THE KAM THEORY OF SYSTEMS WITH SHORT RANGE INTERACTIONS, II

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Abstract

The proof of the results on the KAM theory of systems with short range interactions, stated in [4] is completed. Estimates on the decay of the interactions generated by the iterative procedure in the KAM theorem are proved, as well as the modification of the theorems of [1-3] needed for our results.

1. Introduction

In a previous paper [4], hereafter referred to as I, we presented results on a KAM theory of systems with short range interactions. The proofs of those theorems are completed here. In referring to results from I, we shall precede the equation or theorem number by I, e.g., (I.11) refers to equation (1.1) of I, and Theorem I.1.1 to Theorem 1.1 of the same work. For a general introduction to the problem, and references to previous work in the literature, the reader should see I.

2. Decay Estimates

A sequence of lemmas is proved which we combine in section three to prove Proposition I.5.1. In section 2.A we examine derivatives of the "high frequency" part of the interactions, $f^{k[\geqslant]}$. In section 2.B we study the decay of second derivatives of the functions $\underline{z}(\underline{I}', \underline{z}')$, and in section 2.C we look at derivatives of $\underline{\Xi}(\underline{I}', \underline{z}')$.

2.A

Lemma 2.1. On $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$ one has

$$\sup \left| \frac{\partial^2 f^{k[\geqslant]}}{\partial\phi_i \partial\phi_j} (\underline{I}, \underline{z}) \right| \leqslant 2^4 \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{\beta_k (3/2)^k} (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} , \quad (2.1a)$$

$$\sup \left| \frac{\partial^2 f^{k[\geqslant]}}{\partial\phi_i \partial I_j} (\underline{I}, \underline{z}) \right| \leqslant D_9 \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} , \quad (2.1b)$$

$$\sup \left| \frac{\partial^2 f^{k[\geqslant]}}{\partial I_i \partial I_j} (\underline{I}, \underline{z}) \right| \leqslant D_{10} \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} , \quad (2.1c)$$

$$\sup \left| \frac{\partial f^{k[\geqslant]}}{\partial I_j} (\underline{I}, \underline{z}) \right| \leqslant \epsilon_{k+1} , \quad (2.1d)$$

$$\sup \left| \frac{\partial f^{k[\geqslant]}}{\partial\phi_i} (\underline{I}, \underline{z}) \right| \leqslant \epsilon_{k+1} \rho_{k+1} , \quad (2.1e)$$

with $D_9 = 2^3 (2\rho_k^{-1} \rho_0)^{\beta_k} \left[(2^4 \epsilon_0 \rho_0^{-1})^{\beta_k L_k} + e^{-(\delta - 2 \ln 2) M_k} \right]$

and $D_{10} = 2^3 (2\rho_k^{-1} \rho_0)^{2\beta_k} \left[(2^4 \epsilon_0 \rho_0^{-1})^{\beta_k L_k} + e^{-(\delta - 2 \ln 2) M_k} \right]$.

Proof: The proof is similar to Lemma I.5.2. By the definition of $f^{k[\geqslant]}$,

$$\sup \left| \frac{\partial f^{k[\geqslant]}}{\partial \phi_i}(\underline{I}, \underline{z}) \right| \leqslant \sup \sum_{\underline{\nu} \notin \mathfrak{X}_k} \left| f^k_{\underline{\nu}}(\underline{I})(i\nu_i)\, \underline{z}^{\underline{\nu}} \right|$$

$$\leqslant \sum_{L=L_k+1}^{N} \sum_{M=1}^{\infty} 2^{2M} 2^L N (\epsilon_0 \rho_0)(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)L} e^{-\delta M} \tag{2.2}$$

$$+ \sum_{L=1}^{L_k} \sum_{M=M_k+1}^{\infty} 2^{2M} 2^L N \epsilon_k \rho_k e^{-\delta M} .$$

The last inequality estimated the number of vectors $\underline{\nu}$ with $|\underline{\nu}| = M$ and $d(\operatorname{supp} \underline{\nu}) = L$ by $2^L 2^{2M} N$ and used (I.2.9a) and (I.2.6) to estimate $f^k_{\underline{\nu}}(\underline{I})$ in the first and second terms of (2.2) respectively. Summing the geometric series in (2.2) and using (I.3.31) bound the r.h.s. of (2.2) by $\epsilon_{k+1} \rho_{k+1}$, proving (2.1e).

$$\sup \left| \frac{\partial f^{k[\geqslant]}}{\partial I_j}(\underline{I}, \underline{z}) \right| \leqslant \sup \sum_{\underline{\nu} \in \mathfrak{X}_k} \left| \frac{\partial f^k_{\underline{\nu}}}{\partial I_j}(\underline{I})\, \underline{z}^{\underline{\nu}} \right|$$

$$\leqslant 2 \sum_{L=L_k+1}^{N} \sum_{M=1}^{\infty} 2^{2M} 2^L N \epsilon_0 \rho_0 \rho_k^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)L} e^{-\delta M} \tag{2.3}$$

$$+ 2 \sum_{L=1}^{L_k} \sum_{M=M_k+1}^{\infty} 2^{2M} 2^L N \epsilon_k e^{-\delta M}$$

$$\leqslant \epsilon_{k+1} ,$$

where we combined (I.2.9a) and (I.2.6) with a dimensional estimate to bound the derivative with respect to I_j, then summed the geometric series and used (I.3.31).

$$\sup \left| \frac{\partial^2 f^{k}[\geqslant]}{\partial\phi_i \partial\phi_j} (\underline{I}, \underline{z}) \right| \leqslant \sup \sum_{\underline{\nu} \in \mathfrak{X}_k} \left| f^k_{\underline{\nu}}(\underline{I})(-\nu_i \nu_j) \underline{z}^{\underline{\nu}} \right|$$

$$\leqslant \sum_{L=|i-j|}^{N} \sum_{M=1}^{\infty} \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)L} M^2 N 2^L 2^{2M} e^{-\delta M} \qquad (2.4)$$

$$\leqslant 2\epsilon_0 \rho_0 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} ,$$

where (I.2.9a) and the fact that every nonzero term on the r.h.s. of the first inequality has $i, j \in \operatorname{supp} \underline{\nu}$. Inequality (2.1a) follows by combining (2.4) with (2.1e) and a dimensional estimate

$$\sup \left| \frac{\partial^2 f^{k}[\geqslant]}{\partial\phi_i \partial I_j} (\underline{I}, \underline{z}) \right| \leqslant \sup \sum_{\underline{\nu} \in \mathfrak{X}_k} \left| \frac{\partial f^k_{\underline{\nu}}}{\partial I_j} (\underline{I})(i\nu_i) \underline{z}^{\underline{\nu}} \right|$$

$$\leqslant \sum_{L=L_k+1}^{N} \sum_{M=1}^{\infty} 2^L 2^{2M} N \cdot M \cdot \epsilon_0 (2\rho_0 \rho_k^{-1})^{\beta_k} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)\beta_k L} e^{-\delta M}$$
$$\times (\epsilon_0 \rho_0^{-1})^{(1-\beta_k)(1-\eta_k)|i-j|} \qquad (2.5)$$

$$+ \sum_{L=1}^{L_k} \sum_{M=M_k+1}^{\infty} 2^L 2^{2M} \cdot N \cdot M \cdot \epsilon_0 (2\rho_0 \rho_k^{-1})^{\beta_k} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)\beta_k L} e^{-\delta M}$$
$$\times (\epsilon_0 \rho_0^{-1})^{(1-\beta_k)(1-\eta_k)|i-j|} ,$$

$$\leqslant 2^3 \epsilon_0 (2\rho_k^{-1} \rho_0)^{\beta_k} (\epsilon_0 \rho_0^{-1})^{(1-\beta_k)(1-\eta_k)|i-j|}$$
$$\times \left\{ (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)\beta_k L_k} + e^{-(\delta - 2 \ln 2) M_k} \right\} .$$

In the second line we combined the estimates on $\frac{\partial f^k_{\underline{\nu}}}{\partial I_j}$ from (I.2.9a) and (I.2.9b) with the observation that all nonvanishing terms on the r.h.s. of the first inequality have $i \in \operatorname{supp} \nu$. The last expression results from summing the geometrical series under the assumption that $2(\epsilon_0\rho_0^{-1})^{(1-\eta_k)\beta_k} < 2^{-1}$, which follows from (I.1.15). This proves (2.1b).

Finally,

$$\sup\left|\frac{\partial^2 f^{k[>]}}{\partial I_i \partial I_j}(\underline{I}, \underline{z})\right| \leqslant \sum_{L=L_k+1}^{N} \sum_{M=1}^{\infty} 2^{2M} 2^{L} N(\epsilon_0\rho_0^{-1})(2\rho_k^{-1}\rho_0)^{2\beta_k}$$

$$\times(\epsilon_0\rho_0^{-1})^{(1-\eta_k)\beta_k L}(\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} e^{-\delta M}$$

$$+\sum_{L=1}^{\infty} \sum_{M=M_k+1}^{\infty} 2^{2M} 2^{L} N(\epsilon_0\rho_0^{-1})(2\rho_k^{-1}\rho_0)^{2\beta}(\epsilon_0\rho_0^{-1})^{(1-\eta_k)\beta_k L} e^{-\delta M}$$

$$\times(\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \tag{2.6}$$

$$\leqslant 2^3(\epsilon_0\rho_0^{-1})(2\rho_k^{-1}\rho_0)^{2\beta_k}(\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$$

$$\times\left\{(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)\beta_k L_k} + e^{-(\delta-2\ln 2M_k)}\right\},$$

where we again combined (I.2.6a) and (I.2.6b) with dimensional estimates to control the factors of $\frac{\partial^2 f^k_{\underline{\nu}}}{\partial I_i \partial I_j}(\underline{I})$, appearing in $\frac{\partial^2 f^{k[\geqslant]}}{\partial I_i \partial I_j}$.

