j u(t)\| \leq K_j$, $1 \leq j \leq k$, that is, we will prove $\|D^k u(t)\| \leq K_k$, K_k independant of t . We apply D^α to the equation (2.1), multiply by $D^\alpha u$ and integrate. After summing the result over α such that $|\alpha| = k$, we obtain (D is commuting with P but not with R):

$$(2.31) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=k} \|D^\alpha u\|^2 + \nu \sum_{|\alpha|=k} (P D^\alpha u, D^\alpha u) &= - \sum_{|\alpha|=k} (R D^\alpha u, D^\alpha u) \\ &- \sum_{|\alpha|=k} ([D^\alpha, R]u, D^\alpha u) - \frac{1}{2} \sum_{|\alpha|=k} \sum_{j=1}^n (D^\alpha (D_j u)^2, D^\alpha u) \end{aligned}$$

We shall bound each term of (2.31). With the definition (2.5) of P , we have (as usual $c_0, c_1 \dots$ are absolute constants):

$$(2.33) \quad \sum_{|\alpha|=k} (P D^\alpha u, D^\alpha u) \geq c_0 [D^k u]_{\frac{m}{2}}^2$$

Next, R satisfy (2.6) and therefore, using classical interpolation estimates, we set $\theta = 1 - \frac{m}{m'}$, $\theta' = \frac{m}{m+2}$ and find:

$$(2.33) \quad \begin{aligned} \sum_{|\alpha|=k} (P D^\alpha u, D^\alpha u) &\leq \rho \sum_{|\alpha|=k} \|D^\alpha u\|_{\frac{m'}{2}}^2 \\ &\leq \rho c_1 \sum_{|\alpha|=k} \left(\|D^\alpha u\|^{2(1-\theta)} + [D^\alpha u]_{\frac{m}{2}}^{2(1-\theta)} \right) \|D^\alpha u\|^{2\theta} \\ &\leq \varepsilon \nu [D^k u]_{\frac{m}{2}}^2 + c_2 \|D^k u\|^2 \\ &\leq \varepsilon \nu [D^k u]_{\frac{m}{2}}^2 + c_2 \|D^{k-1} u\|^{2\theta'} [D^k u]_{\frac{m}{2}}^{2(1-\theta')} \\ &\leq 2\varepsilon \nu [D^k u]_{\frac{m}{2}}^2 + c_3 \|D^{k-1} u\|^2 \end{aligned}$$

The commutator $[D^\alpha, R]$ is a linear operator of order $k + m' - 1$; using again (2.6) and the interpolation estimate with $\theta'' = \frac{m-m'+1}{m+2}$, we obtain

$$(2.34) \quad \begin{aligned} - \sum_{|\alpha|=k} ([D^\alpha, R]u, D^\alpha u) &\leq \left| \sum_{|\alpha|=k} (D^\alpha [D^\alpha, R]u, u) \right| \\ &\leq c_4 \|u\|_{k+\frac{m'-1}{2}}^2 \\ &\leq c_5 [D^k u]_{\frac{m}{2}}^{2(1-\theta'')} \|D^{k-1} u\|^{2\theta''} \\ &\leq \varepsilon \nu [D^k u]_{\frac{m}{2}}^2 + c_6 \|D^{k-1} u\|^2 \end{aligned}$$

It remains to handle the nonlinear term in (2.31). By Hölder and Cauchy-Schwarz inequalities, we have first of all:

$$\begin{aligned}
(2.35) \quad & -\frac{1}{2} \sum_{|\alpha|=k} \sum_{j=1}^n (D^\alpha (D_j u)^2, D^\alpha u) \leq \frac{1}{2} \sum_{j=1}^n \sum_{|\alpha|=k} \|D^\alpha (D_j u)^2\| \|D^\alpha u\| \\
& \leq \frac{1}{2} \sum_{j=1}^n \left(\sum_{|\alpha|=k} \|D^\alpha (D_j u)^2\|^2 \right)^{\frac{1}{2}} \left(\sum_{|\alpha|=k} \|D^\alpha u\|^2 \right)^{\frac{1}{2}} \\
& \leq C_7 \|D^k u\| \sum_{j=1}^n \|D^k (D_j u)^2\|
\end{aligned}$$

We now appeal to classical inequalities ([1] for instance) to bound $\|D^k (D_j u)^2\|$ and to the injection $H^{\frac{n}{4}} \subset L^4$ for $n \leq 3$; from (2.35) we thus find:

$$\begin{aligned}
(2.36) \quad & -\frac{1}{2} \sum_{|\alpha|=k} \sum_{j=1}^n (D^\alpha (D_j u)^2, D^\alpha u) \\
& \leq c_8 \|D^k u\| \sum_{j=1}^n \|D_j u\|_{L^4} \|D^k D_j u\|_{L^4} \\
& \leq c_9 \|D^k u\| \sum_{j=1}^n \|D_j u\|_{L^4} \|D^k D_j u\|_{H^{n/4}} \\
& \leq c_{10} \|D^k u\| \|Du\|_{L^4} \|D^k u\|_{H^{1+\frac{n}{4}}} \\
& \leq c_{11} \|D^k u\| \|Du\|_{L^4} (\|D^k u\| + [D^k u]_{1+\frac{n}{4}})
\end{aligned}$$

Now, by hypothesis $[D^k u]_{1+\frac{n}{4}} \leq c_{12} [D^k u]_{\frac{m}{2}}$; interpolating again with $\theta = \frac{m}{m+2}$, we deduce from (2.36):

$$\begin{aligned}
(2.37) \quad & -\frac{1}{2} \sum_{|\alpha|=k} \sum_{j=1}^n (D^\alpha (D_j u)^2, D^\alpha u) \\
& \leq c_{13} \|D^{k-1} u\|^\theta [D^k u]_{\frac{m}{2}}^{1-\theta} \|Du\|_{L^4} \\
& \quad \left(\|D^{k-1} u\|^\theta [D^k u]_{\frac{m}{2}}^{1-\theta} + [D^k u]_{\frac{m}{2}} \right) \\
& \leq \epsilon \nu [D^k u]_{\frac{m}{2}}^2 + c_{14} \|D^{k-1} u\|^2 \left(\|Du\|_{L^4}^{\frac{1}{\theta}} + \|Du\|_{L^4}^{\frac{2}{\theta}} \right)
\end{aligned}$$

Collecting (2.32)–(2.34) and (2.37) into (2.31), for $\epsilon \leq \frac{c_0}{8}$, we finally obtain ($\theta = \frac{m}{m+2}$)

$$\begin{aligned}
(2.38) \quad & \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=k} \|D^\alpha u\|^2 + \frac{c_0 \nu}{2} [D^k u]_{\frac{m}{2}}^2 \\
& \leq c_{15} \|D^{k-1} u\|^2 \left(1 + \|Du\|_{L^4}^{\frac{1}{\theta}} + \|Du\|_{L^4}^{\frac{2}{\theta}} \right).
\end{aligned}$$

Now, by hypothesis, $\|D^j u\| \leq K_j$, $1 \leq j \leq k-1$ ($k \geq 2$); the required bound $\|D^k u\| \leq K_k$ follows then from $\|Du\|_{L^4} \leq \|Du\|^{1-\frac{n}{4}} \|D^2 u\|^{\frac{n}{4}}$ ($n \leq 3$), $\|D^2 u\| \leq \|Du\|^\theta [D^2 u]^{\frac{1}{2}-\theta}$ ($\theta = \frac{m}{m+2}$) and (2.38). The proof is complete. $\square$

Remark 2.5. If $n = 1$, it easily follows from (2.7) and (2.29) that $\int_t^{t+1} [Du(s)]_{\frac{m}{2}}^2 ds$ is uniformly bounded for any $t > 0$. Such a bound authorize to use the uniform Gronwall Lemma (see[8]) We thus improve the hypothesis on m in Theorem 2.4 and find $m \geq 2$, instead of $m \geq \frac{5}{2}$. $\square$

The C^∞ -regularity result is a consequence of Theorem 2.1. We can state:

THEOREM 2.5. *Assume u_0 is given in $C^\infty(\mathbb{T}^n)$, $n \leq 3$, the order m of P satisfy $m \geq 2 + \frac{n}{2}$. Then, the corresponding solution $u = u(x, t)$ of (2.1)–(2.3) belongs to $C^\infty(\mathbb{T}^n \times \mathbb{R}_+)$ As soon (2.29) holds.*

Proof. Due to Theorem 2.4, if $u_0 \in H^k(\mathbb{T}^n)$, then $u \in L^\infty(\mathbb{R}_+, H^k(\mathbb{T}^n))$. If u_0 is given in $C^\infty(\mathbb{T}^n)$, then $u_0 \in H^k(\mathbb{T}^n)$, $k \in \mathbb{N}$; by Theorem 2.4 again $u \in L^\infty(\mathbb{R}_+, H^k(\mathbb{T}^n))$, $k \in \mathbb{N}$ and indeed for $s \in \mathbb{R}_+$, by interpolation. Thus $u \in L^\infty(\mathbb{R}_+, C^\infty(\mathbb{T}^n))$ and using the equation (2.1) $\frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}_+, C^\infty(\mathbb{T}^n))$. Taking the $(k-1)^{\text{th}}$ time derivative ($k \geq 2$) of (2.1), we obtain $\partial_t^k u \in L^\infty(\mathbb{R}_+, C^\infty(\mathbb{T}^n))$ and the conclusion follows.

3. Existence and dimension of the attractor for the generalized K-S equation.

In this section we study the attractor associated to (2.7)–(2.9) with u_0 even and, more generally, associated to (2.1)–(2.3), *assuming that (2.29) holds*.

3.A Existence and regularity; functional setting. Since the space average $\bar{u}(t)$ of u is unbounded as $t \rightarrow \infty$, we are led to define $\dot{H} = H/\mathbb{R}$ the quotient space of $H = L^2(\mathbb{T}^n)$ by $\mathbb{R}$ equipped with the natural norm. (One could also introduce the equation satisfied by $u - \bar{u}(t)$, assuming that $\bar{u}_0 = 0$)

We will denote by $\dot{w}$ the coset in $\dot{H}$ of any element w in H . If u_0 is even, results of Section 2.A allow us to define, for $t > 0$, the operator $S(t) : u_0 \rightarrow S(t)u_0 = u(x, t)$, where $u(x, t)$ is solution of (2.7)–(2.9). More generally, assuming (2.29), $S(t)$ is also defined for u solution of (2.1)–(2.3). In both cases $S(t)$ is an operator in H , and there is a natural associated operator $\dot{S}(t)$ in $\dot{H}$. The ω -limit set for $\dot{S}(t)$ is defined by:

$$(3.1) \quad X = \bigcap_{t>0} \text{cl}\{\dot{u}(s), s \geq t\}$$

Where the closures are taken in $\dot{H}$. Due to (2.25) or (2.29), it follows from Theorem 2.4 that X is bounded in the quotient space $H^2(\mathbb{T}^n)/\mathbb{R}$, thus *compact* in $\dot{H}$. Moreover, due to Theorem 2.5, X is a subset of $C^\infty(\mathbb{R}_+; \mathbb{T}^n)$. One checks that X is a positively invariant

set for $\dot{S}(t)$. Using an abstract result of backward uniqueness of [5] or [18], the mapping is injective² and

$$(3.2) \quad \dot{S}(t)X = X$$

Following the terminology of [14], [51], X is called a *functional invariant set*, for $\dot{S}(t)$; this set may or may not be an attractor. One easily deduces for any solution u ([14], [51]):

$$(3.3) \quad \lim_{t \rightarrow \infty} \text{dist}(\dot{u}(t), X) = \lim_{t \rightarrow \infty} \inf_{\dot{w} \in X} \|\dot{u}(t) - \dot{w}\| = 0.$$

We now briefly recall the functional setting necessary to compute the dimension of X . We refer to [12], [13] for further details. First, we introduce the linearization of $\dot{S}(t)$ around a given solution u of (2.1)–(2.3) or (2.7)–(2.9); that is we consider the following linear evolution equation:

$$(3.4) \quad \frac{\partial U}{\partial t} + \nu P U + R U + \nabla u \cdot \nabla U = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$$

$$(3.5) \quad U(X, 0) = \xi(x) \quad x \in \mathbb{R}$$

$$(3.6) \quad U(x + L e_i, t) = U(x, t) \quad 1 \leq i \leq n$$

Note that the solution $U(t)$ of (3.4)–(3.6) exists for $0 \leq t \leq T(u)$ and defines $DS(t, u_0) : \xi \rightarrow U(t)$ which is a linear and compact operator in $\dot{H}$. Now, given J initial data $\xi_1, \dots, \xi_J$ in $\dot{H}$, we generate J solutions $U_1(t), \dots, U_J(t)$ of (3.4)–(3.6). A direct calculation shows that³:

$$(3.7) \quad \frac{d}{dt} \text{Log} \|\wedge_{j=1}^J U_j(t)\| + \text{Trace}(A(u(t)) \circ P_J(t)) = 0$$

for $u(t) = \dot{S}(t)u_0$ and where by definition:

$$(3.8) \quad A(u(t)) = \nu P + R + \nabla u(t) \cdot \nabla$$

$$(3.9) \quad P_J(t) = \text{orthogonal projection on to span } \{U_1(t), \dots, U_J(t)\}$$

Of course, $P_J(t)$ depends smoothly on $\xi_1, \dots, \xi_J$, n , t and J . We write the equivalent form of (3.7):

$$(3.10) \quad \|\wedge_{j=1}^J U_j(t)\| = \|\wedge_{j=1}^J \xi_j\| \exp - \int_0^t \text{Trace}(A(u(s)) \circ P_J(s)) ds$$

²The results in [5], [18] assume self-ajointness of $\nu P + R$. If $n = 1$, the property is true for R real; for $n \geq 2$, this requires $2m' \leq m$.

³Hereafter $\|\cdot\|$ is the natural norm on $\wedge^J \dot{H} = \dot{H} \wedge \dots \wedge \dot{H}$, where $\wedge$ is the exterior product.

It is thus natural to introduce a norm $\omega_J(u_0, t)$ in $\wedge^J \dot{H}$, which measures the volume variation of a “ J -dimensional cube” $\wedge_{j=1}^J \xi_j$ convected along the orbit $\dot{S}(t)u_0$, precisely:

$$(3.11)_1 \quad \omega_J(u_0, t) \equiv \sup \left\{ \left\| \wedge_{j=1}^J U_j(t) \right\|, \xi_j \in \dot{H}, \|\xi_j\| = 1, 1 \leq j \leq J \right\}$$

For u_0 running over X defined by (3.1), we deduce from (3.11) and (3.10)

$$(3.11)_2 \quad \bar{\omega}_J(t) \equiv \sup_{u_0 \in X} \omega_J(u_0, t) \leq \sup_{u_0 \in X} \exp - \int_0^t \inf_P \text{Trace } A(u(s)) \circ P \, ds,$$

where P is any projector of rank J .

A general result of [12], [13] relates the fractal and Hausdorff dimensions of X , that is $d_F(X)$ and $d_H(X)$, to the first positive integer J_0 such that $\bar{\omega}_{J_0}(t) < 1$ for t large, which is equivalent to the contraction by $\dot{S}(t)$ of any J_0 -dimensional cubes centered in any u_0 of X . We are thus led to find the first $J_0 \in \mathbb{N}$ such that

$$(3.12) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \bar{\omega}_{J_0}(t) < 0$$

since the Hausdorff dimension of X is then estimated [13] by

$$(3.13) \quad d_H(X) \leq J_0 + 1,$$

while the fractal dimension $d_F(X)$ can be also estimated (in a slightly more complicated way) in term of J_0 .

3.D Dimension of the attractor. In the former section, we recalled that (3.13) was a direct consequence of (3.12), that is we are now looking for the first $J_0 \in \mathbb{N}$ such that:

$$(3.13) \quad \liminf_{t \rightarrow \infty} \inf_{u_0 \in X} \inf_P \frac{1}{t} \int_0^t \text{Trace } A(u(s)) \circ P \, ds > 0$$

where P is any projector of rank J_0 , $A(u(s))$ is defined by (3.8) with $u(s) = \dot{S}(s)u_0$. To find J_0 , satisfying (3.13), we shall use a Sobolev inequality for the operator $\wedge_{j=1}^J P$. For $P = -\Delta$ this remarkable result has been obtained by Lieb and Thirring ([30], [31]). We extended it, in the periodic case, to $P = +\Delta^2$ ([39c]) and the proof goes immediately, by direct inspection of the proof in [39c], to P defined by (2.5). To such operator is associated its square root $P^{\frac{1}{2}}$:

$$(3.14) \quad P^{\frac{1}{2}} \left(\frac{\partial}{\partial x} \right) u = \sum_{k \in \mathbb{Z}^n} \left(\frac{2\pi}{L} \right)^{\frac{m}{2}} |k|^{\frac{m}{2}} u_k \exp i \frac{2\pi}{L} k \cdot x, \quad m > 0$$

We thus obtain:

THEOREM 3.1. *Let $P^{\frac{1}{2}}$ be defined by (3.14) and $\{\phi_1, \dots, \phi_J\}$ be any set of J (non constant) functions of $\mathcal{C}^\infty(\mathbb{T}^n)$, which are orthonormal in $L^2(\mathbb{T}^n)$, $n \leq 3$. Then there exists a constant $c_0(n)$, independant of J , such that:*

$$(3.15) \quad \sum_{j=1}^J \|P^{\frac{1}{2}} \phi_j\|^2 \geq c_0(n) \|\phi\|^2_{L^{\alpha_n}},$$

where $|\phi(x)|^2 \equiv \sum_{j=1}^J |\phi_j(x)|^2$ and $\alpha_n \equiv 1 + \frac{m}{n}$.

