**MONOTONE FLOWS IN N-DIMENSIONAL PARTIALLY SATURATED POROUS MEDIA:
LIPSCHITZ CONTINUITY OF THE INTERFACE**

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**Monotone Flows in N-Dimensional Partially Saturated Porous Media:
Lipschitz Continuity of the Interface**

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1. INTRODUCTION

We consider the following equation

$$c(u)_t = \Delta u, \quad (1.1)$$

where $c : \mathbb{R} \rightarrow \mathbb{R}$ is a given function with a graph as in Fig. 1. That is, c is assumed to be identically equal to one on $\mathbb{R}^+$ and to have a positive derivative on $\mathbb{R}^-$.

Fig. 1. The function c .

Equation (1.1) models the partially saturated fluid flow in a porous medium, see e.g. [B]. In this context u stands for the hydrostatic potential due to capillary suction and $c(u)$ for the saturation or relative moisture content. The medium can be saturated, in which case $c(u)$ attains its maximum value, here normalized to be one, and u is positive, or it can be unsaturated, in which case u is negative and $c(u)$ is less than one. This difference reflects itself in (1.1). Writing (1.1) as

$$(1.2) \quad c'(u)u_t = \Delta u,$$

we have

$$(1.3) \quad u_t = \frac{1}{c'(u)} \Delta u \quad (\text{parabolic}) \quad \text{if } u < 0 ,$$

and

$$(1.4) \quad 0 = \Delta u \quad (\text{elliptic}) \quad \text{if } u > 0 .$$

Because of this mixed type (1.1) is usually called elliptic-parabolic. It falls outside the scope of the standard theory for parabolic equations as can be found in [LSU]. Existence and uniqueness of weak solutions to initial boundary value problems for (1.1) have been established by a number of authors, see e.g. [AL], [Ho], [KR], [H2], [vDP], [HP].

The only regularity result obtained so far in several space dimensions is that for bounded weak solutions u the saturation $c(u)$ is continuous ([DBG]), but the continuity of u itself however is still open. One difficulty is the control of u in the saturated part where time only appears as a parameter. The key to regularity of u seems to be the behavior of its zeroset which has been studied in one dimension by [vDP], [vD], [H1] and [BH]. Under certain assumptions on the boundary conditions and the function c , this levelset turned out to be a continuous interface separating the saturated and unsaturated parts of the medium. The continuity of the interface easily implied the continuity of u .

In [BH] the Lipschitz continuity of the interface was established by deriving estimates for the time derivative of level curves which led to an L^∞ -bound for u_t . In the case that c is continuously differentiable weak solutions could be shown to satisfy the equation in the classical sense. However the smoothness of c has one disadvantage: it prevents (1.1) from being uniformly parabolic in the unsaturated region. On the other hand, if one wants to assume that (1.1) is uniformly parabolic in the unsaturated

region, c' has to be discontinuous at zero. In general this will cause Δu to be discontinuous in points where $u = 0$.

In more dimensions however it is not even known whether or not the zero set of u has zero measure and whether or not it is a sharp interface separating the sets $\{u > 0\}$ and $\{u < 0\}$. The main difficulty is the control of the behavior of u in space, especially in $\{u > 0\}$. A situation like this can also be found in the Stefan problem, where so-called mushy regions are known to appear. We note that the nature of the zeroset in our problem is completely different from that in the Stefan problem where one has a jump in the normal derivative across the interface which determines its propagation speed. Here however if the interface is a C^1 -surface, u itself will be C^1 everywhere and no such intrinsic formula for the propagation speed exists.

We consider two specific Dirichlet problems for (1.1). In the first one we impose boundary conditions which will guarantee that u_t is positive and bounded away from zero. We show that the interface separating the saturated and unsaturated regions is a Lipschitz continuous surface $t = S(x)$.

In the second problem we consider a domain with a hole in it with constant positive data on the inner boundary, constant negative data on the outer boundary, and radially decreasing initial data. Here we show that the interface is Lipschitz continuous both in space and time. In both cases the interface coincides with the zeroset of u if (1.1) is uniformly parabolic in the unsaturated region.

Our main tool is a so-called cone argument that has also been applied to the Stefan Problem and the porous media equation, see e.g. [N] and [CVW]. Our techniques are of a global nature whereas the continuity result in [DBG]

is based on local arguments only. It seems to us that results about the zeroset of u can not be obtained without some assumptions on the data, hence the global techniques.

The outline of this paper is as follows. In Section 2 we give a precise statement of the two problems and the results. Section 3 deals with preliminaries that are needed later on. In Sections 4 and 5 we prove the continuity results for the interfaces in the first and second problem respectively. In Section 6 finally, we show that the interface coincides with the zeroset of u if the equation is uniformly parabolic in the unsaturated region.

2. a) STATEMENT OF THE PROBLEMS

In this Section we give a precise statement of the problems we are dealing with.

As we stated in the Introduction we consider the elliptic-parabolic equation

$$(1.1) \quad c(u)_t = \Delta u$$

where the function $c(u)$ is constant (equal to 1) for u positive and strictly increasing for u negative. In addition we suppose that c is a Lipschitz continuous function in $\mathbb{R}$ with continuous bounded first derivative for $u < 0$, and that c is a concave function of u for u larger than a negative constant $(-a)$. We will only consider solutions u in the range $u > -a$. Consequently we may suppose that c is concave everywhere.

We restrict ourselves to monotone flows. In order to guarantee this monotonicity we need to impose some conditions on the initial and boundary data. We consider two situations. In the first one the hydrostatic potential u will be strictly increasing in time. In the second one we will allow constant boundary conditions but we will have some geometric restrictions on the domain and the initial value u_0 . However this will pay off by giving regularity of the interface in space as well.

Problem I

We consider equation (1.1) in $Q_T = \Omega \times (0, T)$ where Ω is a smooth bounded domain in $\mathbb{R}^N$. We impose the following initial and boundary conditions

$$\begin{aligned} c(u(x, 0)) &= v_0(x) && \text{in } \Omega ; \\ u(x, t) &= f(x, t) && \text{on } \partial\Omega \times (0, T) = S_T ; \end{aligned}$$

Here $f > 0$ and v_0 is assumed to be of the form $v_0 = c(u_0)$ for some $u_0 > -a$. We have the following hypothesis on the functions u_0 and f :

- H1) $u_0 \in W^{2,p}(\Omega)$ for some $p > n$;
- H2) $|\{u_0 = 0\}| = 0$;
- H3) $\Delta u_0 > \alpha c'(u_0)$;
- H4) $f : \bar{S}_T \rightarrow \mathbb{R}^+$ is continuous, has continuous first order space derivatives, and bounded second order space and first order time weak derivatives.

$$H5) \quad f_t > \alpha;$$

$$H6) \quad f(x,0) = u_0(x) \quad \text{on } \partial\Omega;$$

$$H7) \quad \Delta u_0 = 0 \quad \text{in a neighborhood of } \partial\Omega;$$

for some $\alpha > 0$. Hypothesis H3 and H5 ensure that u_t stays away from zero. H2 is a nondegeneracy condition on the initial value u_0 , and H6 is a compatibility condition.