<u>2.B</u>

<u>Lemma 2.2</u>: On $W(\tilde{\rho}_k/8, \xi_k - 3\delta, V_k)$

$$\sup\left|z_i^{-1}\frac{\partial z_i}{\partial I'_j}(\underline{I}', \underline{z}')\right| \leqslant D_{11}\rho_0^{-1}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.7a}$$

$$\sup\left|z_n^{-1} z'_j \frac{\partial^2 z_n}{\partial I'_i \partial z'_j}(\underline{I}', \underline{z}')\right| \leqslant D_{12}\rho_0^{-1}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.7b}$$

$$\sup\left|z_n^{-1}\frac{\partial^2 z_n}{\partial I'_i \partial I'_j}(\underline{I}', \underline{z}')\right| \leqslant D_{13}\rho_0^{-2}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.7c}$$

where $D_{11} = 2^{10} N^4 e^{3L_k} \epsilon_0 \rho_0 C^2 \left[(\epsilon_0\rho_0^{-1})^{(3/2)^k - 1} \rho_k^{-1} E_k\right]^{\beta_k}$,

$$D_{12} = \left[2^5(2+D_2)D_{11} + (1+D_2)(2^4 D_{11} N + 2^4 \rho_0 \tilde{\rho}_k^{-1}) + 2^9 \tilde{\rho}_k^{-1} \rho_0 \epsilon_0 E_k C^2 D_{11} N^3 e^{3L_k}\right]$$

and $$D_{13} = 2^4 \Big[2(2^2+D_2)^2 N^2 D_{11}^2 + (2^2+D_2) 2^2 D_{11} \tilde{\rho}_k^{-1} \rho_0$$

$$+ 2^3 \epsilon_0 C \tilde{\rho}_k^{-1} \rho_0 D_{11}^2 N^3 e^{2L_k} \left[\rho_k \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(3/2)^k - 1}\right]^{\beta_k}\Big] .$$

Proof: By (I.3.4)

$$z_i^{-1} \frac{\partial z_i}{\partial I_j'} (\underline{I}', \underline{z}') = z_i^{-1} \frac{\partial}{\partial I_j'} \left(z_i' \exp\left[-i \frac{\partial \Phi^k}{\partial I_i'} (\underline{I}', \underline{z})\right]\right) , \qquad (2.8)$$

on $W(\tilde{\rho}_k/4, \xi_k - 3\delta, V_k)$, where we evaluate the second argument of $\frac{\partial \Phi^k}{\partial I_i'}$ at $\underline{z}(\underline{I}', \underline{z}') = \underline{z}' \exp\left(i\underline{\Delta}(\underline{I}', \underline{z}')\right)$. Applying the chain rule, the l.h.s. of (2.8) becomes

$$-i \frac{\partial^2 \Phi^k}{\partial I_j' \partial I_i'} (\underline{I}', \underline{z}' e^{i\underline{\Delta}}) - i \sum_{m=1}^{N} \frac{\partial^2 \Phi^k}{\partial I_i' \partial \phi_m} (\underline{I}', \underline{z}' e^{i\underline{\Delta}}) \left(-i z_m^{-1} \frac{\partial z_m}{\partial I_j'} (\underline{I}', \underline{z}')\right) . \qquad (2.9)$$

Define vectors $\underline{u}$ and $\underline{v}$ with components

$$u_n = - \frac{\partial^2 \Phi^k}{\partial I_j' \partial I_n'} (\underline{I}', \underline{z}' e^{i\underline{\Delta}}) \quad \text{and} \quad v_n = z_n^{-1} \frac{\partial z_n}{\partial I_j'} (\underline{I}', \underline{z}') ,$$

and a matrix $\hat{M}$ with elements $\hat{M}_{ij} = - \frac{\partial^2 \Phi^k}{\partial I_i' \partial \phi_j} (\underline{I}', \underline{z}' e^{i\underline{\Delta}})$. Then (2.8) and (2.9) may be rewritten as

$$(\mathbb{1} - \hat{M})\underline{v} = \underline{u} , \qquad (2.10)$$

so that

$$z_i^{-1} \frac{\partial z_i}{\partial I_j'} (\underline{I}', \underline{z}') = v_i = \sum_{m=1}^{N} \left[(\mathbb{1} - M)^{-1}\right]_{im} u_m , \qquad (2.11)$$

if the r.h.s. of (2.11) exists. By Lemma I.4.4,

$$(\mathbb{1} - \hat{M})^{-1}_{mn} = \sum_{\Omega : m \to n} \Lambda^{-1}_{mn} \left(\prod_{\ell=1}^{N} \Lambda_{\ell\ell}^{-n(\ell,\Omega)} \right) \left(\prod_{s \in \Omega} M_s \right) , \qquad (2.12)$$

where $\Lambda_{\ell\ell} = 1 + \frac{\partial^2 \Phi^k}{\partial \phi_\ell \partial I'_\ell} (\underline{I}', \underline{z}' e^{i\underline{\Delta}})$ and $M_s = M_{(i,j)} = (\delta_{ij} - 1) \frac{\partial^2 \Phi^k}{\partial \phi_j \partial I'_i} (\underline{I}', \underline{z}' e^{i\underline{\Delta}})$, with (i, j) the endpoints of the steps. Expression (2.12) was estimated in Lemma I.4.3 where we found

$$\sup \left| (\mathbb{1} - \hat{M})^{-1}_{mn} \right| \leqslant 2^3 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k) |m-n|} , \qquad (2.13)$$

and the supremum runs over $W(\tilde{\rho}_k/4, \xi_k - 3\delta, V_k)$. Combining (I.3.9) with a dimensional estimate on $W(\tilde{\rho}_k/4, \xi_k - 2\delta, V_k)$ bounds u_n by

$$\sup |u_n| \leqslant 2^7 \epsilon_k E_k \tilde{\rho}_k^{-1} C^2 N e^{3L_k} . \qquad (2.14)$$

On the other hand

$$\begin{aligned}
\sup \left| \frac{\partial^2 \Phi^k}{\partial I'_j \partial I'_n} (\underline{I}', \underline{z}' e^{i\underline{\Delta}}) \right| = \sup \Bigg| \sum_{\substack{\underline{\nu} \in \mathfrak{X}_k \\ \underline{\nu} \neq \underline{0}}} \Bigg\{ & \frac{\partial^2 f^k_{\underline{\nu}}}{\partial I'_n \partial I'_j} (\underline{I}') \left(i < \underline{\omega}^k(\underline{I}), \underline{\nu} > \right)^{-1} \\
& - \frac{\partial f^k_{\underline{\nu}}}{\partial I'_j} (\underline{I}') \left(i \sum_{\ell=1}^{N} \frac{\partial^2 h^k}{\partial I'_i \partial I'_j} (\underline{I}') \nu_\ell \right) \left(i < \underline{\omega}^k(\underline{I}'), \underline{\nu} > \right)^{-2} \\
& - \frac{\partial f^k_{\underline{\nu}}}{\partial I'_j} (\underline{I}') \left(i \sum_{\ell=1}^{N} \frac{\partial^2 h^k}{\partial I'_\ell \partial I'_n} (\underline{I}') \nu_n \right) \left(i < \underline{\omega}^k(\underline{I}'), \underline{\nu} > \right)^{-2} \qquad (2.15) \\
& - f^k_{\underline{\nu}}(\underline{I}') \left(i \sum_{\ell=1}^{N} \frac{\partial^2 h^k}{\partial I'_\ell \partial I'_n} (\underline{I}') \nu_\ell \right) \left(i \sum_{m=1}^{N} \frac{\partial^2 h^k}{\partial I'_m \partial I'_j} (\underline{I}') \nu_m \right) \left(i < \underline{\omega}^k(\underline{I}), \underline{\nu} > \right)^{-3} \\
& - f^k_{\underline{\nu}}(\underline{I}') \left(i \sum_{\ell=1}^{N} \frac{\partial^3 h^k}{\partial I'_m \partial I'_n \partial I'_j} (\underline{I}') \nu_\ell \right) \left(i < \underline{\omega}^k(\underline{I}'), \underline{\nu} > \right)^{-2} \Bigg\} \underline{z}^{\underline{\nu}} \Bigg| .
\end{aligned}$$

Using (I.2.6), (I.2.8) and (I.2.9) we can extract a factor of $(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|}$ from each of these terms. If we then use our standard estimate to bound the sum over ν, we find, after a little algebra and using (I.1.15),

$$\sup\left|\frac{\partial^2\tilde{\Phi}^k}{\partial I'_j\partial I'_n}(\underline{I}', \underline{z}e^{i\underline{\Delta}})\right| \leqslant 2^5\epsilon_0 C^2 N^3 e^{3L_k}(\epsilon_0\rho_0^{-1})^{(1-\eta_k)|i-j|} , \qquad (2.16)$$

for $(\underline{I}', \underline{z}') \in W(\tilde{\rho}_k/4, \xi_k - 3\delta, V_k)$. Combining (2.14) and (2.16) gives

$$\sup|u_n| = \sup\left|\frac{\partial^2\tilde{\Phi}^k}{\partial I'_j\partial I'_n}\right| \leqslant 2^{-3}N^{-1}D_{11}\rho_0^{-1}(\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|j-n|} , \qquad (2.17)$$

where $D_{11} = 2^{10}N^4 e^{3L_k}(\epsilon_0\rho_0 C^2)\left((\epsilon_0\rho_0^{-1})^{(3/2)^k-1}\tilde{\rho}_k^{-1}E_k\right)^{\beta_k}$. Using (2.13) and (2.17) to extract a factor of $D_{11}N^{-1}\rho_0^{-1}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ from each of the N terms in (2.11) gives (2.7a).

Applying the chain rule to $z_n(I', z') = z'_n \exp\left(i\underline{\Delta}(\underline{I}', \underline{z}')\right)$, and using the fact that $\underline{\Delta}(\underline{I}', \underline{z}') = -\frac{\partial\tilde{\Phi}^k}{\partial \underline{I}'}(\underline{I}', \underline{z}'e^{i\underline{\Delta}})$ on $W(\tilde{\rho}_k/2, \xi_k - 3\delta, V_k)$ we find

$$z_n^{-1} z'_j \frac{\partial^2 z_n}{\partial I'_i \partial z'_j}(\underline{I}', \underline{z}') = \tilde{u}_n - \sum_{\ell=1}^{N} \frac{\partial^2\tilde{\Phi}^k}{\partial I'_n\partial\phi_\ell}(\underline{I}', \underline{z}'e^{i\underline{\Delta}})\left(z_\ell^{-1} z'_j \frac{\partial^2 z_\ell}{\partial I'_i\partial z'_j}(\underline{I}', \underline{z}')\right)$$

with $\tilde{\underline{u}}$ the vector whose components are

$$\begin{aligned}\tilde{u}_n = {} & -i\delta_{nj}\frac{\partial^2\tilde{\Phi}^k}{\partial I'_j\partial I'_i} - \delta_{nj}\sum_{\ell=1}^{N}\frac{\partial^2\tilde{\Phi}^k}{\partial I'_j\partial\phi_\ell}\cdot\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_i}\right) - \sum_{\ell=1}^{N}\frac{\partial^3\tilde{\Phi}^k}{\partial I'_n\partial I'_i\partial\phi_\ell}\cdot\frac{\partial\phi_\ell}{\partial\phi'_j} \\ & + \left(\sum_{\ell=1}^{N}\frac{\partial^2\tilde{\Phi}^k}{\partial I'_n\partial\phi_\ell}\cdot\frac{\partial\phi_\ell}{\partial\phi'_j}\right)\left(i\frac{\partial^2\tilde{\Phi}^k}{\partial I'_n\partial I'_i} + \sum_{\ell=1}^{N}\frac{\partial^2\tilde{\Phi}^k}{\partial\phi_m\partial I'_n}\left(z_m^{-1}\frac{\partial z_m}{\partial I'_i}\right)\right) \\ & + i\sum_{\ell,m=1}^{N}\left(\frac{\partial^3\tilde{\Phi}^k}{\partial\phi_\ell\partial\phi_m\partial I'_n}\right)\left(\frac{\partial\phi_\ell}{\partial\phi'_j}\right)\left(z_m^{-1}\frac{\partial z_m}{\partial I'_i}\right) + \sum_{\ell=1}^{N}\left(\frac{\partial^2\tilde{\Phi}^k}{\partial\phi_\ell\partial I'_n}\right)\left(\frac{\partial\phi_\ell}{\partial\phi'_j}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_i}\right).\end{aligned} \qquad (2.19)$$

