DERIVATION OF THE HYDRODYNAMICAL EQUATION FOR
ONE-DIMENSIONAL GINZBURG-LANDAU MODEL

By

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§0 Introduction

The purpose of the present paper is to investigate the hydrodynamical behavior of a scalar Ginzburg-Landau model on $\mathbb{R}$. The dynamics of the system is described by a stochastic partial differential equation (SPDE). A nonlinear diffusion equation will be derived under appropriate space-time hydrodynamical scaling limit. The result will be formulated as a law of large numbers for the SPDE.

We start with explaining the Ginzburg-Landau model at a formal level. The configuration (or order parameter) $S = \{S(x), x \in \mathbb{R}\}$ over the state space $\mathbb{R}$ is a real-valued function on $\mathbb{R}$. We associate an energy with each configuration S :

$$(0.1) \quad \mathcal{H}(S) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |\nabla S(x)|^2 + U(S(x)) \right\} dx, \quad \nabla = \frac{d}{dx},$$

where the self-potential U is a real-valued function on $\mathbb{R}$. The functional $\mathcal{H}$ is usually called the Ginzburg-Landau-Wilson free energy. The corresponding statistical mechanical system (equilibrium state) is described by a Gibbs distribution and formally given by

$$(0.2) \quad \mu(dS) = Z^{-1} e^{-\mathcal{H}(S)} dS, \quad dS = \prod_{x \in \mathbb{R}} dS(x).$$

More mathematically, μ is a probability measure on the space $C(\mathbb{R})$ of all configurations, which is defined through the DLR equation (see Section 3) or given by taking the thermodynamical limit:

$$(0.3) \quad \mu(dS) = \lim_{\ell \rightarrow \infty} Z_{\ell}^{-1} e^{-\int_{-\ell}^{\ell} U(S(x)) dx} \mu_{-\ell,0;\ell,0}(dS),$$

where Z_{ℓ} is a normalizing constant and $\mu_{-\ell,0;\ell,0}$ is a probability distribution on $C(\mathbb{R})$ of the pinned Brownian motion satisfying $S(-\ell) = S(\ell) = 0$ and $S(x) = 0$, $|x| > \ell$ for convenience.

The dynamics which corresponds to the Ginzburg-Landau-Wilson free energy $\mathcal{H}(S)$ is a time evolution having the Gibbs distribution μ as its equilibrium measure. We can introduce such dynamics by solving a stochastic differential equation:

$$(0.4) \quad dS_t(x) = -A \frac{\delta \mathcal{H}}{\delta S(x)}(S_t) dt + \sqrt{2A} dw_t(x), \quad t > 0, x \in \mathbb{R}.$$

Here $\frac{\delta \mathcal{H}}{\delta S(x)}$ is a functional derivative of $\mathcal{H}$ and w_t is a cylindrical Brownian motion on the space $L^2(\mathbb{R}, dx)$. The fluctuation-dissipation theorem permits us to take a non-negative operator A acting on the variable x freely. In physical literatures especially the cases of $A = \text{identity}$ and $A = -\Delta \equiv -\frac{d^2}{dx^2}$ are discussed. In the latter case the total spin $\int S_t(x) dx$ is (formally) conserved as the time passes, while in the former case there are no such conserved quantities. The equation (0.4) is usually called the time-dependent Ginzburg-Landau equation (TDGL eq.). In this paper we shall discuss the latter case. More precisely, the stochastic dynamics we shall analyze is the conservative TDGL equation:

$$(0.5) \quad dS_t(x) = -\Delta^2 S_t(x) dt + \Delta \{U'(S_t(x))\} dt + \sqrt{2} \nabla dw_t(x), \quad t > 0, x \in \mathbb{R}.$$

Note that $\sqrt{-\Delta} w_t$ and ∇w_t have the same distributions.