Problem 2

We consider equation (1.1) in $Q_T = \Omega \times (0,T)$ where $\Omega = D_1 \setminus D_0$ with $D_0 \subset\subset D_1$ smooth domains that are starshaped with respect to a ball of radius δ contained in D_0 . We place the origin at the center of the ball. The boundary conditions we impose are

$$c(u(x,0)) = v_0(x) \quad \text{in } \Omega;$$

$$u(x,t) = b > 0 \quad \text{on } \partial D_0 \times (0,T);$$

$$u(x,t) = -a < 0 \quad \text{on } \partial D_1 \times (0,T);$$

where a and b are positive constants (and c is concave for $u > -a$).

v_0 is of the form $v_0 = c(u_0)$ with u_0 satisfying,

$$H1') \quad u_0 \in C^{0,1}(\bar{\Omega}) \quad \text{and } u_0 \text{ is } C^2 \text{ in the neighborhood of } \partial\Omega,$$

$$H3') \quad \Delta u_0 > 0 \quad \text{in the sense of distributions,}$$

$$H6') \quad u_0(x) = b \quad \text{on } \partial D_0,$$

$$u_0(x) = -a \quad \text{on } \partial D_1.$$

H7') u_0 is decreasing along the directions $(x-x_0)$, for every $x_0 \in B(0, \delta)$.

That is if $\varepsilon > 0$ and $x_0 \in B(0, \delta)$,

$$u_0(x + \varepsilon(x-x_0)) \leq u_0(x).$$

Note that H7' is verified if $u_0 \in C^1(\bar{\Omega})$, $u_{0r} \leq -\alpha < 0$, and δ is small enough.

b) STATEMENT OF RESULTS

For the weak solution u of Problem I we prove that the interface Γ defined by

$$\Gamma = \partial\{(x,t) \in Q_T : c(u(x,t)) < 1\}$$

is a Lipschitz surface in time. That is there is a Lipschitz function $S(x)$ such that

$$\Gamma : t = S(x)$$

The domain of S is the whole initially unsaturated region

$\{x \in \Omega : c(u_0(x)) < 1\}$ if T is large enough. In fact we show that Ω becomes saturated within finite time.

For the solution of Problem II there is a representation of the interface Γ in spherical coordinates,

$$\Gamma : r = f(\theta, t) \quad \theta \in S^{N-1}, \quad t \in [0, T].$$

where $r = |x|$, $\theta = \frac{x}{|x|}$ and f is Lipschitz continuous in both θ and t .

Here the boundary conditions ensure the existence of an interface for every t .

Finally in both cases we prove that if the problem is uniformly parabolic in the unsaturated set, that is if $c'(u) > \beta > 0$ for $u < 0$, Γ coincides with the zeroset of u . Consequently u is positive on one side of Γ and negative on the other.

3.1. PRELIMINARY RESULTS FOR PROBLEM I

Definition. A function $u \in L^2(0, T; H^1(\Omega))$ is called a weak solution of Problem I if

- i) $u = f$ a.e. on S_T ;
- ii) $c(u) \in C((0, T]; L^2(\Omega))$
- iii) for every test function $\phi \in H^1(Q_T)$ vanishing both at $t = T$ and on S_T the following equality holds:

$$\iint_{Q_T} \{ \nabla u \nabla \phi - c(u) \phi_t \} dx dt = \int_{\Omega} v_0(x) \phi(x, 0) dx .$$

Existence of a weak solution is usually proved by means of parabolic regularization, that is, the weak solution is obtained as a limit of regular solutions u_n of uniformly parabolic problems I_n . Existence and uniqueness of a weak solution was established in [AL]. There one can also find a

comparison principle, however only for so-called strong solutions, i.e. weak solutions with $c(u)_t \in L^2(Q_T)$.

Some of our results will be proved for the regularized problems I_n first. It is therefore necessary to do this regularization all over again with some modifications. Unfortunately, this will be somewhat dry and technical.

Let us first see what we need for the approximating data in order to get the a priori bounds on $\{u_n\}$ that will give compactness and therefore the existence of a limit u of a subsequence. This limit u will be our solution.

Lemma 3.1. Suppose that u is a regular solution of (1.1), where $0 < c' < k$, with

$$u(x,0) = u_0(x) \quad \text{in } \Omega ;$$

$$u(x,t) = f(x,t) \quad \text{on } S_T ;$$

where $u_0 \in H^1(\Omega)$, and f is defined on Q_T , with $f, \nabla f, f_t, \nabla f_t \in L^2(Q_T)$. Then there exists a constant K depending only on these norms and on the $L^2(Q_T)$ -norm of u such that

$$\sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^2(\Omega)} + \|c(u)_t\|_{L^2(Q_T)} \leq K$$

Proof Multiply equation (1.1) by $u-f$ and integrate by parts to get

$$\int_{\Omega} \nabla u \nabla (u-f) = - \int_{\Omega} c(u)_t (u-f).$$

Using Young's inequality, we obtain

$$(3.1) \quad \int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} (c(u)_t)^2 + \int_{\Omega} |\nabla f|^2 + 2 \int_{\Omega} u^2 + 2 \int_{\Omega} f^2$$

In order to bound $\int_{\Omega} (c(u)_t)^2$ we observe that

$$(3.2) \quad \int_{\Omega} (c(u)_t)^2 \leq k \int_{\Omega} u_t c(u)_t.$$

Multiplying equation (1.1) by $u_t - f_t$ and again integrating by parts, we obtain

$$\begin{aligned} (3.3) \quad \int_{\Omega} u_t c(u)_t &= - \int_{\Omega} \nabla u \nabla u_t + \int_{\Omega} c(u)_t f_t + \int_{\Omega} \nabla u \nabla f_t \\ &\leq - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u(t)|^2 + \frac{1}{2k} \int_{\Omega} (c(u)_t)^2 + \frac{k}{2} \int_{\Omega} f_t^2 \\ &\quad + \frac{1}{4k} \int_{\Omega} |\nabla u|^2 + k \int_{\Omega} |\nabla f_t|^2. \end{aligned}$$

