**TWO HYPERFINITE CONSTRUCTIONS OF
THE BROWNIAN BRIDGE**

By

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by

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ABSTRACT

Infinitesimal analysis may be used to give discrete representations to a variety of stochastic processes. These representations are often intuitive and implicit in them we find weak convergence results. In this work we give two hyperfinite representations of the brownian bridge process. The first representation uses as the basic model a hyperfinite tied down random walk B . A brownian bridge $\tilde{B}$ is obtained from B via the standard part map. As a consequence of our construction we are able to proof that the brownian bridge $\tilde{B}$ is the weak limit of a sequence of normalized tied down random walks. The second hyperfinite representation of the brownian bridge process is based on U_n , the uniform empirical process of order n . As in the previous case we obtain a weak convergence result. In particular, we show that the brownian bridge is the weak limit of a sequence of empirical processes. A slight modification of the above construction yields an almost trivial proof of Donsker's Invariance Principle for Empirical Processes.

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INTRODUCTION

The study of empirical processes has been of great importance in statistical theory. In the past ten years, as demonstrated by the abundant new literature in the subject, there has been a great increase in the research done in this field. The brownian bridge process is at the heart of the study of most of the theory of empirical processes. It plays the same role as brownian motion in the study of those processes which are partial sums of independent random variables. Although the brownian bridge has been studied at some length, its literature is by no means as extensive as that on brownian motion.

During the last decade infinitesimal analysis has settled long standing questions and established outstanding new results in several branches of stochastic analysis. For example, nonstandard methods have been applied to study stochastic differential equations, invariance principles, brownian local time, Markov processes, adapted probability spaces, polymer measures and stochastic control. Such a burst of applications stemmed from Loeb's [1975] discovery that "rich" standard probability spaces, the so called Loeb spaces, could be constructed from nonstandard probability spaces. However, the real success and appeal of nonstandard techniques is that in most cases we can represent the object being studied at a "finite" level. (We will explain the exact meaning of the preceding remark in Chapter 1). The advantage of such a representation is that usually the object being studied becomes more computable as

well as more intuitive. We have used infinitesimal analysis to give two new constructions of the brownian bridge process which not only are heuristically very appealing, but give important weak convergence results as corollaries. It is our hope that these constructions will serve as stepping stones for the use of nonstandard methods in the study of problems in empirical processes.

In Chapter 1 we introduce most of the nonstandard results that are needed later. The construction of a hyperfinite approximation to the brownian bridge is used in section 1.1 as motivation. Throughout the chapter as we introduce the nonstandard concepts, we keep coming back to this example. Most of the results in the chapter are well known. The exceptions are Proposition (1.4.3) and Theorem (1.8.1) which are new and needed in Chapter 3.

In Chapter 2 we show how the brownian bridge can be approximated by a hyperfinite tied down random walk. We use this approximation to actually construct a brownian bridge. Our construction is similar to Anderson's [1976] construction of brownian motion and it shows that the brownian bridge is indeed a tied down brownian motion. In Section 2.3 we show that the brownian bridge is the weak limit of a sequence of tied down random walks. To the best of our knowledge, ours seems to be the first written proof of this interesting result. It is certainly the first using these techniques.

In Chapter 3 we give a second hyperfinite approximation to the brownian bridge. This time our basic model is the uniform empirical process. As is usually the case with hyperfinite

constructions, a weak convergence result can be readily obtained. The main result of section 3.3 is Theorem (3.3.4) (Donsker's Theorem for Empirical Processes). The novelty of our result is the simplicity of the proof. Most of the material in this chapter 3 was developed as background for the study of invariance principles for the local time of the empirical process. We hope to give some results in this respect in a future work.

All of the material in this work is part of the author's doctoral thesis at The University of Iowa.

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1 PRELIMINARIES

1.1 Motivation

Consider the statistical experiment of drawing successively and without replacement from an urn containing N balls which are marked "+1" and N balls marked "-1", the balls being otherwise identical. An outcome from such an experiment can be represented as a sequence of +1s or -1s. The set of all possible outcomes will be

$$\Omega_N = \{ \omega: \{1, 2, \dots, 2N\} \rightarrow \{-1, +1\} \mid \#[\omega^{-1}(-1)] = \#[\omega^{-1}(+1)] \},$$

where $\#$ denotes the cardinality of a set.

At times it is more convenient to think that the sampling takes place at equal time intervals of a fixed period of time, say $[0, 1]$. For example the first ball is drawn at time $t = \frac{1}{2N}$, the second at $t = \frac{2}{2N}$, and so on, until the last ball is drawn at $t = 1$. If we let

$$T_N = \{ t \in]0, 1] \mid t = k\delta t, k \in \mathbb{N} \},$$

the regular lattice on $]0, 1]$ of step size $\delta t = \frac{1}{2N}$, then the space of outcomes can be represented by

$$\Omega_N = \{ \omega: T_N \rightarrow \{-1, 1\} \mid \#[\omega^{-1}(-1)] = \#[\omega^{-1}(+1)] \}.$$

Since all the balls are identical we have that each outcome $\omega \in \Omega_N$ is equally likely to occur. Since there are $\binom{2N}{N}$ ways of arranging the N "+1" balls among the $2N$ total number of balls, the probability of each ω is given by $P_N(\omega) = \frac{1}{\binom{2N}{N}}$.

A subset Δ of Ω_N is called an event. For example,

$$\Delta_t = \{ \omega \in \Omega_N \mid \sum_{0 < s \leq t} \omega(s) = 0 \},$$

the set of all outcomes such that the number of "1" balls equals the number of "-1" balls drawn up to time t is an event. We let A_N denote the algebra of all events of Ω_N .

The probability of an event Δ is given by the finite sum of the probabilities of each of the individual outcomes in Δ . Thus

$$P(\Delta) = \sum_{\lambda \in \Delta} P(\lambda) = \frac{\#[\Delta]}{\#[\Omega_N]}.$$

For example if $t = 2m\delta t$ we have that $P(\Delta_t) = \frac{\binom{N}{m} \binom{N}{m}}{\binom{2N}{2m}}.$

Suppose now that each time a "+1" ball is drawn a particle which was originally at rest moves a positive distance $\delta x = \sqrt{\delta t}$, while each time a "-1" ball is drawn it moves a negative distance $-\delta x$. The position of the particle at each time $t \in T_N$ will then be given by the function $B_N: T_N \times \Omega_N \rightarrow \Gamma_N$ defined by

$$B_N(t, \omega) = \sum_{0 \leq s \leq t} \omega(s) \delta x,$$

where $\Gamma_N = \{ x \in \mathbb{R} \mid x = k\delta x, \text{ for some } k \in \mathbb{Z} \cap [-N, N] \}.$

Notation In the above summation we assume that the range of the index of summation is $\{s \in T_N \mid s \leq t\}.$

The function B_N is a discrete time stochastic process. We will call it the "tied down" random walk.

In Chapter 2 we will focus our interest on the process B_N . What we want to do is to let N be infinite in the sense of Abraham Robinson's contemporary Theory of Infinitesimals and to use the above construction of B_N to approximate a brownian

bridge. The details of the above construction tell us that we need to include in our analysis more than just infinite numbers and infinitesimals. We need extensions of functions, sets of functions, functions of functions (like summation), probabilities and even more complex objects. Nonstandard analysis allows us to do that.

1.2 Basic Tools from Nonstandard Analysis.

In this section we state most of the definitions and theorems from nonstandard analysis that will be used in the course of this work. For a detailed study of nonstandard stochastic analysis we refer the reader to the two books "Nonstandard Methods in Stochastic Analysis and Mathematical Physics" by Albeverio, Fenstad, Hoegh-Krohn and Lindstrom [1986] and "Foundations of Infinitesimal Stochastic Analysis" by Stroyan and Bayod [1986]. The latter will be our principal reference throughout this chapter.

For our purpose, all of classical analysis can be described by a superstructure $V(\mathbb{R})$, where $V(S)$ is defined for any ground set S by

$$\begin{aligned} V_0(S) &= S, \\ V_{n+1}(S) &= V_n(S) \cup \mathcal{P}(V_n(S)), \\ V(S) &= \bigcup_{n \in \mathbb{N}} V_n(S), \end{aligned}$$

where $\mathcal{P}(V_n(S))$ denotes the "power set" of all subsets of $V_n(S)$. Our nonstandard universe will then be the superstructure $V(^*\mathbb{R})$ satisfying the following "principles":

There is a mapping $\ast: V(\mathbb{R}) \rightarrow V(\ast\mathbb{R})$ with the following properties:

(1.2.1) if $r \in \mathbb{R}$ then $\ast r = r$, but the set of hyperreal numbers $\ast\mathbb{R}$, is a proper extension of $\mathbb{R}$.

(1.2.2) (Transfer Principle) If $A_1, \dots, A_n \in V(\mathbb{R})$, then a bounded quantifier statement $\varphi(A_1, \dots, A_n)$ holds in $V(\mathbb{R})$ if and only if $\varphi(\ast A_1, \dots, \ast A_n)$ holds in $V(\ast\mathbb{R})$.

(1.2.3) (ω_1 -saturation) If $\{A_n \mid n \in \mathbb{N}\}$ is a decreasing sequence of non-empty internal sets, then $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

An object $A \in V(\ast\mathbb{R})$ is called standard if $A = \ast B$ for some $B \in V(\mathbb{R})$. An object $A \in V(\ast\mathbb{R})$ will be called internal if $A \in \ast B$ for some $B \in V(\mathbb{R})$. All other objects will be called external. For example the set Δ_t of the previous section is internal while the set $\mathbb{N} \subset \ast\mathbb{N}$ of standard natural numbers and the set

$$\Delta = \{ \omega \in \Omega_{\mathbb{N}} \mid B_{\mathbb{N}}(1/2, \omega) \leq n \text{ for each } n \in \mathbb{N} \}$$

defined in terms of $\mathbb{N}$ are external. Every infinite set $A \in V(\ast\mathbb{R})$ is properly extended, $\ast A \supset \{ \ast a \mid a \in A \}$, by (1.2.2).

Notation A set $E \subset V(\ast\mathbb{R})$ is called $\ast$ -finite if it satisfies the $\ast$ -transform of the notion of finiteness, e.g., there exists an internal bijection $f: E \rightarrow \{k \in \ast\mathbb{N} \mid k \leq n\}$ for some $n \in \ast\mathbb{N}$. The set $\Omega_{\mathbb{N}}$ is $\ast$ -finite, but it has uncountably infinite external cardinality.

Some basic external notions are as follows.