Let J_0 be the first positive integer such that:

$$(3.17) \quad J_0 > c_0(n)^{-\frac{1}{\alpha_n-1}} L^{\frac{n}{\beta_n}} \left(\frac{c_4(\nu)}{\nu} + \frac{1}{\nu} L^{n(\frac{n}{\beta_n}-\frac{1}{2})} \mathcal{R}_m^2 \mathcal{Y}^{\frac{1}{2}-\frac{1}{m}} \right)^{\frac{1}{\alpha_n-1}},$$

where by definition (m is the order of P and $u(t) = S(t)u_0$)

$$(3.18)_1 \quad \alpha_n = 1 + \frac{m}{n}, \beta_n = 1 + \frac{n}{m}, c_4(\nu) = c_1 + c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}}$$

$$(3.18)_2 \quad \mathcal{R} = \limsup_{t \rightarrow \infty} \sup_{u_0 \in X} \|\nabla u(t)\|, \mathcal{Y} = \limsup_{t \rightarrow \infty} \sup_{u_0 \in X} \frac{1}{t} \int_0^t [u(s)]_{\frac{m}{2}+1}^2 ds.$$

We are now in position to state our main result

THEOREM 3.2. *Let X (defined by 3.1) be the invariant set (or attractor) associated to (2.1)–(2.3) with $n \leq 3$. Assume that the hypotheses of Theorems 2.4, 2.5 are satisfied. then, the Hausdorff and fractal dimensions $d_H(X)$ and $d_F(X)$ are finite and:*

$$(3.16)_1 \quad d_H(X) \leq J_0 + 1$$

$$(3.16)_2 \quad d_F(X) \leq c_0(J_0 + 1),$$

Where c_0 is an absolute constant.

Proof. In view of (3.13), in order to find $J_0 \in \mathbb{N}$ it is sufficient to derive a lower bound for $\text{Trace } A(u(s)) \circ P$. Therefore, we introduce $\{\phi_j\}_{j \in \mathbb{N}}$ an orthonormal basis of smooth functions of $\dot{H}$ such that $\{\phi_1, \dots, \phi_J\}$ is an orthonormal basis of $P\dot{H}$ and we write, using (3.8) and (3.14)

$$(3.19) \quad \begin{aligned} \text{Trace } A(u(s)) \circ P &= \sum_{j=1}^J \nu(P\phi_j, \phi_j) + (R\phi_j, \phi_j) \\ &\quad + (\nabla u(s) \cdot \nabla \phi_j, \phi_j) \\ &= \sum_{j=1}^J \nu \|P^{\frac{1}{2}} \phi_j\|^2 + (R\phi_j, \phi_j) \\ &\quad - \int \frac{1}{2} \Delta u(s) |\phi_j|^2 dx \end{aligned}$$

Now using (2.6) and classical interpolation with $\theta = 1 - \frac{m}{m'}$ (see the derivation of (2.33)):

$$\begin{aligned}
(R\phi_j, \phi_j) &\leq \rho \|\phi_j\|_{H^{\frac{m'}{2}}}^2 \\
&\leq C_1 \|\phi_j\|^2 + c_2 \|\phi_j\|^{2\theta} [\phi_j]_{\frac{m}{2}}^{2(1-\theta)} \\
(3.20) \quad &\leq \frac{\nu}{2} \|P^{\frac{1}{2}} \phi_j\|^2 + \left(c_1 + c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} \right) \|\phi_j\|^2,
\end{aligned}$$

where we used Cauchy-Schwarz inequality and $[\phi_j]_{\frac{m}{2}}^2 = \|P^{\frac{1}{2}} \phi_j\|^2$ according to (3.14). We define $c_4(\nu) \equiv c_1 + c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}}$, and (3.19), (3.20) combined with (3.15) gives:

$$\begin{aligned}
\text{Trace } A(u(s)) \circ P &\geq \frac{\nu}{2} \sum_{j=1}^J \|P^{\frac{1}{2}} \phi_j\|^2 - \frac{1}{2} \int (\Delta u(s) + c_4(\nu)) |\phi_j|^2 dx \\
(3.21) \quad &\geq \frac{c_0(n)}{2} \nu \|\phi\|^2_{L^{\alpha_n}} - \frac{1}{2} \int (\Delta u(s) + c_4(\nu)) |\phi|^2 dx \\
&\geq \frac{c_0(n)}{2} \nu \|\phi\|^2_{L^{\alpha_n}} - \frac{1}{2} \|(\Delta u(s) + c_4(\nu))\|_{L^{\beta_n}} \|\phi\|^2_{L^{\alpha_n}}
\end{aligned}$$

where we used Hölder inequality and $\beta_n \equiv \frac{\alpha_n}{\alpha_n-1} = 1 + \frac{n}{m}$ (according to (3.15)). Using again Hölder inequality and interpolation with $\theta = \frac{2}{m}$, we have:

$$\begin{aligned}
\|\Delta u(s) + c_4(\nu)\|_{L^{\beta_n}} &\leq c_4(\nu) L^{\frac{n}{\beta_n}} + L^{n(\frac{1}{\beta_n}-\frac{1}{2})} \|\Delta u(s)\| \\
(3.22) \quad &\leq c_4(\nu) L^{\frac{n}{\beta_n}} + L^{n(\frac{1}{\beta_n}-\frac{1}{2})} \|\nabla u(s)\|^{\frac{2}{m}} [u(s)]_{\frac{m}{2}+1}^{1-\frac{2}{m}}
\end{aligned}$$

We recall that $\{\phi_1, \dots, \phi_J\}$ is an orthonormal basis, thus we have $\|\phi\|_{L^{\alpha_n}}^2 \geq L^{\frac{n}{\beta_n}} J$, which gives with (3.21), (3.22):

$$\begin{aligned}
\text{Trace } A(u(s)) \circ P &\geq \frac{1}{2} L^{\frac{n}{\beta_n}} J \left[c_0(n) \nu (L^{\frac{n}{\beta_n}} J)^{\alpha_n-1} - c_4(\nu) L^{\frac{n}{\beta_n}} \right. \\
(3.23) \quad &\quad \left. - L^{n(\frac{1}{\beta_n}-\frac{1}{2})} \|\nabla u(s)\|^{\frac{2}{m}} [u(s)]_{\frac{m}{2}+1}^{1-\frac{2}{m}} \right]
\end{aligned}$$

It is now an easy matter to take the time average of both terms in (3.23) and obtain:

$$\begin{aligned}
&\frac{1}{t} \int_0^t \text{Trace } A(u(s)) \circ P ds \\
&\geq \frac{1}{2} L^{\frac{n}{\beta_n}} J \left[c_0(n) \nu \left(L^{\frac{n}{\beta_n}} J \right)^{\alpha_n-1} - c_4(\nu) L^{\frac{n}{\beta_n}} \right. \\
&\quad \left. - L^{n(\frac{1}{\beta_n}-\frac{1}{2})} \left(\sup_{0 < s < t} \|\nabla u(s)\| \right)^{\frac{2}{m}} \left(\frac{1}{t} \int_0^t [u(s)]_{\frac{m}{2}+1}^2 ds \right)^{\frac{1}{2}-\frac{1}{m}} \right]
\end{aligned}$$

In view of (3.24), the first J_0 such that (3.13) holds, satisfies:

$$J_0 > c_0(n)^{-\frac{1}{\alpha_n-1}} L^{\frac{n}{\beta_n}} \left(\frac{c_4(\nu)}{\nu} + \frac{1}{\nu} L^{n(\frac{1}{\beta_n}-\frac{1}{2})} \mathcal{R}^{\frac{2}{m}} \mathcal{Y}^{\frac{1}{2}-\frac{1}{m}} \right)^{\frac{1}{\alpha_n-1}},$$

where by definition

$$\mathcal{R} \equiv \limsup_{t \rightarrow \infty} \sup_{u_0 \in X} \|\nabla u(s)\| \text{ and } \mathcal{Y} \equiv \limsup_{t \rightarrow \infty} \sup_{u_0 \in X} \frac{1}{t} \int_0^t [u(s)]^{\frac{2}{m}+1} ds.$$

It was mentioned above that a general result of [13a] (Theorem 3.3 i) p. 30) implies (3.16). Finally (3.17) follows from another result of [13a] (Theorem 3.3 ii) p. 31) and a calculation which has been done in several related situations [13a], [12] and that we will not repeat here. The proof is complete. $\square$

Remark 3.3. In space dimension one, for u_0 even, it has been proved in Corollary 2.2 that $\mathcal{R}$ and $\mathcal{Y}$ are bounded explicitly in terms of ν and φ . Since φ depends on L , we thus have bounds in terms of ν and L . We make the bounds precise in the next section.

4. Extension to other nonlinearities and examples..

4.A Subquadratic non linearities in space dimension one. Up to now the non linear term in equation (2.7) was quadratic in $\frac{\partial u}{\partial x}$. For applications ([47]), it is useful to handle subquadratic non linearities of the form $(1 + |\frac{\partial u}{\partial x}|^2)^{\frac{1}{2}} - 1$. We indicate here the precise hypothesis for a class of non linearities which includes this new term and modifications to make in the concerned sections. So, we consider the following equation:

$$\frac{\partial u}{\partial t} + \nu P \left(\frac{\partial}{\partial x} \right) u + R \left(\frac{\partial}{\partial x} \right) u + F \left(\frac{\partial u}{\partial x} \right) = 0 \quad \text{in } \mathbf{R} \times \mathbf{R}$$

$$u(x, 0) = u_0(x) \quad \text{in } \mathbf{R}$$

$$u(x + L, t) = u(x, t) \quad \text{in } \mathbf{R} \times \mathbf{R}_+$$

As before $P(\frac{\partial}{\partial x})$ is defined by (2.5) and $R(\frac{\partial}{\partial x})$ satisfies (2.6). We again restrict the discussion to *even* initial data u_0 . For $F : \mathbf{R} \rightarrow \mathbf{R}$ and smooth we assume:

$$(4.4) \quad F(s) = F(-s), \quad s \in \mathbf{R},$$

$$(4.5) \quad \|w \sup_{0 \leq \sigma \leq 1} |F'' \left(\sigma w + \frac{d\varphi}{dx} \right)|\|_{L^\infty(\mathbf{T})} \leq C(\varphi), \forall w, \varphi,$$

Where the constant $C(\varphi)$ is independant of w (which is obviously true, if for instance, F is subquadratic). Finally we assume (See Remark 2.5) that the order m of $P(\frac{\partial}{\partial x})$ satisfies

$m \geq 2$. Under these hypotheses theorem 2.1 and Corollary 2.2 still hold. We indicate the modifications to the proof.

We start with the change of variable $v = u - \varphi$. By Taylor formula, we have:

$$(4.6) \quad F\left(\frac{\partial u}{\partial x}\right) = F\left(\frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial x}\right) = F\left(\frac{\partial \varphi}{\partial x}\right) + F'\left(\frac{\partial \varphi}{\partial x}\right) \frac{\partial v}{\partial x} + \left|\frac{\partial v}{\partial x}\right|^2 R_2$$

and R_2 satisfies

$$(4.7) \quad |R_2| \leq \sup_{0 < \sigma < 1} \left| F''\left(\sigma \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial x}\right) \right|$$

Now the analog (2.10)–(2.12), satisfied by $v = u - \varphi$ is :

$$(4.8) \quad \begin{aligned} \frac{\partial v}{\partial t} + \nu P\left(\frac{\partial}{\partial x}\right)v + R\left(\frac{\partial}{\partial x}\right)v + F'\left(\frac{\partial \varphi}{\partial x}\right) \frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 R_2 \\ = -\nu P\left(\frac{\partial}{\partial x}\right)\varphi - R\left(\frac{\partial}{\partial x}\right)\varphi - F\left(\frac{\partial \varphi}{\partial x}\right) \end{aligned}$$

$$(4.9) \quad v(x, 0) = u_0(x) - \varphi(x)$$

$$(4.10) \quad v(x + L, t) = v(x, t).$$

Multiplying by $-\frac{\partial^2 v}{\partial x^2}$ equation (4.8) and integrating we obtain instead of (2.17), for $w \equiv \frac{\partial v}{\partial x}$

$$(4.11) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu(Pw, w) + \frac{1}{2} \int \frac{d}{dx} F'\left(\frac{d\varphi}{dx}\right) |w|^2 dx \\ = -(Rw, w) + \left(\nu P\varphi + R\varphi, \frac{\partial w}{\partial x}\right) + \left(F\left(\frac{d\varphi}{dx}\right), \frac{\partial w}{\partial x}\right) \\ + \int \frac{\partial w}{\partial x} |w|^2 R_2 dx \end{aligned}$$

We define the even trigonometric polynomial φ , as the solution of

$$(4.12) \quad -\frac{1}{4} \frac{d}{dx} F\left(\frac{d\varphi}{dx}\right) = q,$$

for q specified by (1.10), (1.27), (1.31). As a result, Theorem 1.1 applies and we find due to (2.25):

$$(4.13) \quad \frac{\nu}{2}(Pw, w) + \frac{1}{2} \int \frac{d}{dx} F'\left(\frac{d\varphi}{dx}\right) |w|^2 dx \geq \frac{\nu}{2} [w]_{\frac{m}{2}}^2 + c_0 \frac{\nu}{2} \tau \|w\|^2$$

In the right hand side of (4.11) only the last two terms are new. Instead of (2.22) we write:

$$(4.14) \quad \begin{aligned} \left(F \left(\frac{\partial \varphi}{\partial x} \right), \frac{\partial w}{\partial x} \right) &= - \left(F' \left(\frac{\partial \varphi}{\partial x} \right) \frac{\partial^2 \varphi}{\partial x^2}, w \right) \\ &\leq \left\| F' \left(\frac{\partial \varphi}{\partial x} \right) \right\|_{L^\infty}^2 \|\varphi''\|^2 + c_0 \|w\|^2 \end{aligned}$$

For the last term we use (4.7) (which requires $m \geq 2$) and standard interpolation

$$(4.15) \quad \begin{aligned} \int \frac{\partial w}{\partial x} |w|^2 R_2 dx &\leq \left\| w \sup_{0 \leq \sigma \leq 1} \left| F \left(\sigma w + \frac{\partial \varphi}{\partial x} \right) \right| \right\|_{L^\infty} \left\| \frac{\partial w}{\partial x} \right\| \|w\| \\ &\leq c(\varphi) \left\| \frac{\partial w}{\partial x} \right\| \|w\| \\ &\leq c(\varphi) [w]_{m/2}^{2/m} \|w\|^{2(1-\frac{1}{m})} \\ &\leq c(\varphi) [w]_{\frac{m}{2}}^2 + c_{11} \left(\frac{1}{\nu} \right)^{\frac{1}{m-1}} \|w\|^2 \end{aligned}$$

The rest of the proof is then unchanged, collecting (4.13)–(4.15) into (4.11). The definitions (2.15) and (2.16) are now:

$$(4.16) \quad \lambda = \frac{1}{2} c_0 \nu \left(c_3 \left(\frac{1}{\nu} \right)^{\frac{m'}{m-m'}} + c_{11} \left(\frac{1}{\nu} \right)^{\frac{1}{m-1}} + c_9 \right)$$

$$(4.17) \quad S(\varphi) = 2 \left(c_4 \nu [\varphi]_{\frac{m}{2}+1}^2 + \left(1 + \frac{c_6}{\nu} \right) \|\varphi\|_{m'+1-\frac{m}{2}}^2 + \|F' \left(\frac{\partial \varphi}{\partial x} \right)\|_{L^\infty}^2 \|\varphi''\|^2 \right)$$

We proved the:

THEOREM 4.1. Assume (4.4), (4.5) and $m \geq 2$. Let $u_0 \in H^1(\mathbb{T})$, and even and let φ defined in (4.12). Then, for u solution of (4.1)–(4.3) and $v = u - \varphi$, the conclusion of Theorem 2.1 and Corollary 2.2 holds, where $\lambda, S(\varphi)$ are defined by (4.16), (4.17).