Combining (3.1), (3.2) and (3.3) yields

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 &\leq -2k \frac{d}{dt} \int_{\Omega} |\nabla u(t)|^2 + 4 \int_{\Omega} u^2 + 4 \int_{\Omega} f^2 + 2 \int_{\Omega} |\nabla f|^2 \\ &\quad + 2k^2 \int_{\Omega} f_t^2 + 4k^2 \int_{\Omega} |\nabla f_t|^2, \end{aligned}$$

hence

$$(3.4) \quad \int_{\Omega} |\nabla u(t)|^2 \leq \int_{\Omega} |\nabla u_0|^2 + \frac{2}{k} \int_{\Omega} u^2 + \frac{2}{k} \int_{\Omega} f^2 + \frac{1}{k} \int_{\Omega} |\nabla f|^2 \\ + k \int_{\Omega} f_t^2 + 2k \int_{\Omega} |\nabla f_t|^2 = K_1 .$$

Replacing (3.3) into (3.2) and using (3.4) an integration from 0 to T yields

$$\iint_{Q_T} (c(u)_t)^2 \leq K_2 ,$$

where K_2 has the desired properties. Thus we may set $K = K_1 + K_2$, which completes the proof.

Remark 3.1. If instead of the hypothesis in Lemma 3.1 we have $u_0 \in H^1(\Omega)$, $f, \nabla f \in L^2(Q_T)$ and $f_t \in L^2(0, T; L^2(\partial\Omega))$ there is a constant K depending only on these norms, the L^2 norm of u in Q_T and $\sup \left| \frac{\partial u}{\partial \nu} \right|$ on $\partial\Omega \times (0, T)$ such that

$$\sup_{0 \leq t \leq T} \int |\nabla u|^2 + \iint (c(u)_t)^2 \leq K$$

The proof follows the same lines. The only difference is that in order to bound $\int u_t c(u)_t$ we multiply equation (1.1) by u_t and not by $u_t - f_t$.

Corollary 3.1. Let $c_n \rightarrow c$ uniformly, $\frac{1}{n} \leq c'_n \leq k$, $c_n(u_0^n) \rightarrow c(u_0)$ in $L^2(\Omega)$ and $f_n \rightarrow f$ in $L^2(0, T; L^2(\partial\Omega))$ be such that

$$i) \quad \int_{\Omega} |\nabla u_0^n|^2 \leq C$$

$$ii) \quad \|f_n\|_{L^2(Q_T)}; \|\nabla f_n\|_{L^2(Q_T)} \leq C$$

Let u_n be the solution of

$$c_n(u_n)_t = \Delta u_n \quad \text{in } \Omega \times (0, T);$$

$$u_n(x, t) = f_n(x, t) \quad \text{on } \partial\Omega \times (0, T);$$

$$u_n(x, 0) = u_0^n(x) \quad \text{in } \Omega.$$

If $\|u_n\|_{L^2(Q_T)} \leq C$ and either $\|\nabla f_{nt}\|_{L^2(Q_T)} \leq C$ and $\|f_{nt}\|_{L^2(Q_T)} \leq C$ or

$\|f_{nt}\|_{L^2(0, T; L^2(\partial\Omega))} \leq C$ and $\left\| \frac{\partial u_n}{\partial \nu} \right\|_{L^\infty(\partial\Omega \times (0, T))} \leq C$, we have that

$$u_n \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)),$$

$$c_n(u_n) \rightharpoonup c(u) \quad \text{in } H^1(Q_T),$$

where u is the solution of equation (1.1) with

$$u(x, t) = f(x, t) \quad \text{on } \partial\Omega \times (0, T);$$

$$c(u(x, 0)) = c(u_0(x)) \quad \text{in } \Omega.$$

Proof. The proof is based on a standard compactness argument added to the uniqueness of weak solution.

From the estimates in Lemma 3.1 one finds a subsequence $\{u_{n_j}\}$ and functions u and v such that

$$u_{n_j} \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)),$$

$$c_{n_j}(u_{n_j}) \rightharpoonup v \quad \text{in } H^1(Q_T).$$

The proof that v is actually $c(u)$ follows as in [Ho], page 168. So u is the weak solution of Problem I and the whole sequence $\{u_n\}$ converges to u in the stated way.

Proposition 3.1. Let $c \in C^{0,1}(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be concave. $c' > 0$ on $\mathbb{R}^-$ and $c \equiv 1$ on $\mathbb{R}^+$. Suppose that c, u_0 and f satisfy H1-7. Then there exist sequences $c_n \in C^\infty(\mathbb{R})$, $u_0^n \in C^{2+\gamma}(\bar{\Omega})$, and $f_n \in C^{2+\gamma, 1+\gamma/2}(\bar{Q}_T)$ for some $0 < \gamma < 1$, with $\frac{1}{n} \leq c'_n \leq k < \infty$ and c_n concave, satisfying

$$(3.5) \quad \Delta u_0^n \geq \alpha c'_n(u_0^n) \quad \text{in } \Omega;$$

$$(3.6) \quad \{f_n\} \text{ is uniformly bounded in } W_\infty^{2,1}(Q_T);$$

$$(3.7) \quad f_{nt} \geq \alpha - \frac{1}{n} \quad \text{on } S_T;$$

$$(3.8) \quad u_0^n(x) = f_n(x, 0) \quad \text{on } \partial\Omega;$$

$$(3.9) \quad c'_n(u_0^n(x)) f_{nt}(x, 0) = \Delta u_0^n(x) \quad \text{on } \partial\Omega;$$

$$(3.10) \quad \{u_0^n\} \text{ is uniformly bounded in } W^{1,\infty}(\Omega).$$

and such that $c_n \rightarrow c$ uniformly on compact sets, $c_n(u_0^n) \rightarrow c(u_0) = v_0$ in $L^2(\Omega)$, and $f_n \rightarrow f$ in $L^2(0, T; L^2(\partial\Omega))$.

Proof. Let $\{c_n\}$ be a sequence of smooth concave functions converging to c uniformly on compact sets, with $c'_n \rightarrow c'$ on compact sets of $\mathbb{R} \setminus \{0\}$ and $\frac{1}{n} \leq c'_n \leq k$. In addition we may ask that $c'_n \equiv \frac{1}{n}$ on $(\frac{1}{2}\delta, \infty)$, where $\delta = \inf_{S_T} f > 0$.