(1.2.4) Definition

An element $r \in {}^*\mathbb{R}$ is infinite if $|r| > n$ for each $n \in \mathbb{N}$, and it is infinitesimal ($r \approx 0$) if $r < 1/n$ for each standard natural number $n \in \mathbb{N}$. We say that $x \approx y$ (x is infinitely near y) if $x - y \approx 0$.

The existence of infinite and infinitesimal hyperreals follows from (1.2.1) and (1.2.2), in particular ${}^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$. Each finite hyperreal is infinitely near a standard real while an infinite hyperreal is not. The compact internal interval ${}^*[0,1]$ has all its elements near standard points but ${}^*\mathbb{R}$ does not.

We can define the notion of $\approx$ in an arbitrary metric space M with metric d in the following way.

(1.2.5) Definition

Let (M, d) be a standard metric space. For x and y in *M we say that $y \approx x$ if ${}^*d(x, y) \approx 0$. An element $y \in {}^*M$ is near-standard for d if there exists an x in M such that $x \approx y$. In this case we define $st_d(y) = x$ (x is the standard part of y for the metric d). We denote by $ns_d({}^*M)$ the set of near-standard points in *M .

1.3 Continuity and the Space $C[0,1]$

Let $(C[0,1], \kappa)$ be the metric space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the metric κ of uniform convergence. Recall that $(C[0,1], \kappa)$ is a Polish space, that is, the topology on $C[0,1]$ is given by a complete separable metric. By $*$ -transfer there is a $*$ metric space $({}^*C[0,1], {}^*\kappa)$. A function f

$\epsilon^*\mathbb{C}[0,1]$ will be a $*$ -continuous function. In the same way as it is true that $^*\mathbb{R}$ contains elements that are not near-standard, $^*\mathbb{C}[0,1]$ contains elements that are not near any element of $\mathbb{C}[0,1]$. It is clear then that if we want to study standard continuous functions by means of elements of $^*\mathbb{C}[0,1]$ we need more than just the $*$ -transform property of continuity.

(1.3.1) Definition

Let $\Gamma \subset ^*[0,1]$ and $g: \Gamma \rightarrow ^*\mathbb{R}$ be a function. We say that g is S-continuous relative to Γ if $g(x) \approx g(y)$ whenever $x \approx y$ for each $x, y \in \Gamma$.

Suppose that in the previous definition Γ is S-dense (i.e. $\text{st}(\Gamma) = [0,1]$), and that $g(\Gamma) \subset \text{ns}(^*\mathbb{R})$, then we can define a function $\tilde{g}: [0,1] \rightarrow \mathbb{R}$ by

$$\tilde{g}(r) = \text{st } g(\text{st}_\Gamma^{-1}(r)),$$

where st_Γ is the restriction of st to Γ . That is, if $x, y \in \text{st}_\Gamma^{-1}(r)$ then $x \approx y$ so that $\text{st}(g(x)) = \text{st}(g(y))$ by S-continuity, i.e., $\tilde{g}$ is well defined by any choice of $\text{st}_\Gamma^{-1}(r)$.

(1.3.2) Proposition

If $g: \Gamma \rightarrow ^*\mathbb{R}$ is internal, then the function $\tilde{g}: [0,1] \rightarrow \mathbb{R}$ defined above is a standard real valued ϵ - δ -continuous function.

(See Stroyan and Bayod [1986, (APP.1.5)])

The next theorem tells us that the class of finite valued S-continuous functions on $^*[0,1]$ are precisely the functions which are infinitely near a standard continuous function.

(1.3.3) Theorem ($\approx$ Ascoli-Arzela Theorem)

A function $f \in {}^*\mathbb{C}[0,1]$ is in $ns_\kappa({}^*\mathbb{C}[0,1])$ if and only if f is finite and S -continuous. Moreover, for each $r \in [0,1]$

$$(st_\kappa f)(r) = st f(r).$$

(See Stroyan and Bayod [1986.(5.1.30)])

If we have a finite S -continuous internal function $g: T_N \rightarrow {}^*\mathbb{R}$ defined on our discrete time axis T_N , then the function $\tilde{g}$ is the standard part with respect to κ of the internal function $f: {}^*[0,1] \rightarrow {}^*\mathbb{R}$ obtained by linearly interpolating g between the values of T_N . It is easy to see that f is finite valued and S -continuous.

Many times it is convenient to think of such an object as our process $B_N: T_N \times \Omega_N \rightarrow {}^*\mathbb{R}$ as a random element, $B_N: \Omega_N \rightarrow \{ f: T_N \rightarrow {}^*\mathbb{R} \}$. In this case we have that if N is infinite, the set T_N is S -dense in $[0,1]$, and so, if for a particular $\omega \in \Omega_N$ $B_N(\cdot, \omega)$ is finite and S -continuous then $\tilde{B}_N(\cdot, \omega)$, the standard part of the path for this particular ω , is continuous. Notice that the internal process is defined on T_N while $\tilde{B}_N$ is defined on the standard set $[0,1]$.

1.4 The Space $\mathcal{D}[0,1]$

We turn next to a characterization of $\mathcal{D}[0,1]$ the space of right continuous functions with left limits. Recall that $\mathcal{D}[0,1]$ with Kolmogorov's metric for Skorohod's J_1 topology is a complete separable metric space. This means that $(\mathcal{D}[0,1], J_1)$ is a Polish space. (See Stroyan and Bayod, [1986, 5.3]).

(1.4.1) Definition

Let $\Gamma \subset {}^*\mathbb{R}$ be an S-dense set, that is, $\text{st}(\text{ns}(\Gamma)) = \mathbb{R}$. Let $f: \Gamma \rightarrow {}^*\mathbb{R}$ be a function, then we say that $b \in \mathbb{R}$ is the right S-limit of f at $r \in \mathbb{R}$ relative to Γ ,

$$\text{S-lim}_{x \downarrow r} f(x) = b,$$

if for every positive ϵ in $\mathbb{R}$ there is a positive η , such that $x \in \Gamma$ and $r \ll x < r + \eta$ implies $|f(x) - b| < \epsilon$.

($r \ll x$ means that $r < x$ but x is not infinitely near r).

(1.4.2) Definition

Let $\Gamma \subset {}^*[0,1]$ and $f: \Gamma \rightarrow {}^*\mathbb{R}$ be a function. We say that f has S-separated jumps relative to Γ if for each standard r in $]0,1[$ there exists $s \approx r$ in Γ such that if $r_1 \approx r_2 \approx r$ and $r_1 < r_2 < s$, then $f(r_1) \approx f(r_2)$ and if $s_1 \approx r$ and $s < s_1$ then $f(s) \approx f(s_1)$. Also, if η and ι are positive infinitesimals $f(0) \approx f(\eta)$ and $f(1-\eta) \approx f(1-\iota)$.

(1.4.3) Theorem

A function $f \in {}^*\mathcal{D}[0,1]$ is in $\text{ns}_J({}^*\mathcal{D}[0,1])$ if and only if it takes finite values and has S-separated jumps. Moreover,

$$(\text{st}_J f)(r) = \text{S-lim}_{s \downarrow r} f(s).$$

(See Stroyan and Bayod, [1986, (5.3.7)])

As in the case of S-continuous functions it is enough for f to have S-separated jumps relative to an S-dense subset $\Gamma \subset {}^*[0,1]$.

The next proposition is a hyperfinite version of Lemma 3 in Cramer [1966]. It is basically a technical result, which will be used in the proof of Theorem (1.8.1).

(1.4.4) Proposition

Let T be a discretization of ${}^*[0,1]$ with step size $\delta t = 2^{-\kappa}$ for some $\kappa \in {}^*\mathbb{N} \setminus \mathbb{N}$. For each hypernatural $n \leq \kappa$ and for each hypernatural k , $0 \leq k \leq 2^n - 1$, let $B(n,k)$ be the set of all internal functions $f: T \rightarrow {}^*\mathbb{R}$ such that at least one of the following inequalities holds,

$$|f(k2^{-n}) - f((k-1)2^{-n})| \leq g(2^{-(n-1)}), \text{ or} \\ |f(k2^{-n}) - f((k+1)2^{-n})| \leq g(2^{-(n-1)}),$$

where g a function such that $g(h) \downarrow 0$ as $h \downarrow 0$, and $\sum_{n \in \mathbb{N}} g(2^{-n}) < \infty$.

If there exist a standard natural number n_0 such that

$$(1.4.5) \quad f \in C = \bigcap_{n=n_0}^{\kappa} \bigcap_{k=1}^{2^n-1} B(n,k),$$

then f has S -separated jumps relative to T .

We will first prove two preliminary lemmas.

(1.4.6) Lemma

If $f \in C$ and if $|f[k2^{-n}] - f[(k+1)2^{-n}]| < g(2^{-n})$ then for each $t \in [k2^{-n}, (k+1)2^{-n}] \cap T$

$$(1.4.7.a) \quad |f[k2^{-n}] - f[t]| < \sum_n^{\kappa} g[2^{-j}], \text{ and}$$

$$(1.4.7.b) \quad |f[(k+1)2^{-n}] - f[t]| < \sum_n^{\kappa} g[2^{-j}]$$

Proof

Bisect the interval $[k2^{-n}, (k+1)2^{-n}]$ at the point $(2k+1)2^{-n-1}$. Since $f \in B(n+1, 2k+1)$ at least one of the two inequalities below must hold,

$$(1.4.8.a) \quad |f[(2k+1)2^{-n-1}] - f[k2^{-n}]| < g[2^{-n-1}]$$

$$(1.4.8.b) \quad |f[(k+1)2^{-n}] - f[(2k+1)2^{-n-1}]| < g[2^{-n-1}].$$

Since $|f[k2^{-n}] - f[(k+1)2^{-n}]| < g(2^{-n})$ it follows that

$$\begin{aligned} |f[(2k+1)2^{-n-1}] - f[k2^{-n}]| &< g[2^{-n-1}] + g[2^{-n}], \text{ and} \\ |f[(k+1)2^{-n}] - f[(2k+1)2^{-n-1}]| &< g[2^{-n-1}] + g[2^{-n}]. \end{aligned}$$

Again we can bisect each interval and use the fact that $f \in B(n+2, 4k+1) \cap B(n+2, 4k+3)$ to obtain that

$$\begin{aligned} |f[r2^{-n-2}] - f[(k)2^{-n}]| &< g[2^{-n}] + g[2^{-n-1}] + g[2^{-n-2}], \text{ and} \\ |f[(k+1)2^{-n}] - f[r2^{-n-2}]| &< g[2^{-n}] + g[2^{-n-1}] + g[2^{-n-2}], \end{aligned}$$

for $r = 4k, 4k+1, \dots, 4k+4$. We obtain inequalities (1.4.7.a) and (1.4.7.b) by repeating the bisection procedure until it is no longer possible to bisect an interval by an element of T . $\square$