4.B Examples. We quote here three examples (for $n = 1$) which motivated the present study. These examples correspond respectively to:

$$(I) \quad P \left(\frac{\partial}{\partial x} \right) = \frac{\partial^4}{\partial x^4}; R \left(\frac{\partial}{\partial x} \right) = + \frac{\partial^2}{\partial x^2}; F(s) = \frac{s^2}{2}$$

$$(II) \quad P \left(\frac{\partial}{\partial x} \right) = - \frac{\partial^2}{\partial x^2}; R \left(\frac{\partial}{\partial x} \right) = \nu p \frac{1}{x} * \frac{\partial}{\partial x}; F(s) = \frac{s^2}{2}$$

$$(III) \quad P \left(\frac{\partial}{\partial x} \right) = \varepsilon \frac{\partial^4}{\partial x^4}; R \left(\frac{\partial}{\partial x} \right) = (2\varepsilon - 1) \frac{\partial^2}{\partial x^2} + (1 - \varepsilon) \nu p \frac{1}{x} * \frac{\partial}{\partial x} \quad 0 \leq \varepsilon \leq 1; F(s) = \frac{s^2}{2}$$

The first example corresponds to the K-S equation. The second one has been introduced by Sivashinsky [48a], to take into account hydrodynamic effects for the propagation of a flame. The third has been studied numerically by Pumir [42b]. (The extension to space dimension 2 and 3 is immediate). Here, we will give a more explicit formulation of Corollary 2.2 and Theorem 3.2 in terms of the characteristic length L and the parameter ν . First of all, we remark that in example (I) (resp. (II)), $R(\frac{\partial}{\partial x})$ is exactly of order 2 (resp 1). As a consequence, (2.6) is in fact sharpened in $(R(\frac{\partial}{\partial x})u, u) = -[u]_{\frac{m'}{2}}^2 m' = 2$ or 1; in other words, we can take $c_2 = 0$ in (2.19) and replace $\|\varphi\|_{m'+1-\frac{m}{2}} \|w\|_{\frac{m}{2}}$ by $[\varphi]_{m'+1-\frac{m}{2}} [w]_{\frac{m}{2}}$ in (2.21). Following again the proof in Theorem 2.1, the expression of $S(\varphi)$ is now:

$$(4.18) \quad \tilde{S}(\varphi) = 2c'_4 \nu + 2\frac{c'_6}{\nu} [\varphi]_{m'-\frac{m}{2}+1}^2 + \|\varphi'\|_{L^\infty}^2 \|\varphi''\|^2$$

To compute $\mathcal{R}_0$ and $\mathcal{Y}_0$ in Corollary, we need the expression of $[\varphi]_s^2$ for φ defined by (2.13); we obtain by direct computation:

$$(4.19) \quad [\varphi]_s^2 \sim L^{\alpha(s)}, \quad \alpha(s) = 5 - 2s + \frac{2m}{m-1} \text{Max}(2s - 3, 0).$$

We recall ([39b]) also that

$$(4.20) \quad \|\varphi'\|_{L^\infty} \sim L$$

We have, by direct computation:

LEMMA 4.2. *Let $\alpha(s) \equiv 5 - 2s + \frac{m+1}{m-1} \text{Max}(2s - 3, 0)$ and $\tilde{S}(\varphi)$ be defined by (4.18), then the following holds:*

$$(4.21) \quad S(\varphi) \sim L^\beta, \quad \beta = \text{Max} \left(5, 1 + \frac{2m}{m-1}, \alpha \left(m' - \frac{m}{2} + 1 \right) \right)$$

$$(4.22) \quad \tilde{S}(\varphi) \sim L^{\tilde{\beta}}, \quad \tilde{\beta} = \text{Max} \left(1 + \frac{2m}{m-1}, \alpha \left(m' - \frac{m}{2} + 1 \right) \right)$$

As a consequence we obtain $\mathcal{R}_0$, $\mathcal{Y}_0$, J_0 (defined in (2.25), (2.26), (3.17)) as a function of L for the examples (I)–(II) (with $S(\varphi)$ replaced by $\tilde{S}(\varphi)$ in (I), (II)). The results are summarized in a table.

	$\mathcal{R}_0$	$\mathcal{Y}_0$	$\mathcal{J}_0$
(I) $m = 4 \ m' = 2$	$L^{\frac{11}{6}}$	$L^{\frac{5}{2}}$	$L^{\frac{4}{3}}$
(II) $m = 2 \ m' = 1$	$L^{\frac{5}{2}}$	L^5	$L^{\frac{3}{2}}$
(III) $m = 4 \ m' = 2$	$L^{\frac{5}{2}}$	L^5	$L^{\frac{3}{2}}$

Remark 4.1. In space dimension 1, we can consider the ω -limit set of the absorbing set $\mathcal{A}$ introduced in Remark 2.4 i.e. ([51]):

$$(4.23) \quad \omega(\mathcal{A}) = \bigcap_{s>0} \overline{\bigcup_{t>s} \dot{S}(t)\mathcal{A}}$$

This set denoted $\tilde{X}$ is an attractor, its basin of attraction is the whole space $\dot{H}$. It is the largest *bounded* invariant set, the largest attractor, it contains in particular the ω -limit sets associated to a particular trajectory like (3.1). All that has been said for X applies as well to $\tilde{X}$ and in particular the bounds tabulated above for the norms and the dimensions. $\square$

PART II : THE CAHN-HILLIARD EQUATION

1. Non linear stability analysis for the Cahn-Hilliard equation.

In this section, we study the Cahn-Hilliard (C-H) equation with an arbitrary non linearity of polynomial type. Precisely we consider the following non linear evolution equation in space dimension $n \leq 3$, for real u :

$$(1.1) \quad \frac{\partial u}{\partial t} - \Delta \mathcal{A}(u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+$$

$$(1.2) \quad u(x, 0) = u_0 \quad \text{in } \Omega$$

The non linear operator $\mathcal{A}(u)$ is defined by:

$$(1.3) \quad \mathcal{A}(u) = -\Delta u + f(u) = -\Delta u + \sum_{j=1}^{2p-1} a_j u^j,$$

where $p \in \mathbb{N}, p \geq 2$. The boundary conditions are either of Neuman or periodic type:

$$(1.4)_1 \quad \frac{\partial u}{\partial n} = \frac{\partial}{\partial n} \mathcal{A}(u) = 0, \quad \text{if } \Omega \text{ is a bounded set of } \mathbb{R}^n$$

$$(1.4)_2 \quad u(x + Le_i, t) = u(x, t), \text{ if } \Omega \text{ is the torus } \left[-\frac{L}{2}, \frac{L}{2}\right]^n$$

We denote by $g(u)$ the primitive of f vanishing at $u = 0$:

$$(1.5) \quad \begin{cases} g(u) = \sum_{j=2}^{2p} b_j u^j, g'(u) = f(u), \\ j b_j = a_{j-1} \quad 2 \leq j \leq 2p \end{cases}$$

and we assume that the leading coefficient of f (and g) is positive

$$(1.6) \quad a_{2p-1} = 2p b_{2p} > 0.$$

This means that the leading term of f is dissipative while the lower order terms may not be dissipative, a situation which is typical of the Cahn-Hilliard equation. For instance for $p = 2$ we recover the usual Cahn-Hilliard equation ($f(u) = -\alpha u + \beta u^3$, $\alpha, \beta > 0$, see [7][8]). No other assumption will be made on the coefficients of the polynomial $f(u)$ so that the study of the long time behavior of the solutions of (1.1) is not straightforward.

A. Following ideas of [38] we now derive a priori estimates, uniform in time, for the solution u of the problem (1.1) (1.2) (1.4). Using standard techniques these estimates insure global existence, uniqueness and regularity of the solution; we will not develop this classical aspect here and we will derive the a priori estimates in a formal way, assuming that the solutions are sufficiently regular. We start with the following:

THEOREM 1.1. *Assume that u_0 is in $H^1(\Omega) \cap L^{2p}(\Omega)$ and let g be defined by (1.5). Then for u solution of (1.1)–(1.4), the following holds*

$$(1.7) \quad \|\nabla u(t)\| \leq r_0(u_0), \|u(t)\|_{L^{2p}(\Omega)} \leq r_1(u_0), \forall t \geq 0,$$

where r_0 and r_1 depend on the initial value u_0 (and the data, f , g , Ω).

Furthermore, if we assume that

$$(1.8) \quad \frac{1}{|\Omega|} \left| \int_{\Omega} u_0(x) dx \right| \leq \rho,$$

then there exists $t_* = t_*(u_0)$ and two constants : K_1, K_2 depending only on ρ (and the data f , g , Ω) such that

$$(1.9) \quad \|\nabla u(t)\| \leq K_1(\rho), \|u(t)\|_{L^{2p}(\Omega)} \leq K_2(\rho), \quad \forall t \geq t_*(u_0).$$

Proof. The first part (i.e. (1.7)) is a direct consequence of (1.5). We multiply (1.1) by $\mathcal{A}(u)$ and integrate over Ω :

$$(1.10) \quad \left(\frac{\partial u}{\partial t}, \mathcal{A}(u) \right) + \|\nabla \mathcal{A}(u)\|^2 = 0$$

Let us define

$$(1.11) \quad V(u) = \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} g(u) dx.$$

From (1.3.) (1.5) (1.10) (11), we obtain,

$$(1.12) \quad \frac{d}{dt} V(u) \leq 0$$

and after integration in time from 0 to t

$$(1.13) \quad \begin{cases} V(u) \leq V(u_0) \\ \frac{1}{2} \|\nabla u\|^2 + \int_{\Omega} g(u) dx \leq \frac{1}{2} \|\nabla u_0\|^2 + \int_{\Omega} g(u_0) dx \end{cases}$$

Using the Young inequality, we infer from (1.5) (1.6) the existence of two constants c' , c'_2 (depending only on the coefficients of g) such that

$$(1.14) \quad \frac{1}{2}b_{2p}s^{2p} - c'_1 \leq g(s) \leq \frac{1}{2}b_{2p}s^{2p} + c'_2, \forall s \in \mathbb{R}.$$

Then (1.7) follows readily from (1.13) (1.14).

We proceed now to the proof of (1.9) which is more delicate.

We first observe that by integration of (1.1) over Ω and using the boundary condition (1.4) we find that the average value of u is independent of t ,

$$(1.15) \quad \frac{\partial \bar{u}}{\partial t} = 0, \bar{u}(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx.$$

Thus (1.1) can be rewritten

$$(1.16) \quad \frac{\partial}{\partial t}(u - \bar{u}) - \Delta \mathcal{A}(u) = 0.$$

Now let $\psi = N\varphi$ denote the solution to the Neuman problem,

$$\begin{cases} -\Delta \psi = \varphi \\ + \text{Boundary condition } \frac{\partial \psi}{\partial n} = 0 \text{ in the case (1.4)}_1 \text{ or} \\ \text{the periodicity in the case (1.4)}_2. \end{cases}$$

We notice that $(N\varphi, \psi)$ is a scalar product hereafter denoted $(\varphi, \psi)_{-1}$, while $|\varphi|_{-1} = \{(\varphi, \varphi)_{-1}\}^{\frac{1}{2}}$.⁴ We apply the operator N to both sides of (1.16) and multiply the resulting equation by $u - \bar{u}$ and integrate on Ω , to obtain:

$$(1.17) \quad \frac{1}{2} \frac{d}{dt} |u - \bar{u}|_{-1}^2 + (\mathcal{A}(u), u - \bar{u}) = 0$$

We have

$$(1.18) \quad \begin{aligned} (\mathcal{A}(u), u - \bar{u}) &= -(\Delta u, u - \bar{u}) + (f(u), u - \bar{u}) \\ &= \|\nabla u\|^2 + \int_{\Omega} f(u)u dx - \bar{u}_0 \int_{\Omega} f(u) dx \end{aligned}$$

Thanks to (1.6) we can see, by application of the Young inequality that there exists two constants c_3, c_4 , such that

$$\begin{cases} f(s)s \geq pb_{2p}s^{2p} - c'_3 \\ |f(s)| \leq \varepsilon b_{2p}s^{2p} + c'_4(\varphi), \forall s \in \mathbb{R} \\ \forall \varepsilon > 0 \end{cases}$$

⁴This is the scalar product and the norm of $H^{-1}(\Omega)$, which can be identified to a superspace of $L^2(\Omega)$, but this point of view is not necessary here.

Thus with (1.8), (1.15) (1.18) and taking $\varepsilon = \frac{1}{\rho}(p - \frac{1}{2})$

$$\begin{aligned}
(\mathcal{A}(u), u - \bar{u}) &\geq \|\nabla u\|^2 + \frac{1}{2}b_{2p} \int_{\Omega} u^{2p} dx - (c'_3 + \rho c'_4)|\Omega| \\
&\geq \text{(with (1.14))} \\
&\geq \|\nabla u\|^2 + \int_{\Omega} g(u) dx - k_3(\rho) \\
&\geq V(u) - k_3(\rho)
\end{aligned}$$

where k_3 depends only on ρ and f, g, Ω .

By the Poincaré inequality technique one can check that there exist constants c_5, c_6 depending on Ω , such that

$$(1.19) \quad |\varphi - \bar{\varphi}|_{-1} \leq c'_5 |\varphi - \bar{\varphi}| \leq c'_6 |\nabla \varphi|, \forall \varphi \in H^1(\Omega)$$

which gives another lower bound

$$\begin{aligned}
(1.20) \quad (\mathcal{A}(u), u - \bar{u}) &\geq V(u) - k_3(\rho) \\
&\geq \frac{1}{2(c'_6)^2} |u - \bar{u}|_{-1}^2 \leq k_4(\rho)
\end{aligned}$$

It then follows from (1.17) that

$$(1.21) \quad \frac{1}{2} \frac{d}{dt} |u - \bar{u}|_{-1}^2 + \frac{1}{2(c'_6)^2} |u - \bar{u}|_{-1}^2 \leq k_4(\rho)$$

$$(1.22) \quad \frac{1}{2} \frac{d}{dt} |u - \bar{u}|_{-1}^2 + V(u) \leq k_3(\rho)$$

By Gronwall lemma, (1.21) shows that $|u - \bar{u}|_{-1}^2$ is bounded by

$$\begin{aligned}
&|u_0 - \bar{u}_0|_{-1}^2 \exp(-c'_7 t) + \frac{2k_4(\rho)}{c_7} (1 - \exp(-c'_7 t)) \\
&\leq \frac{2k_4(\rho)}{c_7} + 1 \text{ for } t \geq t_*(u_0),
\end{aligned}$$

where $c'_7 = 1/(c'_6)^2$. Then (1.22) shows that $\int_t^{t+1} V(u(s)) ds$ is bounded independently of t (and of u_0 , except through ρ) for $t \geq t_*(u_0)$; more precisely:

$$(1.23) \quad \int_t^{t+1} V(u(s)) ds \leq \frac{k_4(\rho)}{c_7} + k_3(\rho) + \frac{1}{2}, \text{ for } t \geq t_*(u_0)$$

Now integrating (1.12) between $t \geq t_*(u_0)$ and s , where $t \leq s \leq t+1$, then using (1.23) we find:

$$(1.24) \quad V(u(t)) \leq k_5(\rho) \text{ for } t \geq t_*(u_0).$$

Coming back to (1.11) and (1.14), we finally obtain (1.9). $\square$

Remark 1.2. Inequality (1.2) together with (1.20), (1.11) states that $V(u)$ is a *Lya-punov functional* associated to (1.1). Another way to obtain (1.12) is to apply again N to both sides of (1.16), multiply by $\frac{\partial}{\partial t}(u - \bar{u})$ and integrate on Ω . One thus recover the classical way to obtain energy estimates for parabolic equations; the difference just lies in the choice of scalar product. $\square$

We now address the question of higher order estimates for the solution u . We restrict ourselves to the case of periodic boundary conditions (1.4)₂: in the sequel, Ω will denote the n -dimensional torus $\mathbb{T}^n = [-\frac{L}{2}, \frac{L}{2}]^n$, $n \leq 3$. For $n = 1$, one checks that the results below still holds with the Neuman boundary conditions (1.4)₁.

THEOREM 1.2. *For $\Omega = \mathbb{T}^n$, assume u_0 is in $H^2(\Omega)$ and $p \in \mathbb{N}$ satisfies respectively $2 \leq p < \infty$ if $n = 1, 2$ and $p = 2$ if $n = 3$. Then, for u solution of (1.1)–(1.4), the following holds:*

$$(1.25)_1 \quad \|D^2 u(t)\| \leq r_2(u_0, \Omega), \forall t > 0.$$

Here $r_2(u_0, \Omega)$ depends only on $r_1(u_0)$ as defined in (1.7).

Furthermore, if we assume (1.8) : then there exists $t_* = t_*(u_0)$ such that

$$(1.25)_2 \quad \|D^2 u(t)\| \leq R_2(\rho), \forall t > t_*.$$

Here $R_2(\rho)$ depends only on ρ and the data f, Ω .