If we let $\hat{M}$ be the matrix defined following (2.9) and $\tilde{\underline{v}}$ the vector with components $\tilde{v}_n = z_n^{-1} z'_j \frac{\partial^2 z_n}{\partial I'_i\partial z'_j}(\underline{I}', \underline{z}')$, (2.18) can be rewritten as

$$(\mathbb{1} - \hat{M})\tilde{\underline{v}} = \tilde{\underline{u}} \tag{2.20}$$

or

$$\tilde{v}_n = \sum_{m=1}^{N} (\mathbb{1} - \hat{M})^{-1}_{nm} \tilde{u}_m \tag{2.21}$$

By combining (2.16), (2.7a), (I.5.24), and Lemmas I.4.1 and I.4.4 with dimensional estimates on $W(\tilde{\rho}_k/4, \xi_k-3\ell, V_k)$ we can once again pull a factor of $(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ from each term in the expression for $\tilde{u}_n$. Combining these bounds gives

$$\begin{aligned} \sup|\tilde{u}_n| \leqslant \Big[& 2^6\tilde{\rho}_k^{-1}E_k C^2 N^3 e^{3L_k}\epsilon_0 2^3 D_{11} + 2^4(2+D_2)D_{11}\rho_0^{-1} + 2^3 D_{11}\rho_0^{-1} \\ & + (1+D_2)(2^4\tilde{\rho}_k^{-1} + 2^4 N D_{11}\rho_0^{-1})\Big](2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \end{aligned} \tag{2.22}$$

By (2.13) and (I.1.15) we have

$$\sup \sum_n \left| (\mathbb{1} - \hat{M})^{-1}_{mn} \right| \leqslant 2^3 + 2^4(2^4\epsilon_0\rho_0^{-1})^{1/8} \leqslant 2^4 , \tag{2.23}$$

for all $m = 1,\ldots,N$ which when combined with (2.21) and (2.22) yields (2.7b). (Once again, the suprema in (2.21) -(2.23) run over $W(\tilde{\rho}_k/4, \xi_k-3\delta, V_k)$.)

Finally, use the chain rule to write

$$z_n^{-1} \frac{\partial^2 z_n}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}') = \hat{u}_n - \sum_{\ell=1}^{N} \frac{\partial^2 \Phi^k}{\partial\phi_\ell \partial I_n'} \cdot \left(z_\ell^{-1} \frac{\partial^2 z_\ell}{\partial I_i' \partial I_j'} \right) , \tag{2.24}$$

where $\hat{\underline{u}}$ is the vector with components

$$\hat{u}_n = \left(-i\frac{\partial^2\Phi^k}{\partial I'_n \partial I'_j} - \sum_{\ell=1}^{N}\left(\frac{\partial^2\Phi^k}{\partial\phi_\ell \partial I'_n}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_j}\right)\right)\left(-i\frac{\partial^2\Phi^k}{\partial I'_n \partial I'_i} - \sum_{\ell=1}^{N}\left(\frac{\partial^2\Phi^k}{\partial\phi_\ell \partial I'_n}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_i}\right)\right)$$

$$- \sum_{\ell=1}^{N}\left(\frac{\partial^3\Phi^k}{\partial\phi_\ell \partial I'_n \partial I'_j}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_i}\right) - \sum_{\ell=1}^{N}\left(\frac{\partial^3\Phi^k}{\partial\phi_\ell \partial I'_n \partial I'_i}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_j}\right) - i\left(\frac{\partial^3\Phi^k}{\partial I'_n \partial I'_i \partial I'_j}\right)$$

$$- \sum_{\ell, m=1}^{N}\left(\frac{\partial^3\Phi^k}{\partial\phi_m \partial\phi_\ell \partial I'_n}\right)\left(z_m^{-1}\frac{\partial z_m}{\partial I'_i}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_j}\right) \tag{2.25}$$

$$+ i\sum_{\ell=1}^{N}\left(\frac{\partial^2\Phi^k}{\partial\phi_\ell \partial I'_n}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_i}\right)\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I'_j}\right) .$$

The arguments of the functions in (2.25) have been omitted to save space. Defining the matrix $\hat{M}$ as we did following (2.9), and the vector $\hat{\underline{v}}$ by $\hat{v}_n = z_n^{-1}\frac{\partial^2 z_n}{\partial I'_i \partial I'_j}(\underline{I}', \underline{z}')$ we can rewrite (2.24) as

$$(\mathbb{1} - \hat{M})\hat{\underline{v}} = \hat{\underline{u}} \tag{2.26}$$

so that

$$\hat{v}_n = \sum_{\ell=1}^{N}\left[(\mathbb{1} - \hat{M})^{-1}\right]_{n\ell}\hat{u}_\ell . \tag{2.27}$$

From $\hat{u}_\ell$ we extract a factor of $(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ by combining (2.17), (2.7a), (I.4.1), (I.4.30), and dimensional estimates on $W(\tilde{\rho}_k/8, \xi_k - 3\delta, V_k)$. These estimate give

$$\sup|\hat{u}_\ell| \leqslant \Big\{(2^2 + D_2)^2 N^2 D_{11}^2 \rho_0^{-2} + (2^2 + D_2) 2^2 D_{11} \tilde{\rho}_k^{-1} \rho_0^{-1}$$

$$+ (2 + D_2) N D_{11}^2 \rho_0^{-1} + 2^3 \epsilon_0 C N^3 D_{11}^2 \tilde{\rho}_k^{-1} \rho_0^{-1} e^{2L_k} \tag{2.28}$$

$$\times\left[\rho_k \rho_0^{-1}(\epsilon_0\rho_0^{-1})^{(3/2)^k - 1}\right]^{\beta_k}\Big\}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} .$$

Combining (2.28) with (2.23) and (2.27) yields (2.7c).

Lemma 2.3. On $W(\rho_{k+1}, \xi_{k+1}, V_k)$

$$\sup\left|\frac{\partial \Xi_m}{\partial I_i'}(\underline{I}', \underline{z}')\right| \leqslant D_{14}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-m|}, \tag{2.29a}$$

$$\sup\left|\frac{\partial^2 \Xi_m}{\partial\phi_i'\partial I_j'}(\underline{I}', \underline{z}')\right| \leqslant D_{15}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.29b}$$

$$\sup\left|\frac{\partial^2 \Xi_m}{\partial I_i'\partial I_j'}(\underline{I}', \underline{z}')\right| \leqslant \rho_0^{-1} D_{16}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.29c}$$

where $D_{14} = \left[D_2 + 2^2\epsilon_0 CN^2 e^{2L_k}\left[(\rho_k\rho_0^{-1})(\epsilon_0\rho_0^{-1})^{(3/2)^k-1}\right]^{\beta_k} D_{11}\right]$,

$$D_{15} = \left[2^{10}\epsilon_0 CN^3 e^{2L_k} D_{11} + 2^2(2^4 D_{11} + D_{12})\epsilon_0 CN^2 e^{2L_k} + 2^{12}\epsilon_0 CN^2 e^{3L_k}\right],$$

and
$$D_{16} = \Big[2^6(2+D_2)D_{11}N + (D_{11}^2 + D_{13})(2^6\epsilon_k\rho_k\rho_0^{-1}CN^2 e^{2L_k}) +$$
$$+ 2^6\epsilon_0 CD_{11}^2 N^2 e^{2L_k}\left[(\rho_k\rho_0^{-1})(\epsilon_0\rho_0^{-1})^{(3/2)^k-1}\right]^{\beta_k}\Big].$$

Proof: By (I.3.15), $\Xi_m(\underline{I}', \underline{z}') = \frac{\partial\bar{\Phi}^k}{\partial\phi_m}(\underline{I}', \underline{z}' e^{i\underline{\Delta}})$. Thus the chain rule gives

$$\frac{\partial \Xi_m}{\partial I_i'}(\underline{I}', \underline{z}') = \frac{\partial^2\bar{\Phi}^k}{\partial I_i'\partial\phi_m}(\underline{I}', \underline{z}' e^{i\underline{\Delta}}) + \sum_{j=1}^{N}\frac{\partial^2\bar{\Phi}^k}{\partial\phi_j\partial\phi_m}(\underline{I}', \underline{z}' e^{i\underline{\Delta}})\left(-iz_j^{-1}\frac{\partial z_j}{\partial I_i'}(\underline{I}', \underline{z}')\right). \tag{2.30}$$

Inequality (I.4.1) bounds the first of these terms by $D_2(\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-m|}$. (If $i = m$ this bound follows from (I.3.7) and a dimensional estimate.) Bound the factor of $\frac{\partial^2\bar{\Phi}^k}{\partial\phi_j\partial\phi_m}$ using (I.4.30) and the factor of $z_j^{-1}\frac{\partial z_j}{\partial I_i'}$ with (2.7a). Combining these estimates we have

$$\sup\left|\frac{\partial \Xi_m}{\partial I_i'}\right| \leqslant \left[D_2 + N\cdot\left(2^2\epsilon_0\rho_0 CN\, e^{2L_k}\left[(\rho_k\rho_0^{-1})(\epsilon_0\rho_0^{-1})^{(3/2)^k-1}\right]^{\beta_k}\right)D_{11}\rho_0^{-1}\right]$$
$$\times(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-m|}, \tag{2.31}$$

from which (2.29a) follows. Applying the chain rule to $\dfrac{\partial^2\Xi_m}{\partial\phi_i'\partial I_j'}$ gives

$$\frac{\partial^2\Xi_m}{\partial\phi_i'\partial I_j'}(\underline{I}',\underline{z}') = -i\sum_{\ell,n=1}^{M}\left(\frac{\partial^3\Phi^k}{\partial\phi_\ell\partial\phi_m\partial\phi_n}\right)\left(z_i'z_\ell^{-1}\frac{\partial z_\ell}{\partial z_i'}\right)\left(z_n^{-1}\frac{\partial z_n}{\partial I_j'}\right)$$
$$-\sum_{\ell=1}^{N}\left(\frac{\partial^2\Phi^k}{\partial\phi_\ell\partial\phi_m}\right)\left\{\left(z_\ell^{-1}\frac{\partial z_\ell}{\partial I_j'}\right)\left(z_i'z_\ell^{-1}\frac{\partial z_\ell}{\partial z_i'}\right) - z_i'z_\ell^{-1}\frac{\partial^2 z_\ell}{\partial z_i'\partial I_j'}\right\}$$
$$+\sum_{\ell=1}^{N}\left(\frac{\partial^3\Phi^k}{\partial\phi_\ell\partial\phi_m\partial I_j'}\right)\left(z_i'z_\ell^{-1}\frac{\partial z_\ell}{\partial z_i'}\right). \tag{2.32}$$