(1.4.9) Lemma

Suppose that $f \in C$ then for each $n \geq n_0$ and each $k = 1, \dots, 2^n - 1$, there exists a $t(n, k) \in [(k-1)2^{-n}, (k+1)2^{-n}] \cap T$ such that

$$\begin{aligned} |f[t] - f[(k-1)2^{-n}]| &< \sum_n^k g(2^{-j}) \quad \text{if } (k+1)2^{-n} \leq t < t(n, k), \\ |f[(k+1)2^{-n}] - f(t)| &< \sum_n^k g(2^{-j}) \quad \text{if } t(n, k) \leq t \leq (k+1)2^{-n}. \end{aligned}$$

Proof

Let $L(n,k)$ be the set of all $t \in T \cap [(k-1)2^{-n}, (k+1)2^{-n}]$ such that the first inequality is satisfied, and $U(n,k)$ the set where the second inequality is satisfied. Since $f \in B(n,k)$ at least one of the inequalities

$$\begin{aligned} |f[k2^{-n}] - f[(k-1)2^{-n}]| &< g(2^{-n}), \quad \text{or} \\ |f[(k+1)2^{-n}] - f[k2^{-n}]| &< g(2^{-n}) \end{aligned}$$

must hold. If both inequalities hold then by Lemma (1.4.6) we can choose $t(n,k) = k2^{-n}$. Assume that the first inequality holds but not the second. Then by Lemma (1.4.6), $[(k-1)2^{-n}, k2^{-n}] \cap T \subset L(n,k)$. Bisect the interval $[k2^{-n}, (k+1)2^{-n}]$ at the point $(2k+1)2^{-n-1}$. Since $f \in B(n+1, 2k+1)$ at least one of the inequalities (1.4.8) must be satisfied. If both are satisfied then choose $t(n,k) = (2k+1)2^{-n-1}$. Now assume that (1.4.8.b) is satisfied but not (1.4.8.a). Then $[(2k+1)2^{-n-1}, (k+1)2^{-n}] \cap T \subset U(n,k)$. Split the interval $[(2k+1)2^{-n-1}, k2^{-n}]$ at the point $(4k+1)2^{-n-2}$, and procede as above until a bisection is no longer needed or possible with a point in T . In the last case we can choose $t(n,k) \pm \delta t$ as the last point of bisection according to whether the last bisection was done from the right or from the left. □

Proof of Proposition (1.4.3)

Assume that f does not have S -separated jumps. Then there exist $t_1 \approx t_2 \approx t_3$, with $t_1 < t_2 < t_3$ such that $f(t_1) \neq f(t_2)$ and $f(t_2) \neq f(t_3)$. Choose an infinite n such that $(k-1)2^{-n} \leq t_1$

$< t_3 \leq (k+1)2^{-n}$. Since $f \in C$ and n is infinite, by Lemma (1.4.9) there exist $t(n,k) \in [(k-1)2^{-n}, (k+1)2^{-n}] \cap T$ such that

$$\begin{aligned} |f[(k-1)2^{-n}] - f[t]| &\approx 0 \quad \text{if } (k-1)2^{-n} \leq t < t(n,k), \text{ and} \\ |f[t] - f[(k+1)2^{-n}]| &\approx 0 \quad \text{if } t(n,k) \leq t \leq (k+1)2^{-n}. \end{aligned}$$

Hence, if t_1 and t_2 are both to the left or both to the right of $t(n,k)$, for example to the left, then $f[t_1] \approx f[(k-1)2^{-n}] \approx f[t_2]$, which is a contradiction. Similarly, if t_2 and t_3 are both to the left or both to the right of $t(n,k)$, we obtain a contradiction. So f must have S -separated jumps. $\square$

1.5 Loeb Probability Theory

So far the theory above shows us how to deal with the time aspects of the process B_N , changing internal times and values to standard times and values provided $B_N(\cdot, \omega)$ is "nice". Our next step is to introduce the relevant notions for the treatment of the probabilistic aspects of the problem, that is, to show that B_N is in some sense near a standard stochastic process. Although $(\Omega_N, \mathcal{A}_N, P_N)$ is sufficient to deal with all the internal probabilistic aspects of B_N , it is not a standard probability space. First, P_N takes its values in $^*[0,1]$. We can correct this by taking $\text{st}P_N(A)$ for $A \in \mathcal{A}_N$. However, $\mathcal{A}_N$ is not a σ -algebra. The next theorem due to Loeb [1975] tells us that there is a unique probability space extending $\text{st}P_N$ from $\mathcal{A}_N$ to the σ -algebra it generates.

(1.5.1) Theorem

Let $(\Omega, \mathcal{A}, P)$ be an internal probability space and let $L(A)$ be the external σ -algebra generated by A . Then there exists a unique probability $\tilde{P}$ on $L(A)$ such that, if $A \in \mathcal{A}$ then

$$\tilde{P}(A) = \text{st}P(A).$$

Moreover, for each $B \in L(A)$ we have that $\tilde{P}(A \Delta B) = 0$ for some $A \in \mathcal{A}$.

(See Stroyan and Bayod [1986, (1.1.6)])

Notation. We will call $L(A)$ the Loeb σ -algebra generated by A , and call $\tilde{P}$ the Loeb measure induced by P . Our notation differs from the custom of letting $L(A)$ denote the $\tilde{P}$ -completion of $L(A)$. We will say that a subset of Ω is $\tilde{P}$ -measurable if it belongs to the $\tilde{P}$ -completion of $L(A)$.

So far we know that $\tilde{B}_N(\cdot, \omega)$ is a continuous function if $B_N(\cdot, \omega)$ is finite and S -continuous. Our next problem is to show that the external set

$A = \{ \omega \in \Omega_N \mid B_N(\cdot, \omega) \text{ is finite and } S\text{-continuous} \}$
is $L(A_N)$ -measurable.

If $(V_1, \mathcal{F}_1)$ and $(V_2, \mathcal{F}_2)$ are two measurable spaces we say that the function $f: V_1 \rightarrow V_2$ is $\mathcal{F}_1/\mathcal{F}_2$ -measurable if $f^{-1}(F) \in \mathcal{F}_1$ for each $F \in \mathcal{F}_2$. We will indicate this by writing $f: (V_1, \mathcal{F}_1) \rightarrow (V_2, \mathcal{F}_2)$. If M is a metric space, $\mathcal{B}(M)$ (or just $\mathcal{B}$) will denote its Borel σ -algebra. The next theorem asserts the measurability of the standard part map.

(1.5.2) Theorem

Let V be a Polish space, that is, the topology on V is given by a complete separable metric. Let $\mathcal{B}$ be its Borel σ -algebra.

Suppose that Γ is an internal subset of *V such that $\text{st}_\Gamma(\text{ns}(\Gamma)) = V$. Then if $\mathcal{F}$ is the algebra of internal sets of Γ , the map $\text{st}_\Gamma: \Gamma \rightarrow V$ is $L(\mathcal{F})/\mathcal{B}$ -measurable. In particular, $\text{ns}(\Gamma)$ is $L(\mathcal{F})$ measurable.

(See Stoll [1985, 2.1] or Stroyan and Bayod [1986, (2.1.2)] for a special case.)

(1.5.3) Definition

Let Ω be a $*$ -finite set, $\mathcal{A}$ be the internal algebra of all internal subsets of Ω and P be an internal probability. Then $(\Omega, L(\mathcal{A}), \tilde{P})$ is called a hyperfinite probability space.

(1.5.4) Definition

Let $(\Omega, L(\mathcal{A}), \tilde{P})$ be a hyperfinite probability space and M a standard metric space. An internal function $X: \Omega \rightarrow {}^*M$ is called an internal random element.

(1.5.5) Definition

Let $(\Omega, L(\mathcal{A}), \tilde{P})$ be a hyperfinite probability space, M a metric space and $f: \Omega \rightarrow M$. A lifting of f is an internal function $F: (\Omega, \mathcal{A}) \rightarrow ({}^*M, {}^*\mathcal{B})$ such that $\text{st}_M F = f$ $\tilde{P}$ -a.s.. An internal F that $\tilde{P}$ -a.s. has a standard part is automatically a lifting of $f = \text{st}_M F$.

(1.5.6) Remark Let T be S -dense in ${}^*[0,1]$, $f: \Omega \rightarrow \mathbb{C}[0,1]$ and F be an internal function defined on Ω such that $F(\omega)$ is $\tilde{P}$ -a.s. a finite and S -continuous function defined on T . We can think then

of F as a lifting of f in the sense that for each $r \in [0,1]$ $\text{st}F(\text{st}_T^{-1}(r)) = f(r)$ $\tilde{P}$ -a.s.. It is clear that by interpolating each $F(\omega)$ linearly we can get a lifting in ${}^*\mathbb{C}[0,1]$.

(1.5.7) Theorem

If M is a Polish space then a function $f: \Omega \rightarrow M$ has a lifting if and only if it is $\tilde{P}$ -measurable.

(See Stroyan and Bayod [1986, 5.1.4 & 5.1.6])

If in Definition (1.5.5) M is $\mathbb{R}^n$ then X is called an internal random variable. In this case we can define the internal cumulative distribution function of X as the internal function $F: {}^*\mathbb{R}^n \rightarrow {}^*[0,1]$ given by

$$F_X(x) = P(X \leq x),$$

where the inequality is taken componentwise. Similarly the internal characteristic function of X is the internal function $\phi_X: {}^*\mathbb{R} \rightarrow {}^*\mathbb{C}$ given by

$$\phi_X(u) = E(e^{iuX}) = \int e^{iux} dF(x).$$

(1.5.8) Theorem

For X and F_X as above, define for each x in $\mathbb{R}^n$

$$\hat{F}_X(x) = S\text{-}\lim_{y \downarrow x} F_X(y).$$

If X is $\tilde{P}$ -a.s. finite then $\text{st}X: (\Omega, \mathcal{L}(\mathcal{A})) \rightarrow (\mathbb{R}^n, \mathcal{B})$ is a random variable and we have that

$$\hat{F}_X(x) = \tilde{P}(\text{st}X \leq x) = F_{\text{st}X}(x).$$

The function $\text{st}X$ is arbitrarily defined on the null set where $X(\omega)$ is not finite.