Next we construct u_0^n . This will be done in two steps. First let $u_\star^n$ be a sequence of smooth functions converging to u_0 in $W^{2,p}(\Omega)$. Since

$\Delta u_0 = 0$ in a neighborhood of $\partial\Omega$, the sequence $u_\star^n$ can be chosen to satisfy $\Delta u_\star^n = 0$ in a neighborhood of $\partial\Omega$ which does not depend on n . In addition because of H6, we can take $u_\star^n$ to satisfy $u_\star^n > \frac{\delta}{2}$ on this same neighborhood for all n .

We have to change this $u_\star^n$ a little bit in order to get (3.5). Let $F_n = (\alpha c'_n(u_\star^n) - \Delta u_\star^n)^+$. Then by H3, $F_n \rightarrow 0$ in $L^p(\Omega)$.

Let $\tilde{u}^n$ be the solution of

$$\Delta \tilde{u}^n = F_n \quad \text{in } \Omega ;$$

$$\tilde{u}^n = 0 \quad \text{on } \partial\Omega .$$

Then $\tilde{u}^n \rightarrow 0$ in $W^{2,p}(\Omega)$, so if $\gamma = 1 - \frac{n}{p}$

$$\tilde{u}^n \rightarrow 0 \quad \text{in } C^{1+\gamma}(\bar{\Omega}) .$$

Now let $u_0^n = u_\star^n + \tilde{u}^n + \|\tilde{u}^n\|_\infty$. In our fixed neighborhood of $\partial\Omega$ we have

$$\Delta u_0^n \equiv \frac{\alpha}{n} \quad \text{and} \quad c'_n(u_0^n) \equiv \frac{1}{n} ,$$

whereas

$$\Delta u_0^n = \Delta u_\star^n + (\alpha c'_n(u_\star^n) - \Delta u_\star^n)^+ > \alpha c'_n(u_\star^n) > \alpha c'_n(u_0^n) ,$$

because of the concavity of c_n . Note that $u_0^n \in C^{2+\gamma}(\bar{\Omega})$ for every n and is uniformly bounded in $C^{1+\gamma}(\bar{\Omega})$.

Finally we have to construct a sequence f_n in such a way that (3.6)-(3.9) are satisfied.

First we extend f as a function $f : \bar{Q}_T \rightarrow [\delta, \infty)$ with continuous first order space derivatives and bounded second order space and first order time weak derivatives. Furthermore we can take $f_t > \alpha$ on $\bar{Q}_T$.

Now choose a sequence of functions f_n in $C^{2+\gamma, 1+\gamma/2}(\bar{Q}_T)$ uniformly bounded in $C^{2,1}(\bar{Q}_T)$ with $\tilde{f}_{nt} > \alpha - \frac{1}{n}$, and $\tilde{f}_n(x, 0) = u_0^n(x)$ on $\partial\Omega$, such that $\tilde{f}_n \rightarrow f$ in $L^2(S_T)$. It remains to adjust $\tilde{f}_n$ in such a way that (3.9) will hold.

So let $f_n = \tilde{f}_n + g_n$. Then we want g_n to satisfy

$$g_{nt}(x, 0) = \alpha - \tilde{f}_{nt}(x, 0) \quad \text{on } \partial\Omega ,$$

$$g_n(x, 0) = 0 \quad \text{on } \partial\Omega ,$$

since

$$\frac{\Delta u_0^n}{c_n'(u_0^n)} = \alpha \quad \text{near } \partial\Omega .$$

Let

$$g_{nt} = (\alpha - \tilde{f}_{nt}(x, t))\phi_n(t)$$

where ϕ_n is smooth, $\phi_n(0) = 1$, $\phi_n' < 0$, and $\phi_n \equiv 0$ on $[\frac{1}{n}, T]$. Then f_{nt} is between α and $\tilde{f}_{nt}$, so (3.7) is not affected. Furthermore f_n satisfies (3.9) by construction. The only thing we have to check is (3.6).

We have

$$\begin{aligned} g_n(x, t) &= \int_0^t (\alpha - \tilde{f}_{nt}(x, s))\phi_n(s)ds \\ &= (\alpha t - \tilde{f}_n(x, t))\phi_n(t) + \tilde{f}_n(x, 0) + \int_0^t (\tilde{f}_n(x, s) - \alpha s)\phi_n'(s)ds , \end{aligned}$$

which has uniformly bounded first and second order space derivatives, because $\tilde{f}_n$ does. By construction g_{nt} is uniformly bounded, and $g_n \rightarrow 0$ in $L^2(S_T)$. This completes the proof.

Corollary 3.2. Let c_n, u_0^n, f_n be the sequences constructed in Proposition 3.1. Let u_n be the solution of

$$\begin{aligned} c_n(u_n)_t &= \Delta u_n && \text{in } Q_T; \\ (I_n) \quad u_n(x, t) &= f_n(x, t) && \text{on } S_T; \\ u_n(x, 0) &= u_0^n(x) && \text{in } \Omega. \end{aligned}$$

Then u_n converges to the unique weak solution u of Problem I weakly in $L^2(0, T; H^1(\Omega))$.

Proof. It is a consequence of the properties of these sequences and Corollary 3.1. Note that by the maximum principle $\{u_n\}$ is uniformly bounded in $L^\infty(Q_T)$.

The only condition that has yet to be verified is the uniform bound of $\|\partial u_n / \partial \nu\|_{L^\infty(S_T)}$. This is part of the proof of Lemma 4.1.

Proposition 3.2. For every n , $u_{nt} > \alpha - \frac{1}{n}$.

Proof. u_n is a $C^{2+\gamma, 1+\gamma/2}(\bar{Q}_T)$ solution of

$$c'_n(u_n)u_{nt} = \Delta u_n$$

We differentiate this equation with respect to t to get,

$$c'_n(u_n)(u_{nt})_t + c''_n(u_n)(u_{nt})^2 = \Delta u_{nt} \quad \text{in } Q_T;$$

$$u_{nt}(x,t) = f_{nt}(x,t) > \alpha - \frac{1}{n} \quad \text{on } \partial\Omega \times (0,T);$$

$$u_{nt}(x,0) = \frac{\Delta u_0^n}{c'_n(u_0^n)} > \alpha \quad \text{in } \Omega.$$

Therefore as $c''_n > 0$, $u_{nt} > \alpha - \frac{1}{n}$ in Q_T .

Corollary 3.3. The weak solution u of Problem I satisfies $u_t > \alpha$ in the sense of distributions.