Since the equation (0.5) has a conserved quantity, it carries as its invariant measures not only μ but also a one-parameter family of probability distributions $\{\mu_\lambda\}_{\lambda \in \mathbb{R}}$ which are formally given by

$$(0.6) \quad \mu_\lambda(dS) = Z_\lambda^{-1} e^{-\mathcal{H}_\lambda(S)} dS,$$

$$(0.7) \quad \mathcal{H}_\lambda(S) = \mathcal{H}(S) - \lambda \int S(x) dx, \quad \lambda \in \mathbb{R}.$$

These measures can be defined again by taking the thermodynamical limit:

$$(0.8) \quad \mu_\lambda(dS) = \lim_{\ell \rightarrow \infty} Z_{\ell, \lambda}^{-1} e^{-\int_{-\ell}^{\ell} (U(S(x)) - \lambda S(x)) dx} \mu_{-\ell, 0; \ell, 0}(dS).$$

The parameter λ represents the chemical potential or the strength of the external field. See Hohenberg and Halperin [19], Kawasaki and Ohta [26] and Spohn [39] for more detailed expositions on the Ginzburg-Landau model. Similar model appears also in the context of quantum field theory, see Parisi and Wu [31] and Jona-Lasinio and Mitter [23].

Now we introduce the hydrodynamical scaling:

$$(0.9) \quad t \longrightarrow t/\varepsilon^2, \quad x \longrightarrow x/\varepsilon, \quad \varepsilon > 0,$$

for the TDGL equation (0.5). After this scaling the process $S_t^\varepsilon(x) = S_{t/\varepsilon^2}(x/\varepsilon)$ solves the following scaled TDGL equation:

$$(0.10) \quad dS_t^\varepsilon(x) = -\varepsilon^2 \Delta^2 S_t^\varepsilon(x) dt + \Delta(U'(S_t^\varepsilon(x))) dt + \sqrt{2\varepsilon} \nabla dw_t(x), \quad t > 0, \quad x \in \mathbb{R}.$$

We note that $w_t(x)$ and $\sqrt{\varepsilon} w_{t/\varepsilon^2}(x/\varepsilon)$ are equivalent in law. The main purpose of the present paper is to investigate the asymptotic behavior of $S_t^\varepsilon(x)$ as ε tends to 0. Assuming that the self-potential function U is a small perturbation from a quadratic function $\gamma s^2/2$, $\gamma > 0$, and also assuming some regularity condition on an initial value S of the equation (0.10) we shall prove that

S_t^ε converges to a non-random function $\rho_t = \rho_t(x)$ which is a solution of the nonlinear diffusion equation:

$$(0.11) \quad \frac{\partial \rho_t}{\partial t} = \frac{\partial}{\partial x} \left(d(\rho_t) \frac{\partial \rho_t}{\partial x} \right) .$$

This equation may be thought of as a correspondence to the Euler equation in hydrodynamics. The diffusion coefficient $d(\rho)$ of the equation (0.11) is positive and determined in the following manner:

Let $H_\lambda = -\frac{1}{2} \frac{d^2}{ds^2} + \{U(s) - \lambda s\}$, $\lambda \in \mathbb{R}$, be a self-adjoint operator on the space $L^2(\mathbb{R}, ds)$ and let Ω_λ be a positive and normalized eigenfunction of H_λ corresponding to its minimal eigenvalue, that is, Ω_λ is the ground state of the operator H_λ . The mean spin function $\bar{\rho}$ is defined by

$$(0.12) \quad \bar{\rho}(\lambda) = \int s \Omega_\lambda^2(s) ds, \quad \lambda \in \mathbb{R},$$

which is real analytic and strictly increasing in λ . The function $d(\rho)$ is now given by the derivative

$$(0.13) \quad d(\rho) = \bar{\lambda}'(\rho),$$

of an inverse function $\bar{\lambda} = \bar{\lambda}(\rho)$ of $\bar{\rho} = \bar{\rho}(\lambda)$.

It is worth making a remark that the diffusion coefficient is not $U''(\rho)$. If it holds that both the first and the third terms in the right hand side (RHS) of (0.10) disappear as ε tends to 0, then one might get this conclusion; however this is not true.

Heuristically the equation (0.11) can be derived according to the following idea. We assume the principle of hydrodynamics: There exists a function $\lambda(t, x)$ giving a chemical profile such that for each $(t, x) \in (0, \infty) \times \mathbb{R}$ the distribution of $S_t^\varepsilon(x)$ converges weakly to that of $S(0)$ under $\mu_{\lambda(t, x)}$ as ε tends to 0. This

assumption seems to be true, since the hydrodynamical scaling makes the system evolve rapidly and therefore $S_t^\varepsilon(x)$ may converge to one of the equilibrium states $\{\mu_\lambda\}_{\lambda \in \mathbb{R}}$. Physicists sometimes call this principle the local equilibrium approximations, see Kawasaki [24], [25]. Let us consider only the behavior of the mean spin $\rho_t^\varepsilon(x) = E[S_t^\varepsilon(x)]$ for simplicity. Then from the equation (0.10) one might have

$$(0.14) \quad \frac{\partial}{\partial t} \rho_t^\varepsilon(x) = \frac{\partial^2}{\partial x^2} E[U'(S_t^\varepsilon(x))] + O(\varepsilon), \quad \varepsilon \downarrow 0.$$