(See Stroyan and Bayod [1986, (4.1.2)])

(1.5.9) Definition

Let $(\Omega, \mathcal{A}, P)$ be an internal probability space and $X: (\Omega, \mathcal{A}) \rightarrow (*\mathbb{R}^n, *\mathcal{B})$ be an internal random variable. We say that X is S -integrable if $E[X]$ is finite and $E[I_A X] \approx 0$ whenever $A \in \mathcal{A}$ and $P(A) \approx 0$. If $1 \leq p < \infty$ we define

$$SL^P(\Omega, \mathcal{A}, P) = \{ X: (\Omega, \mathcal{A}) \rightarrow (*\mathbb{R}^n, *\mathcal{B}) \mid |X|^P \text{ is } S\text{-integrable} \}.$$

1.6 Stochastic Processes

Our main reference for the standard definitions of this section is Shorak and Wellner [1986]. For an account of the infinitesimal theory of stochastic processes we recommend Stroyan and Bayod [1986].

Let F be a collection of real-valued functions defined on a set T . Usually T will be $[0, 1]$ or $\mathbb{R}$, so one usually thinks of T as "time". However there are cases when T is a more general index set. For each $k \geq 1$, and all $t_1, \dots, t_k$ in T let

$$\pi_{t_1, \dots, t_k}(x) = (x(t_1), \dots, x(t_k))$$

be the projection mapping of F into $\mathbb{R}^k$. For any $B \in \mathcal{B}(\mathbb{R}^k)$ the set $\pi_{t_1, \dots, t_k}^{-1}(B)$ is called a finite-dimensional subset of F .

The collection $\mathcal{F}_0$ of all finite-dimensional subset of F is an algebra of sets. Let $\mathcal{F} = \sigma(\mathcal{F}_0)$ and let $(\Omega, \mathcal{A}, P)$ a probability space, then the random element

$$X: (\Omega, \mathcal{A}) \rightarrow (F, \mathcal{F})$$

will be called a stochastic process with paths on F .

The above definition of a stochastic process agrees with the usual definition due to Doob. That is, $X: T \times \Omega \rightarrow \mathbb{R}$ is a stochastic process on F if for each $\omega \in \Omega$ $X(\cdot, \omega) \in F$, and for each $t \in T$ $X(t, \cdot)$ is a random variable.

We say that two processes X and Z are indistinguishable if $X = Z$ P -a.s., that is, for P -almost all ω , $X(t, \omega) = Z(t, \omega)$ for each $t \in T$.

If all the finite dimensional distributions of a process X on $(F, \mathcal{F})$ are multivariate normal, then we call X a gaussian process on F . A gaussian process S on F is called a brownian motion if its mean function $E[S(t)]$ and covariance function $\text{Cov}[S(s), S(t)]$ are given by

$$E[S(t)] = 0 \quad \text{and} \quad \text{Cov}[S(t), S(s)] = \min(s, t) \quad \text{for all } s, t.$$

(1.6.1) Definition

A brownian bridge or a tied down brownian motion on F is a gaussian process B on $(F, \mathcal{F})$ such that $E[B(t)] = 0$ and $\text{Cov}[B(s), B(t)] = \min(s, t) - st$ for all s, t .

In this work we will be interested only in stochastic processes with paths either in $\mathcal{C}[0, 1]$ or $\mathcal{D}[0, 1]$. The next two propositions tell us that a random element with values in $(\mathcal{C}[0, 1], \mathcal{B}_K)$ or in $(\mathcal{D}[0, 1], \mathcal{B}_J)$ is a stochastic process.

(1.6.2) Proposition

The Borel σ -algebra $\mathcal{B}_K$ on $(\mathcal{C}[0, 1], \kappa)$ is generated by the finite dimensional sets of $\mathcal{C}[0, 1]$.

(See Billingsley [1968. p. 19-20])

(1.6.3) Proposition

The Borel σ -algebra $\mathfrak{B}_J$ on $(\mathfrak{D}[0,1], J)$ is generated by the finite dimensional sets of $\mathfrak{D}[0,1]$.

(See Billingsley [1968. Theorem 14.5])

(1.6.4) Remark If $A \in \mathfrak{B}(\mathbb{R}^n)$ then the set $\{f \in \mathfrak{D}[0,1] \mid [f(t_1), \dots, f(t_n)] \in A\}$ is $\mathfrak{B}_J$ measurable. Since $\mathcal{C}[0,1]$ is closed in $(\mathfrak{D}[0,1], J)$ it is $\mathfrak{B}_J$ measurable. We have then that $\{f \in \mathfrak{D}[0,1] \mid [f(t_1), \dots, f(t_2)] \in A\} \cap \mathcal{C}[0,1] = \{f \in \mathcal{C}[0,1] \mid [f(t_1), \dots, f(t_2)] \in A\} \in \mathfrak{B}_K$. Thus $\mathfrak{B}_K$ is the trace of $\mathcal{C}[0,1]$ on $\mathfrak{B}_J$.

Recall that our goal is to obtain from the internal process B_N a standard brownian bridge. So far we know that if N is infinite and if $B_N(\cdot, \omega)$ is finite and S -continuous then $\tilde{B}_N(\cdot, \omega)$ is in $\mathcal{C}[0,1]$. Moreover suppose that B_N is $\tilde{P}$ -a.s. finite and S -continuous. Since B_N is a lifting in the sense of Remark (1.5.6) it follows from Theorem (1.5.7) and Proposition (1.6.2) that $\tilde{B}_N$ is indistinguishable from a stochastic process on $\mathcal{C}[0,1]$. In the next section we give sufficient conditions for the process B_N to be finite and S -continuous.

1.7 An S -continuity Theorem

(1.7.1) Theorem ($\approx$ Kolmogorov's Continuity Theorem)

Let Γ be a hyperfinite lattice in ${}^*\mathbb{R}^n$. $(\Omega, \mathcal{L}(A), \tilde{P})$ a hyperfinite probability space and let $Z: \Gamma \times \Omega \rightarrow {}^*\mathbb{R}$ be an internal process. If there exist positive real numbers K, p, r

such that

$$E[|Z(x)-Z(y)|^p] \leq K |x-y|^{n+r}$$

for all $x, y \in \Gamma$, then Z is $\tilde{P}$ -a.s. finite and S -continuous.

(See Albeverio, Fenstad, Hoegh-Krohn and Linstrom [1986.4.8.5]).

We will see in Chapter 2 that our internal process B_N satisfies the conditions of the above theorem. We will also show there that the finite dimensional distributions of $\tilde{B}_N$ are those of a brownian bridge. We can conclude then, that $\tilde{B}_N$ is indistinguishable from a brownian bridge on $C[0,1]$. We will see in Chapter 2 that we not only have constructed a brownian bridge, but that in fact we have shown that the brownian motion is the weak limit of a sequence of tied down random walks.

In the remaining two sections of this chapter we develop some technical material that will be used in the last sections of chapters 2 and 3.

1.8 An S -separated Jumps Theorem

The theorem below gives sufficient conditions for an internal process to have S -separated jumps. It is a hyperfinite version of a theorem by Cramer [1966]. We will use this theorem in Chapter 3 when we study the weak convergence of empirical processes. We believe that this theorem can be useful in establishing weak convergence when the limiting process is a Poisson process.

(1.8.1) Theorem ($\approx$ Cramer)

Let T be a hyperfinite discretization of ${}^*[0,1]$. $(\Omega, \mathcal{A}, P)$ a hyperfinite probability space and $X: T \times \Omega \rightarrow {}^*\mathbb{R}$ an internal stochastic process such that $X(\delta t)$ is finite. If there exist standard positive numbers K, p and r such that for each $0 \leq t < s < t+h \leq 1$

$$E(|X(s)-X(t)|^p |X(t+h)-X(s)|^p) \leq K h^{1+r}$$

then $\tilde{P}$ -a.s. X is finite and has S -separated jumps.

Proof

For notational simplicity assume that the lattice has step size $\delta t = 2^{-\kappa}$ for some $\kappa \in {}^*\mathbb{N} \setminus \mathbb{N}$. Fix a real number α , $0 < \alpha < \frac{r}{p}$. Then for $0 \leq t < s < t+h \leq 1$

$$\begin{aligned} P(|X(t)-X(s)| \geq h^\alpha, |X(s)-X(t+h)| \geq h^\alpha) &\leq \\ P(|X(t)-X(s)| |X(s)-X(t+h)| \geq h^{2\alpha}) &\leq \\ \frac{E[|X(t)-X(s)|^p |X(s)-X(t+h)|^p]}{h^{2\alpha p}} &\leq K h^{1+\epsilon}, \end{aligned}$$

where $\epsilon = r - 2\alpha p$. For each $m \leq \kappa$ and each $1 \leq k \leq 2^m - 1$, let $B(m, k)$ be the set of all internal functions $Y: T \rightarrow {}^*\mathbb{R}$ such that either $|Y(k2^{-m}) - Y((k-1)2^{-m})| \leq 2^{-\alpha(m-1)}$ or $|Y((k+1)2^{-m}) - Y(k2^{-m})| \leq 2^{-\alpha(m-1)}$. Then

$$\begin{aligned} \sum_{n \in \mathbb{N}} P\left[\bigcup_{m=n}^{\kappa} \bigcup_{k=1}^{2^m-1} \{\omega \mid X(\omega) \notin B(m, k)\} \right] &\leq \\ K \sum_{n \in \mathbb{N}} \sum_{m=n}^{\kappa} \sum_{k=1}^{2^m-1} P(X(\omega) \notin B(m, k)) &\leq \\ K \sum_{n \in \mathbb{N}} \sum_{m=n}^{\kappa} 2^{-(m-1)\epsilon} &\leq \sum_{n \in \mathbb{N}} \frac{2^{-\epsilon(n-1)}}{(1-2^{-\epsilon})} < \infty \end{aligned}$$

Thus, by the Borel-Cantelli Lemma there exists a set $\Delta \in \mathcal{L}(\mathcal{A})$

with $\tilde{P}(\Delta) = 0$ such that whenever $\omega \notin \Delta$ there exists a standard natural number $\eta(\omega)$ such that

$$X(\omega) \in \bigcap_{n=\eta(\omega)}^{\infty} \bigcap_{k=1}^{2^n-1} B(n,k).$$

By Proposition (1.4.3) it follows that for each $\omega \notin \Delta$, $X(\omega)$ has S -separated jumps relative to T . $\square$

1.9 Weak Convergence

We say that a sequence of probability measures $\{ \mu_n \mid n \in \mathbb{N} \}$ on a Polish space $(M, \mathcal{B})$ converges weakly to a probability measure μ on $(M, \mathcal{B})$ ($\mu_n \Rightarrow \mu$) if $\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$ for each bounded continuous function $f: M \rightarrow \mathbb{R}$.