Lemma 2.2, Lemma 4.3, (I.5.25), (I.5.33), and (I.5.35) allow us to extract from each of these terms a factor of $(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$, bounding (2.30) by

$$\sup\left|\frac{\partial^2\Xi_m}{\partial\phi_i'\partial I_j'}(\underline{I}',\underline{z}')\right| \leqslant \left\{2^{10}\epsilon_0 CN^3 e^{2L_k}D_{11} + 2^{12}\epsilon_0 CN^2 e^{3L_k}\right.$$
$$\left. + 2^2(2^4 D_{11} + D_{12})\epsilon_0 CN^2 e^{2L_k}\right\}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.33}$$

verifying (2.29b). Similarly

$$\frac{\partial^2 \Xi_m}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}') = \left(\frac{\partial^3 \Phi^k}{\partial I_i' \partial I_j' \partial \phi_m}\right) - i \sum_{\ell=1}^{N} \left(\frac{\partial^3 \Phi^k}{\partial \phi_\ell \partial \phi_m \partial I_j'}\right)\left(z_\ell^{-1} \frac{\partial z_\ell}{\partial I_i'}\right)$$

$$- i \sum_{\ell=1}^{N} \left(\frac{\partial^3 \Phi^k}{\partial \phi_\ell \partial \phi_m \partial I_i'}\right)\left(z_\ell^{-1} \frac{\partial z_\ell}{\partial I_j'}\right) \tag{2.34}$$

$$- \sum_{\ell, n=1}^{N} \left(\frac{\partial^3 \Phi^k}{\partial \phi_\ell \partial \phi_m \partial \phi_n}\right)\left(z_\ell^{-1} \frac{\partial z_\ell}{\partial I_i'}\right)\left(z_n^{-1} \frac{\partial z_n}{\partial I_j'}\right)$$

$$+ i \sum_{i=1}^{N} \left(\frac{\partial^2 \Phi^k}{\partial \phi_\ell \partial \phi_m}\right)\left[\left(z_\ell^{-1} \frac{\partial z_\ell}{\partial I_i'}\right)\left(z_\ell^{-1} \frac{\partial z_\ell}{\partial I_j'}\right) - z_\ell^{-1} \frac{\partial^2 z_\ell}{\partial I_i' \partial I_j'}\right] .$$

Once again we use Lemma 2.2, (I.4.30), (I.3.7), (2.17) and dimensional estimates on $W(\gamma_k/8, \xi_k - 4\delta, V_k)$ to bound each of the terms on the r.h.s. of (2.32) resulting in

$$\sup \left|\frac{\partial^2 \Xi_m}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}')\right| \leqslant \Big\{ 2N^{-1} D_{11} \rho_0^{-1} + 2^5 (2 + D_2) D_{11} N \rho_0^{-1} + 2^6 \epsilon_0 \rho_0^{-1} C D_{11}^2 N^2 e^{2L_k}$$

$$\times \left[(\rho_k \rho_0^{-1})(\epsilon_0 \rho_0^{-1})^{(3/2)^k - 1}\right]^{\beta_k} + \tag{2.35}$$

$$+ (D_{11}^2 + D_{13})(2^6 \epsilon_k \rho_k \rho_0^{-2} C N^2 e^{2L_k}) \Big\} (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} ,$$

which yields (2.29b).

3. The Proof of Proposition I.5.1

In the present section we complete the proof of Proposition I.5.1. The procedure consists of applying the chain rule to the expressions for the derivatives of $f^{I}(\underline{I}',\underline{z}')$, $f^{II}(\underline{I}',\underline{z}')$, and $f^{III}(\underline{I}',\underline{z}')$, and then using the decay estimates of the previous section, and of sections 4 and 5 of I to bound the factors that arise.

3.A. Derivatives of $f^{I}(\underline{I}',\underline{z}')$.

Applying the chain rule to (I.3.23) gives

$$\begin{aligned}
\frac{\partial^2 f^I}{\partial\phi_i'\partial\phi_j'}(\underline{I}',\underline{z}') = \int_0^1 dt \int_0^t ds \Bigg\{ & \sum_{\ell,m,n,p=1}^{N} \left(\frac{\partial^4 h^k}{\partial I_\ell \partial I_m \partial I_n \partial I_p}\right)\left(s\,\frac{\partial \Xi_n}{\partial\phi_j'}\right)\left(s\,\frac{\partial \Xi_p}{\partial \phi_i'}\right)\Xi_\ell \Xi_m \\
& + \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(s\,\frac{\partial^2 \Xi_n}{\partial\phi_i'\partial\phi_j'}\right)\Xi_\ell \Xi_m \\
& + \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(s\,\frac{\partial \Xi_n}{\partial\phi_j'}\right)\left[\frac{\partial \Xi_\ell}{\partial\phi_i'}\Xi_m + \frac{\partial \Xi_m}{\partial\phi_i'}\Xi_\ell\right] \qquad (3.1)\\
& + \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(s\,\frac{\partial \Xi_n}{\partial\phi_i'}\right)\left[\frac{\partial \Xi_\ell}{\partial\phi_j'}\Xi_m + \frac{\partial \Xi_m}{\partial\phi_j'}\Xi_\ell\right] \\
& + \sum_{\ell,m=1}^{N} \left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_m}\right)\left[\Xi_m\left(\frac{\partial^2 \Xi_\ell}{\partial\phi_i'\partial\phi_j'}\right) + \left(\frac{\partial \Xi_\ell}{\partial\phi_i'}\right)\left(\frac{\partial \Xi_m}{\partial\phi_j'}\right) + \left(\frac{\partial \Xi_m}{\partial\phi_i'}\right)\left(\frac{\partial \Xi_\ell}{\partial\phi_j'}\right) + \Xi_\ell\left(\frac{\partial^2 \Xi_m}{\partial\phi_i'\partial\phi_j'}\right)\right]\Bigg\}.
\end{aligned}$$

We have omitted (and will continue to do so) the arguments of the functions on the r.h.s. of (3.1) to save space. They are in each case equal to the argument of the corresponding factor in (I.3.23). Bound each term on the r.h.s. of (3.1), taking care to extract from each one a factor of $(2^8\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$. Throughout this section we will assume that $i \neq j$, since in that case the bounds of (I.5.1) follow from (I.3.27)-(I.3.31) and dimensional estimates. Use (I.3.7) and (I.3.14) to bound the factors Ξ_ℓ and Ξ_m in the first term, (2.8) and a dimensional estimate on $W(\rho_{k+1}, \xi_{k+1}, V_k)$ to control $\left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_m \partial I_n \partial I_p}\right)$, and Lemma I.4.5 to bound the factors of $\frac{\partial \Xi_n}{\partial \phi'_j}$ and $\frac{\partial \Xi_p}{\partial \phi'_i}$. Bounding the number of terms in the sum over ℓ, m, n, p by N^4 gives an estimate on this first term of

$$\epsilon_0 \rho_0^{-1} N^4 \cdot 2(\epsilon_0 D_3)^2 (2\tilde{\rho}_k^{-1})^2 (2+N)(2^2 \epsilon_k \rho_k C N e^{2L_k})^2 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.2}$$

In like fashion the second term is bounded, using (I.3.14), (I.2.8) and Lemma I.5.4, by

$$\epsilon_0 \rho_0^{-1} N^3 \cdot (2\tilde{\rho}_k^{-1})(2+N)(\epsilon_0 D_8)(2^2 \epsilon_k \rho_k C N e^{2L_k})^2 (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.3}$$

The third and fourth terms may each be bounded by

$$\epsilon_0 \rho_0^{-1} N^3 (2\tilde{\rho}_k^{-1})(2+N)(\epsilon_0 D_3)^2 (2^3 \epsilon_k \rho_k C N e^{2L_k})(2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.4}$$

Finally, the last term is bounded by

$$\epsilon_0 \rho_0^{-1} N^2 (2+N) \left[2(2^2 \epsilon_k \rho_k C N e^{2L_k})(\epsilon_0 D_8) + 2(\epsilon_0 D_3)^2\right] (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \tag{3.5}$$

Combining these estimates we find

$$\begin{aligned} \sup\left|\frac{\partial^2 f^I}{\partial \phi'_i \partial \phi'_j}(\underline{I}', \underline{z}')\right| \leq \Big\{ & 2^7 (2+N) N^8 D_3^2 (\epsilon_0 \rho_0^{-1})^2 (\tilde{\rho}_k^{-1} \rho_k)^2 \epsilon_k^2 C^2 e^{4L_k} \\ & + 2^7 (2+N)(\epsilon_0 \rho_0^{-1}) N^7 D_8 (\rho_0 \tilde{\rho}_k)^{-1} \rho_k^2 \epsilon_k^2 C^2 e^{4L_k} + 2^5 (2+N) N^2 D_3^2 (\epsilon_0 \rho_0^{-1})^2 \\ & \qquad \times (\tilde{\rho}_k^{-1} \rho_k) \epsilon_k C e^{2L_k} \\ & + 2(2+N)(\epsilon_0 \rho_0^{-1}) N \left[2^3 \epsilon_k \rho_k C e^{2L_k} N \rho_0^{-1} D_8 + 2(\epsilon_0 \rho_0^{-1}) D_3^2\right] \Big\} \\ & \qquad \times (\epsilon_0 \rho_0)(2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \end{aligned} \tag{3.6}$$

Throughout this section suprema will be taken over $W(\rho_{k+1}, \xi_{k+1}, V_k)$. Some algebra, and (I.1.15) imply that the quantity in braces is bounded by (1/3). Also, (I.1.15) implies

$$\begin{aligned}(2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} &\leqslant (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)\left(1-(3/32)\ell n(3/2)/\ell n N\right)} \\ &\leqslant (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})}\end{aligned} \tag{3.7}$$

These two observations yield the $m = I$ case of (I.5.1a).

Applying the chain rule again we find

$$\begin{aligned}\frac{\partial^2 f^I}{\partial I_i' \partial \phi_j'}(\underline{I}', \underline{z}') = \int_0^1 dt \int_0^t ds \Bigg\{ &\sum_{\ell, m, n, p=1}^{N} \left(\frac{\partial^4 h^k}{\partial I_\ell \partial I_m \partial I_n \partial I_p}\right)\left(s \frac{\partial \Xi_n}{\partial \phi_j'}\right) \\ &\times\left(\delta_{p,i} + \frac{\partial \Xi_p}{\partial I_i'}\right) \Xi_\ell \Xi_m \\ &+ \sum_{\ell, m, n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(s \frac{\partial^2 \Xi_n}{\partial I_i' \partial \phi_j'}\right) \Xi_\ell \Xi_m \\ &+ \sum_{\ell, m, n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(s \frac{\partial \Xi_n}{\partial \phi_j'}\right)\left[\frac{\partial \Xi_\ell}{\partial I_i'} \Xi_m + \Xi_\ell \frac{\partial \Xi_m}{\partial I_i'}\right] \\ &+ \sum_{\ell, m, n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n}\right)\left(\delta_{n,i} + s \frac{\partial \Xi_n}{\partial I_i'}\right)\left[\frac{\partial \Xi_\ell}{\partial \phi_j'} \Xi_m + \Xi_\ell \frac{\partial \Xi_m}{\partial \phi_j'}\right] \\ &+ \sum_{\ell, m=1}^{N} \left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_m}\right)\left[\Xi_m \frac{\partial^2 \Xi_\ell}{\partial I_i' \partial \phi_j'} + \frac{\partial \Xi_\ell}{\partial \phi_j'} \frac{\partial \Xi_m}{\partial I_i'} + \frac{\partial \Xi_\ell}{\partial I_i'} \cdot \frac{\partial \Xi_m}{\partial \phi_j'} + \Xi_\ell \frac{\partial^2 \Xi_m}{\partial I_i' \partial \phi_j'}\right]\Bigg\}\end{aligned} \tag{3.8}$$