Proof. We shall use frequently the notation, for $\ell \in \mathbb{N}$,

$$[u]_\ell = \|D^\ell u\| = \left(\sum_{|\alpha|=\ell} \|D^\alpha u\|^2 \right)^{\frac{1}{2}}$$

and the following usual inequalities (assuming $n \leq 3$ and u of zero mean):

$$(1.26) \quad \|u\|_{L^\infty} \leq \gamma_n [u]_1^{1-\alpha_n} [u]_4^{\alpha_n}$$

with $\alpha_1 = 0$, $\alpha_2 = \text{any } \varepsilon > 0$, $\alpha_3 = \frac{1}{6}$.

$$(1.27) \quad \|u\|_{L^{2p}} \leq \|u\|^{1-\frac{n}{2p'}} \|Du\|^{\frac{n}{2p'}}, \quad p' = \frac{p}{p-1}$$

$$(1.28) \quad [u]_\ell \leq [u]_{\ell'}^{1-\theta} [u]_{\ell''}^\theta, \quad \ell = (1-\theta)\ell' + \theta\ell'', \quad \ell' \leq \ell \leq \ell''.$$

To prove (1.25), we apply D^α , with $|\alpha| = 2$ to (1.1), multiply the result by $D^\alpha u$ and integrate by parts, using the definition (1.3) of $\mathcal{A}(u)$ and Hölder inequality:

$$(1.29) \quad \frac{1}{2} \frac{d}{dt} \|D^2 u\|^2 + \|D^4 u\|^2 \leq \|D^2 f(u)\| \cdot \|D^4 u\|$$

To estimate $\|D^2 f(u)\|$, we compute

$$(1.30) \quad D_j D_k f(u) = f'(u) D_{jk} u + f''(u) D_j u D_k u$$

and remark that $f(u)$ being polynomial of order $2p - 1$:

$$(1.31) \quad |f'(u)| \leq \varphi_1(1 + |u|^{2p-2})$$

$$(1.32) \quad |f''(u)| \leq \varphi_2(1 + |u|^{2p-3})$$

From (1.30)–(1.32) we thus get⁵

$$(1.33) \quad \begin{aligned} \|D^2 f(u)\|^2 &\leq c_0 [\|f'(u)\|_{L^\infty}^2 \|D^2 u\|^2 + \|f''(u)\|_{L^\infty}^2 \|Du\|_{L^4}^4] \\ &\leq c_1 \left[(1 + \|u\|^{4p-4}) \|D^2 u\|^2 \right. \\ &\quad \left. + (1 + \|u\|_{L^\infty}^{4p-6}) \|Du\|_{L^4}^4 \right] \end{aligned}$$

First of all, we use (1.27) with $p = 2$ and (1.28) with $\ell = 2$, $\ell' = 1$, $\ell'' = 4$, $\theta = \frac{1}{3}$ to obtain

$$(1.34) \quad \|D^2 u\|^2 \leq \|Du\|^{\frac{4}{3}} \|D^4 u\|^{\frac{2}{3}}$$

$$(1.35) \quad \begin{aligned} \|Du\|_{L^4}^4 &\leq \|Du\|^{4-n} \|D^2 u\|^n \\ &\leq \|Du\|^{4-n} (\|Du\|^{\frac{2}{3}} \|D^4 u\|^{\frac{1}{3}})^n \\ &\leq \|Du\|^{4-\frac{n}{3}} \|D^4 u\|^{\frac{n}{3}} \end{aligned}$$

Then, we use (1.26), (1.15) and find

$$(1.36) \quad \begin{aligned} \|u\|_{L^\infty} &\leq \|u - \bar{u}\|_{L^\infty} + |\bar{u}| \\ &\leq \gamma_n [u]_1^{1-\alpha_n} [u]_4^{\alpha_n} + |\bar{u}_0|. \end{aligned}$$

To avoid heavy expressions, we set

$$(1.37) \quad X \equiv \|Du\| = [u]_1, \quad Y \equiv \|D^4 u\| = [u]_4,$$

and rewrite (1.36) as

$$(1.38) \quad \|u\|_{L^\infty} \leq \gamma_n X^{1-\alpha_n} Y^{\alpha_n} + |\bar{u}_0| \equiv Z$$

⁵As usual $c_0, c_1 \dots$, are absolute constants.

We now collect (1.34), (1.35) and (1.38) into (1.33) with using the notations (1.37); this yields:

$$(1.39) \quad \|D^2 f(u)\|^2 \leq c_2 [(1 + Z^{4p-4})X^{\frac{4}{3}}Y^{\frac{2}{3}} + (1 + Z^{4p-6})X^{4-\frac{n}{3}}Y^{\frac{n}{3}}]$$

Next, by using (1.7) i.e. $X \leq r_0(u_0)$ we derive upper bounds for the right hand side of (1.38), (1.39) but only in term of $Y = \|D^4 u\|$. Precisely, we obtain:

$$(1.40) \quad \|D^2 f(u)\|^2 \leq c_3 (1 + \|D^4 u\|^\beta + \|D^4 u\|^\gamma)$$

where the constant c_3 depends only on u_0, Ω and $\beta \equiv 4(p-1)\alpha_n + \frac{2}{3}, \gamma \equiv (4p-6)\alpha_n + \frac{n}{3}$. Finally, notice that (1.29) implies:

$$(1.41) \quad \frac{d}{dt} \|D^2 u\|^2 + \|D^4 u\|^2 \leq \|D^2 f(u)\|^2$$

Thus if $\beta < 2$ and $\gamma < 2$, that is if $p < +\infty$ for $n = 1, 2$ and $p < 3$ for $n = 3$, we can apply Cauchy-Young inequality to (1.40) and deduce with (1.41), the following

$$(1.42) \quad \frac{d}{dt} \|D^2 u\|^2 + \|D^4 u\|^2 \leq c_4(u_0, \Omega)$$

This yields (1.25)₁, by Gronwall's Lemma. The proof of (1.25)₂ is identical; the only difference occurs in using (1.8) to bound $|\bar{u}_0|$ in (1.36) and (1.9), to bound X . $\square$

Remark 1.3. Since $\alpha_1 = 0$, inequalities become simpler if $n = 1$ and the way the bounds (1.25) depend on $|\Omega|$ can be made more precise. $\square$

From Theorem 1.2, we deduce more regularity for u by a reiteration argument.

THEOREM 1.3. *For $\Omega = \mathbb{T}^n$, Assume u_0 is in $H^k(\Omega)$, $k \in \mathbb{N}$, and $p \in \mathbb{N}$ satisfy the conditions of Theorem 1.2. Then, for u solution of (1.1)–(1.4), the following holds:*

$$(1.43)_1 \quad \|D^j u(t)\| \leq r_j(u_0, \Omega), \quad 1 \leq j \leq k, \quad \forall t \geq 0.$$

Furthermore, if we assume (1.8), then there exists $t_* = t_*(u_0)$ such that:

$$(1.43)_2 \quad \|D^j u(t)\| \leq R_j(\rho), \quad 1 \leq j \leq k, \quad \forall t \geq t_*.$$

Here $R_j(\rho)$ depends only on ρ and the data f, Ω .

Proof. We proceed by an induction argument, therefore the proof of (1.33)₁ or (1.33)₂ is the same. For $k = 1, 2$, the result follows from Theorem 1.1, 1.2. We thus start with $k \geq 3$ assuming $\|D^j u(t)\| \leq r_j$, $1 \leq j \leq k-1$, we will prove $\|D^k u(t)\| \leq r_k$. For $|\alpha| = k$, we apply D^α to the equation (1.1) and take the scalar product in $L^2(\Omega)$ with $D^\alpha u$. After

integrating by parts, and taking the sum over α such that $|\alpha| = k$, we get (using notations of Theorem 1.2):

$$(1.44) \quad \frac{1}{2} \frac{d}{dt} \|D^k u\|^2 + [u]_{k+2}^2 = \sum_{|\alpha|=k} (D^\alpha f(u), \Delta D^\alpha u)$$

For the right handside of (1.44), we write

$$(1.45) \quad \sum_{\alpha=k} (D^\alpha f(u), \Delta D^\alpha u) \leq c_4 \|D^k f(u)\| [u]_{k+2}$$

using Hölder inequality. Now, by hypothesis, we have at least $u \in H^2(\Omega)$, with $\|u\|_{H^2(\Omega)}$ uniformly bounded in time. For $n \leq 3$, this implies that $\|u\|_{L^\infty(\Omega)}$ is also bounded uniformly in time; we can thus use a classical result (see [25] for instance) and write:

$$(1.46) \quad \|D^k f(u)\| \leq c_5 \|D^k u\|,$$

where c_5 depends on $\|u\|_{L^\infty(\Omega)}$. From (1.46), and interpolation inequality (1.28), we then deduce:

$$(1.47) \quad \begin{aligned} \|D^k f(u)\| &\leq c_6 \|D^{k-1} u\|^{\frac{2}{3}} \|D^{k+2} u\|^{\frac{1}{3}} \\ &\leq c_7 [u]_{k+2}^{\frac{1}{3}} \end{aligned}$$

after using the induction hypothesis.

Finally (1.45), (1.47) are collected with (1.44) and we obtain:

$$(1.48) \quad \frac{1}{2} \frac{d}{dt} \|D^k u\|^2 + [u]_{k+2}^2 \leq c_7 [u]_{k+2}^{\frac{4}{3}}$$

and use of Cauchy-Young inequality complete the proof. $\square$

An argument similar to that of Theorem 2.5 in Part I gives C^∞ -regularity in space and time. We therefore omit the proof and only state:

COROLLARY 1.4. *For $\Omega = \mathbb{T}^n$, $n \leq 3$ let $u_0 \in C^\infty(\Omega)$. Then under the hypothesis of Theorem 1.2 on p ; the solution $u = u(x, t)$ of (1.1), (1.2), (1.4)₂ belongs to $C^\infty(\mathbb{R}_+ \times \mathbb{T}^n)$.*

Remark 1.5. If $y(t) \geq 0$ satisfies the differential inequality $y' + \lambda y \leq f$ for $\lambda > 0$ and f is time independant, then the interval $[0, \alpha + \frac{f}{\lambda}]$ is absorbing and invariant for every $\alpha > 0$. By combining this elementary observation with inequalities (1.21)₁, (1.42)₂, (1.48) we can construct invariant and absorbing sets in $H_\rho \equiv \{u \in L^2(\Omega), |\bar{u}| \leq \rho\}$ in various norms, depending on the smoothness of u_0 . Such a construction will be useful in Section 2 and Part III. $\square$

B. We now address the question of the asymptotic behavior of the solution $u = u(t)$ of (1.1)–(1.4). Given u_0 , the (smooth, unique) solution $u(t)$ defines a family $\{S(t)\}_{t>0}$ of applications in $L^2(\Omega)$, by the formula $S(t)u_0 = u(t)$. In the sequel, we will frequently use the following subspace of $L^2(\Omega)$ defined for $\rho > 0$ by:

$$(1.49) \quad H_\rho = \{u \in L^2(\Omega), \quad |\bar{u}| \leq \rho\}.$$

Using Theorem 1.1, one checks in the classical way that $\{S(t)\}_{t>0}$ is a continuous semi-group on $H_\rho \cap H^1(\Omega)$. Classical results for dynamical systems extended to Hilbert spaces (see [21a], [40] for instance) are thus applicable. In particular, introducing $\omega(u_0)$ —the ω -limit set associated to u_0 —these results and Theorem 1.1, 1.2 imply that $\omega(u_0)$ is *nonempty, invariant for $S(t)$, connected and compact in $H^1(\Omega)$; moreover*

$$\text{dist}(S(t)u_0, \omega(u_0)) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Here, due to the existence of the Lyapunov functional $V(u)$, we prove more, that is $u(t)$ converges, in an appropriate way, as $t \rightarrow \infty$ to a stationary solution of (1.1). More precisely, we will first establish that $\frac{\partial u}{\partial t}(t) \rightarrow 0$ in $H^{-1}(\Omega)$ as $t \rightarrow \infty$, for either Neuman or periodic boundary conditions. We state:

PROPOSITION 1.6. *For $n \leq 3$, let $u = u(x, t)$ be a (smooth) solution of (1.1)–(1.4). Then the following holds:*

$$(1.50) \quad \int_{\frac{1}{2}}^{\infty} \left| \frac{\partial u}{\partial t}(t) \right|_{-1}^2 dt + \sup_{t > \frac{1}{2}} V(u(t)) \leq c_1$$

$$(1.51) \quad \sup_{t \geq 1} \left| \frac{\partial}{\partial t} u(t) \right|_{-1}^2 + \int_{-1}^{+\infty} \left\| \nabla \frac{\partial u}{\partial t}(t) \right\|^2 dt \leq c_2,$$

where c_1, c_2 are constants depending only on Ω and u_0 or Ω and ρ if $|\bar{u}_0| \leq \rho$.

Proof. We follow classical arguments, so we omit the details. According to the Remark 1.2, (1.10) is equivalent to:

$$(1.52) \quad \left| \frac{\partial}{\partial t} u(t) \right|_{-1}^2 + \frac{d}{dt} V(u(t)) = 0$$

Similarly, after taking the time derivative of (1.1), we obtain:

$$(1.53) \quad \frac{1}{2} \frac{d}{dt} \left| \frac{\partial u}{\partial t}(t) \right|_{-1}^2 + \left\| \nabla \frac{\partial u}{\partial t}(t) \right\|^2 + \int_{\Omega} f'(u(x, t)) \left| \frac{\partial u}{\partial t}(x, t) \right|^2 dx = 0$$

Inequalities (1.50), (1.51) are thus obtained after multiplication of (1.52), (1.53) by an appropriate cut-off function in t , integration over time and use of Theorem 1.1, (1.31), (1.43). $\square$

Let us introduce the set $\mathcal{F}$ of fixed points (or equilibrium points) of $S(t)$, that is the set of stationary solutions for (1.1)–(1.4). For $D(\mathcal{A}) = \{u \in H^2(\Omega) \text{ with either } (1.4)_1 \text{ or } (1.4)_2\}$ and $\mathcal{A}$ defined by (1.3), we have:

$$(1.54) \quad \mathcal{F} = \{u \in D(\mathcal{A}), \mathcal{A}(u) = q, q \in \mathbb{R}\}.$$

COROLLARY 1.7. *For any u_0 given in $H_\rho \cap H^1(\Omega)$, we have $\omega(u_0) \subset \mathcal{F}$. Furthermore, the Lyapunov functional V is constant on $\omega(u_0)$. If, in addition, the cardinal of $\mathcal{F}$ is finite, then:*

$$(1.55) \quad \omega(u_0) = \{u_s\} \text{ where } u_s \in \mathcal{F}$$

for each (u_0) in $H_\rho \cap H^1(\Omega)$.

Proof. The first part is a direct consequence of the Proposition 1.6 and of the continuity of $V(u)$ on $H^1(\Omega)$ (which result, since $n \leq 3$, of the Sobolev imbedding theorem). Therefore, if $\mathcal{F}$ is finite, $\omega(u_0)$ is also finite and, being connected, is thus reduced to a singleton. $\square$

Finite cardinality of $\mathcal{F}$ has been shown, for Neuman boundary conditions, by several authors *assuming the space dimension is one* (see [36a], [9]), which authorizes the use of Sturmian theory of differential equations. For space dimension greater than one, the result is still true ([19c], [19b]) under the following hypothesis on the self adjoint operator $\mathcal{A}'(u_s) \equiv -\Delta + f'(u_s)$:

$$(1.56) \quad \dim \text{Ker } \mathcal{A}'(u_s) \leq 1, \quad \forall u_s \in \mathcal{F}.$$

This assumption is *generically true* with respect to the domain Ω , for boundary conditions (1.4)₁, or with respect to the metric on torus Ω (which amounts to work with $-\sum_{ij} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ instead of $-\Delta$) for boundary conditions (1.4)₂. Proof of the above assertion relies on an extension of a transversality Theorem of Quinn which can be found in [43], [52], [47].

2. Structure and dimension of attractors for the C-H equation.

A. We already noticed several times that $\bar{u}(t)$, the spatial average of any solution of (1.1)–(1.4), is constant i.e. $\bar{u}(t) = \bar{u}_0$, $\forall t > 0$. Consequently, there is no absorbing set for $S(t)$ in the ordinary sense. One circumvents this technical difficulty by restricting the size of $\bar{u}_0$; this amounts to make $S(t)$ operate in the space H_ρ defined by (1.49) instead of H . Let us introduce the set:

$$(2.1) \quad \mathcal{A}_\rho = \{u \in H_\rho \cap H^1(\Omega), \|\nabla u\| \leq k_1(\rho) + 1\}$$

where $k_1(\rho)$ is the same constant as in (1.9). Associated to $\mathcal{A}_\rho$ is its ω -limit set $\omega(\mathcal{A}_\rho) = \bigcap_{t>0} \overline{\bigcap_{s \geq t} S(t)\mathcal{A}_\rho}$, where the closures are taken in $L^2(\Omega)$. We are now in position to state

THEOREM 2.1. *Let $S(t)$ operate in H_ρ . Then, $X_\rho \equiv \omega(\mathcal{A}_\rho)$ is a compact maximal attractor in H_ρ .*

Proof. According to (2.1) and Theorem 1.1, the set $\mathcal{A}_\rho$ attracts each bounded set of $H_\rho \cap H^1(\Omega)$. Therefore, due to a general result (see [13], [51]), $\omega(\mathcal{A}_\rho)$ is a maximal attractor. Compactness in H_ρ follows from the boundedness of $\mathcal{A}$ in $H^1(\Omega)$. $\square$

The structure of X_ρ can be made more precise, due to the existence of the Lyapunov functional defined by (1.11). One checks the continuity of V , for the $H^1(\Omega)$ -topology, on $\omega(\mathcal{A}_\rho)$. Therefore, using Theorem 2.1 and a general result of [4c] (see also [19b]), we can state:

THEOREM 2.2. *Define $\mathcal{F}_\rho = \mathcal{F} \cap H_\rho$, for $\mathcal{F}$ given by (1.54). Then, $X_\rho \equiv \omega(\mathcal{A}_\rho) = W^u(\mathcal{F}_\rho)$, where $W^u(\mathcal{F}_\rho)$ is the unstable set of $\mathcal{F}_\rho$.*

In other words, for $\rho > 0$ given, X_ρ is the set of the stationary solutions for (1.1)–(1.4) together with their associated unstable sets. Under additional assumption concerning the spectrum of $\mathcal{A}'(u) = -\Delta + f'(u)$, it can be shown ([21a], [4b]) that the unstable set $W^u(\mathcal{F}_\rho)$ is in fact a manifold. On the other hand, if $\mathcal{F}$ is assumed finite, then (see [4b]), we have the more precise result: $X_\rho = \cup_{u^* \in \mathcal{F}_\rho} W^u(u^*)$.