Let X be an M -valued random element. For each $A \in \mathcal{B}$ define $\mu_X(A) = P(X^{-1}(A))$. Then μ_X is a probability measure on $(M, \mathcal{B})$. We call μ_X the law of X on M . If $\{ X_n \mid n \in \mathbb{N} \}$ and X are M -valued random elements we say that X_n converges weakly to X ($X_n \Rightarrow X$) if $\mu_n \Rightarrow \mu$, where μ_n and μ are the laws on $(M, \mathcal{B})$ of X_n and X respectively.

The next two theorems are powerful tools in proving weak convergence results.

(1.9.1) Theorem

If M is a Polish space and $\{ \mu_n \mid n \in \mathbb{N} \}$ is a sequence of probability measures on $(M, \mathcal{B})$, then $\mu_n \Rightarrow \mu$ if and only if for all $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$ and all $A \in \mathcal{B}$

$$\tilde{\mu}_\eta(st^{-1}(A)) = \mu(A).$$

(See Anderson and Rashid [1978, Theorem 4])

(1.9.2) Definition

Let $(M, \mathcal{B})$ be a metric space together with its Borel σ -algebra. We say that an internal probability measure μ on $({}^*M, {}^*\mathcal{B})$ is S -tight if for each internal set $\Delta \in {}^*\mathcal{B}$, such that $\Delta \cap ns({}^*M) = \emptyset$, $\mu(\Delta) \approx 0$.

The next result is our own variation on Theorem (1.9.1). It can be proved in a similar way.

(1.9.3) Theorem

Let M be a Polish space, N an infinite hypernatural number and $\{\mu_n \mid n \leq N\}$ a family of internal probability measures on $({}^*M, {}^*\mathcal{B})$. For each n define a measure on $\hat{\mu}_n = \tilde{\mu}_n \circ st_M^{-1}$ on $(M, \mathcal{B})$. If μ_N is S -tight and if for infinite n the measures $\hat{\mu}_n$ are the same, then the common measure $\hat{\mu}$ is a probability measure and $\hat{\mu}_n \Rightarrow \hat{\mu}$.

2 TIED DOWN RANDOM WALKS AND THE BROWNIAN BRIDGE

2.1 Introduction

In this chapter we give a new construction of the brownian bridge as the standard part of a hyperfinite random walk conditioned to return to the origin at time $t=1$. This construction is similar to Anderson's [1976] construction of brownian motion. The construction below has at least two advantages when compared with other classical constructions of the brownian bridge. First, it formalizes the heuristic characterization of the brownian bridge as a normalized brownian motion conditioned to return to the origin at time $t=1$. Second, it implicitly contains an invariance principle. More specifically we will show that the brownian bridge is the weak limit of a sequence of conditioned random walks which have been properly normalized.

2.2 The Construction

For each $N \in {}^*\mathbb{N}$ let T_N be a discretization of step size $\delta t = \frac{1}{2N}$ of the time interval ${}^*]0,1]$.

$$T_N = \{ t \in {}^*]0,1] \mid t = k\delta t, \ k \in {}^*\mathbb{N} \}.$$

Now consider the probabilistic experiment where at each t in T_N we sample without replacement from an urn containing $2N$ identical balls, N of which are marked "+1" and the remaining N "-1". Let Ω_N then be the set of all possible outcomes of

such an experiment. We can think formally of Ω_N as follows:

$$\Omega_N = \{\omega: T_N \rightarrow \{-1, 1\} \mid \#\{\omega^{-1}(-1)\} = \#\{\omega^{-1}(1)\}\},$$

where "#" denotes the internal cardinality of a set.

Since all the outcomes ω in Ω are equally likely, we have that the internal probability of ω is $P_N(\omega) = \frac{1}{\binom{2N}{N}}$.

(2.2.1) Definition

The internal process $B_N : T_N \times \Omega_N \rightarrow \Gamma_N$, such that

$$B_N(t, \omega) = \sum_{0 < s \leq t} \omega(s) \delta x,$$

where $\delta x = \sqrt{\delta t}$ and $\Gamma_N = \{x \in {}^*\mathbb{R} \mid x = k\delta x, k \in {}^*\mathbb{Z} \cap [-N, N]\}$ will be called a hyperfinite "tied down" random walk.

We will now state some properties of the process B_N . For simplicity we will carry out the computations below only at time points which are even multiples of $\frac{1}{2N}$. The calculations for the odd multiples are similar.

For each $t = 2m\delta t$ in T_N and for each $x = 2k\delta x$ in Γ_N

$$(2.2.2) \quad P_N(B_N(t) = x) = \frac{\binom{N}{m+k} \binom{N}{m-k}}{\binom{2N}{2m}} \quad \text{if } |k| \leq m$$

and 0 if $|k| > m$,

$$(2.2.3) \quad E[B_N(t)] = 0,$$

$$(2.2.4) \quad E[B_N^2(t)] = t(1-t)\left(1 + \frac{1}{2N-1}\right).$$

The equality (2.2.2) follows at once if we think in terms of our urn model. If $B_N(t) = x$ then the difference between the number of "+1" balls and the number of "-1" balls is $2k$. That is, of the total $2m$ balls sampled up to time t , $m+k$ are "+1" balls while $m-k$ are "-1" balls. Since $P_N(B_N(t) = x) = P_N(B_N(t) = -x)$ (2.2.3) follow at once.

Formula (2.2.4) is obtained from a direct calculation as follows:

$$\begin{aligned}
 E[B_N^2(t)] &= \sum_{k=-m}^m (2k\delta x)^2 P(B_N(2m\delta t) = 2k\delta x) \\
 &= (2\delta x)^2 \sum_{k=-m}^m k^2 \frac{\binom{N}{m+k} \binom{N}{m-k}}{\binom{2N}{2m}} \\
 (2.2.5) \quad &= 4\delta t \sum_{n=0}^{2m} (n-m)^2 \frac{\binom{N}{n} \binom{N}{2m-n}}{\binom{2N}{2m}}.
 \end{aligned}$$

Noting that $\frac{\binom{N}{n} \binom{N}{2m-n}}{\binom{2N}{2m}} = P(H = n)$, where H is a random variable with hypergeometric distribution with parameters $(2m, N, 2N)$, we see at once that the sum in (2.2.5) is just the variance of H , which is equal to $\frac{(N-m)m}{(2N-1)}$. (See Johnson and Kotz [1969, p.144]). It follows then that,

$$E[B_N^2(t)] = \frac{4\delta t(N-m)m}{(2N-1)} = (1-t)t\left(1 + \frac{1}{2N-1}\right).$$

With a little bit more of effort we can show that, if t is as above and $s = t + 2q\delta t$ then

$$(2.2.6) \quad E[B_N(t)B_N(s)] = (1-s)t\left(1 + \frac{1}{2N-1}\right).$$

To prove this we use the double expectation formula to write

$$(2.2.7) \quad E[B_N(t)B_N(s)] = E[B_N(t)E[B_N(s)|B_N(t)]]$$

To compute $E[B_N(s)|B_N(t)]$, let $B_N(t) = 2p\delta x$, so that

$$\begin{aligned} E[B_N(s) | B_N(t)=2p\delta x] &= \sum_{k=p-q}^{p+q} (2k\delta x) P(B_N(s)=2k\delta x | B_N(t)=2p\delta x) \\ &= 2\delta x \sum_{k=p-q}^{q+p} k \frac{\binom{N-m-p}{q+k-p} \binom{N-m+p}{q-k+p}}{\binom{2N-2m}{2q}} \\ &= 2\delta x \sum_{n=0}^{2q} ((p-q)+n) \frac{\binom{N-m-p}{n} \binom{N-m+p}{2q-n}}{\binom{2N-2m}{2q}} \\ &= 2\delta x(p-q + E[H]), \end{aligned}$$

where H is now a random variable with hypergeometric distribution with parameters $(2q, N-m-p, 2N-2m)$. Since $E[H] = \frac{q(N-m-p)}{N-m}$, we get

$$E[B_N(s)|B_N(t)] = B_N(t) \frac{1-s}{1-t}$$

Substituting the above result in (2.2.7) and then using (2.2.4) we get

$$E[B_N(s)B_N(t)] = (1-s)t(1 + \frac{1}{2N-1}).$$

Before we proceed with our construction, we need to show the following property of B_N .

(2.2.8) Proposition

For each s and t in T_N

$$E[|B_N(s) - B_N(t)|^4] \leq 3(s-t)^2 \frac{(2N)^2}{(2N-1)(2N-3)}$$

Proof

Let $t = 2m\delta t$ and $s = 2(m+q)\delta t$. The case when either t or s is an odd multiple of δt is similar. We compute first

$$P(B_N(s) - B_N(t) = 2k\delta x)$$

$$P(B_N(s) - B_N(t) = 2k\delta x) = \sum_{p=-m}^m P(B_N(s) - B_N(t) = 2k\delta x | B_N(t) = 2p\delta x) \\ \times P(B_N(t) = 2p\delta x)$$

$$= \sum_{p=-m}^m \frac{\binom{N-m-p}{q+k} \binom{N-m+p}{q-k}}{\binom{2N-2m}{2q}} \frac{\binom{N}{m+p} \binom{N}{m-p}}{\binom{2N}{2m}}$$

$$= \frac{\binom{2q}{q-k}}{\binom{2N}{N}} \sum_{p=-m}^m \binom{2N-2n-2q}{N-q-k-m-p} \binom{2m}{m+p},$$

after rearranging the binomial coefficients;

$$= \frac{\binom{2q}{q+k}}{\binom{2N}{N}} \sum_{n=0}^{2m} \binom{2N-2n-2q}{N-q-k-n} \binom{2m}{n}$$

$$= \frac{\binom{2q}{q+k}}{\binom{2N}{N}} \sum_{n=0}^{N-p-k} \binom{2N-2n-2q}{N-q-k-n} \binom{2m}{n}$$

$$= \frac{\binom{2q}{q+k} \binom{2N-2q}{N-q-k}}{\binom{2N}{N}}.$$

We obtained the limits of the last sum by noting that if $2m > N-q-k$ then $\binom{2N-2m-2q}{N-q-k-n} = 0$ for $n > N-q-k$; and if

$2m < N-q-k$, then $\binom{2m}{n} = 0$ for $n > 2m$.