Proof. Immediate from Proposition 3.2 and Corollary 3.2.

3.II. PRELIMINARY RESULTS FOR PROBLEM II.

The definition of a weak solution of Problem II is of course similar to Definition 3.1 and therefore we omit it. In fact the only difference between Problem II and Problem I is that the adjustments of the standard parabolic regularization we need here are only minor. We will not be needing a uniform gradient bound on the lateral boundary. Also the time independence of the lateral boundary conditions will allow us to use the first condition in Corollary 3.1 instead of the second.

Proposition 3.3. Let c be as in Proposition 3.1. Suppose c , u_0 and f satisfy $H1'$, $H3'$ and $H6'$. Then there exist sequences $c_n \in C^\infty(\mathbb{R})$, $u_0^n \in C^{2+\gamma}(\bar{\Omega})$, $f_n \in C^{2+\gamma, 1+\gamma/2}(\bar{Q}_T)$, for some $0 < \gamma < 1$ with $\frac{1}{n} < c'_n < k < \infty$ and c_n concave, such that

$$(3.11) \quad \Delta u_0^n > 0 \quad \text{in } \Omega ;$$

$$(3.12) \quad f_n, \nabla f_n, f_{nt} \quad \text{and} \quad \nabla f_{nt} \quad \text{are uniformly bounded in } L^2(Q_T) ;$$

$$(3.13) \quad f_{nt} > 0 \quad \text{on } \bar{Q}_T ;$$

such that (3.8) and (3.9) hold, and such that $c_n \rightarrow c$ uniformly on compacts, $c_n(u_0^n) \rightarrow c(u_0) = v_0$ in $L^2(\Omega)$, $f_n \rightarrow b$ in $L^2(0, T; L^2(\partial D_0))$, and $f_n \rightarrow -a$ in $L^2(0, T; L^2(\partial D_1))$.

Proof: Choose c_n as in the proof of Proposition 3.2, but not necessarily satisfying $c'_n = \frac{1}{n}$ on a positive interval. Next choose a sequence of smooth functions u_0^n converging in $C^{0,1}(\bar{\Omega})$ to u_0 . We may take u_0^n to satisfy $\Delta u_0^n > 0$. Finally, for the construction of f_n , let $\phi_n(t)$ be a sequence of smooth nondecreasing concave functions with $\phi_n(0) = 0$, whose derivatives ϕ'_n have support $[0, \varepsilon_n]$ ($\varepsilon_n > 0$ to be chosen later on) satisfying $\phi'_n(0) = 1$. Note that $0 \leq \phi_n \leq \varepsilon_n$. Set

$$(3.14) \quad f_n = u_0^n + \frac{\Delta u_0^n}{c'_n(u_0^n)} \phi_n(t)$$

Then the compatibility conditions (3.8) and (3.9) are satisfied, and (3.13) is also obvious.

It remains to show that we can choose $\varepsilon_n \rightarrow 0$ in such a way that (3.12) holds and that f_n converges in $L^2(S_T)$ to the corresponding boundary data.

The first term on the right hand side of (3.14) has the desired properties since it is time independent and uniformly bounded in $C^{0,1}(\bar{\Omega})$. Note also that $u_0^n(x)$ converges uniformly to b on ∂D_0 and to $-a$ on ∂D_1 .

The second term will converge to zero in $L^2(0,T; H^1(\Omega))$ if ϵ_n goes to zero fast enough. As for the time derivatives, note that

$$f_{nt} = \frac{\Delta u_0^n}{c'_n(u_0^n)} \phi'_n(t)$$

and

$$\nabla f_{nt} = \nabla \left(\frac{\Delta u_0^n}{c'_n(u_0^n)} \right) \phi'_n(t)$$

have support contained in $\Omega \times [0, \epsilon_n]$. So again these functions can be made to go to zero in $L^2(Q_T)$ by letting ϵ_n go to zero fast enough.

Proposition 3.4 For every n , $u_{nt} > 0$.

Proof. As proposition 3.2.

Corollary 3.4. u_n converges to the weak solution u of Problem II weakly in $L^2(0,T; H^1(\Omega))$ and $u_t > 0$ in the sense of distributions.

Proof. Same as that of Problem I.

4. LIPSCHITZ CONTINUITY OF THE INTERFACE IN PROBLEM I.

Theorem 4.1. Let $c \in C^{0,1}(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ be concave, with $c'(u) > 0$ for $u < 0$, and $c(u) \equiv 1$ for $u > 0$. Let u_0 and f satisfy conditions H1 through H7. Let u be the solution of

$$c(u)_t = \Delta u \quad \text{in } Q_T;$$

$$u(x,t) = f(x,t) \quad \text{on } \partial\Omega \times (0,T);$$

$$c(u(x,0)) = c(u_0(x)) \quad \text{in } \Omega$$

Let $\Gamma = \partial\{(x,t) : c(u(x,t)) < 1\}$ be the boundary of the unsaturated region. There is a Lipschitz continuous function $S(x)$ defined on the initially unsaturated region such that Γ is represented by $t = S(x)$.

Remark 4.1. Since $f > \delta$ on $\partial\Omega$, the region Ω will become saturated within finite time; therefore if T is large enough the domain of S will be the whole set $\{x : c(u_0(x)) < 1\}$. This can be seen as follows. Since Ω is bounded it is contained in $\{x \in \mathbb{R}^N : -m < x_1 < m\}$ for some m large enough. Choose a Lipschitz continuous function $u_0^* : [-m, m] \rightarrow \mathbb{R}$ such that $u_0^*(x_1) < \min(\delta, u_0(x))$ for all x in Ω and with $u_0^*(-m) = u_0^*(m) = \delta$. Now let u^* be the solution of the one dimensional problem,

$$c(u)_t = u_{xx} \quad -m < x < m, \quad t > 0;$$

$$u(-m, t) = u(m, t) = \delta \quad t > 0;$$

$$c(u(x,0)) = c(u_0^*(x)) \quad -m < x < m.$$

From a comparison result in [AL] it follows that $c(u) > c(u^*)$ ($u^* < \delta < f$ on $\partial\Omega \times (0,T)$ and $c(u_0^*) < c(u_0)$ in Ω). We claim that $c(u^*)$ becomes identically equal to one within finite time, which proves the remark.