The only bound we need (in addition to those used to bound (3.1)) is (2.29). One then finds that the first term is bounded by

$$N^4(\epsilon_0\rho_0^{-1})(2\tilde{\rho}_k^{-1})^2(2+N)(\epsilon_0 D_3)(1+D_{14})(2^2\epsilon_k\rho_k CN e^{2L_k})^2(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.9}$$

The second term we bound by

$$N^3(\epsilon_0\rho_0^{-1})(2\tilde{\rho}_k^{-1})(2+N)D_{15}(2^2\epsilon_k\rho_k CN e^{2L_k})^2(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.10}$$

The third and fourth terms we control with

$$N^3(\epsilon_0\rho_0^{-1})(2\tilde{\rho}_k^{-1})(2+N)(1+D_{14})(2\epsilon_0 D_3)(2^2\epsilon_k\rho_k CN e^{2L_k})(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} , \tag{3.11}$$

and the fifth term is bounded by

$$N^2(\epsilon_0\rho_0^{-1})(2+N)\left[2(2^2\epsilon_k\rho_k CN e^{2L_k})D_{15} + 2D_{14}(\epsilon_0 D_3)\right](2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \tag{3.12}$$

Combining (3.9) - (3.12) gives

$$\begin{aligned} \sup\left|\frac{\partial^2 f^I}{\partial I_i'\partial\phi_j'}(\underline{I}',\underline{z}')\right| \leqslant \Big\{ & 2^8(\epsilon_0\rho_0^{-1})(\epsilon_k C)^2(\tilde{\rho}_k^{-1}\rho_k)^2 N^7 D_3(1+D_{14})e^{4L_k} \\ & + 2^6(\epsilon_k C)^2(\tilde{\rho}_k^{-1}\rho_k)(\rho_k\rho_0^{-1})N^6 D_{15}e^{4L_k} + 2^6(\epsilon_0\rho_0^{-1})(\epsilon_k C)(\tilde{\rho}_k^{-1}\rho_k)N^6 D_3(1+D_{14})e^{2L_k} \\ & + 2^5 D_{15}(\epsilon_k\rho_0^{-1})(\rho_k C)N^6 e^{3L_k} + 2^2 N^3\epsilon_0\rho_0^{-1}D_3 D_{14}\Big\} \\ & \times \epsilon_0(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} . \end{aligned} \tag{3.13}$$

Once again (I.1.15) implies the quantity in braces is bounded by (1/3) and $(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} \leqslant (\epsilon_0\rho_0^{-1})^{(1-\eta_{k+1})}$, verifying (5.1b) for $m = I$.

Again using the chain rule we have

$$\frac{\partial^2 f^I}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}') = \int_0^1 dt \int_0^t ds \Bigg\{ \sum_{\ell,m,n,p=1}^{N} \left(\frac{\partial^4 h^k}{\partial I_\ell \partial I_m \partial I_n \partial I_p} \right) \left(\delta_{n,i} + \frac{\partial \Xi_n}{\partial I_i'} \right) \times \left(\delta_{p,j} + \frac{\partial \Xi_p}{\partial I_j'} \right) \Xi_\ell \Xi_m$$

$$+ \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n} \right) \left(s \frac{\partial^2 \Xi_n}{\partial I_i' \partial I_j'} \right) \Xi_\ell \Xi_m$$

$$+ \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n} \right) \left(\delta_{n,j} + s \frac{\partial \Xi_n}{\partial I_i'} \right) \left[\frac{\partial \Xi_\ell}{\partial I_i'} \Xi_m + \Xi_\ell \frac{\partial \Xi_m}{\partial I_j'} \right]$$

$$+ \sum_{\ell,m,n=1}^{N} \left(\frac{\partial^3 h^k}{\partial I_\ell \partial I_m \partial I_n} \right) \left(\delta_{n,i} + s \frac{\partial \Xi_n}{\partial I_i'} \right) \left[\frac{\partial \Xi_\ell}{\partial I_j'} \Xi_m + \Xi_\ell \frac{\partial \Xi_m}{\partial I_i'} \right] \tag{3.14}$$

$$+ \sum_{\ell,m=1}^{N} \left(\frac{\partial^2 h^k}{\partial I_\ell \partial I_m} \right) \left[\frac{\partial^2 \Xi_\ell}{\partial I_i' \partial I_j'} \Xi_m + \frac{\partial \Xi_\ell}{\partial I_i'} \frac{\partial \Xi_m}{\partial I_j'} + \left(\frac{\partial \Xi_\ell}{\partial I_j'} \right) \left(\frac{\partial \Xi_m}{\partial I_i'} \right) + \Xi_\ell \frac{\partial^2 \Xi_m}{\partial I_i' \partial I_j'} \right] \Bigg\} .$$

Each of these terms may be controlled with the decay estimates of I, and the previous section, yielding

$$\sup \left| \frac{\partial^2 f^I}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}') \right| \leqslant \Big\{ 2N^5 (2\tilde{\rho}_k^{-1})^2 (2^2 \epsilon_k \rho_k C N e^{2L_k})^2 (1 + D_{14})^2$$

$$+ N^3 (2\tilde{\rho}_k^{-1})(\rho_0^{-1} D_{16})(2^2 \epsilon_k \rho_k C N e^{2L_k})^2 + 2N^4 (2\tilde{\rho}_k^{-1})(1 + D_{14})(2^3 \epsilon_k \rho_k C N e^{2L_k}) D_{14}$$

$$+ N \left[2^3 \epsilon_k \rho_k C e^{2L_k} N \rho_0^{-1} D_{16} + 2D_{14}^2 \right] \Big\} (\epsilon_0 \rho_0^{-1})(2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \tag{3.15}$$

$$\leqslant (1/3)(\epsilon_0 \rho_0^{-1})(\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} .$$

The remainder of the theorem is proved in like fashion. We bounded $\sup\left|\frac{\partial^2 f^{II}}{\partial\phi'_i\partial\phi'_j}\right|$ in section 5 of I. Consider

$$
\begin{aligned}
\frac{\partial^2 f^{II}}{\partial\phi'_i\partial I'_j}(\underline{I}',\underline{z}') = \int_0^1 dt \Bigg\{ & \sum_{\ell,m,n=1}^{N} \Bigg[\left(\frac{\partial^3 f^{k[\leqslant]}}{\partial I_\ell \partial I_m \partial I_n}\right)\left(\delta_{n,j} + t\,\frac{\partial \Xi_n}{\partial I'_j}\right) \\
& + \left(\frac{\partial^3 f^{k[\leqslant]}}{\partial\phi_n \partial I_\ell \partial I_m}\right)\left(-iz_n^{-1}\,\frac{\partial z_n}{\partial I'_j}\right)\Bigg] \left(t\,\frac{\partial \Xi}{\partial\phi'_i}\right)\Xi_\ell \\
& + \sum_{\ell,m=1}^{N} \left(\frac{\partial^2 f^{k[\leqslant]}}{\partial I_\ell \partial I_m}\right)\left[\left(t\,\frac{\partial^2 \Xi_\ell}{\partial I'_j \partial\phi'_i}\right)\Xi_m + \left(t\,\frac{\partial \Xi_\ell}{\partial\phi'_i}\right)\left(\frac{\partial \Xi_m}{\partial I'_j}\right)\right] \\
& + \sum_{\ell,m,n=1}^{N} \Bigg[\left(\frac{\partial^3 f^{k[\leqslant]}}{\partial I_\ell \partial\phi_m \partial I_n}\right)\left(\delta_{n,j} + t\,\frac{\partial \Xi_n}{\partial I'_j}\right) + \left(\frac{\partial^3 f^{k[\leqslant]}}{\partial\phi_n \partial\phi_m \partial I_\ell}\right)\left(-iz_n^{-1}\,\frac{\partial z_n}{\partial I'_j}\right)\Bigg] \\
& \qquad \times \left(\frac{\partial\phi_m}{\partial\phi'_i}\right)\Xi_\ell \\
& + \sum_{\ell,m=1}^{N} \left(\frac{\partial^2 f^{k[\leqslant]}}{\partial\phi_m \partial I_\ell}\right)\Bigg[\left(\frac{\partial\phi_m}{\partial\phi'_i}\right)\left(\frac{\partial \Xi_\ell}{\partial I'_j}\right) + \Xi_\ell \Bigg(z_m^{-1} z'_i\,\frac{\partial^2 z_m}{\partial z'_i \partial I'_j} \\
& \qquad - z'_i z_m^{-2}\left(\frac{\partial z_m}{\partial I'_j}\right)\left(\frac{\partial z_m}{\partial z'_i}\right)\Bigg)\Bigg] \\
& + \sum_{\ell,m=1}^{N} \Bigg[\left(\frac{\partial^2 f^{k[\leqslant]}}{\partial I_\ell \partial I_m}\right)\left(\delta_{m,j} + t\,\frac{\partial \Xi_m}{\partial I'_j}\right) + \left(\frac{\partial^2 f^{k[\leqslant]}}{\partial\phi_m \partial I_\ell}\right)\left(-iz_m^{-1}\,\frac{\partial z_m}{\partial I'_j}\right)\Bigg]\left(\frac{\partial \Xi_k}{\partial\phi'_i}\right) \\
& + \sum_{\ell=1}^{N} \left(\frac{\partial f^{k[\leqslant]}}{\partial I_\ell}\right)\left(\frac{\partial^2 \Xi_\ell}{\partial I'_j \partial\phi'_i}\right)\Bigg\}
\end{aligned} \tag{3.16}
$$

Extracting from each of these terms a factor of $(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ we bound (3.16) by

$$
\begin{aligned}
\sup\left|\frac{\partial^2 f^{II}}{\partial\phi_i'\partial I_j'}\right| \leqslant \Big\{ & N^3\left[(2\tilde{\rho}_k^{-1})(\epsilon_0\rho_0^{-1})(1+D_{14})+(2\tilde{\rho}_k^{-1})(\epsilon_0\rho_0^{-1})2^3 D_{11}\right](\epsilon_0 D_3) \\
& \times(2^2\epsilon_k\rho_k CN e^{2L_k}) \\
& + N^2(\epsilon_0\rho_0^{-1})\left[(2^2\epsilon_k\rho_k CN e^{2L_k})+\epsilon_0 D_3 D_{14}\right] \\
& + N^3\left[(2\tilde{\rho}_k^{-1})(\epsilon_0)(1+D_{14})+(2\tilde{\rho}_k^{-1})(\epsilon_0 D_{11})\right](2^5\epsilon_k\rho_k CN e^{2L_k}) \\
& + N^3(\epsilon_0)\left[2^3 D_{14}+(2^2\epsilon_k\rho_k CN e^{2L_k})(2^3\rho_0^{-1}D_{11}+D_{12}\rho_0^{-1})\right] \qquad (3.17) \\
& + N^2\left[(\epsilon_0\rho_0^{-1})(1+D_{14})+\epsilon_0 D_{11}\rho_0^{-1}\right](\epsilon_0 D_3) \\
& + N\epsilon_{k+1}D_{15}\Big\}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\
\leqslant\ & (1/3)\epsilon_0(\epsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|} .
\end{aligned}
$$