B. We will now derive an upper bound for the dimension of the attractor X_ρ of $S(t)$, defined in Theorem 2.1. The necessary framework has already been described in section 3.B. of Part I. Here, due to the specific form of the equation (1.1)₂, it will be easier to compute the dimension by choosing $H^{-1}(\Omega)$ instead of $L^2(\Omega)$ as a reference space for X_ρ . We are thus led to prove that the dimension of X_ρ is independant of the choice of the reference space. For that purpose we first prove a series of Lemmas.

LEMMA 2.3. *Let E, F be two metric spaces and $S : E \rightarrow F$ a Lipschitz map. Then, for any set $A \subset E$, the following holds:*

$$(2.2) \quad \dim S(A) \leq \dim A,$$

for both Hausdorff and fractal dimensions. Furthermore, if A is compact, (2.2) still holds assuming S is a locally Lipschitz map.

The proof is a straightforward application of the definitions of the dimensions (see for instance [46]).

In what follows, we will denote by $\dim^E A$ the dimension of the set A , for the topology of the space E .

LEMMA 2.4. *Let $E \subset F$ be two metric spaces, the injection i being continuous. Let $T : F \rightarrow E$ a locally Lipschitz map such that $T(A) = A$ for $A \subset E$ and is compact. Then:*

$$(2.4) \quad \dim^E A = \dim^F A,$$

for both Hausdorff and fractal dimension.

Proof. First, we apply Lemma 2.3, with $S = i : E \rightarrow F$ and obtain $\dim^F A \leq \dim^E A$. Then, we apply again the Lemma, with $S = T$ and find $\dim^E A = \dim^E T(A) \leq \dim^F A$. For technical purpose, we will restrict our analysis to “sections” of X_ρ . $\square$

Precisely, we state:

THEOREM 2.5. : Let X_ρ (defined in Theorem 2.1) be the attractor associated to $S(t)$ operating in H_ρ . Define $X_\rho^\mu = \{u \in X_\rho | \bar{u} = \mu\}$, $|\mu| \leq \rho$. Then:

$$(2.5) \quad \dim^{H^{-1}(\Omega)}(X_\rho^\mu) = \dim^{L^2(\Omega)}(X_\rho^\mu),$$

for both Hausdorff and fractal dimensions.

Proof. We apply the Lemma 2.4 with $T = S(t)$, $E = \{u \in L^2(\Omega) | \bar{u} = \mu\}$, $F = \{u \in H^{-1}(\Omega) | \bar{u} = \mu\}$, $A = X_\rho^\mu$, where $|\mu| \leq \rho$. Obviously $E \subset F$, the injection being continuous, and A is compact and invariant by $S(t)$, according to Theorem 2.1. It remains to check that $S(4) : F \rightarrow E$ and is locally Lipschitz, which is a consequence of the smoothing effect of $S(t)$. The precise result is stated in the next Lemma. $\square$

LEMMA 2.5. Assume that u_0, v_0 are in $H^{-1}(\Omega)$ and $\bar{u}_0 = \bar{v}_0 = \mu$, $|\mu| \leq \rho$. Let $u(t) = S(t)u_0$, $v(t) = S(t)v_0$ be the corresponding solutions of (1.1). Then, there exists $s > 0$ and c_0 such that:

$$(2.6) \quad \|u(t_1) - v(t_1)\| \leq c_0 |t_1 - t_2|^{-s} \|u(t_2) - v(t_2)\|_{-1}, \quad t_1 > t_2 > 0,$$

where the constant c_0 depends on t_1, t_2, u_0, v_0, ρ .

Proof of Lemma 2.5. We use a classical, but lengthy argument. First, we prove that $S(t)$ is locally Lipschitz in $H^{-1}(\Omega)$. Set $\omega(t) \equiv u(t) - v(t)$. By assumption $\bar{\omega}(t) = 0$ and ω do satisfy:

$$(2.7) \quad \frac{\partial \omega}{\partial t} + \Delta^2 \omega - \Delta(f(u) - f(v)) = 0$$

As in the proof of Theorem 1.1, we take the scalar product in $H^{-1}(\Omega)$ of this equation with ω :

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{-1}^2 + \|\nabla \omega\|^2 + \int_{\Omega} (f(u) - f(v)) \omega \, dx = 0$$

Thanks to (1.6), we can see, by application of the Young's inequality that there exists a constant c_1 such that:

$$(2.8) \quad f'(s) \geq b_{2p} |s|^{2(p-1)} - c_1$$

From the mean value formula applied to $f(u) - f(v)$ and (2.8), we thus deduce:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{-1}^2 + \|\nabla \omega\|^2 &\leq c_1 \|\omega\|^2 \\ &\leq c_1 \|\omega\|_{-1} \|\nabla \omega\| \end{aligned}$$

We conclude, using Cauchy-Schwarz inequality and Gronwall Lemma, that:

$$(2.9)_1 \quad \|u(t_1) - v(t_1)\|_{-1}^2 \leq c_2(T') \|u(t_2) - v(t_2)\|_{-1}^2$$

$$(2.9)_2 \quad \int_{t_2}^{t_1} \|\nabla u(s) - \nabla v(s)\|^2 ds \leq C_3(T) \|u(t_2) - v(t_2)\|_{-1}^2$$

where $0 \leq t_2 \leq t_1 \leq T < \infty$.

Next, we need to show that $S(t)$ is a smoothing operator; precisely to prove (2.6) we shall show that $S(t)$ is smoothing from $H^{-1}(\Omega)$ into $H^2(\Omega)$. First of all, for $V(u)$ defined by (1.11), the inequality (1.12) yields:

$$(2.10) \quad \frac{d}{dt}(tV(u)) \leq V(u)$$

Integrating in time for $0 \leq t \leq T$, we obtain

$$(2.11) \quad tV(u) \leq \int_0^T V(u(s)) ds$$

But, in view of (1.22), the right hand side of (2.11) is bounded from above as soon as u_0 belongs to H^{-1} . On the other hand, we recall (see (1.11) and (1.14)) that $V(u)$ is bounded from below:

$$(2.12) \quad V(u) \geq \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} b_{2p} \|u\|_{L^{2p}}^{2p} - c'_1 |\Omega|$$

Thus, in view of the two foregoing observations, we deduce that, for u_0 given in $H^{-1}(\Omega)$

$$(2.13) \quad t^{\frac{1}{2}} u(t) \in L^\infty(0, T; H^1(\Omega)), \quad t^{\frac{1}{2p}} u(t) \in L^\infty(0, T; L^{2p}(\Omega))$$

and the corresponding norms are bounded in terms of ρ , T , $\|u_0 - \bar{u}_0\|_{-1}$.

The above results will be used to prove the appropriate modification of (1.25)₁. Precisely, (2.13) says that there exists a constant r'_0 , depending only on ρ , T , $|\Omega|$ and $\|u_0 - \bar{u}_0\|_{-1}$, such that

$$(2.14) \quad \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|\nabla u(t)\| \leq r'_0.$$

We now proceed as for the proof of Theorem 1.2 but changing the definition (1.37) of X into $t^{\frac{1}{2}} \|\nabla u(t)\|$. In view of (2.14), the appropriate expression of (1.40) reads now

$$(2.15) \quad \|D^2 f(u)\| \leq c_3(1 + t^{-\frac{\beta'}{2}} \|D^4 u\|^\beta + t^{-\frac{\gamma'}{2}} \|D^4 u\|^\gamma)$$

where $\beta' \equiv 4(p-1)(1-\alpha_n) + \frac{2}{3}$, $\gamma' \equiv (4p-6)(1-\alpha_n) + \frac{n}{3}$ and β, γ, α_n are defined as in the proof of Theorem 1.2.

Again assuming the hypothesis of Theorem 1.2 insuring $\beta, \gamma < 2$, we deduce from (2.15) and (1.41), by Cauchy-Young inequality, the equivalent of (1.42). But here the constant c_4 in (1.42) depends on t and we have:

$$(2.16) \quad \frac{d}{dt} \|D^2 u\|^2 + \|D^4 u\|^2 \leq c'_4(1 + t^{\frac{\beta'}{2-\beta}} + t^{-\frac{\gamma'}{2-\gamma}})$$

where the constant c'_4 now depends only on $\rho, T, |\Omega|$ and $\|u_0 - \bar{u}_0\|_{-1}$. As an immediate consequence we obtain boundedness of $t^\delta u(t)$ in $L^\infty(0, T; H^2(\Omega))$ for an appropriate $\delta > 0$. For example, if $n = 3$ $p = 2$ (the most disfavorable case), we have:

$$(2.17) \quad t^{\frac{3}{2}} u(t) \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^4(\Omega))$$

and the corresponding norm is bounded in terms of $\rho, T, |\Omega|, \|u_0 - \bar{u}_0\|_{-1}$.

We are now in position to prove (2.6). Since $n \leq 3$, $H^2(\Omega) \subset L^\infty(\Omega)$ and the last result just implies:

$$(2.18) \quad t^\delta u(t) \in L^\infty(0, T; L^\infty(\Omega))$$

for an appropriate $\delta > 0$. We take the scalar product in $L^2(\Omega)$ of (2.7) with u and obtain:

$$(2.19) \quad \frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \|\Delta \omega\|^2 = \int R(u, v) \omega \Delta \omega \, dx$$

where we introduced

$$(2.20) \quad R \equiv R(u, v) = \int_0^1 f'(su + (1-s)v) \, ds$$

The identity (2.19) easily yields to:

$$(2.21) \quad \frac{d}{dt} \|\omega\|^2 + \|\Delta \omega\|^2 \leq \|R\|_{L^\infty}^2 \|\omega\|^2$$

Due to the definition of R , (1.31) and (2.18):

$$\|R\|_{L^\infty}^2 \leq c_5(1 + \|u\|_{L^\infty}^{4(p-1)} + \|v\|_{L^\infty}^{4(p-1)}) \leq c_6(1 + t^{-4(p-1)\delta})$$

Hence, if $\zeta(t) \equiv t^s, s \geq 4(p-1)\delta$, we deduce from the two inequalities (2.21), (2.22) and Poincaré Lemma:

$$\begin{aligned} \frac{d}{dt} \|\sqrt{\zeta}\omega\|^2 &\leq \left[c_6(1 + \zeta) + \frac{d\zeta}{dt} \right] \|\omega\|^2 \\ &\leq c_7 \left[c_6(1 + \zeta) + \frac{d\zeta}{dt} \right] \|\nabla\omega\|^2 \end{aligned}$$

Finally, we integrate this inequality between t_1 and t_2 , $0 \leq t_2 \leq t_1 \leq T$. The term in brackets is uniformly bounded on $[0, T]$; since $\delta > 0$, $\zeta(0) = 0$ and using (2.9)₂, we obtain (2.6). The proof is complete. $\square$

Remark 2.1. Uniqueness also follows from the argument proving that $S(t)$ is locally Lipschitz in $H^{-1}(\Omega)$. $\square$

Define J_0 as the first positive integer such that:

$$J_0 > c_1 L^{\frac{(p-1)^2}{p}+1} (\overline{\lim}_{t \rightarrow \infty} \sup_{u_0 \in X_\rho^\mu} \|u\|_{L^{2p}})^{4p(p-1)} (\overline{\lim}_{t \rightarrow \infty} \sup_{u_0 \in X_\rho^\mu} \|Du\|)^{2(p-1)}$$

for $u(s) = S(s)u_0$. We have:

THEOREM 2.6. *Let X_ρ (defined in Theorem 2.1) be an attractor associated to (1.1)–(1.4), with $n \leq 3$ and assume that the hypothesis of Theorem 1.3 are satisfied. Define $X_\rho^\mu = \{u \in X_\rho | \bar{u} = \mu\}$ for $|\mu| \leq \rho$. Then the Hausdorff and fractal dimensions $d_H(X_\rho^\mu)$ and $d_F(X_\rho^\mu)$ of X_ρ^μ are finite and:*

$$(2.25) \quad d_H(X_\rho^\mu) \leq J_0 + 1, \quad d_F(X_\rho^\mu) \leq c_0(J_0 + 1),$$

where c_0 is an absolute constant.

Proof. Due to Theorem 2.5, we can use the topology of $H^{-1}(\Omega)$ instead of that of $L^2(\Omega)$. We now define the linearized operator associated to the orbit $u = u(t)$ after application of the operator $(-\Delta)^{-1}$ (compare to equation (2.1)):

$$(2.26) \quad \mathcal{A}'(u) = -\Delta + f'(u)$$

Then, following the framework described in Section 3.A of Part I, we consider $U_1(t), \dots, U_J(t)$ defined as the space periodic solutions of

$$(2.27) \quad \frac{\partial U_j}{\partial t} - \Delta(\mathcal{A}'(u)U_j) = 0 \quad \text{in } \Omega \times \mathbf{R}_+$$

$$(2.28) \quad U_j(x, 0) = \xi_j \quad \text{in } \Omega,$$

where the initial data $\xi_j (1 \leq j \leq J)$ are given in $H^{-1}(\Omega)$ and such that $\bar{\xi}_j = 0$. Notice that $\bar{U}_j = 0$, for $1 \leq j \leq J$. Using (2.27), a direct computation gives the equivalent of (3.7) Part I, that is⁶:

$$(2.29) \quad \frac{d}{dt} \log \|\wedge_{j=1}^J U_j(t)\|_{\wedge^J H^{-1}(\Omega)} + \text{Trace } \mathcal{A}'(u(t)) \circ P_J(t) = 0$$

for $u(t) = S(t)u_0$, $\mathcal{A}'$ defined by (2.26) and $P_J(t)$ as in (3.9) Part I. Defining $\omega_J(u_0, t)$ as in (3.11)₁ Part I, except that the norm is now the norm of $\wedge^J H^{-1}(\Omega)$, we get from (2.29) the equivalent of (3.11)₂ Part I:

$$\bar{\omega}_J(t) \equiv \sup_{u_0 \in X_p} \omega_J(u_0, t) \leq \sup_{u_0 \in X_p} \exp - \int_0^t \inf_P \text{Trace } \mathcal{A}'(u(s)) \circ P \, ds,$$

where P is any projector of rank J . Since J_0 is the first positive integer such that $\bar{\omega}_{J_0}(t) < 1$, J_0 is characterized by the following inequality:

$$(2.30) \quad \liminf_{t \rightarrow \infty} \inf_{u_0 \in X_p} \inf_P \frac{1}{t} \int_0^t \text{Trace } \mathcal{A}'(u(s)) \circ P \, ds > 0$$

We are thus led to estimate $\text{Trace } \mathcal{A}'(u(s)) \circ P$; for any set $\{\phi_j\}_{j=1}^J$ of orthonormal functions in $H^{-1}(\Omega)$, we have, from the definition (2.26) of $\mathcal{A}'(u)$

$$(2.31) \quad \begin{aligned} \text{Trace } \mathcal{A}'(u(s)) \circ P &= \sum_{j=1}^J (\mathcal{A}'(u(s)) \phi_j, \phi_j) \\ &= \sum_{j=1}^J \left\{ \|\nabla \phi_j\|^2 + \int_{\Omega} f'(u) |\phi_j|^2 \, dx \right\} \end{aligned}$$

Introducing the set $\{\psi_j\}_{j=1}^J$ where $(-\Delta)^{\frac{1}{2}} \psi_j = \phi_j$, which provides an orthonormal basis in $L^2(\Omega)$, we obtain from (2.31):

$$(2.32) \quad \text{Trace } \mathcal{A}'(u(s)) \circ P = \sum_{j=1}^J \left\{ \|\nabla (-\Delta)^{\frac{1}{2}} \psi_j\|^2 + \int_{\Omega} f'(u) |(-\Delta)^{\frac{1}{2}} \psi_j|^2 \, dx \right\}$$

If $\{X_j\}_{j=1}^J$ are the orthonormalized eigenfunctions of Δ^2 corresponding to the first J eigenvalues $\lambda_1, \dots, \lambda_J$, there exists an orthogonal $J \times J$ matrix O such that $\sum_{j=1}^J O_{ij} X_j = \psi_i$:

$$(2.33) \quad \sum_{j=1}^J \|\nabla (-\Delta)^{\frac{1}{2}} \psi_j\|^2 = \sum_{j=1}^J (\Delta^2 \psi_j, \psi_j) = \sum_{j=1}^J \lambda_j$$

⁶Notice that the trace is taken in the usual sense, since the operators have a finite rank.