So consider the solution u^* and suppose that the unsaturated region is a nonempty set for all positive t . By [HP] it has to be an open interval of

the form $(\xi^-(t), \xi^+(t))$ with

$$-m < \xi^-(t) < \xi^+(t) < m \quad \text{for all } t > 0.$$

From a variant of the continuity proof in [H1] it follows that ξ^- and ξ^+ are continuous. Since $u_{xx}^* = 0$ a.e. in the saturated region we have that

$$u_x^*(-m, t) = \frac{-\delta}{\xi^-(t) + m} < \frac{-\delta}{2m}$$

and

$$u_x^*(m, t) = \frac{\delta}{m - \xi^+(t)} > \frac{\delta}{2m}.$$

Consequently

$$\frac{d}{dt} \int_{-m}^m c(u^*(x, t)) dx = \int_{-m}^m u_{xx}^*(x, t) dx = u_x^*(x, t) \Big|_{-m}^m > \frac{\delta}{m}$$

Since $c(u)$ is bounded this is a contradiction.

Let us now prove the proposition. We first need to prove a lemma.

Lemma 4.1. Let u_0^n, f_n be the approximation functions found in Proposition 3.1. Let u_n be the classical solution of I_n . Then there exists a constant $\tilde{C}$ independent of n such that

$$u_{nt} > \tilde{C} |\nabla u_n| \quad \text{in } Q_T$$

Proof: Let us see that there exists a constant C independent of n such that

$$\left\| \frac{\partial u_n}{\partial v} \right\|_{L^\infty(\partial\Omega \times (0,T))} \leq C.$$

This will prove that $\|\nabla u_n\|_{L^\infty(\partial\Omega \times (0,T))} \leq C$. In order to do this we will construct some barriers. First let $w_n^+(\cdot, t)$ be the solution of the equation

$$\begin{aligned} \Delta w_n^+(\cdot, t) &= 0 & \text{in } \Omega; \\ w_n^+(\cdot, t) &= f_n(\cdot, t) & \text{on } \partial\Omega. \end{aligned}$$

As $u_{nt} > 0$ we have

$$\begin{aligned} \Delta(u_n - w_n^+) &> 0 & \text{in } \Omega; \\ u_n - w_n^+ &= 0 & \text{on } \partial\Omega; \end{aligned}$$

and therefore $u_n(\cdot, t) - w_n^+(\cdot, t) < 0$ in Ω . This together with the fact that they are equal on the boundary gives,

$$\frac{\partial u_n}{\partial v}(\cdot, t) > \frac{\partial w_n^+}{\partial v}(\cdot, t) \quad \text{on } \partial\Omega.$$

Because of (3.6), w_n^+ is uniformly bounded (both in t and n) in $C^{1+\gamma}(\bar{\Omega})$. Therefore

$$\frac{\partial u_n}{\partial v} > C_1 \quad \text{on } \partial\Omega \times (0, T).$$

In order to get a bound from above we need to construct another barrier.

This one will be the solution of a parabolic equation.

Let $w_n^-(x,t)$ be the solution of

$$kw_{nt}^- = \Delta w_n^- \quad \text{in } Q_T ;$$

$$w_n^-(x,t) = f_n(x,t) \quad \text{on } \partial\Omega \times (0,T) ;$$

$$w_n^-(x,0) = u_0^n(x) \quad \text{in } \Omega .$$

As $u_{nt} > 0$,

$$k(u_{nt} - w_{nt}^-) > \Delta(u_n - w_n^-) \quad \text{in } Q_T ;$$

$$u_n(x,t) = w_n^-(x,t) \quad \text{on } \partial\Omega \times (0,T) ;$$

$$u_n(x,0) = w_n^-(x,0) \quad \text{in } \Omega .$$

Therefore $u_n(x,t) > w_n^-(x,t)$ in Q_T and they coincide on $\partial\Omega \times (0,T)$.

Thus,

$$\frac{\partial u_n}{\partial \nu} < \frac{\partial w_n^-}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T) .$$

By Lemma 3.1 in Chapter 6 of [LSU] $\partial w_n^- / \partial \nu$ is bounded on $\bar{S}_T$ uniformly both in x,t and n . Therefore

$$\frac{\partial u_n}{\partial \nu} < C \quad \text{on } \partial\Omega \times (0,T)$$

with C independent of n . So $|\nabla u_n| < C$ on $\partial\Omega \times (0,T)$ and on $\Omega \times \{0\}$

with C independent of n .

We already know that $u_{nt}(x,t) = f_{nt}(x,t) > \alpha - \frac{1}{n}$ on $\partial\Omega \times (0,T)$. Let us see that an inequality like this one also holds in Ω at time $t = 0$.

We know that

$$c'_n(u_0^n)u_{nt}(x,0) = \Delta u_n(x,0) \quad \text{in } \Omega ,$$

therefore $u_{nt}(x,0) = \frac{\Delta u_0^n}{c'_n(u_0^n)} > \alpha$ in Ω . So there is a constant $\tilde{C}$ such that

$$u_{nt} > \tilde{C}|\nabla u_n|$$

on $\partial\Omega \times (0,T)$ and on $\Omega \times \{0\}$. We will show that this inequality holds in Q_T .

As in the proof of Prop. 3.2 let us differentiate with respect to t and also with respect to x_i the equation

$$c_n(u_n)_t = \Delta u_n .$$

We get,

$$c'_n(u_n)(u_{nt})_t + (c''_n(u_n)u_{nt})u_{nt} = \Delta u_{nt} ,$$

$$c'_n(u_n)(u_{nx_i})_t + (c''_n(u_n)u_{nt})u_{nx_i} = \Delta u_{nx_i} .$$

That is both u_{nt} and u_{nx_i} are solutions of the same uniformly parabolic linear equation. As $u_{nt} > \tilde{C}u_{nx_i}$ on the parabolic boundary of Q_T we have,

$$u_{nt} > \tilde{C}u_{nx_i} \quad \text{in } Q_T .$$

We can replace x_i by $-x_i$ to get,

$$u_{nt} > -\tilde{C}u_{nx_i} \quad \text{in } Q_T.$$

Therefore,

$$u_{nt} > \tilde{C}|\nabla u_n| \quad \text{in } Q_T.$$

Proof of Thm. 4.1: By Lemma 4.1 we know that there exists a constant $\tilde{C}$ independent of n s.t.

$$u_{nt} > \tilde{C}|\nabla u_n|.$$

This means that there exists a cone of directions

$K = \{\mu = (\mu_x, \mu_t) \neq 0 : |\mu_x| < \tilde{C}\mu_t\}$ such that u_n is increasing in these directions, that is for any $\lambda > 0$

$$u_n((x,t) + \lambda\mu) > u_n(x,t)$$

if $\mu \in K$.