With the above results at hand it is easy to estimate $E[|B(t)-B(s)|^4]$.

$$\begin{aligned} E[|B_N(t)-B(s)|^4] &= \sum_{k=-q}^q (2k\delta x)^4 P(B_N(t)-B_N(s) = 2k\delta x) \\ &= (2\delta x)^4 \sum_{k=-q}^q k^4 \frac{\binom{2q}{q+k} \binom{2N-2q}{N-q-k}}{\binom{2N}{N}} \\ &= (2\delta x)^4 \sum_{n=0}^{2q} (n-q)^4 \frac{\binom{2q}{n} \binom{2N-2q}{N-n}}{\binom{2N}{N}} \\ &= (2\delta x)^4 \mu_4, \end{aligned}$$

where μ_4 is the fourth central moment of a hypergeometric distribution with parameters $(N, 2q, 2N)$. The computation of μ_4 is simplified by the use of a recursive relation for the central moments of a hypergeometric distribution (see Johnson and Kotz [1969, p.145]). After some algebraic manipulation we have that

$$\mu_4 = \frac{3q^2(N-q)^2}{(2N-1)(2N-3)} - \frac{Nq(N-q)}{(2N-1)(2N-3)},$$

hence $E[|B(s)-B(t)|^4] \leq 3|s-t|^2 \frac{(2N)^2}{(2N-1)(2N-3)}$.

The above results hold regardless of whether N is infinite or finite. However if N is infinite then

$$E[|B(t)-B(s)|^4] \leq 4|t-s|^2 \quad \text{for each } s, t \in T_N.$$

(2.2.9) Proposition

If N is infinite then,

- (a) $B_N^2(t)$ is in SL^1 for each $t \in T_N$
- (b) B_N has its paths $\tilde{P}$ -a.s. finite and S -continuous.

Proof

To prove (a) note that for each $A \in \mathcal{A}$

$$E[I_A B_N^2(t)] \leq \sqrt{E[I_A]} \sqrt{E[B_N^4(t)]}.$$

If $P(A) \approx 0$ then $E[I_A B_N^2(t)] \approx 0$ since $E[B_N^4(t)] < \infty$ by (2.2.8). It follows that $B_N^2(t) \in SL^1$ for each $t \in T_N$.

The results on (b) follow at once from Theorem (1.7.1) and Proposition (2.2.8). □

We are now ready to state the main result of this section.

(2.2.10) Theorem

For each infinite N , there exists a set $\Delta_N \in L(A_N)$ with $\tilde{P}(\Delta_N) = 0$ such that the process $\tilde{B}_N: [0,1] \times \Omega_N \rightarrow \mathbb{R}$ defined for each $r \in [0,1]$ by

$$\tilde{B}_N(r, \omega) = st[B_N(st_T^{-1}(r), \omega) I_{\Omega_N \setminus \Delta_N}(\omega)]$$

is a brownian bridge on $(\mathcal{C}[0,1], \kappa)$ defined on the space $(\Omega_N, L(A_N), \tilde{P}_N)$.

Proof

Let Δ_N be the $\tilde{B}_N$ -null set of (2.2.9.b). (we can take $\Delta_N \in L(A_N)$). For each $\omega \notin \Delta_N$, $B_N(\cdot, \omega)$ is S-continuous and finite so by (1.3.2) $\tilde{B}_N$ is well defined, and we have that $\tilde{B}_N(\cdot, \omega) \in C[0,1]$ for each $\omega \in \Omega_N$. Since B_N is a lifting in the sense of Remark (1.5.6), it follows from Theorem (1.5.7) and Proposition (1.6.2) that $\tilde{B}_N$ is a standard stochastic process with paths on $C[0,1]$.

From (2.2.3), (2.2.4) and the S-integrability of $B_N^2(t, \cdot)$ it follows that for each r_1 and r_2 in $[0,1]$ $E[\tilde{B}_N(r_1)] = 0$ and $E[\tilde{B}_N(r_1)\tilde{B}_N(r_2)] = \min(r_1, r_2) - r_1 r_2$. All that is left to show is that the finite dimensional distributions of $\tilde{B}_N$ are gaussian.

The proposition below concludes the proof of the theorem.

(2.2.11) Proposition

For each $0 < t_1 < t_2 < \dots < t_n < 1$, $n \in \mathbb{N}$, and for each $x_1, x_2, \dots, x_n \in ns(\Gamma_N)$, there exists an infinitesimal η such that

$$P(B_N(t_1)=x_1, \dots, B_N(t_n)=x_n) = \frac{\exp(-\bar{x}' D^{-1} \bar{x})}{\int (2\pi)^n \det(D)} (1 + \eta)(2\delta x)^n,$$

where $\bar{x}' = (x_1, \dots, x_n)$ and D is a covariance matrix with elements $(t_i(1-t_j))$, $1 \leq i \leq j \leq n$.

Proof

For notational convenience we will give the proof when $n = 2$. For other values of n the proof is essentially the same.

Let $t = 2m\delta t$, $s = 2(m+q)\delta t$, $x = 2k\delta x$ and $y = 2p\delta x$, with $0 \ll t \ll s \ll 1$ and $x, y \in ns(\Gamma_N)$. Then

$$P(B_N(t)=y, B_N(s)=x) = P(B_N(s)=x | B_N(t)=y) P(B_N(t)=y)$$

$$\begin{aligned} &= \frac{\binom{N-m-p}{q+k-p} \binom{N-m+p}{q-k+p}}{\binom{2N-2m}{2q}} \frac{\binom{N}{m+p} \binom{N}{m-p}}{\binom{2N}{2m}} \\ &= K \frac{\binom{N-m-p}{q+k-p} \binom{N-m+p}{q-k+p}}{\binom{N-m}{q}^2} \frac{\binom{N}{m+p} \binom{N}{m-p}}{\binom{N}{m}^2}, \end{aligned}$$

where $K = P(B_N(t)=0, B_N(s)=0)$. We compute first the value of K .

$$\begin{aligned} K &= \frac{\binom{N-m}{q}^2 \binom{N}{m}^2}{\binom{2N-2m}{2q} \binom{2N}{2m}} = \\ (2.2.12) \quad &= \frac{\binom{2N-2m-2q}{N-m-q} \binom{2m}{m} \binom{2q}{q}}{\binom{2N}{N}}. \end{aligned}$$

Using the *-transform of Stirling's inequality (see Feller [1968, p.54])

$$(2.2.13) \quad \sqrt{2\pi n} n^n \exp(-n + \frac{1}{12n+1}) \leq n! \leq \sqrt{2\pi n} n^n \exp(-n + \frac{1}{12n+1})$$

we have that,

$$(2.2.14) \quad \binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}} (1 + \epsilon),$$

where $\epsilon \approx 0$ if n is infinite. Replacing the above formula in (2.2.12) we get that

$$K = \frac{(2\delta x)^2}{\sqrt{(2\pi)^2 t(s-t)(1-s)}} (1 + \epsilon),$$

where $\epsilon \approx 0$.

Next we need to estimate the expression

$$\frac{\binom{N-m-p}{q+k-p} \binom{N-m+p}{q-k+p}}{\binom{N-m}{q}^2} \times \frac{\binom{N}{m+p} \binom{N}{m-p}}{\binom{N}{m}^2}.$$

If we expand the binomial coefficients the above expression is equal to

$$\frac{[(N-m-q)!]^2 [q!]^2 [m!]^2}{(N-m-q-k)! (q+k-p)! (N-m-q+k)! (q-k-p)! (m+p)! (m-p)!}.$$

The conditions on s, t, x and y make all the factorials above infinite. An application of Stirling's Inequalities yields that the above expression is equal to

$$\begin{aligned} & \frac{1}{\sqrt{1 - (\frac{k}{N-m-q})^2}} \times \frac{1}{\sqrt{1 - (\frac{p}{m})^2}} \times \frac{1}{\sqrt{1 - (\frac{k-p}{q})^2}} \times \\ & \times (1 - (\frac{k}{N-m-q})^2)^{-(N-m-q)} (1 - (\frac{p}{m})^2)^{-m} (1 - (\frac{k-p}{q})^2)^{-q} \times \\ & \times (\frac{N-m-q-k}{N-m-q+k})^k (\frac{m-p}{m+p})^p (\frac{q-(k-p)}{q+(k-p)})^{k-p}. \end{aligned}$$

We note that $(1 - (\frac{p}{m})^2) = 1 - \frac{y^2}{2t} \frac{1}{m} = 1 - \eta_1$, where $\eta_1 \approx 0$.

Similarly $(1 - (\frac{k}{N-m-q})^2) = 1 - \eta_2$ and $(1 - (\frac{k-p}{q})^2) = 1 - \eta_3$, where $\eta_2 \approx \eta_3 \approx 0$.

Next we have that

$$(1 - (\frac{p}{m})^2)^m = (1 - \frac{y^2}{2t} \frac{1}{m})^m = \exp(-\frac{y^2}{2t}) (1 - \epsilon_1).$$

Similarly $(1 - (\frac{k}{N-m-q})^2)^{N-m-q} = \exp(\frac{-x^2}{2(1-t)}) (1 - \epsilon_2)$ and

$$(1 - (\frac{k-p}{q})^2)^q = \exp(\frac{-(x-y)^2}{2(s-t)}) (1 - \epsilon_3), \text{ where } \epsilon_1 \approx \epsilon_2 \approx \epsilon_3 \approx 0.$$

If p is infinite then

$$(\frac{m-p}{m+p})^p = (1 - \frac{2p}{m+p})^p = (1 - \frac{y^2}{t+2p\delta t} \frac{1}{p})^p = \exp(\frac{-y^2}{t}) (1+\epsilon_1)$$

Similarly if k and $k-p$ are infinite we have

$$(\frac{N-m-q-k}{M-m-q+k})^k = \exp(\frac{-x^2}{1-s}) (1 + \epsilon_2), \text{ and}$$

$$(\frac{q-k+p}{q+k-p})^{k-p} = \exp(\frac{-(x-y)^2}{s-t}) (1 + \epsilon_3), \text{ where } \epsilon_1 \approx \epsilon_2 \approx \epsilon_3 \approx 0.$$

If p is finite then $(\frac{m-p}{m+p})^p \approx 1$. Similar results hold for the corresponding terms when k and $k-p$ are finite.

Putting all the above estimates together we have that

$$P(B_N(t)=x, B_N(t)=y) = \frac{(2\delta x)^2 (1-\eta)}{\int (2\pi)^n \det D} \exp(-\frac{1}{2}(y,x)D^{-1}(y,x)') ,$$

$$\text{where } \eta \approx 0 \text{ and } D = \begin{bmatrix} t(1-t) & t(1-s) \\ t(1-s) & s(1-s) \end{bmatrix}.$$

□

Going back to the proof of Theorem (2.2.10) we have that according to Theorem (1.5.7) the finite dimensional distributions of $\tilde{B}_N$ are gaussian. Hence $\tilde{B}_N$ is indeed a brownian bridge.

2.3 An Invariance Principle for Tied Down Random Walks

One of the advantages of the construction of the brownian bridge given in the previous section is that it implicitly

contains a weak convergence result. Explicitly we have that the brownian bridge is the weak limit of a sequence of tied down random walks.