On the other hand, we have:

$$\begin{aligned}
(2.34) \quad \int_{\Omega} f'(u) |(-\Delta)^{\frac{1}{2}} \psi_j|^2 dx &\geq -\|f'(u)\|_{L^\infty} \|(-\Delta)^{\frac{1}{2}} \psi_j\|^2 \\
&\geq -\|f'(u)\|_{L^\infty} \|\psi_j\| \|\nabla(-\Delta)^{\frac{1}{2}} \psi_j\| \\
&\geq -\frac{1}{2} \|f'(u)\|_{L^\infty}^2 - \frac{1}{2} \|\nabla(-\Delta)^{\frac{1}{2}} \psi_j\|^2
\end{aligned}$$

after using Hölder inequality, interpolation and Cauchy-Schwarz inequality. Collecting (2.33), (2.34) into (2.32) we obtain a lower bound for the trace of $\mathcal{A}'(u(s)) \circ P$:

$$(2.35) \quad \text{Trace } \mathcal{A}'(u(s)) \circ P \geq \frac{1}{2} \left(\sum_{j=1}^J \lambda_j - J \|f'(u)\|_{L^\infty}^2 \right)$$

Thus, from (2.30), a sufficient criterion to characterize J_0 is:

$$(2.36) \quad \frac{1}{J_0} \sum_{j=1}^{J_0} \lambda_j - \overline{\lim}_{t \rightarrow \infty} \sup_{u_0 \in X_\rho^\mu} \frac{1}{t} \int_0^t \|f'(u(s))\|_{L^\infty}^2 ds > 0$$

and this given (2.24), (2.25). $\square$

It remains to relate $\frac{1}{t} \int_0^t \|f'(u(s))\|_{L^\infty}^2 ds$, where u is solution of (1.1), (1.2) with $\bar{u}_0 = \mu$, $|\mu| \leq \rho$, with the uniform estimates for the same u . This is an easy consequence of Theorem 1.1 together with (1.31), (1.38) and Hölder inequality. Here, we quote the result in the simplest case $n = 1$, where $\Omega = [-\frac{L}{2}, \frac{L}{2}]$. Let J_0 the first positive integer such that:

$$J_0 > c_1 L^{\frac{(p-1)^2}{p} + 1} \left(\overline{\lim}_{t \rightarrow \infty} \sup_{u_0 \in X_\rho^\mu} \|u\|_{L^{2p}} \right)^{4p(p-1)} \left(\overline{\lim}_{t \rightarrow \infty} \sup_{u_0 \in X_\rho^\mu} \|Du\| \right)^{2(p-1)}$$

where c_1 is an absolute constant.

COROLLARY 2.5. *Under the assumption of Theorem 2.4, then*

$$(2.24) \quad \dim_H(X_\rho^\mu) \leq J_0 + 1, \dim_F(X_\rho^\mu) \leq c_0(J_0 + 1)$$

For example if $n = 1$, $p = 2$, we find $J_0 > c_1 \rho^{\frac{3}{4}} L^{\frac{39}{8}}$.

PART III: INERTIAL MANIFOLD FOR THE CAHN-HILLIARD EQUATION

Here we are concerned with the construction of an inertial manifold for the Cahn-Hilliard equation. This concept has been first introduced in [16b], [17b], (see [16b] for additional references).

1. The framework for inertial manifolds.

We consider the Cahn-Hilliard equation studied in Part II, which we rewrite as an abstract evolution equation

$$(1.1) \quad \frac{du}{dt} + A(Au + f(u)) = 0$$

$$(1.2) \quad u(0) = u_0$$

Here, the above operator A is defined by

$$(1.3) \quad A = -\Delta, \quad D(A) = H^2(\Omega).$$

From now on, Ω will denote the torus $[-\frac{L}{2}, \frac{L}{2}]^n$, and we restrict ourselves to space dimension $n \leq 2$. The reason for this will be explained in Section 5, Remark 5.3. It is strongly related to the so called gap condition for A . The case $n = 3$ could be handled in the same way as in [34] by replacing A by A^m ($m \geq 2$) in (1.1).

Let us finally recall the definition of f :

$$(1.4) \quad f(u) = \sum_{j=1}^{2p-1} a_j u^j$$

where $p \in \mathbb{N}$, $p \geq 2$ and $a_{2p-1} > 0$. The operation A , defined by (1.3) is positive self-adjoint and $\text{Ker} A = \text{coker} A = \mathbb{R}$. The corresponding projection on $\text{Ker} A$ is :

$$(1.5) \quad u \longrightarrow \bar{u} \equiv \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx.$$

It has been noticed in Part II that any solution of (1.1), (1.2) satisfies $\frac{d}{dt} \bar{u}(t) = 0$; as a consequence, there is no absorbing set for the semi group $S(t)$ defined by (1.1), (1.2). in $L^2(\Omega)$.

To circumvent this difficulty, from now on we will make $S(t)$ operate in the space:

$$(1.6) \quad H_{\rho} = \{u \in L^2(\Omega) \mid |\bar{u}| \leq \rho\}$$

where $\rho \geq 0$ is fixed.

For technical purpose, appearing later, we finally introduce the operator Λ depending on the dimension n :

$$\Lambda = \begin{cases} (1 + A)^{\frac{1}{2}}, & D(\Lambda) = D(A^{\frac{1}{2}}) & \text{if } n = 1 \\ (1 + A), & D(\Lambda) = D(A) & \text{if } n = 2 \end{cases}$$

Notice that Λ is a strictly positive self-adjoint operator commuting with A . Moreover, $\|\Lambda \cdot\|$ is a natural norm on $D(\Lambda)$.

PROPOSITION 1.1. *There exists a constant $r > 0$, depending only on ρ and the data f, Ω such that the set B_ρ :*

$$B_\rho = \{u \in H_\rho \cap D(\Lambda) \mid \|\Lambda u\| \leq r\}$$

is absorbing for $S(t)$ acting in H_ρ .

Proof. We remark that $\|\Lambda u\|$ is a norm equivalent to $(\|A^{\frac{1}{2}}u\|^2 + |\bar{u}|^2)^{\frac{1}{2}}$ if $n = 1$ and $(\|Au\|^2 + |\bar{u}|^2)^{\frac{1}{2}}$ if $n = 2$. Hence the result is a consequence Theorem 1.1, 1.2 of Part II if $n = 1, 2$. $\square$

Remark 1.1. The set B_ρ is even invariant by $S(t)$. $\square$

As in [16b], the construction of an inertial manifold amounts to work with a modification of (1.1). Let $\theta : \mathbf{R}^+ \rightarrow [0, 1]$ denote a smooth non increasing function such that $\theta(s) = 1$ if $0 \leq s \leq 1$ and $\theta(s) = 0$ for $s > 2$. For r defined by the Proposition 1.1, we define (compare with [13]), since $f(0) = 0$

$$(1.9) \quad F(u) = f\left(\theta\left(\frac{\|\Lambda u\|}{r}\right)u\right)$$

From now on we shall work with the following evolution equation:

$$(1.10) \quad \frac{du}{dt} + A(Au + F(u)) = 0$$

$$(1.11) \quad u(0) = u_0$$

Due to Proposition 1.1, for t large, (1.10) is equivalent to (1.1). Moreover, $F(u)$ shares nice properties given by the:

LEMMA 1.2. *There exists two absolute constants K_1, K_2 , depending only on r and the data f, Ω such that, for any u, v in $D(\Lambda)$:*

$$(1.12)_1 \quad \|\Lambda F(u) - \Lambda F(v)\| \leq K_1 \|\Lambda u - \Lambda v\|,$$

The proof of this Lemma is given in Appendix. With the aid of it, one easily checks (see the proofs of theorem 1.1–1.3 in Part II) that (1.10)(1.11) admits a unique smooth solution (if $u_0 \in H_\rho$ is smooth) for all $t \geq 0$ and defines a semi-group, still denoted $S(t)$, acting in H_ρ for which B_ρ is invariant and absorbing.

2. Existence of an inertial manifold.

Since $\Omega = [-\frac{L}{2}, \frac{L}{2}]^n$, we choose $\mathcal{W} = \{w_k\}_{k \in \mathbf{Z}^n} = \{\exp i \frac{2\pi}{L} k \cdot x\}_{k \in \mathbf{Z}^n}$ as a Hilbert basis of $H \equiv L^2(\Omega)$. For a given $N \in \mathbf{N}$, we have

$$(2.1) \quad \mathcal{W} = \{W_k\}_{|k| \leq N} \oplus \{w_k\}_{|k| > N+1}$$

and this induces an orthogonal decomposition of $L^2(\Omega)$ and H_ρ . We will denote by P and $Q = I - P$ the corresponding projectors in all what follows. If $u(t)$ is solution of (1.10), (1.11), then $p(t) \equiv Pu(t)$, $q(t) \equiv Qu(t)$ satisfy⁷:

$$\begin{aligned}\frac{dp}{dt} + A^2p + APF(p + q) &= 0 \\ \frac{dq}{dt} + A^2q + AQF(p + q) &= 0\end{aligned}$$

Conversely, we would find a mapping Φ such that $\Phi(p) = q$, $p + \Phi(p)$ is solution of (1.10), (1.11) for p, q solutions of the foregoing equations.

First, for Φ given in a set $\mathcal{F}_{b,\ell}$ defined below, we shall introduce an implicit nonlinear transformation of Φ through the following system of evolution equations:

$$(2.2) \quad \frac{dp}{dt} + A^2p + APF(p + \Phi(p)) = 0$$

$$(2.3) \quad \frac{dq}{dt} + A^2q + APF(p + \Phi(p)) = 0$$

Such a mapping Φ will be sought in the set $\mathcal{F}_{b,\ell}$ parametrized by given $b, \ell > 0$. Precisely, $\Phi \in \mathcal{F}_{b,\ell}$ if and only if:

$$(2.4) \quad \Phi : PH_\rho \rightarrow QD(\Lambda)$$

$$(2.5) \quad \|\Lambda\Phi(p)\| \leq b, \quad \forall p \in PH_\rho,$$

$$(2.6) \quad \|\Lambda\Phi(p_1) - \Lambda\Phi(p_2)\| \leq \ell \|\Lambda p_1 - \Lambda p_2\|, \quad \forall p_1, p_2 \in PH_\rho(\Lambda),$$

$$(2.7) \quad \text{supp } \Phi \subset \{p \in PH_\rho, \|\Lambda p\| \leq 4r + \rho\},$$

where r is the constant appearing in Proposition 1.1 and ρ is introduced in (1.6).

Now, $p(0) = p_0$ given in PH_ρ , (2.2) admits an unique solution $p(t) = p(t; \Phi, p_0)$ defined for $t \in \mathbf{R}$. Also, (2.3) admits an unique solution $q(t)$, bounded as $t \rightarrow -\infty$. In particular, proceeding as in [16b], the value $q(0)$ is given by:

$$(2.8) \quad q(0) = - \int_{-\infty}^0 e^{sA^2Q} AQF(p(s) + \Phi(p(s))) ds \equiv T\Phi(p_0)$$

The right hand side of (2.8) defines the value at p_0 of the image of Φ by a non linear functional operator T and we require as in [16b], that Φ satisfies $q(0) = \Phi(q_0)$:

$$\Phi(p_0) = T\Phi(p_0), \quad \forall p_0 \in PH_\rho.$$

Summarizing, an inertial manifold $\mathcal{M}$ for $S(t)$, acting in H_ρ , will be sought as the graph of a mapping Φ belonging to $\mathcal{F}_{b,\ell}$ and solution of the fixed point equation (2.9). *Our main result states the existence of an inertial manifold for (1.10), (1.11) (or (1.1), (1.2)) with A defined by (1.3) and F by (1.9), (1.4).*

⁷ P and Q commute with A .

THEOREM 2.1. *Let $0 < \ell < 1$. There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$, one can find $b > 0$ such that:*

$$(2.10) \quad T : \mathcal{F}_{b,\ell} \longrightarrow \mathcal{F}_{b,\ell}$$

$$(2.11) \quad T \text{ is a compact strict contraction}$$

$$(2.12) \quad T \text{ admits a fixed point } \Phi \text{ in } \mathcal{F}_{b,\ell} \text{ and } \mathcal{M} = \text{Graph}(\Phi) \text{ is an inertial manifold associated to (1.10), (1.11).}$$

The proof of this result is lengthy. The main difficulty consists in checking (2.10) and (2.11). Then, the fixed point contraction Theorem or the Schauder theorem provides a fixed point $\Phi \in \mathcal{F}_{b,\ell}$ of T . Finally, one has to prove that $\mathcal{M} = \text{graph } \Phi$ is invariant and exponentially attracting for the orbits in H_ρ of (1.10), (1.11). This program is achieved in the next sections.

We close this section by introducing the set $\{\lambda_N\}_{N \in \mathbb{N}}$ of distinct eigenvalues of the operator A , where:

$$(2.13) \quad \lambda_N = \# \{k \in \mathbb{Z}^n \mid |k| = N\} \cdot \left(\frac{2\pi}{L}N\right)^2$$

This defines an increasing sequence of positive numbers and the following inequalities will be useful in the next sections:

$$(2.14) \quad \|AQw\| \geq \lambda_{N+1}\|Qw\|, \quad w \in D(A),$$

$$(2.15) \quad \|APw\| \leq \lambda_N\|Pw\|, \quad w \in D(A).$$

We also remark that (see (1.3) and [1]):

$$(2.16) \quad \lambda_N \sim N^{\frac{2m}{n}} \text{ as } N \rightarrow +\infty.$$

3. Invariance properties of the functional operator T .

We prove (2.10), that is $T\Phi \in \mathcal{F}_{b,\ell}$ if $\Phi \in \mathcal{F}_{b,\ell}$, proceeding by a series of Lemmas. First we establish (2.4) and (2.5).

LEMMA 3.1. *Let $b > 0$. There exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$, then $T\Phi(p)$ belongs to $QD(\Lambda)$ and:*

$$(3.1) \quad \|\Lambda T\Phi(p)\| \leq b, \quad \forall p \in PH_\rho(\Lambda)$$

The constant N_0 depends only on b, ρ and the data f, Ω .

Proof. For every p_0 in PH_ρ , $T\Phi(p_0) = q(0)$ is the right hand side of (2.3) and therefore belongs to the image of Q . Now $\Lambda T\Phi(p_0) = \Lambda q(0)$ and by definition:

$$(3.2) \quad \|\Lambda T\Phi(p_0)\| \leq \int_{-\infty}^0 \|\Lambda e^{sA^2} Q A Q F(u(s))\| ds$$

$$(3.3) \quad \|\Lambda T\Phi(p_0)\| \leq \int_{-\infty}^0 \|AQe^{sA^2Q}\Lambda QF(u(s))\| ds \leq \int_{-\infty}^0 \|AQe^{sA^2Q}\|_{op} \|\Lambda QF(u(s))\| ds$$

As in [16b], one easily check that:

$$(3.4) \quad \int_{-\infty}^0 \|AQe^{sA^2Q}\|_{op} ds \leq \frac{2}{\sqrt{e}} \lambda_{N+1}^{-\frac{1}{2}}$$

where λ_{N+1} is defined in (2.13). From (3.3), (3.4) and (1.12)₂, we finally deduce:

$$\|\Lambda T\Phi(p_0)\| \leq \frac{2}{\sqrt{e}} \lambda_{N+1}^{-\frac{1}{2}} K_2,$$

and so far $b > 0$, N_0 is characterized by $\frac{2}{\sqrt{e}} \lambda_{N_0+1}^{-\frac{1}{2}} K_2 \leq b$. The proof is complete. $\square$

Next, we have the analog of (2.6):

LEMMA 3.2. *Let $0 < \ell < 1$. There exists $N_1 \in \mathbb{N}$ such that if $N \geq N_1$, then for Φ in $\mathcal{F}_{b,\ell}$ one has:*

$$(3.5) \quad \|\Lambda T\Phi(p_1) - \Lambda T\Phi(p_2)\| \leq \ell \|\Lambda p_1 - \Lambda p_2\|, \quad \forall p_1, p_2 \in PH_\rho$$

The constant N_1 depends only on ℓ, ρ and the data f, Ω .

The property (3.5) is a direct consequence of another remarkable one concerning the squeezing of the trajectories (1.10), (1.11) and some cone invariance property. This point will be elaborated in Section 5. (See Theorem 5.2). Next we establish (2.7)

LEMMA 3.3. *For every Φ in $\mathcal{F}_{b,\ell}$ the support of $T\Phi$ satisfies:*

$$(3.6) \quad \text{supp } T\Phi \subset \{p \in PH_\rho, \|\Lambda p\| \leq 4r + \rho\}$$

Proof. We argue by contradiction as in [16b]. We start with p_0 in PH_ρ such that $\|\Lambda p_0\| > 4r + \rho$ and show $T\Phi(p_0) = 0$. If $\|\Lambda p_0\| > 4r + \rho$, then there is a neighborhood $\mathcal{O}$ of 0, such that the corresponding solution of (2.2) satisfies $\|\Lambda p(t)\| > 2r + \rho$ for any t in $\mathcal{O}$. Thus, this implies $\|\Lambda(p(t) + \Phi(p(t)))\| > 2r$, $F(p(t)) + \phi(p(t)) = 0$ and therefore:

$$(3.7) \quad \frac{d}{dt} p(t) + A^2 p(t) = 0, \quad \forall t \in \mathcal{O}.$$

Obviously, we have $\frac{d}{dt} \bar{p}(t) = 0$; by taking the scalar product of (3.7) with $\Lambda^2(p - \bar{p})$, noticing $\overline{\Lambda p} = \Lambda \bar{p}$, we obtain.