Passing to the (weak) limit as $(n \rightarrow \infty)$ we see that this is true also for u . Let $(x_0, t_0) \in \Gamma = \partial\{(x,t) : c(u(x,t)) < 1\}$ and let $\mu \in K$, then as $c(u(x_0, t_0)) = 1$ ($c(u)$ is continuous by [DBG]) we get

$$c(u((x_0, t_0) + \lambda\mu)) = 1 \quad \text{for every } \lambda > 0.$$

That is $c(u) \equiv 1$ in $K + (x_0, t_0)$. We claim that $c(u) < 1$ in $(x_0, t_0) - K$. Suppose that $(x_1, t_1) \in (x_0, t_0) - K$ is such that $c(u(x_1, t_1)) = 1$. As before we see that $c(u) \equiv 1$ in $(x_1, t_1) + K$. But (x_0, t_0) is in the interior of this cone. This is a contradiction. Therefore $c(u) < 1$ in $(x_0, t_0) - K$. Hence

$$(4.1) \quad \Gamma \cap \{((x_0, t_0) - K) \cup ((x_0, t_0) + K)\} = \emptyset.$$

In particular there is a function $S(x)$ such that Γ is given by $t = S(x)$. The relation (4.1) implies that S is Lipschitz continuous and estimates the Lipschitz constant by $1/\tilde{C}$.

The proof is complete.

5. LIPSCHITZ CONTINUITY OF THE INTERFACE IN PROBLEM II.

Theorem 5.1: Let $\Omega = D_1 \setminus D_0$ where $D_0 \subset\subset D_1$ are open bounded domains that are starshaped with respect to a ball of radius δ contained in D_0 . We place the origin of coordinates at the center of this ball.

Let $c \in C^{0,1}(\mathbb{R})$, $c(u) \equiv 1$ for $u > 0$; and $c'(u) > 0$ for $u < 0$.

Let u_0 satisfy hypothesis $H1'$, $H3'$, $H6'$, $H7'$ and let u be the solution of the problem

$$\text{II} \quad \left\{ \begin{array}{ll} c(u)_t = \Delta u & \text{in } Q_T ; \\ u(x,t) = b & \text{on } \partial D_0 \times (0,T) ; \\ u(x,t) = -a & \text{on } \partial D_1 \times (0,T) ; \\ c(u(x,0)) = c(u_0(x)) & \text{in } \Omega . \end{array} \right.$$

Again we define Γ by $\Gamma = \partial\{(x,t) : c(u(x,t)) < 1\}$. Then there is a locally Lipschitz continuous function $f(\theta,t)$, $\theta \in S^{N-1}$, $t \in [0,T]$ such that Γ is represented in spherical coordinates by

$$r = f(\theta,t), \quad \theta \in S^{N-1}, \quad t \in [0,T]$$

where $r = |x|$, $\theta = x/|x|$.

Proof. By H7'

$$(5.1) \quad u_0(x + \varepsilon(x-x_0)) \leq u_0(x), \quad \varepsilon > 0$$

for every $x_0 \in B(0,\delta)$ and $x \in \Omega_{\varepsilon,x_0}$ where

$$\Omega_{\varepsilon,x_0} = D_{\varepsilon,x_0} \setminus D_0, \quad D_{\varepsilon,x_0} = \{x \in D_1 : x_0 + \varepsilon(x-x_0) \in D_1\}$$

Let $u_\varepsilon(x,t) = u(x + \varepsilon(x-x_0),t)$. We have,

$$u(x,t)|_{\partial D_{\varepsilon,x_0} \times (0,T)} \geq -a = u_\varepsilon(x,t)|_{\partial D_{\varepsilon,x_0} \times (0,T)}$$

Also

$$u(x,t)|_{\partial D_0 \times (0,T)} = b > u_\varepsilon(x,t)|_{\partial D_0 \times (0,T)} .$$

This is because u remains between $-a$ and b in $\bar{Q}_T$.

For $t = 0$ (5.1) is telling us that

$$u(x,0) > u_\varepsilon(x,0) \quad \text{in } \Omega_{\varepsilon, x_0} .$$

It is easy to verify that u_ε is a subsolution in the sense of [AL] if ε is nonnegative. This is a consequence of the fact that $c(u)_t$ is nonnegative as we saw in Prop. 3.2. Since both $c(u)_t$ and $c(u_\varepsilon)_t$ are L^2 -integrable on $\Omega_{\varepsilon, x_0} \times (0,T)$ we can apply the comparison principle for strong super and subsolutions in [AL] and conclude

$$(5.2) \quad u(x,t) > u_\varepsilon(x,t) \quad \text{in } \Omega_{\varepsilon, x_0} \times (0,T) .$$

Hence u is nonincreasing in directions close to the radial one in the interior of Ω . Writing

$$\Gamma(t) = \partial\{x \mid c(u(x,t)) < 1\} ,$$

we claim that there is a representation of $\Gamma(t)$ as

$$r = f(\theta, t) \quad \theta \in S^{N-1}$$

where f is a Lipschitz continuous function of θ uniformly in t .

Obviously, since u is radially nonincreasing,

$$f(\theta, t) = \inf\{r : u(r\theta, t) < 0\} .$$

We have,

$$c(u(r\theta, t)) < 1 \quad \text{if } r > f(\theta, t)$$

$$c(u(r\theta, t)) = 1 \quad \text{if } r \leq f(\theta, t)$$

Let K_x be the cone of directions $\lambda \frac{x-x_0}{|x-x_0|}$, $\lambda > 0$ with $x_0 \in B(0, \delta)$. By (5.2) u is nonincreasing in all the directions of K_x at the point x for every t . It is easy to see that the tangent of the aperture of this cone varies continuously with x .

As in Thm. 4.1 it follows that if $(x_1, t_1) \in \Gamma$,

$$(5.3) \quad \begin{aligned} c(u(x, t)) &< 1 & \text{if } x \in (x_1 + K_{x_1}) \cap \Omega , \\ c(u(x, t)) &= 1 & \text{if } x \in (x_1 - K_{x_1}) \cap \Omega . \end{aligned}$$

The difference in this case is that the aperture and axis of K_x vary with x but one can still see that (5.3) is true because if $x \in (x_1 + K_{x_1}) \cap \Omega$ then $x_1 \in (x - K_x)$.

From this we deduce that for $(x_1, t_1) \in \Gamma$,

$$\Gamma(t_1) \cap [(x_1 + K_{x_1}) \cup (x_1 - K_{x_1})] = \emptyset .$$

Since the aperture of K_x is nonzero, varies continuously with x and does not depend on t , $f(\theta, t)$ is Lipschitz in $\theta \in S^{N-1}$ uniformly in t .