We start by noticing that the construction of the space $(\Omega_n, \mathcal{A}_n, P_n)$ and of the process B_n itself are valid for each $n \in \mathbb{N}$. Since B_n is only defined on T_n we extend it to $[0,1]$ by means of the following linear interpolation:

$$B_n(r, \omega) = (k+1-nr)B_n\left(\frac{k}{n}, \omega\right) + (nr-k)B_n\left(\frac{k+1}{n}, \omega\right) \quad \text{if } \frac{k}{n} \leq r \leq \frac{k+1}{n}.$$

We can now consider B_n as a random element with values in the space $(\mathcal{C}[0,1], \mathcal{B})$ defined in Section 1.3. B_n induces a measure μ_n on $\mathcal{B}$ defined for each $A \in \mathcal{B}$ by $\mu_n(A) = P_n(\omega \mid B_n(\omega) \in A)$.

It is convinient to consider all of the above construction as a funtion F defined on $\mathbb{N}$ as follows:

$$F(n) = B_n: (\Omega_n, \mathcal{A}_n, P_n) \rightarrow (\mathcal{C}[0,1], \mathcal{B}, \mu_n).$$

The $*$ -transform of F is the function *F defined on ${}^*\mathbb{N}$ by:

$${}^*F(n) = B_n: (\Omega_n, \mathcal{A}_n, P_n) \rightarrow ({}^*\mathcal{C}[0,1], {}^*\mathcal{B}, \mu_n)$$

If n is finite then for each $\omega \in \Omega_n$ $B_n(\cdot, \omega)$ is in $ns_K({}^*\mathcal{C}[0,1])$. If n is infinite then there exist a $\tilde{P}_n$ -null set $\Delta_n \in \mathcal{L}(\mathcal{A}_n)$ outside of which $B_n(\cdot, \omega)$ is in $ns_K({}^*\mathcal{C}[0,1])$. The last remark follows from the fact that the restriction of B_n to T_n is $\tilde{P}_n$ -a.s. finite and S -continuous according to Proposition (2.2.9.b) and the fact that the linear interpolation of an S -continuous function with respect to T_n is an S -continuous element of ${}^*\mathcal{C}[0,1]$. For each $n \in {}^*\mathbb{N}$ we can now define the function $\tilde{B}_n: \Omega_n \rightarrow \mathcal{C}[0,1]$ as follows

$$\tilde{B}_n(\cdot, \omega) = \begin{cases} \text{st}_K B_n(\cdot, \omega) & \text{if } \omega \notin \Delta_n \\ 0 & \text{if } \omega \in \Delta_n \end{cases}$$

Since B_n is a lifting of $\tilde{B}_n$ it follows from Theorem (1.5.7) that $\tilde{B}_n$ is a random element on $\mathcal{C}[0,1]$. Hence by Proposition (1.6.2) $\tilde{B}_n$ is a standard stochastic process.

Although the next theorem is a natural extension of Donsker's Invariance Principle for brownian motion we have not been able to find any reference to it in the literature.

(2.3.1) Theorem

The sequence of tied down random walks $\{ B_n \mid n \in \mathbb{N} \}$ converges weakly on $(\mathcal{C}[0,1], \mathcal{B})$ to a brownian bridge B .

Proof

For each $n \in \mathbb{N}$ let μ_n be the measure on $\mathcal{B}$ induced by B_n . For n infinite the measure $\hat{\mu}_n$ define on each $A \in \mathcal{B}$ by

$$\hat{\mu}_n(A) = \tilde{\mu}_n(\text{st}_K^{-1}(A))$$

is independent of n , since

$$\mu_n(A) = \tilde{P}(B_n \in \text{st}_K^{-1}(A)) = \tilde{P}(\tilde{B}_n \in A)$$

and $\tilde{B}_n$ is a brownian bridge. It follows by Theorem (1.9.1) that $B_n \Rightarrow B$, a brownian bridge. We will call the law of B on $\mathcal{C}[0,1]$ the tied down Wiener measure on $\mathcal{C}[0,1]$.

3 THE HYPERFINITE EMPIRICAL PROCESS

3.1 Introduction

In this chapter we give a new construction of the brownian bridge as the standard part of a hyperfinite uniform empirical process. This construction implicitly contains an invariance principle. We will show that the brownian bridge is the weak limit of a sequence of uniform empirical processes. This result was conjectured by Doob [1949], yet a formal proof was not given until three years later by Donsker [1952]. Our construction makes the proof of this result almost trivial.

3.2 The Construction

For each $N \in {}^*\mathbb{N}$, let $T_N = \{t \in {}^*\mathbb{R} \mid 0 \leq t \leq 1 \mid t = k\delta t, k \in {}^*\mathbb{N}\}$ be a discretization of $[0, 1]$ with step size $\delta t = N^{-1}$. Let $\Omega_N = \{\omega: T_N \rightarrow T_N\}$ be the set of internal functions from T_N into itself, and let $\mathcal{A}_N$ be the algebra of internal subsets of Ω_N . For each $\omega \in \Omega_N$ define the internal uniform probability $P_N(\omega) = N^{-N}$. It is convenient to think of ω as the result of sampling N independent random variables with uniform distribution on T_N . We let $\delta x = \sqrt{\delta t}$ and $\Gamma_N = \{x \in {}^*\mathbb{R} \cap [-\sqrt{N}, \sqrt{N}] \mid x = k\delta x, k \in {}^*\mathbb{N}\}$ be a discretization of ${}^*\mathbb{R} \cap [-\sqrt{N}, \sqrt{N}]$ with step size δx .

(3.2.1) Definition

The internal process $U_N: T_N \times \Omega_N \rightarrow \Gamma_N$ defined by

$$U_N(t, \omega) = \frac{1}{\sqrt{N}} \sum_{s \in T_N} [I(\omega(s) \leq t) - t]$$

will be called the hyperfinite uniform empirical process of order N .

The following are some important properties of the process U_N .

(3.2.2) Proposition

For each $s, t \in T_N$

- (a) $E[U_N(t)] = 0$
- (b) $E[U_N(s) U_N(t)] = \min(s, t) - st$
- (c) $E[|U_N(s) - U_N(t)|^4] \leq 3(1 + \frac{1}{N|t-s|})|t-s|^2$

Moreover if $t \neq 1$ then for each $x \in \Gamma_N$

- (d) $\left| P_N\left(\frac{U_N(t)}{\sqrt{t(1-t)}} \leq x\right) - \Phi(x) \right| \leq 24 \delta x$

where Φ is the distribution function of a standard normal random variable.

Proof

Fix $N \in {}^*\mathbb{N}$. For convenience we will drop the subscript N unless there is the possibility of confusion. Let $Y(t) = \sum_{s \in T} I(\omega(s) \leq t)$ then $Y(t)$ has a binomial distribution with parameters (N, t) , ($Y(t) \sim \text{Bi}(N, t)$). We have that

$$E[U(t)] = E[\delta x(Y(t) - Nt)] = 0.$$

It follows at once that $E[U^2(t)] = t(1-t)$.

To show that (b) holds assume that $s \geq t$ and use the double expectation formula to write

$$(3.2.3) \quad E[U(t)U(s)] = E[U(t)E[U(s)|U(t)]]$$

We can compute $E[U(s)|U(t)]$ as follows:

Let $U(t) = x$, then

$$E[U(s)|U(t) = x] = \delta x E[Y(s)-sN | Y(t) = x\sqrt{N} + tN].$$

Since the conditional distribution of $Y(s)$ given that

$Y(t) = x\sqrt{N} + tN$ is the same as the random variable $x\sqrt{N} + tn + Y$,

where $Y \sim \text{Bi}(N-x\sqrt{N}-tN, \frac{s-t}{1-t})$, it follows that

$$E[U(s)|U(t)] = \frac{1-s}{1-t} U(s).$$

If we substitute this in (3.2.3) we get:

$$\begin{aligned} E[U(s)U(t)] &= \frac{1-s}{1-t} E[U^2(s)] \\ &= (1-s)t. \end{aligned}$$

The following direct computation gives (c).

$$E[|U(s) - U(t)|^4] = \delta x^4 E[Y(s)-Y(t) - N(s-t)]^4.$$

Noting that $Y(t)-Y(s) \sim \text{Bi}(N, s-t)$ we have that

$$\begin{aligned} E[|U(s)-U(t)|^4] &= \left(\frac{1}{N}\right)^2 \{3[(s-t)(1-(s-t)N)]^2 + \\ &\quad + N(s-t)(1-s+t)[1-6(s-t)(1-(s-t))]\} \\ &\leq 3(s-t)^2 \left(1 + \frac{1}{N(s-t)}\right) \end{aligned}$$

The approximation in (d) follows by *-transfer of a Berry-Esseen type of approximation (See Shorak & Wellner [1986, p. 11]). □

Remark The key ingredient of the above proof is the fact that the $I(\omega(s) \leq t)$ are independent Bernoulli trials. Thus we

can argue as above to show that the theorem is true when we extend t in the definition of $U_N(t)$ to be in $^*[0,1]$.

Now we are ready to prove the main theorem of this section.

(3.2.4) Theorem

For each infinite $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ there exists a set $\Delta_N \in L(A)$ with $\tilde{P}(\Delta_N) = 1$ such that the process $\tilde{U}: [0,1] \times \Omega_N \rightarrow \mathbb{R}$ defined by

$$\tilde{U}_N(r, \omega) = st[U_N(st_T^{-1}(r), \omega) I_{\Delta_N}(\omega)]$$

is a brownian bridge in $\mathcal{C}[0,1]$.

Proof

When N is infinite we can bound the right hand side of (3.2.2.d) by $6|s-t|^2$, hence U_N satisfies the conditions of Theorem (1.7.1) with $p = 4$, $K = 6$ and $r = 1$. It follows that there exists a set $\Delta_N \in L(A)$ with $\tilde{P}(\Delta_N) = 1$ on which the paths of U_N are finite and S -continuous. Hence $\tilde{U}_N$ is well defined, and for each $\omega \in \Omega$, $\tilde{U}_N(\cdot, \omega) \in \mathcal{C}[0,1]$.