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Lambda p - \overline{\Lambda p}\|^2 + \lambda_1^2 \|\Lambda p - \overline{\Lambda p}\|^2 \\ & \leq \frac{1}{2} \frac{d}{dt} \|\Lambda p - \overline{\Lambda p}\|^2 + \|A\Lambda p\|^2 = 0, \quad \forall t \in \mathcal{O} \end{aligned}$$

where λ_1 is defined by (2.13). Consequently, for $t < 0$, we find from Gronwall Lemma:

$$(3.9) \quad \|\Lambda p(0) - \overline{\Lambda p(0)}\| \leq \|\Lambda p(t) - \overline{\Lambda p(t)}\| e^{\lambda^2 t}$$

But $|\overline{\Lambda p(0)}| = |\overline{u(0)}| = |\overline{u_0}| \leq \rho$ since u_0 belongs to H_ρ ; on the other hand,

$$\|\Lambda p(0)\|^2 = \|\Lambda p(0) - \overline{\Lambda p(0)}\|^2 + |\overline{\Lambda p(0)}|^2 > (2r + \rho)^2$$

and therefore $\|\Lambda p(0) - \overline{\Lambda p(0)}\|^2 > (2r + \rho)^2 - \rho^2 > 4r^2$. From this observation and (3.9), we deduce:

$$(3.10) \quad \|\Lambda p(t) - \overline{\Lambda p(t)}\| > 2r \quad \forall t < 0,$$

and, a fortiori, $\|\Lambda(p(t) + \Phi(p(t)))\| > 2r$, hence $F(p(t) + \Phi(p(t))) = 0$ for any $t < 0$. Due to (2.8), we finally obtain

$T\Phi(p_0) = 0$, if $\|\Lambda p_0\| > 4r + \rho$ for any Φ in $\mathcal{F}_{b,\ell}$. The proof is complete. $\square$

4. Existence and properties of a fixed point.

Here we are concerned with the properties (2.11), (2.12) of T . First, let us introduce a distance d on $\mathcal{F}_{b,\ell}$ defined by:

$$(4.1) \quad d(\Phi_1, \Phi_2) = \sup_{p \in PH_\rho} \|\Lambda \Phi_1(p) - \Lambda \Phi_2(p)\|$$

Since $QD(\Lambda)$ is complete in the natural norm of $D(\Lambda)$, $\mathcal{F}_{b,\ell}$ is a complete metric space when equipped with d . Now, T is a strict contraction:

LEMMA 4.1. *Let $0 < \eta < 1$. Then there exists $N_2 \in \mathbb{N}$ such that if $N \geq N_2$, then:*

$$(4.2) \quad d(T\Phi_1, T\Phi_2) \leq \eta d(\Phi_1, \Phi_2), \quad \forall \Phi_1, \Phi_2 \in \mathcal{F}_{b,\ell}$$

The constant N_2 depends only on η, ℓ, r and the data f, Ω .

The proof of (4.2) proceeds essentially the same way as in [16b] and therefore will be omitted. (One could also proceed as for (3.5) using an appropriate set of two differential inequalities as in [17b]).

Next, we have the:

LEMMA 4.2. *Let the conditions of Lemmas 3.1–3.3 be satisfied. There exists $N_3 \in \mathbb{N}$ such that if $N \geq N_3$, then T is a compact mapping in $\mathcal{F}_{b,\ell}$.*

Proof. We show that the image by T of $\mathcal{F}_{b,\ell}$ is compact in $\mathcal{F}_{b,\ell}$ (for the metric d) by using Arzela-Ascoli Theorem. Thus, we consider the set $T\mathcal{F}_{b,\ell} = \{T\Phi, \Phi \in \mathcal{F}_{b,\ell}\}$. This set

is bounded in $\mathcal{F}_{b,\ell}$ (Lemma 3.1). Moreover any element of $T\mathcal{F}_{b,\ell}$ is a continuous mapping with compact support (Lemma 3.2 and 3.3). The family $T\mathcal{F}_{b,\ell}$ is also equicontinuous (Lemma 3.2). Finally $\{T\Phi(p), \Phi \in \mathcal{F}_{b,\ell}\}$ is compact in $QD(\Lambda)$ for any p given in PH_ρ . This last property follows from the observation that $\|A^{\frac{1}{2}}\Lambda T\Phi(p)\|$ is uniformly bounded in Φ and p (this is proved the same way as (3.1)) and from the compactness of $(AQ)^{-\frac{1}{2}}$. The proof is complete. $\square$

We conclude this section by proving (2.12).

Proof of (2.12). We have just seen that T is a compact strict contraction in $\mathcal{F}_{b,\ell}$. Thus, a direct application of either the fixed point contraction Theorem or the Schauder Theorem gives existence of a fixed point Φ in $\mathcal{F}_{b,\ell}$ of the functional operator T . Let $\mathcal{M} = \text{graph } \Phi$ be the corresponding manifold. Of course, $\mathcal{M}$ is a finite-dimensional lipschitz manifold. Since (2.9) holds, proceeding as in [17b], we find the invariance of $\mathcal{M}$:

$$(4.3) \quad S(t)\mathcal{M} \subset \mathcal{M}, \quad \forall t \geq 0.$$

On the other hand we shall prove in the next section (see Theorem 5.1) strong squeezing and cone invariance properties which implies (see Corollary 5.1) the alternative used in [16b]. Therefore, we can rely to the argument of [16b] for proving:

$$(4.4) \quad \text{dist}(S(t)u_0, \mathcal{M}) \rightarrow 0 \text{ as } t \rightarrow +\infty, \quad \forall u_0 \in H_\rho$$

However two differences occur here. First of all, the distance we use is defined in terms of the norm of $D(\Lambda)$:

$$\text{dist}(u, \mathcal{M}) = \inf_{v \in \mathcal{M}} \|\Lambda u - \Lambda v\|$$

Next, we introduce (see (1.5)) for $|\mu| \leq \rho$.

$$(4.6) \quad \mathcal{M}_\mu = \{u = p + \Phi(p) \mid \overline{u} = \mu\}$$

That is a section of $\mathcal{M}$ by functions of mean μ . Then, we first prove, as in [16b], by using Lemma 56.2, Corollary 5.1, the following:

$$(4.7) \quad \text{dist}(S(t)u_0, \mathcal{M}_\mu) \rightarrow 0 \text{ as } t \rightarrow +\infty,$$

for any u_0 in H_ρ such that $\overline{u_0} = \mu$. In (4.7) the rate of decay is exponential. Finally, (4.4) follows from (4.7), since $\mathcal{M} = \cup_{|\mu| \leq \rho} \mathcal{M}_\mu$ and we conclude from (4.3), (4.4) that $\mathcal{M}$ is an inertial manifold for $S(t)$ in H_ρ . $\square$

5. The strong squeezing property and its consequences.

A. We begin with a simple Lemma. Let us consider an interval I of $\mathbb{R}$ and $Y, Z : I \rightarrow \mathbb{R}^+$ two locally integrable functions. We also assume we have a system of two differential inequalities:

$$(5.1) \quad \frac{dy}{dt} + ay + bz \geq 0,$$

$$(5.2) \quad \frac{dz}{dt} - cy + dz \leq 0,$$

satisfied, for $t \in I$, in the set $\mathcal{C}_\gamma$

$$(5.3) \quad \mathcal{C}_\gamma = \{(y, z) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid z \geq \gamma y\}, \quad \gamma > 0.$$

LEMMA 5.1. Assume $a, b, c, d, \gamma > 0$ and moreover:

$$(5.4) \quad d - \gamma^{-1}c > 0, d - a - \gamma b - \gamma^{-1}c > 0$$

Then, every point of the boundary $\Gamma = \{(y, z) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid z = \gamma y\}$ of $\mathcal{C}_\gamma$ is a strict egress point.⁸ Moreover, this boundary attracts at an exponential rate every trajectory of (5.1), (5.2). In particular, if $(y(t_0), z(t_0)) \in \mathcal{C}_\gamma$ for some t_0 , then

$$(5.5) \quad z(t) \leq z(t_0) \exp[-(d - \gamma^{-1}c)(t - t_0)]$$

for any $t \geq t_0$ such that $(y(t), z(t)) \in \mathcal{C}_\gamma$.

Remark 5.1. Both $\mathcal{C}_\gamma$ and its complement $\mathcal{C}'_\gamma$ are cones in $\mathbb{R}^+ \times \mathbb{R}^+$. The strict egress property implies that $\mathcal{C}'_\gamma$ is an invariant cone. $\square$

Proof. First, we prove (5.5). Since (5.2) holds in $\mathcal{C}_\gamma$, we also have:

$$(5.6) \quad \frac{dz}{dt} - c\gamma^{-1}z + dz < 0$$

and therefore we conclude using (5.4) and Gronwall Lemma. Exponential decay follows from (5.5). Next, given a trajectory of (5.1), (5.2), we compute $\frac{d}{dt}(z - \gamma y)$ on Γ . Since $z = \gamma y$ on Γ , we find from (5.1) and (5.2):

$$(5.7) \quad \frac{d}{dt}(z - \gamma y) + [d - a - \gamma b - \gamma^{-1}c]z \leq 0$$

Therefore, due to (5.3), $z - \gamma y$ is strictly decreasing on Γ , which implies that every point of Γ is a strict egress point. $\square$

⁸For the definition of egress points, [20] is relevant.

We will prove a somewhat related result for the system (2.2), (2.3). Precisely, let u_0, v_0 be given in the (absorbing and invariant) set B_ρ defined by (1.8). The corresponding solutions of (1.10) are $u(t) = S(t)u_0$ and $v(t) = S(t)v_0$ and we look at the flow:

$$(5.8) \quad (u_0, v_0) \rightarrow (u(t), v(t)) \text{ and } B_\varrho \times B_\varrho$$

Of course both images of $u(t), v(t)$ by the projectors P, Q satisfy (2.2), (2.3). Our result concerns a property of the subset $\mathcal{S}_\gamma$ of $B_\varrho \times B_\varrho$:

$$(5.9) \quad \mathcal{S}_\gamma = \{(u, v) \in B_\varrho \times B_\varrho \mid \|\Lambda Q(u - v)\| \leq \gamma \|\Lambda P(u - v)\|\}$$

where $\gamma > 0$. Such a property, stated in the next Theorem, is known as the “strong squeezing” property and the cone invariance property (see [13], [19a] for related results).

THEOREM 5.1. *Let $\gamma > 0$. There exists $N_4 \in \mathbb{N}$ such that if $N \geq N_4$, then the set $\mathcal{S}_\gamma$ is invariant and attracting at an exponential rate for the flow defined by (5.8). In particular, if $(v_0, v_0) \notin \mathcal{S}_\gamma$, then:*

$$(5.10) \quad \|\Lambda Q(u(t) - v(t))\| \leq \exp(-Kt) \|\Lambda Q(u_0 - v_0)\|$$

for any $t \geq 0$ such that $(u(t), v(t)) \notin \mathcal{S}_\gamma$. The constants N_4 and $K > 0$ depend only on ϱ, γ and the data f, Ω .

Proof. Since $u(t), v(t)$ are solutions of (1.10) $w(t) \equiv u(t) - v(t)$ satisfies:

$$(5.11) \quad \frac{d}{dt}w + A^2w + A(F(u) - F(v)) = 0.$$

Then, $p(t) \equiv Pw(t)$ and $q(t) \equiv Qw(t)$ are respectively solutions of

$$(5.12) \quad \frac{d}{dt}p + A^2p + AP(F(u) - F(v)) = 0$$

$$(5.13) \quad \frac{d}{dt}q + A^2q + AQ(F(u) - F(v)) = 0.$$

Now, we take the $L^2(\Omega)$ -scalar product of (5.12) (resp. (5.13)) with $\Lambda^2 p$ (resp. $\Lambda^2 q$); since A and Λ commute, we obtain:

$$(5.14) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 + \|A\Lambda p\|^2 + (\Lambda P(F(u) - F(v)), A\Lambda p) = 0$$

$$(5.15) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\|^2 + (\Lambda Q(F(u) - F(v)), A\Lambda q) = 0$$

To bound the non linear term in (5.14), (5.15) we just apply Hölder inequality and inequality (1.12)₁ of Lemma 1.2; since $w = p + q$, we find:

$$(5.15) \quad \left| \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 + \|A\Lambda p\|^2 \right| \leq +K_1 \|\Lambda(p+q)\| \cdot \|A\Lambda p\|$$

$$(5.16) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\|^2 \leq K_1 \|\Lambda(p+q)\| \cdot \|A\Lambda q\|$$

But $\|\Lambda(p+q)\| \leq \|\Lambda p\| + \|\Lambda q\|$; also $A\Lambda p = AP\Lambda w$, $A\Lambda q = AQ\Lambda w$ and we can apply (2.14), (2.15) with Λw in place of u . We can therefore deduce from (5.15):

$$(5.17) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 \geq -\lambda_N^2 \|\Lambda p\|^2 - K_1 \lambda_N (\|\Lambda p\| + \|\Lambda q\|) \|\Lambda p\|$$

or

$$(5.18) \quad \frac{d}{dt} \|\Lambda p\| + \lambda_N^2 \|\Lambda p\| + K_1 \lambda_N (\|\Lambda p\| + \|\Lambda q\|) \geq 0$$

Considering (5.16) in the complementary set $\mathcal{S}'_\gamma$ of $\mathcal{S}_\gamma$ in $\mathcal{B}_e \times \mathcal{B}_e$, where $\|\Lambda q\| > \gamma \|\Lambda p\|$, we obtain:

$$(5.19) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\|^2 \leq -K_1 (\|\Lambda p\| + \|\Lambda q\|) \|A\Lambda q\| < K_1 (1 + \gamma^{-1}) \|\Lambda q\| \|A\Lambda q\|$$

Next, we choose N such that

$$(5.20) \quad \lambda_{N+1} \geq K_1 (1 + \gamma^{-1}),$$

and thus, from (2.14), (5.19), (5.20), we deduce:

$$(5.21) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \lambda_{N+1} (\lambda_{N+1} - K_1 (1 + \gamma^{-1})) \|\Lambda q\|^2 \leq \\ & \leq \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\| (\|A\Lambda q\| - K_1 (1 + \gamma^{-1}) \|\Lambda q\|) \leq 0 \end{aligned}$$

or

$$(5.22) \quad \frac{d}{dt} \|\Lambda q\| + \lambda_{N+1} (\lambda_{N+1} - K_1 (1 + \gamma^{-1})) \|\Lambda q\| \leq 0$$

Finally, we consider both (5.13) and (5.22) in the complementary set $\mathcal{S}'_\gamma$ of $\mathcal{S}_\gamma$ and apply the Lemma 5.1. Here $y(t) = \|\Lambda p(t)\|$, $z(t) = \|\Lambda q(t)\|$ and $\mathcal{S}'_\gamma$ reads $\mathcal{S}_\gamma$; moreover $a = \lambda_N^2 + K_1 \lambda_N$, $b = K_1 \lambda_N$, $c = 0$ and $d = \lambda_{N+1} + 1(\lambda_{N+1} - K_1 (1 + \gamma^{-1}))$ and conditions (5.4) reads:

$$(5.23) \quad \lambda_{N+1}^2 (\lambda_{N+1} - K_1 (1 + \gamma^{-1})) > 0$$

$$(5.24) \quad \lambda_{N+1}^2 - \lambda_N^2 - K_1 [(1 + \gamma^{-1}) \lambda_{N+1} + (1 + \gamma) \lambda_N] > 0$$

Due to (2.16), if $n = 1$, and a result of Richards [45], if $n = 2$, these conditions ((5.23) implies (5.20)) defines N_4 for which they are true as soon as $N \geq N_4$. Thus Lemma 5.1 implies (5.10) with $K = \lambda_{N+1}(\lambda_{N+1} - K_1(1 + \gamma^{-1}))$. Also, every point of the boundary $\{\|\Lambda q\| = \gamma\|\Lambda p\|\}$ of $\mathcal{S}'_\gamma$ is a strict egress point, that is any trajectory of (5.12), (5.13) starting from $\mathcal{S}_\gamma$ cannot leave it, which implies invariance of $\mathcal{S}_\gamma$. The proof is complete. $\square$

Remark 5.2. We could have obtained the same result for equation (1.1), since $\mathcal{B}_\rho$ is also invariant and absorbing for it. $\square$

Remark 5.3. Notice that (5.24) also reads:

$$(5.24) \quad \lambda_{N+1} - \lambda_N \geq \text{Max}(1 + \gamma^{-1}, 1 + \gamma)K_1$$

Such a condition is the “weak gap condition” or the operator A . (see [16b], [34]). It leads to the choice of N_4 , using (2.16) if $n = 1$ and a result of Richard [45], if $n = 2$. $\square$

A useful reformulation of Theorem, particularly to prove that $\mathcal{M} = \text{graph } \Phi$ is an invariant manifold (see Section 4), is the

COROLLARY 5.1. *Under the hypothesis of Theorem 5.1, for a given $t_0 \geq 0$, the following alternative holds, as soon as $N \geq N_4$:*

$$(5.25) \quad u(t_0), v(t_0) \in \mathcal{S}_\gamma \text{ and thus } u(t), v(t) \in \mathcal{S}_\gamma, \quad \forall t \geq t_0$$

$u(t_0), v(t_0) \notin \mathcal{S}_\gamma$ and thus for any $t \geq t_0$ such that $u(t), v(t) \notin \mathcal{S}_\gamma$ we have:

$$(5.26) \quad \|\Lambda(u(t) - v(t))\| \leq (1 + \gamma^{-1}) \exp(-K(t - t_0)) \|\Lambda(u(t_0) - v(t_0))\|$$

Corollary 5.1 enable us to demonstrate the inertial property of the invariant manifold $\mathcal{M}$ —namely $\mathcal{M}$ is attracting any orbit at a uniform exponential rate— by following lines of [16b], [17b]. Also needed is the following Lemma, whose proof is standard, stating a lipschitz continuity property for $S(t)$.