So in order to finish the proof we just have to see that f is also Lipschitz in time. Let

$$u_{\varepsilon}(x,t) = u((1+\varepsilon)x, (1+\varepsilon)t) \quad \text{in } \Omega_{\varepsilon,0} \times (0, T_{\varepsilon})$$

$$T_{\varepsilon} = T/(1+\varepsilon); \quad \varepsilon > 0.$$

u_{ε} is a subsolution of equation (1.1) in $\Omega_{\varepsilon,0} \times (0, T_{\varepsilon})$. For $(x,t) \in \partial D_0 \times (0, T_{\varepsilon})$,

$$u_{\varepsilon}(x,t) \leq b = u(x,t);$$

for $(x,t) \in \partial D_{\varepsilon,0} \times (0, T_{\varepsilon})$,

$$u_{\varepsilon}(x,t) = -a \leq u(x,t);$$

and initially

$$u_{\varepsilon}(x,t) = u((1+\varepsilon)x, 0) \leq u(x, 0) \quad x \in \Omega_{\varepsilon,0}$$

because u is radially decreasing. Therefore, again by [AL],

$$u_{\varepsilon}(x,t) \leq u(x,t) \quad \text{in } \Omega_{\varepsilon,0} \times (0, T_{\varepsilon}); \quad \varepsilon > 0.$$

That is for every $\varepsilon > 0$,

$$u((1+\varepsilon)x, (1+\varepsilon)t) \leq u(x,t).$$

Let $(x_0, t_0) \in \Gamma$. So $x_0 = r_0 \theta_0$ with $r_0 = f(\theta_0, t_0)$. For $r > r_0$ we have

$$c(u((1+\epsilon)r\theta_0, (1+\epsilon)t_0)) < c(u(r\theta_0, t_0)) < 1.$$

Let $t = (1+\epsilon)t_0$, we have

$$f(\theta_0, t) < (1+\epsilon)r = r \frac{t}{t_0} \quad \text{for } r > f(\theta_0, t_0).$$

This means that

$$f(\theta_0, t) < f(\theta_0, t_0) \frac{t}{t_0}.$$

Thus

$$f(\theta_0, t) - f(\theta_0, t_0) < \frac{f(\theta_0, t_0)}{t_0} (t - t_0).$$

As $u_t > 0$, the left hand side is nonnegative. Hence f is Lipschitz in time, uniformly in sets of the form $(t > \bar{t} > 0)$.

The proposition is proved.

6. FURTHER REGULARITY

Theorem 6.1 Let c as in theorems 4.1 and 5.1. Let u_0 and f satisfy either the conditions of Thm. 4.1 or those of Thm. 5.1. In the latter suppose that $\Omega = D_1 \setminus D_0$ with D_0 and D_1 as in Thm. 5.1. If in addition $c'(u) > \beta > 0$ for $u < 0$, then

$$\Gamma = \partial\{(x,t) : c(u(x,t)) < 1\} = \{(x,t) : u(x,t) = 0\}$$

That is Γ separates the sets where $u > 0$ and $u < 0$.

Proof: In the case of Problem I we know there is a cone K such that u increases in the directions in K . This cone has as axis the t axis and it has a uniform aperture.

For the solution of Problem II we conclude, just as in the proof of Thm.

5.1, that

$$u_\varepsilon(x,t) = u(x + \varepsilon(x-x_0), (1+\varepsilon)t)$$

with $\varepsilon > 0$ and $x_0 \in B(0,\delta)$, satisfies

$$u_\varepsilon(x,t) < u(x,t) \quad \text{in } \Omega_{\varepsilon,x_0} \times (0,T_\varepsilon),$$

that is

$$u(x+\varepsilon(x-x_0), (1+\varepsilon)t) < u(x,t)$$

for $\varepsilon > 0$ if $x_0 \in B(0,\delta)$.

For (x,t) fixed these directions form a cone in space and time with an oblique axis that coincides with the direction (x,t) . These directions are the lines through (x,t) and $(x_0,0)$ with $x_0 \in B(0,\delta)$. Let us call this cone $K_{(x,t)}$. As in Thm. 5.1 we can see that for $(x_1,t_1) \in \Gamma$,

$$(6.1) \quad c(u) < 1 \quad \text{in } ((x_1, t_1) + K_{(x_1, t_1)}) \cap \bar{Q}_T;$$

$$(6.2) \quad c(u) \equiv 1 \quad \text{in } ((x_1, t_1) - K_{(x_1, t_1)}) \cap \bar{Q}_T.$$

So the only difference between Problem I and Problem II is that the cone K in Problem I is independent of the point (x_1, t_1) . We will prove that

$$(6.3) \quad u < 0 \quad \text{in } ((x_1, t_1) + K_{(x_1, t_1)}) \cap \bar{Q}_T;$$

$$(6.4) \quad u > 0 \quad \text{in } ((x_1, t_1) - K_{(x_1, t_1)}) \cap \bar{Q}_T.$$

for any $(x_1, t_1) \in \Gamma$.

(6.3) is an immediate consequence of (6.1) and the definition of c , whereas (6.2) only implies that u is nonnegative in $((x_1, t_1) - K_{(x_1, t_1)}) \cap Q_T$. We want to see that it cannot be zero. Suppose that there is a point (x_2, t_2) in $((x_1, t_1) - K_{(x_1, t_1)}) \cap Q_T$ such that $u(x_2, t_2) = 0$. Consider the cone $K_{(x_2, t_2)}$; we have that $u \leq 0$ in $((x_2, t_2) + K_{(x_2, t_2)}) \cap Q_T$. Also (x_1, t_1) is in its interior. So there exists a cylinder $Q_\mu : \{|x - x_1| < \mu, |t - t_1| < \mu^2\}$ contained in $((x_2, t_2) + K_{(x_2, t_2)}) \cap Q_T$ for a $\mu > 0$ small enough.

Since $u \leq 0$ in Q_μ equation (1.1) is uniformly parabolic in this set. In particular u is a classical solution. As $u \leq 0$ in the parabolic boundary of Q_μ , the strong maximum principle states that either $u < 0$ in Q_μ or $u \equiv 0$ in Q_μ . But this contradicts the fact that (x_1, t_1) is a

boundary point of the unsaturated region. Therefore $u > 0$ in

$$((x_1, t_1) - K_{(x_1, t_1)}) \cap Q_T.$$

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