Since U_N is a lifting of $\tilde{U}_N$ in the sense of Remark (1.5.6) it follows from Theorem (1.5.7) that $\tilde{U}_N$ is a random element on $(\mathcal{C}[0,1], \mathcal{B}_K)$. Hence by Proposition (1.6.2) $\tilde{U}$ is a standard process on $\mathcal{C}[0,1]$. Moreover by Proposition (3.2.2) together with Theorem (1.5.8) the finite dimensional distributions of the process $\tilde{U}_N$ are gaussian with mean zero and covariance structure $\min(r_1, r_2) - r_1 r_2$. Hence $\tilde{U}_N$ is a brownian bridge on $\mathcal{C}[0,1]$. $\square$

3.3 Invariance Principles

If $\{ \xi_i \mid i \in \mathbb{N} \}$ is a sequence of independent random variables defined on a common probability space Ω , each with uniform distribution on $[0,1]$, then $\tilde{V}_n$, the uniform empirical process of order n , is the process defined on $[0,1]$ as follows:

$$(3.3.1) \quad \tilde{V}_n(r) = \frac{1}{\sqrt{n}} \sum_{i=1}^n [I(\xi_i \leq r) - r].$$

It follows at once from the Central Limit Theorem that the limiting finite-dimensional distributions are jointly normal with mean zero and covariance structure $\min(r_1, r_2) - r_1 r_2$. (The same finite-dimensional distributions as those of a brownian bridge.)

In a landmark paper Doob [1949] conjectured the weak convergence of $\tilde{V}_n$ to a brownian bridge V . In fact in that same paper he made use of this conjecture to compute the asymptotic distributions of statistics which are functionals of $\tilde{V}_n$ by computing the distribution of the functional evaluated at V . A proof for this conjecture was given by Donsker [1952]. In this section we give a simple proof of this very important result.

We start by proving a simpler version of the above statement.

For $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{A}_n, P_n)$ and U_n be as in section 3.2. Extend U_n to $[0,1]$ as follows:

$$U_n(r) = \frac{1}{\sqrt{n}} \sum_{s \in T_n} [I(\omega(s) \leq r) - r].$$

We can think of U_n as a random element with values in the space $\mathcal{D}[0,1]$, the metric space of right continuous functions with left limits with Kolmogorov's metric for Skorohod's J_1 topology. Let $\mathcal{B}$ be the Borel σ -algebra on $\mathcal{D}[0,1]$, and let μ_n be the measure

induced on $\mathfrak{B}$ by U_n . (For $A \in \mathfrak{B}$, $\mu_n(A) = P_n(\omega \mid U_n(\omega) \in A)$). It is convenient to think of the above construction in terms of a function F defined on $\mathbb{N}$ by

$$F(n) = U_n : (\Omega_n, \mathcal{A}_n, P_n) \rightarrow (\mathfrak{D}[0,1], \mathfrak{B}, \nu_n)$$

and then take its $*$ -transform.

For each $n \in {}^*\mathbb{N}$ we can define a random element $\tilde{U}_n$ on $(\Omega_n, \mathcal{L}(\mathcal{A}_n), \tilde{P}_n)$ with values on $\mathfrak{D}[0,1]$ such that $\tilde{U}_n = \text{st}_J^* U_n$ $\tilde{P}_n$ -a.s. We point out that, if n is finite then $\tilde{U}_n = U_n$, while if n is infinite Theorem (3.2.4) and the remark preceding it, imply that $\tilde{U}_n$ is indistinguishable from a brownian bridge on $\mathcal{C}[0,1]$. As in the previous section we will let $\tilde{U}_n$ be identically zero on the set where *U_n is not in $\text{ns}_J({}^*\mathfrak{D}[0,1])$.

(3.3.2) Theorem

The sequence of processes $\{ \tilde{U}_n \mid n \in \mathbb{N} \}$ converges weakly on $(\mathfrak{D}[0,1], \mathfrak{B})$ to a brownian bridge $\tilde{U}$ on $\mathcal{C}[0,1]$.

Proof

By Theorem (1.5.2), when n is infinite the measure $\hat{\mu}_n$ defined on $\mathfrak{B}$ by $\hat{\mu}_n(A) = \tilde{\mu}_n(\text{st}_J^{-1}(A))$ is such that

$$\hat{\mu}_n(A) = \tilde{P}_n(\omega \mid \tilde{U}_n(\omega) \in A).$$

By Theorem (3.2.4) $\tilde{U}_n$ is a brownian bridge for each n in ${}^*\mathbb{N} \setminus \mathbb{N}$, so that $\hat{\mu}_n$ is independent of n . Hence by Theorem (1.9.1) we have that $\tilde{U}_n \Rightarrow \tilde{U}$, a brownian bridge on $\mathfrak{D}[0,1]$. Since $\mathcal{C}[0,1]$ is a measurable subset of $\mathfrak{D}[0,1]$ and since for infinite n the paths of $\tilde{U}_n$ are in $\mathcal{C}[0,1]$, $\tilde{U}$ is a brownian bridge on $\mathcal{C}[0,1]$. $\square$

We now modify the construction given in Section 3.2 to prove the following theorem.

(3.3.3) Theorem

There exists a sequence $\{\xi_n \mid n \in \mathbb{N}\}$ of independent random variables each with uniform distribution on $[0,1]$ and a brownian bridge $\tilde{V}$ on $C[0,1]$ all defined on a hyperfinite probability space $(\Omega, L(A), \tilde{P})$, such that the sequence of uniform empirical processes $\{\tilde{V}_n \mid n \in \mathbb{N}\}$ converges weakly on $\mathcal{D}[0,1]$ to $\tilde{V}$.

Let N be a fixed infinite hypernatural number and let $(\Omega, A, P) = (\Omega_N, A_N, P_N)$, the internal probability space defined at the beginnig of Section 3.2. For each $n \leq N$ define the internal random variable $X_n(\omega) = \omega(n\delta t)$. So defined $\{X_n \mid n \leq N\}$ is a family of internal random variables each with uniform distribution on T_N .

For each $n \leq N$ define the internal process V_n on T_N by

$$(3.3.4) \quad V_n(t) = \frac{1}{\sqrt{n}} \sum_1^n [I(X_i \leq t) - t]$$

Since for each $t \in T_N$, $\{I(X_i \leq t) \mid i \leq N\}$ is a family of independent Bernoulli random variables with mean t , Proposition (3.2.2) holds with U_N replaced by V_n and in (d) we can let x range over all of ${}^*\mathbb{R}$. Unfortunately the conditions for the S -continuity of V_n are no longer satisfied for all infinite $n \leq N$. For example if $n = \sqrt{N}$ and if $s-t = \delta t = N^{-1}$ then the right hand side of (3.2.2.d) is equal to $3(1 + \sqrt{N}) |s-t|^2$, which gives an infinite K in Theorem (1.7.1). We will show below that the conditions of Theorem (1.8.1) are satisfied and hence that for n infinite the paths of V_n are $\tilde{P}$ -a.s. finite and with S -separated jumps.

(3.3.5) Proposition

For each $n \leq N$ and for each $0 \leq t < s < t+h \leq 1$ in T

$$E[|V_n(s)-V_n(t)|^2 |V_n(t+h)-V_n(s)|^2] \leq 6h^2.$$

Proof

Note that $V_n(s)-V_n(t) = \frac{1}{\sqrt{n}} \sum_1^n A_i$, where the A_i s are independent centered Bernoulli random variable with parameter $p_1 = (s-t)$. Similarly we have that $V(t+h)-V(s) = \frac{1}{\sqrt{n}} \sum_1^n B_i$, where the B_i s are independent centered Bernoulli random variables with parameter $p_2 = (t+h-s)$. Since A_i and B_i depend only on X_i it follows that the vectors (A_i, B_i) form an independent family. It follows then that

$$\begin{aligned} E[|V_n(s)-V_n(t)|^2 |V_n(t+h)-V_n(s)|^2] &= \\ n^{-2} E\left[\left(\sum_1^n A_i\right)^2 \left(\sum_1^n B_i\right)^2\right] &= \\ n^{-2} \{n^2 E[A_1^2 B_1^2] + n(n-1)E[A_1^2 B_1^2] + 2n(n-1)E[A_1 B_1]^2\} &\leq \\ p_1(1-p_1)^2 p_2^2 + p_2 p_1(1-p_2)^2 + (1-p_1-p_1)p_1^2 p_2^2 + & \\ + p_1(1-p_1)p_2(1-p_2) + p_1^2 p_2^2 &\leq 6p_1 p_2 \leq 6h^2. \quad \square \end{aligned}$$

Proof of Theorem (3.3.3)

By Proposition (3.3.5) and Theorem (1.8.1) we have that for each $n \leq N$, there exists a set $\Delta_n \in L(A)$ with $\tilde{P}(\Delta_n) = 0$ such that for each $\omega \notin \Delta_n$, $V_n(\omega)$ is finite and has S -separated jumps. As before we will redefine V_n to be identically 0 on Δ_n . Extend V_n to $^*[0,1]$ by linear interpolation except at those points where the S -jumps take place; on those points define V_n so that it is

-right continuous. So defined, $V_n(\omega) \in {}^\mathcal{D}[0,1]$ for each $\omega \in \Omega$. Moreover, by Theorem (1.4.3) $V(\omega) \in ns_J({}^*\mathcal{D}[0,1])$. Hence $\tilde{V}_n = st_J V_n$ is a stochastic process with paths in $\mathcal{D}[0,1]$.

Let ν_n be the law on $({}^*\mathcal{D}[0,1], {}^*\mathcal{B})$ induced by V_n . As stated above for infinite n the distributions of V_n are nearly those of a brownian bridge. Hence $\tilde{V}_n$ is a brownian bridge on $\mathcal{D}[0,1]$. It follows then that for infinite n , the measure $\hat{\nu}_n$ on $(\mathcal{D}[0,1], \mathcal{B})$ defined by

$$\hat{\nu}_n(A) = \tilde{\nu}_n(st_J^{-1}(A)) = \tilde{P}(\omega \mid \tilde{V}_n(\omega) \in A)$$

does not depend on n . Since $\tilde{P}$ -a.s. $V_n \in ns({}^*\mathcal{D}[0,1])$, ν_n is S -tight. Thus, by Theorem (1.9.3) we have that $\tilde{V}_n \Rightarrow \tilde{V}$, a brownian bridge on $\mathcal{D}[0,1]$. We note however that if $n = N$ then $V_N = U_N$. Thus V_N is finite and S -continuous, so that $\hat{\nu}_N(\mathcal{C}[0,1]) = 1$. It follows then that $\tilde{V}$ must be a brownian bridge on $\mathcal{C}[0,1]$.

If we now let $\xi_i(\omega) = st(X_i(\omega))$, we have that the ξ_i s form an independent family of random variables with uniform distribution on $[0,1]$. We conclude the proof by observing that for each $r \in [0,1]$ and $n \in \mathbb{N}$

$$\begin{aligned} \tilde{V}_n(r) &= S\text{-}\lim_{t \uparrow r} \frac{1}{\sqrt{n}} \sum_{i=1}^n [I(X_i \leq t) - t] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n [I(\xi_i \leq r) - r]. \end{aligned}$$

□

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