LEMMA 5.2. *Let u_{01}, u_{02} be given in $\mathcal{B}_\rho$ and such that $\bar{u}_{01} = \bar{u}_{02} = \mu$ ($|\mu| \leq \rho$) and let u_1, u_2 by the corresponding solutions of (1.1), (1.2). Then, there exist $K > 0$, depending only on ρ and the data f, Ω such that:*

$$\|u_1(t) - u_2(t)\| \leq \|u_{01} - u_{02}\| e^{kt}$$

for any $t \geq 0$.

Notice that the Lemma is false if $\bar{u}_{01} \neq \bar{u}_{02}$.

B. The same approach as above will allow us to prove Lemma 3.2. For Φ given in $\mathcal{F}_{b,\ell}$ and p_{01}, p_{02} given in PH_ρ , we consider the corresponding solutions of (2.2), (2.3)

$$(5.27) \quad p_i(t) = p(t; \Phi, p_{0i}), \quad p_i(0) = p_{0i}, \quad i = 1, 2,$$

$$(5.28) \quad q_i(t), \lim_{t \rightarrow -\infty} q_i(t), \text{ is bounded,} \quad i = 1, 2$$

We also define, for the same Φ as above

$$(5.29)_1 \quad u_i(t) = p_i(t) + q_i(t), \quad i = 1, 2,$$

$$(5.29)_2 \quad U_i(t) = p_i(t) + \Phi(p_i(t)), \quad i = 1, 2.$$

Lemma 3.2 states that, for N large enough⁹:

$$\|\Lambda q_1(0) - \Lambda q_2(0)\| \leq \ell \|\Lambda p_{01} - \Lambda p_{02}\|.$$

In fact, we will investigate the geometric properties of the trajectories $(u_1(t), u_2(t))$ in $H_\rho \times H_\rho$ and make precise the behavior of $\|\Lambda q_1(t) - \Lambda q_2(t)\|$ for $t \in \mathbb{R}$. Let us introduce (see (5.9)) The set:

$$(5.30) \quad \sum_\ell = \{(u, v) \in H_\rho \times H_\rho \mid \|\Lambda Q(u - v)\| \leq \ell \|\Lambda P(u - v)\|\}$$

where $\ell > 0$.

THEOREM 5.2. *Let $\ell > 0$. There exists $N_5 \in \mathbb{N}$ such that if $N \geq N_5$, then the set $\sum_\ell$ is invariant and attracting at an exponential rate with respect to the trajectories (5.29)₁. In particular if $(u_1(t_0), v_2(t_0)) \notin \sum_\ell$ at t_0 , then:*

$$(5.31) \quad \|\Lambda Q(u_1(t) - u_2(t))\| \leq \exp -K(t - t_0) \|\Lambda Q(u_1(t_0) - u_2(t_0))\|$$

for any $t \geq t_0$ such that $(u_1(t), u_2(t)) \notin \sum_\ell$. The constants N_5 and K depends only on ρ, ℓ and the data f, Ω .

Proof. We proceed as for Theorem 5.1. We set $w(t) \equiv u_1(t) - u_2(t)$ and $p(t) \equiv Pw(t) = p_1(t) - p_2(t)$, $q(t) \equiv Qw(t) = q_1(t) - q_2(t)$, in view of (5.29)₁. Thus ((5.29)₂) $p(t)$, $q(t)$ are solutions of (5.12), (5.13) where u (resp. v) is replaced by U_1 (resp. U_2) and t belongs now to $\mathbb{R}$ (instead of $\mathbb{R}^+$).

Therefore, the two differential inequalities (5.15) and (5.16) holds for $t \in \mathbb{R}$, provided that we replace $\|\Lambda(p + q)\|$ by $\|\Lambda(U_1 - U_2)\|$. Now, since Φ belongs to $\mathcal{F}_{b,\ell}$, we have in view of (5.29)₂:

$$(5.32) \quad \|\Lambda(U_1 - U_2)\| \leq \|\Lambda p\| + \|\Lambda(\Phi(p_1) - \Phi(p_2))\| \leq (1 - \ell) \|\Lambda p\|.$$

⁹We assume $p_{01} \neq p_{02}$, otherwise the result is void.

Collecting (5.32) into (5.15), (5.16)—modified as previously indicated—we find:

$$(5.33) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda p\|^2 + \|A\Lambda p\|(\|A\Lambda p\| + K_1(1 + \ell)\|\Lambda p\|) \geq 0$$

$$(5.34) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\|(\|A\Lambda q\| - K_1(1 + \ell)\|\Lambda p\|) \leq 0$$

Next, we consider (5.33) and (5.34) in the complementary set $\sum'_\ell$ of $\sum_\ell$ in $H_\rho \times H_\rho$, where $\|\Lambda q\| > \ell\|\Lambda p\|$. Thus (5.34) becomes:

$$(5.35) \quad \frac{1}{2} \frac{d}{dt} \|\Lambda q\|^2 + \|A\Lambda q\|(\|A\Lambda q\| - K_1(1 + \ell^{-1})\|\Lambda q\|) \leq 0$$

By choosing N such that

$$(5.36) \quad \lambda_{N+1} \geq K_1(1 + \ell^{-1}),$$

and proceeding as in the proof of Theorem 5.1, we deduce from (5.33), (5.35):

$$(5.37) \quad \frac{d}{dt} \|\Lambda p\| + \lambda_N(\lambda_N + K_1(1 + \ell))\|\Lambda p\| \geq 0$$

$$(5.38) \quad \frac{d}{dt} \|\Lambda q\| + \lambda_{N+1}(\lambda_{N+1} - K_1(1 + \ell^{-1}))\|\Lambda q\| \leq 0,$$

for $t \in \mathbb{R}$.

We are thus in position to apply Lemma 5.1 with $I = \mathbb{R}$. Here, γ is replaced by ℓ , $\sum_\ell$ reads $\mathcal{C}_\ell$, $y(t) = \|\Lambda p(t)\|$, $z(t) = \|\Lambda q(t)\|$, $b = c = 0$ and $a = \lambda_N(\lambda_N + K_1(1 + \ell))$, $d = \lambda_{N+1}(\lambda_{N+1} - K_1(1 + \ell^{-1}))$ and conditions (5.4) reads

$$(5.39) \quad \lambda_{N+1}(\lambda_{N+1} - K(1 + \ell^{-1})) > 0,$$

$$(5.40) \quad \lambda_{N+1}^2 - \lambda_N^2 - K_1[(1 + \ell^{-1})\lambda_{N+1} + (1 + \ell)\lambda_N] > 0.$$

These conditions are the same as (5.23), (5.24) (with ℓ replacing γ) and we conclude as in the proof of Theorem 5.1. Notice that $K = d$. $\square$

As a Corollary, we have the

Proof of Lemma 3.2. It amounts to show there exists $t_0 < 0$ such that for any $t \geq t_0$, we have:

$$(5.41) \quad \|\Lambda q_1(t) - \Lambda q_2(t)\| \leq \ell\|\Lambda p_1(t) - \Lambda p_2(t)\|$$

since then, for $t = 0$, we obtain (3.5) in view of the invariance, the definition (2.8) and (5.27), (5.28). If (3.41) were false, there would exist $t_0 < 0$ such that for any $t \geq t_0$:

$$(5.42) \quad \|\Lambda q_1(t) - \Lambda q_2(t)\| > \ell\|\Lambda p_1(t) - \Lambda p_2(t)\|$$

On the other hand, since $(u_1(t_0), u_2(t_0)) \notin \sum_\ell$, then (5.31) is true as soon $N \geq N_5$ and with (5.42) we find, for $t \geq t_0$:

$$(5.43) \quad \begin{aligned} \ell \|\Lambda p_1(t) - \Lambda p_2(t)\| &< \|\Lambda q_1(t) - \Lambda q_2(t)\| \\ &\leq \exp -K(t - t_0) \|\Lambda q_1(t_0) - \Lambda q_2(t_0)\| \end{aligned}$$

Taking $t = 0$ and keeping in mind (5.28), the limit $t_0 \rightarrow -\infty$ in (5.42) gives $\Lambda p_1(0) = \Lambda p_2(0)$ or $p_1(0) = p_2(0)$ that is $p_{01} = p_{02}$. But $p_{01} \neq p_{02}$, so we reach a contradiction and (5.41) is proved. We complete the proof of lemma 3.2 by taking $N_1 = N_5$. $\square$

We conclude by quoting some detailed estimates of $\dim \mathcal{M}$, which are derived in an Indiana Report to appear by B. Nicolaenko for $\mathcal{M}_\mu, \forall \mu$ such that $|\mu| \leq \rho$:

THEOREM 5.3. *Let $\Omega = [-\frac{1}{2}, +\frac{1}{2}]$ and $\tilde{L} \equiv L/2\pi$; then*

$$\dim \mathcal{M} \geq C(\rho, r, f) \tilde{L}^{6p-4} + 1;$$

Let $\Omega = [-\frac{L}{2}, +\frac{L}{2}]^2$ and restrict the Cahn-Hilliard equation to $p = 2$ (cubic nonlinearity); then $\dim \mathcal{M} \geq C(\rho, r, f) |\Omega|^{10} + 1$; C denotes a generic constant, taking different values for $n = 1, 2$. The above estimates are lengthy and rely on Agmon's and Poincaré's inequalities, rather than using (1.12), and (1.12)₂; the latter do not yield optimal estimates.

APPENDIX

The purpose of this Appendix is to prove Lemma 1.2. Let us recall that, by definition, $f(u) = \sum_{j=1}^{2p-1} a_j u^j$ and:

$$(A.1) \quad F(u) = f\left(\theta\left(\frac{\|\Lambda u\|}{r}\right)u\right) = \sum_{j=1}^{2p-1} a_j \theta\left(\frac{\|\Lambda u\|}{r}\right)^j u^j.$$

Let u_1, u_2 be given in $D(\Lambda)$; we will set for the sequel $\theta_i \equiv \theta(\frac{\|\Lambda u_i\|}{r})$. We begin by the proof of (1.12)₁. We have

$$(A.2) \quad \begin{aligned} \Lambda F(u_1) - \Lambda F(u_2) &= \Lambda f(\theta_1 u_1) - \Lambda f(\theta_2 u_2) \\ &= \Lambda \left(\int_0^1 \frac{d}{d\sigma} f(\sigma \theta_1 u_1 + (1-\sigma) \theta_2 u_2) d\sigma \right) \\ &= \Lambda \left((\theta_1 u_1 - \theta_2 u_2) \int_0^1 f'(\sigma \theta_1 u_1 + (1-\sigma) \theta_2 u_2) d\sigma \right) \end{aligned}$$

For $n \leq 3$, $D(\Lambda)$ endowed with the natural norm is an algebra and thus we deduce from (A.2)

$$(A.3)_1 \quad \|\Lambda F(u_1) - \Lambda F(u_2)\| \leq \|\Lambda(\theta_1 u_1 - \theta_2 u_2)\| \int_0^1 \|\Lambda f'(w_\sigma)\| d\sigma$$

where $w_\sigma \equiv \sigma\theta_1u_1 + (1 - \sigma)\theta_2u_2$ and $0 \leq \sigma \leq 1$. But f' is polynomial of order $2(p - 1)$; using again the algebra property, we find:

$$(A.3_2) \quad \begin{aligned} \|\Lambda f'(w_\sigma)\| &\leq \sum_{j=0}^{2p-1} |b_j| \|\Lambda w_\sigma^j\| \\ &\leq \sum_{j=0}^{2p-1} |b_j| \|\Lambda w_\sigma\|^j \end{aligned}$$

On the other hand, due to the definition of θ_1, θ_2 , we have:

$$(A.4) \quad \begin{aligned} \|\Lambda w_\sigma\| &\leq \sigma\theta_1\|\Lambda u_1\| + (1 - \sigma)\theta_2\|\Lambda u_2\| \\ &\leq \sigma\theta_1 2r + (1 - \sigma)\theta_2 2r \\ &\leq 2r \end{aligned}$$

Combining (A.3)₂ and (A.4), we obtain

$$(A.5) \quad \int_0^1 \|\Lambda f'(w_\sigma)\| d\sigma \leq \sum_{j=0}^{2p-1} |b_j| (2r)^j \equiv C(r)$$

Next, we estimate $\|\Lambda(\theta_1u_1 - \theta_2u_2)\|$ in (A.3)₁. We remark that $\theta_1u_1 - \theta_2u_2 = \frac{1}{2}(\theta_1 - \theta_2)(u_1 + u_2) + \frac{1}{2}(\theta_1 + \theta_2)(u_1 - u_2)$ and thus:

$$(A.6) \quad \|\Lambda(\theta_1u_1 - \theta_2u_2)\| \leq \frac{1}{2}|\theta_1 - \theta_2|(\|\Lambda u_1\| + \|\Lambda u_2\|) + \|\Lambda(u_1 - u_2)\|.$$

It remains to estimate $R \equiv \frac{1}{2}|\theta_1 - \theta_2|(\|\Lambda u_1\| + \|\Lambda u_2\|)$. First of all, we notice that the definition of θ insures:

$$(A.7) \quad |\theta_1 - \theta_2| \leq \frac{2}{r} \|\Lambda(u_1 - u_2)\|.$$

Then, we distinguish four cases for u_1, u_2 in $D(\Lambda)$:

$$1) \quad \|\Lambda u_1\| > 2r, \|\Lambda u_2\| > 2r$$

By definition of θ_i , we have $\theta_1 = \theta_2 = 0$ and therefore $R = 0$

$$2) \quad \|\Lambda u_1\| \leq 2r, \|\Lambda u_2\| \leq 2r$$

Using (A.7), we find:

$$R \leq \frac{2}{r} \|\Lambda(u_1 - u_2)\| 4r = 8 \|\Lambda(u_1 - u_2)\|$$

3) $\|\Lambda u_1\| \leq 2r$, $\|\Lambda u_2\| > 2r$

Observing that $\|\Lambda u_2\| \leq \|\Lambda u_1\| + \|\Lambda(u_1 - u_2)\|$, we estimate R as follows:

$$\begin{aligned} R &\leq \frac{1}{2}|\theta_1 - \theta_2|(2\|\Lambda u_1\| + \|\Lambda(u_1 - u_2)\|) \\ &\leq |\theta_1 - \theta_2|\|\Lambda u_1\| + \frac{1}{2}|\theta_1 - \theta_2|\|\Lambda(u_1 - u_2)\| \\ &\leq \frac{2}{r}\|\Lambda(u_1 - u_2)\|2r + \|\Lambda(u_1 - u_2)\| \\ &= 5\|\Lambda(u_1 - u_2)\|, \end{aligned}$$

where we used (A.7) for the third inequality.

4) $\|\Lambda u_1\| > 2r$, $\|\Lambda u_2\| \leq 2r$

We proceed as in 3).

Summarizing, we found:

$$(A.8) \quad \frac{1}{2}|\theta_1 - \theta_2|(\|\Lambda u_1\| + \|\Lambda u_2\|) \leq 8\|\Lambda(u_1 - u_2)\|.$$

Finally, collecting (A.8), (A.6), (A.5) in (A.3)₁, we deduce (1.12)₁, that is:

$$\|\Lambda F(u_1) - \Lambda F(u_2)\| \leq 9c(r)\|\Lambda(u_1 - u_2)\|$$

The proof of (1.12)₂ is even simpler:

$$\begin{aligned} \Lambda F(u) &= \Lambda f(\theta u) \\ &= \Lambda\left(\sum_{j=1}^{2p-1} a_j \theta^j u^j\right), \end{aligned}$$

where $\theta \equiv \theta(\frac{\|\Lambda u\|}{r})$. Thus, using the algebra property of $D(\Lambda)$ and the fact that $\|\Lambda(\theta u)\| = \theta\|\Lambda u\| \leq 2r$, we obtain:

$$\begin{aligned} \|\Lambda F(u)\| &\leq \sum_{j=1}^{2p-1} |a_j| \theta^j \|\Lambda u^j\| \\ &\leq \sum_{j=1}^{2p-1} |a_j| \theta^j \|\Lambda u\|^j \\ &\leq \sum_{j=1}^{2p-1} |a_j| (2r)^j \end{aligned}$$

The proof is complete. $\square$

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