Next one has

$$
\begin{aligned}
\frac{\partial^2 f^{II}}{\partial I_i'\partial I_j'}(\underline{I}',\underline{z}') = \int_0^1 dt \Bigg\{ & \sum_{\ell,m,n=1}^{N}\left[\left(\frac{\partial^2 f^{k[\leqslant]}}{\partial I_\ell\partial I_m\partial I_n}\right)\left(\delta_{n,j}+t\frac{\partial\Xi_n}{\partial I_j'}\right)\right. \\
& \left.+\left(\frac{\partial^3 f^{k[\leqslant]}}{\partial\phi_n\partial I_\ell\partial I_m}\right)\left(-iz_n^{-1}\frac{\partial z_n}{\partial I_j'}\right)\right]\left(\delta_{m,i}+t\frac{\partial\Xi_m}{\partial I_j'}\right)\Xi_\ell \\
& + \sum_{\ell,m,n=1}^{N}\left[\left(\frac{\partial^3 f^{k[\leqslant]}}{\partial I_n\partial\phi_m\partial I_\ell}\right)\left(\delta_{n,j}+t\frac{\partial\Xi_n}{\partial I_j'}\right)+\left(\frac{\partial^3 f^{k[\leqslant]}}{\partial\phi_n\partial\phi_m\partial I_\ell}\right)\left(-iz_n^{-1}\frac{\partial z_n}{\partial I_i'}\right)\right] \qquad (3.18) \\
& \times\left[\left(-iz_m^{-1}\frac{\partial z_m}{\partial I_i'}\right)\Xi_\ell\right]
\end{aligned}
$$

(cont'd)

(3.18 cont'd)

$$+ \sum_{\ell, m=1}^{N} \left(\frac{\partial^2 f^k[\leqslant]}{\partial I_\ell \partial I_m}\right) \left[\left(t \frac{\partial^2 \Xi_m}{\partial I_i' \partial I_j'}\right) \Xi_\ell + \left(t \frac{\partial \Xi_m}{\partial I_i'}\right)\left(\frac{\partial \Xi_\ell}{\partial I_j'}\right)\right]$$

$$+ \sum_{\ell, m=1}^{N} \left(\frac{\partial^2 f^k[\leqslant]}{\partial \phi_m \partial I_\ell}\right) \left[\left(-iz_m^{-2} \frac{\partial z_m}{\partial I_i'} \cdot \frac{\partial z_m}{\partial I_j'} - iz_m^{-1} \frac{\partial^2 z_m}{\partial I_i' \partial I_j'}\right) \Xi_\ell + \left(-iz_m^{-1} \frac{\partial z_m}{\partial I_i'}\right)\left(\frac{\partial \Xi_\ell}{\partial I_j'}\right)\right]$$

$$+ \sum_{\ell, m=1}^{N} \left[\left(\frac{\partial^2 f^k[\leqslant]}{\partial I_\ell \partial I_m}\right)\left(\delta_{m,j} + t \frac{\partial \Xi_m}{\partial I_j'}\right) + \left(\frac{\partial^2 f^k[\leqslant]}{\partial \phi_m \partial I_\ell}\right)\left(-iz_m^{-1} \frac{\partial z_m}{\partial I_j'}\right)\right]\left(\frac{\partial \Xi_\ell}{\partial I_i'}\right)$$

$$+ \sum_{\ell=1}^{N} \left(\frac{\partial f^k[\leqslant]}{\partial I_\ell}\right)\left(\frac{\partial^2 \Xi_\ell}{\partial I_i' \partial I_j'}\right) ,$$

which implies

$$\sup\left|\frac{\partial^2 f^{II}}{\partial I_i' \partial I_j'}\right| \leqslant \Big\{N^3 \left[(2\tilde{\rho}_k^{-1})(\epsilon_0 \rho_0^{-1})(1+D_{14}) + (2\tilde{\rho}_k^{-1})(\epsilon_0 \rho_0^{-1}) D_{11}\right] \times (1+D_{14})(2^2 \epsilon_k \rho_k C N e^{2L_k})$$

$$+ N^3 \left[(2\tilde{\rho}_k^{-1})\epsilon_0(1+D_{14}) + (2\tilde{\rho}_k^{-1})(\epsilon_0 D_{11})\right] \rho_0^{-1} D_{11} (2^2 \epsilon_k \rho_k C N e^{2L_k})$$

$$+ N^2 (\epsilon_0 \rho_0^{-1}) \left[(2^2 \epsilon_k \rho_k C N e^{2L_k})(\rho_0^{-1} D_{16}) + D_{14}^2\right] \tag{3.19}$$

$$+ N\epsilon_0 \left[(2^2 \epsilon_k \rho_k C N e^{2L_k})(D_{11}^2 \rho_0^{-2} + D_{13} \rho_0^{-2}) + D_{11} D_{14} \rho_0^{-1}\right]$$

$$+ N^2 \left[(\epsilon_0 \rho_0^{-1})(1+D_{14}) + (\epsilon_0 \rho_0^{-1}) D_{11}\right] D_{14} + N \epsilon_{k+1} \rho_0^{-1} D_{16}\Big\} \times (2^4 \epsilon_0 \rho_0)^{(1-\eta_k)(1-\beta_k)|i-j|}$$

$$\leqslant (1/3)(\epsilon_0 \rho_0^{-1})(\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} .$$

Finally, for the sake of completeness, we write out the expressions for the derivatives of $f^{III}(\underline{I}', \underline{z}')$, and the bounds on these quantities. The reason for providing these details is that it is then easy to verify that the bound of (I.1.15) is qualitatively sufficient to prove Proposition I.5.1. The only difficulty comes in the large amount of algebra necessary to compute the constants $B, \ldots, \gamma$ appearing in that expression, and that we omit

$$\frac{\partial^2 f^{III}}{\partial\phi_i'\partial\phi_j'}(\underline{I},\underline{z}) = \sum_{\ell,m=1}^{N}\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_\ell \partial I_m}\right)\left(\frac{\partial \Xi_m}{\partial\phi_i'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial\phi_m \partial I_\ell}\right)\left(\frac{\partial\phi_m}{\partial\phi_i'}\right)\Bigg]\left(\frac{\partial\Xi_\ell}{\partial\phi_j'}\right)$$
$$+ \sum_{\ell,m=1}^{N}\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_m \partial\phi_\ell}\right)\left(\frac{\partial \Xi_m}{\partial\phi_i'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial\phi_m \partial\phi_\ell}\right)\left(\frac{\partial\phi_m}{\partial\phi_i}\right)\Bigg]\left(\frac{\partial\phi_\ell}{\partial\phi_j'}\right) \tag{3.20}$$
$$+ \sum_{\ell=1}^{N}\left[\left(\frac{\partial f^{k[\geqslant]}}{\partial I_\ell}\right)\left(\frac{\partial^2\Xi_\ell}{\partial\phi_i'\partial\phi_j'}\right) + \left(\frac{\partial f^{k[\geqslant]}}{\partial\phi_\ell}\right)\left(\frac{\partial^2\phi_\ell}{\partial\phi_i'\partial\phi_j'}\right)\right]$$

with $\left(\frac{\partial\phi_m}{\partial\phi_i'}\right)$ and $\left(\frac{\partial^2\phi_m}{\partial\phi_i'\partial\phi_j'}\right)$ the functions defined in I. Thus,

$$\sup\left|\frac{\partial^2 f^{III}}{\partial\phi_i'\partial\phi_j'}(\underline{I}',\underline{z}')\right| \leqslant \Big\{N^2\left[(\epsilon_0\rho_0^{-1})\epsilon_0 D_3 + \epsilon_0 2^3 D_9\right]\epsilon_0 D_3$$
$$+ N^2\left[\epsilon_0^2 D_9 D_3 + 2^8\epsilon_0\rho_0(\epsilon_0\rho_0^{-1})^{\beta_k(3/2)^k}\right]2^3 + N\left[\epsilon_{k+1}\epsilon_0 D_8 + \epsilon_{k+1}\rho_{k+1}D_7\right]\Big\} \tag{3.21}$$
$$\times(2^8\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$$

$$\leqslant (1/3)(\epsilon_0\rho_0)(\epsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|} .$$

Next,

$$\frac{\partial^2 f^{III}}{\partial I_i'\partial\phi_j'}(\underline{I}',\underline{z}') = \sum_{\ell,m=1}^{N}\left[\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_\ell \partial I_m}\right)\left(\delta_{m,i} + \frac{\partial \Xi_m}{\partial I_i'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_\ell \partial \phi_m}\right)\left(-iz_m^{-1}\frac{\partial z_m}{\partial I_i'}\right)\right] \times \left(\frac{\partial \Xi_\ell}{\partial \phi_j'}\right)$$

$$+ \sum_{\ell,m=1}^{N}\left[\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_m \partial \phi_\ell}\right)\left(\delta_{m,i} + \frac{\partial \Xi_m}{\partial I_i'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial \phi_m \partial \phi_\ell}\right)\left(-iz_m^{-1}\frac{\partial z_m}{\partial I_i'}\right)\right]\left(\frac{\partial \phi_\ell}{\partial \phi_j'}\right) \tag{3.22}$$

$$+ \sum_{\ell=1}^{N}\left[\left(\frac{\partial f^{k[\geqslant]}}{\partial I_\ell}\right)\left(\frac{\partial^2 \Xi_\ell}{\partial I_i'\partial \phi_j'}\right) + \left(\frac{\partial f^{k[\geqslant]}}{\partial \phi_\ell}\right)\left(iz_j z_\ell^{-1}\frac{\partial^2 z_\ell}{\partial I_i' \partial z_j'} - iz_j z_\ell^{-2}\frac{\partial z_\ell}{\partial I_i'}\cdot\frac{\partial z_\ell}{\partial z_j'}\right)\right],$$

so that

$$\sup\left|\frac{\partial^2 f^{III}}{\partial I_i'\partial \phi_j'}\right| \leqslant \Big\{N^2\left[(\epsilon_0\rho_0^{-1})D_{10}(1+D_{14}) + (\epsilon_0 D_9)(\rho_0^{-1}D_{11})\right]\epsilon_0 D_3$$

$$+ N^2\left[(\epsilon_0 D_9)(1+D_{14}) + 2^4(\epsilon_0\rho_0)(\epsilon_0\rho_0^{-1})^{\beta_k(3/2)^k}\rho_0^{-1}D_{11}\right]2^3 \tag{3.23}$$

$$+ N\left[\epsilon_{k+1}D_{15} + \epsilon_{k+1}\rho_{k+1}(D_{12}\rho_0^{-1} + 2^3 D_{11}\rho_0^{-1})\right]\Big\}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)\ i-j}$$

$$\leqslant (1/3)\epsilon_0(\epsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|}.$$

Finally,

$$\frac{\partial^2 f^{III}}{\partial I_i'\partial I_j'}(\underline{I}',\underline{z}') = \sum_{\ell,m=1}^{N}\left[\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_\ell \partial I_m}\right)\left(\delta_{m,j} + \frac{\partial \Xi_m}{\partial I_j'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_\ell \partial \phi_m}\right)\left(-iz_m^{-1}\frac{\partial z_m}{\partial I_j'}\right)\right] \times \left(\delta_{\ell,i} + \frac{\partial \Xi_\ell}{\partial I_i'}\right) \tag{3.24}$$

$$+ \sum_{\ell,m=1}^{N}\left[\left(\frac{\partial^2 f^{k[\geqslant]}}{\partial I_m \partial \phi_\ell}\right)\left(\delta_{m,j} + \frac{\partial \Xi_m}{\partial I_j'}\right) + \left(\frac{\partial^2 f^{k[\geqslant]}}{\partial \phi_\ell \partial \phi_m}\right)\left(-iz_m^{-1}\frac{\partial z_m}{\partial I_j'}\right)\right]\left(-iz_\ell^{-1}\frac{\partial z_\ell}{\partial I_i'}\right)$$

(cont'd)

(3.24 cont'd)

$$+ \sum_{\ell=1}^{N} \left(\frac{\partial f^{k[\geqslant]}}{\partial I_\ell}\right)\left(\frac{\partial^2 \Xi_\ell}{\partial I_i' \partial I_j'}\right) + \left(\frac{\partial f^{k[\geqslant]}}{\partial \phi_\ell}\right)\left(iz_i^{-2} \frac{\partial z_\ell}{\partial I_i'} \cdot \frac{\partial z_\ell}{\partial I_j'} - iz_\ell^{-1} \frac{\partial^2 z_\ell}{\partial I_i' \partial I_j'}\right)\Bigg].$$

Thus,

$$\begin{aligned}
\sup\left|\frac{\partial^2 f^{III}}{\partial I_i' \partial I_j'}\right| \leqslant \Bigg\{ & N^2(1+D_{14})\left[(D_{10}\epsilon_0\rho_0^{-1})D_{14} + \epsilon_0\rho_0^{-1}D_9 D_{11}\right] \\
& + N^2\left[\epsilon_0 D_9(1+D_{14}) + 2^4\epsilon_0(\epsilon_0\rho_0^{-1})^{\beta_k(3/2)^k} D_{11}\right]\rho_0^{-1}D_{11} \\
& + N\left[\epsilon_{k+1}\rho_0^{-1}D_{16} + \epsilon_{k+1}\rho_{k+1}(\rho_0^{-2}D_{13} + \rho_0^{-2}D_{11}^2)\right]\Bigg\}(2^4\epsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\
& \leqslant (1/3)(\epsilon_0\rho_0^{-1})(\epsilon_0\rho_0^{-1})^{(1-\eta_{k+1})|i-j|} .
\end{aligned} \tag{3.25}$$

4. The Proof of Theorem 1.2

Large parts of the proof of Theorem 1.2 are identical to the proofs of the main theorems in [1] and [2]. We don't repeat those parts here but instead emphasize the differences. The idea is to construct an infinite sequence of canonical transformations, which at each stage reduce the size of the perturbation of the Hamiltonian, resulting finally in an integrable system. Consider the Hamiltonian $H \circ C(\underline{I}, \underline{z}) \equiv H^0(\underline{I}, \underline{z}) = h^0(\underline{I}) + f^0(\underline{I}, \underline{z})$ constructed in Theorem I.1.1. If

$$\sup \left\{ \left| \frac{\partial \hat{f}^0}{\partial \underline{I}} \right| + \hat{\rho}_0^{-1} \left| \frac{\partial \hat{f}^0}{\partial \phi} \right| \right\} \leqslant \hat{\epsilon}_0 \quad , \tag{4.1}$$

$$\sup \left| \frac{\partial \hat{h}^0}{\partial \underline{I}} \right| \leqslant \hat{E}_0 \quad , \tag{4.2}$$

$$\sup \left| \left(\frac{\partial \hat{h}^0}{\partial \underline{I} \partial \underline{I}} \right)^{-1} \right| \leqslant \hat{\eta}_0 \quad , \tag{4.3}$$

and

$$\frac{\partial^2 \hat{h}^0}{\partial I_i \partial I_j}(\underline{I}) \equiv \frac{\partial \hat{\omega}_i^0}{\partial I_j}(\underline{I}) = \delta_{ij}(1 + \hat{\chi}_i^0(\underline{I})) + (1 - \delta_{ij}) \frac{\partial \hat{\omega}_i^0}{\partial I_j}(\underline{I}) \quad , \tag{4.4}$$

then (I.1.16) - (I.1.20) insure that we can take $\hat{\epsilon}_0 = \rho_0(\epsilon_0 \rho_0^{-1})^N$, $\hat{E}_0 = 2E_0$, and $\hat{\rho}_0 = \rho_{k_0}$, where $k_0 = \left[(\ell n\, N)(\ell n\, 3/2)^{-1}\right] + 1$, and ρ_{k_0} is defined inductively by (I.2.1). Here, the quantities with no hats (e.g., ϵ_0, ρ_0) refer to the Hamiltonian $H(\underline{I}, \underline{z})$ of I. Also, $\sup |\hat{\chi}_i^0(I)| \leqslant (2^2 N)^{-1}$ and $\sup \left| (1 - \delta_{ij}) \frac{\partial \hat{\omega}_i^0}{\partial I_j} \right| \leqslant (\epsilon_0 \rho_0^{-1})(\epsilon_0 \rho_0^{-1})^{(1/8)|i-j|}$, and all suprema are evaluated over the region $W(\hat{\rho}_0, \hat{\xi}_0; \hat{V})$ where $\hat{V} = V^{k_0 - 1}$. From the proof of Theorem I.1.1 we see we may pick $\hat{\xi}_0 = 1$. Define inductively the sequence of parameters

$$\hat{\delta}_j = \hat{\xi}_0 / 16(1 + j^2)$$

$$\hat{C}_k = (1 + k^2) \hat{C}_0$$

$$\hat{\xi}_{k+1} = \hat{\xi}_k - 4 \hat{\delta}_k$$

$$\hat{E}_{k+1} = \hat{E}_k + \hat{\epsilon}_k$$

$$\hat{\eta}_{k+1} = \hat{\eta}_k(1 + 4N^2 \hat{\eta}_k \hat{\epsilon}_k \hat{\rho}_k^{-1}) \tag{4.5}$$

$$\hat{\rho}_{k+1} = \hat{\rho}_k/(2^8 N 2^N \hat{E}_k \hat{C}_k \hat{\delta}_k^{-(N+1)}) \left[\ell n(\hat{\epsilon}_k \hat{C}_k \delta_k^N)^{-1}\right]^{-N+1}$$

$$\hat{\epsilon}_{k+1} = (\hat{\epsilon}_k \hat{C}_k)^{(3/2)^k} \hat{E}_k (\hat{E}_k \hat{C}_k) \hat{M}_k^{N+1} \hat{\delta}_k^{-(N+1)}$$

$$\hat{M}_k = 2\hat{\delta}_k^{-1} \ell n(\hat{\epsilon}_k \hat{C}_k \hat{\delta}_k^N)^{-1} ,$$

with $\hat{C}_0$ a constant to be determined later. Defining $\tilde{\hat{\rho}}_k = \hat{\rho}_k(2^4 N\hat{C}_k E_k M_k^{N+1})^{-1}$, we define a sequence of regions $\hat{V} \supset \hat{V}_0 \supset \hat{V}_1 \ \dots$. Set

$$\hat{R}(k, \hat{h}^k, \hat{V}_{k-1}) = \left\{\underline{I} \middle| \underline{I} \in \hat{V}^{k-1} ; \left|\langle \hat{\underline{\omega}}^k(\underline{I}), \underline{\nu} \rangle\right| < \hat{C}_k |\underline{\nu}|^N \text{ for some } \underline{\nu} \text{ with } 0 < |\underline{\nu}| < M_k \right\} . \tag{4.6}$$

(By convention take $\hat{V}_{-1} = \hat{V}$.) Then

$$\bar{\hat{V}}_k \equiv \left\{\underline{I} \middle| \underline{I} \in \hat{V}_{k-1} ; \operatorname{dist}(\underline{I}, \partial\hat{V}_{k-1}) \geqslant \tilde{\hat{\rho}}_k/2\right\} \setminus \hat{R}(k, \hat{h}^k, \hat{V}_{k-1}) ,$$

and $\hat{V}_k \equiv \bigcup_{\underline{I} \in \bar{\hat{V}}_k} S(\underline{I}, \tilde{\hat{\rho}}_k/2)$.

Following [1] or [2] one constructs canonical transformations

$$\hat{C}^k : (\underline{I}', \underline{z}') \longrightarrow (\underline{I}, \underline{z})$$
$$\tilde{\hat{C}}^k : (\underline{I}, \underline{z}) \longrightarrow (\underline{I}', \underline{z}') \tag{4.7}$$

where both transformations are defined on $W(\tilde{\hat{\rho}}_k/2, \hat{\xi}_k - 2\hat{\delta}_k ; \hat{V}^k)$ and map $W(\tilde{\hat{\rho}}_k/4, \hat{\xi}_i - 3\hat{\delta}_k ; \hat{V}^k)$ into $W(\tilde{\hat{\rho}}_k/2, \hat{\xi}_k - 2\hat{\delta}_k ; \hat{V}_k)$. Defining

$$H^k(\underline{I}', \underline{z}') = \hat{H}^{k-1} \circ \hat{C}^{k-1}(\underline{I}', \underline{z}') \equiv \hat{h}^k(\underline{I}') + \hat{f}^k(\underline{I}', \underline{z}') \tag{4.8}$$

one has

$$\sup \left| \frac{\partial \hat{h}^k}{\partial \underline{I}} \right| \leqslant \hat{E}_k \tag{4.9}$$

$$\sup \left| \left(\frac{\partial^2 \hat{h}^k}{\partial \underline{I} \partial \underline{I}} \right)^{-1} \right| \leqslant \eta_k \tag{4.10}$$

and

$$\sup \left\{ \left| \frac{\partial \hat{f}^k}{\partial \underline{I}} \right| + \hat{\rho}_k^{-1} \left| \frac{\partial \hat{f}^k}{\partial \underline{\phi}} \right| \right\} \leqslant \hat{\epsilon}_k \ . \tag{4.11}$$

The conditions which guarantee that the iterative procedure of (4.7) - (4.11) may be arbitrarily repeated can be combined in the single inequality

$$\hat{\epsilon}_0 < \hat{C}_0^{-1} \hat{B}_2 \hat{\eta}_0^{-2} (\hat{\rho}_0 \hat{E}_0^{-1})^2 (\hat{E}_0 \hat{C}_0)^{-16} \ \hat{g}'(N) \ , \tag{4.12}$$

where $\hat{g}'(N) = (N!)^{-4} N^{-10N} e^{-152N}$ and $\hat{B}_2 = e^{-220}$ (This is essentially inequality (3.58) of [2] and we do not repeat its derivation.) Given this sequence of canonical transformations we must estimate the value of $\hat{C}_0$ which insures that not too much phase space gets thrown away. We will need the fact that the integrable part of the Hamiltonians constructed above is given by

$$\hat{h}^{k+1}(\underline{I}) = \hat{h}^0(\underline{I}) + \sum_{j=0}^{k} \hat{f}_{\underline{0}}^j(\underline{I}) \ . \tag{4.13}$$

As in I note that

$$\mathrm{vol}(\hat{V}^{k-1} \setminus \hat{V}^k) \leqslant \mathrm{vol}\, \hat{B} + \mathrm{vol}\, \hat{R}(k, \hat{h}^k, \hat{V}_{k-1}) \tag{4.14}$$

with

$$\hat{B} = \left\{ \underline{I} \,|\, \underline{I} \in \hat{V}^{k-1} \text{ and } \mathrm{dist}(\underline{I}, \partial \hat{V}^{k-1}) \leqslant \hat{\rho}_k / 2 \right\} \ . \tag{4.15}$$

As in section 6 of I,

$$\mathrm{vol}\, \hat{B} \leqslant \left| 1 - (1 + \tilde{\hat{\rho}}_k / \hat{\rho}_k)^N \right| \mathrm{vol}\, \hat{V} \ . \tag{4.16}$$

It is easy to show that $\hat{\hat{\rho}}_k / \hat{\rho}_k \leqslant 2^{-4}(1+k^2)^{-1} N^{-1} 2^{-N} (\hat{E}_0 \hat{C}_0)^{-1}$. Thus,

$$\mathrm{vol}\,\hat{B} \leqslant 2^{-2} 2^{-N} (1+k^2)^{-1} (\hat{E}_0 \hat{C}_0)^{-1} \mathrm{vol}\,\hat{V} \,. \tag{4.17}$$

Since $\underline{\hat{\omega}}^k(\underline{I})$ is single-valued (a fact we verify below),

$$\mathrm{vol}\left(\hat{R}(k, \hat{h}^k, \hat{V}_{k-1})\right) = \int_{\underline{\hat{\omega}}^k\left(\hat{R}(k, \hat{h}^k, \hat{V}_{k-1})\right)} \left|\det\left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}}\right)^{-1}\right| d\underline{\omega} \,. \tag{4.18}$$

Bound $\left|\det\left(\frac{\partial \underline{\hat{\omega}}^k}{\partial \underline{I}}\right)^{-1}\right|$ by 2^3 using (I.6.8) - (I.6.10), with $B_1 = (1/2)$. Since $\underline{\hat{\omega}}^k(\underline{I}) = \underline{\hat{\omega}}^0(\underline{I}) + \sum_{j=0}^{k-1} \frac{\partial \hat{f}_0^j}{\partial \underline{I}}(\underline{I})$, verifying that the matrix $\mathbb{D}$ defined there has diagonal entries bounded by $B_1 N^{-1} + 2 \sum_{j=0}^{k-1} \hat{\epsilon}_j \hat{\rho}_j^{-1}$, and off diagonal entries bounded by $(\epsilon_0 \rho_0^{-1}) N^{-1}$ is an easy exercise. Definitions (4.5) and inequality (4.12) imply $\hat{\epsilon}_j \hat{\rho}_j^{-1} \leqslant (\hat{\epsilon}_0 \hat{C}_0)^{(1/16)(3/2)^j}$, and we then bound the sum over j by a geometric series in $(\hat{\epsilon}_0 \hat{C}_0)$. The single-valuedness of $\underline{\hat{\omega}}^k$ follows by noting that (I.6.7) implies $\left|\underline{\hat{\omega}}^0(\underline{I}') - \underline{\hat{\omega}}^0(\underline{I})\right| \geqslant (1/2)|\underline{I} - \underline{I}'|$, while Appendix H of [2] guarantees that

$$\left|\sum_{j=0}^{k-1} \frac{\partial \hat{f}_0^j}{\partial \underline{I}}(\underline{I}') - \frac{\partial \hat{f}_0^j}{\partial \underline{I}}(\underline{I})\right| \leqslant 3|\underline{I} - \underline{I}'| \sum_{j=0}^{k-1} \hat{\epsilon}_j \hat{\rho}_j^{-1} \leqslant (1/4)|\underline{I} - \underline{I}'| \,.$$

Thus,

$$\begin{aligned} \left|\underline{\hat{\omega}}^k(\underline{I}') - \underline{\hat{\omega}}^k(\underline{I})\right| &\geqslant \left|\underline{\hat{\omega}}^0(\underline{I}) - \underline{\hat{\omega}}^0(\underline{I}')\right| - \left|\sum_{j=0}^{k-1} \frac{\partial \hat{f}_0^j}{\partial \underline{I}}(\underline{I}') - \frac{\partial \hat{f}_0^j}{\partial \underline{I}}(\underline{I})\right| \\ &\geqslant (1/4)|\underline{I}' - \underline{I}'| \,. \end{aligned} \tag{4.19}$$

Thus

$$\mathrm{vol}\left(\hat{R}(k,\hat{h}^k,\hat{V}_{k-1})\right) \leqslant 2^3 \int_{\hat{\omega}^k\left(\hat{R}(k,\hat{h}^k\hat{V}_{k-1})\right)} d\underline{\omega}$$

$$\leqslant 2^3 \sum_{\underline{\nu}\neq\underline{0}} \int_{\substack{|<\underline{\omega},\underline{\nu}>| < \hat{C}_k^{-1}|\underline{\nu}|^{-(N+1)} \\ \|\underline{\omega}-\underline{\omega}^k(\underline{0})\|\leqslant r(1+\tau)}} d\underline{\omega} \tag{4.20}$$

This estimate follows by noting that (I.6.13) implies $\|\hat{\underline{\omega}}^0(\underline{I}') - \hat{\underline{\omega}}^0(\underline{0})\| \leqslant (1+\eta)r$ for all $\underline{I}\in\hat{V}$. By the estimate of [2] used just above $\left|\sum_{j=0}^{k-1} \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}) - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{0})\right| \leqslant 3|\underline{I}|\sum_{j=0}^{k-1}\hat{\epsilon}_j\hat{\rho}_j^{-1}$. For any vextor $\underline{x}$, $\|\underline{x}\| \leqslant |\underline{x}| \leqslant N\|\underline{x}\|$, so for any $\underline{I}\in R(k,h^k,\hat{V}_{k-1})$

$$\|\hat{\underline{\omega}}^k(\underline{I}) - \hat{\underline{\omega}}^k(\underline{0})\| \leqslant \|\hat{\underline{\omega}}^0(\underline{I}) - \hat{\underline{\omega}}^0(\underline{0})\| + \left\|\sum_{j=0}^{k-1} \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}) - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{0})\right\| \leqslant (1+\tau)r\ , \tag{4.21}$$

with $\tau = \eta + 3N\sum_{j=0}^{k-1}\hat{\epsilon}_j\hat{\rho}_j^{-1} = (2N)^{-1} + 6\epsilon_0\rho_0^{-1}N + N\sum_{j=0}^{k-1}\hat{\epsilon}_j\hat{\rho}_j^{-1}$.

A simple geometrical argument gives

$$\int_{\substack{|<\underline{\omega},\underline{\nu}>| \leqslant \hat{C}_k|\underline{\nu}|^{-(N+1)} \\ \|\underline{\omega}-\underline{\omega}^k(\underline{0})\|\leqslant r(1+\tau)}} d\underline{\omega} \leqslant \hat{C}_k^{-1}\pi^{(N-1)/2}\left[r(1+\tau)\right]^{N-1}\left[\Gamma\left(1+(N-1)/2\right)\right]^{-1} \times|\underline{\nu}|^{-(N+1)} \tag{4.22}$$

Also, $\mathrm{vol}\,\hat{V} \geqslant (1-\lambda)\mathrm{vol}\,V = (1-\lambda)\pi^{N/2}r^N/\Gamma(1+N/2)$, and $\Gamma(1+N/2)/\Gamma\left(1+(N-1)/2\right) \leqslant 2^2\left[(N/4)+1\right]^{1/2}$, so

$$\sum_{\underline{\nu}\neq\underline{0}} \int_{\substack{|<\underline{\omega},\underline{\nu}>| \leqslant \hat{C}_k^{-1}|\underline{\nu}|^{-(N+1)} \\ \|\underline{\omega}-\underline{\omega}^k(\underline{0})\|\leqslant r(1+\tau)}} d\underline{\omega} \leqslant 2^N(1-\lambda)^{-1}(1+k^2)^{-1}N^{1/2}(\rho_0\hat{C}_0)^{-1}\,\mathrm{vol}\,\hat{V}\ , \tag{4.23}$$

where the last inequality used the fact that $\rho_0 < r$. Combining (4.16), (4.20) and (4.23) we find

$$\sum_{k=0}^{\infty} \mathrm{vol}(\hat{V}_{k-1} \setminus \hat{V}_k) \leqslant \sum_{k=0}^{\infty} \Big\{ 2^{-(N+1)}(1+k^2)^{-1}(\hat{E}_0\hat{C}_0)^{-1} + 2^{3+N}(1-\lambda)^{-1}(1+k^2)^{-1}N^{1/2}(\rho_0\hat{C}_0)^{-1} \Big\} \mathrm{vol}\,\hat{V}$$
$$\leqslant \hat{\lambda}\,\mathrm{vol}\,\hat{V}\,, \tag{4.24}$$

provided $\hat{C}_0^{-1} = 2^{5+N}(1-\lambda)^{-1}N^{1/2}(\rho_0\hat{\lambda})^{-1}$.

Finally we estimate the parameter $\hat{\eta}_0$. Writing $\left(\frac{\partial^2 \hat{h}^0}{\partial \underline{I}\partial \underline{I}}\right)^{-1} = (\hat{\mathbb{D}} - \hat{\mathbb{M}})^{-1}$, where $\hat{\mathbb{D}}$ is a diagonal matrix and $\mathbb{M}$ a purely off diagonal one, (I.1.18) - (I.1.20) allow one to estimate the elements of $\hat{\mathbb{D}}$ and $\hat{\mathbb{M}}$ by $|\hat{D}_{ii}| > (1/2)$, and $|\hat{M}_{ij}| < \epsilon_0\rho_0^{-1}$ respectively. Using this information it is easy to bound $\sup \left|\left(\frac{\partial^2 \hat{h}^0}{\partial \underline{I}\partial \underline{I}}\right)^{-1}_{ij}\right|$ by 2^2 by estimating the Neuman series for $(\hat{\mathbb{D}} - \hat{\mathbb{M}})^{-1}_{ij}$. Thus,

$$\sup \left|\left(\frac{\partial^2 \hat{h}^0}{\partial \underline{I}\partial \underline{I}}\right)^{-1}\right| \leqslant 2^2 N^2 \equiv \hat{\eta}_0\,. \tag{4.25}$$

Inserting the expressions for $\hat{\eta}_0$ and $\hat{C}_0$ into (4.12), the inequality which allows arbitrarily many canonical transformations to be performed becomes

$$\hat{\epsilon}_0 < \rho_0 B_3 (1-\lambda)^{17} (\rho_0 \hat{E}_0^{-1})^2 (\rho_0 \hat{E}_0^{-1})^{16} \hat{\lambda}^{17}\, \hat{g}(N) \tag{4.26}$$

with $B_3 = e^{-283}$, and $\hat{g}(N) = (N!)^{-4} N^{-10N} e^{-185N}$. This yields inequality (I.1.21). Given that one may iterate the canonical transformation arbitrarily often while losing arbitrarily little phase space, the remainder of the statements in Theorem 1.2 follow word-for-word from [1] or [2] and we don't repeat the proofs here.

References

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