Now the principle of hydrodynamics implies

$$(0.15) \quad \lim_{\varepsilon \downarrow 0} E[U'(S_t^\varepsilon(x))] = \int U'(s) \Omega_{\lambda(t,x)}^2(s) ds,$$

since, under the distribution μ_λ , $S(x)$ having $x \in \mathbb{R}$ as its time parameter is a symmetric diffusion process with reversible measure $\Omega_\lambda^2(s)ds$, see Section 3. However, using integration by parts it is not difficult to prove the RHS of (0.15) is equal to $\lambda(t,x)$.

Therefore, if the limit $\rho_t(x) = \lim_{\varepsilon \downarrow 0} \rho_t^\varepsilon(x)$ exists, then ρ_t must solve the equation (0.11) since $\lambda(t,x) = \bar{\lambda}(\rho_t(x))$.

The actual proof to derive the nonlinear diffusion equation (0.11) will not be given in this manner, since it seems to be very difficult to prove the principle of hydrodynamics. We shall follow and extend the method due to Fritz [13],[14], who discussed the problem for the discretized version of the Ginzburg-Landau model. See the references in the papers of Fritz as an introduction to the problem of hydrodynamical limit; we especially refer to De Masi, Ianiro, Pellegrinotti and Presutti [6], De Masi, Ferrari and Lebowitz

[7] and Rost [33],[34]. Funaki [16] gave a probabilistic interpretation for the derivation of the Euler equation from the Boltzmann equation in terms of Markov processes of McKean type. Zhu [43] discussed the central limit theorem for the TDGL equation (0.5).

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§1 Main result and idea of the proof

We assume the following conditions on the self-potential function U :

$$(1.1) \quad U(s) = \frac{\gamma}{2}s^2 + V(s) ; \gamma > 0, V \in C_b^3(\mathbb{R}),$$

and

$$(1.2) \quad \gamma_0 \equiv \|V''\|_\infty < \min(\gamma, \gamma_1) ,$$

where γ_1 is an absolute constant which will be introduced in Section 10. Let $C_b^{2+\beta}(\mathbb{R})$, $0 < \beta < 1$, be the class of all $S \in C_b^2(\mathbb{R})$ satisfying

$$\sup \{ |S''(x) - S''(y)| / |x - y|^\beta ; x, y \in \mathbb{R}, |x - y| < 1 \} < \infty .$$

Our main result is the following.

Theorem 1.1 Let S_t^ε and ρ_t be the solutions of the scaled

TDGL equation (0.10) and the nonlinear diffusion equation (0.11), respectively, with same initial value S ; $S_0^\varepsilon = \rho_0 = S$. We assume $S \in C_b^{2+\beta}(R)$. Then S_t^ε converges to ρ_t as ε tends to 0 in the following sense:

$$(1.3) \quad \lim_{\varepsilon \downarrow 0} P(|\int_R (S_t^\varepsilon(x) - \rho_t(x)) \varphi(x) dx| > \delta) = 0$$

uniformly in $t \in [0, T]$ for every $\delta > 0$, $T > 0$ and $\varphi \in C_0^\infty(R)$.

The existence and uniqueness theorems for the equations (0.10) and (0.11) will be discussed in Sections 5 and 12, respectively.

Now we expose the strategy for the proof of Theorem 1.1 somewhat formally. The generator of the scaled process S_t^ε may be written as

$$(1.4) \quad \mathcal{G}^\varepsilon = -\varepsilon \int \left(\frac{\delta}{\delta S(x)} \Delta \frac{\delta}{\delta S(x)} - \Delta \frac{\delta \mathcal{H}^\varepsilon}{\delta S(x)} \frac{\delta}{\delta S(x)} \right) dx ,$$

where

$$(1.5) \quad \mathcal{H}^\varepsilon(S) = \frac{1}{\varepsilon} \int \left(\frac{\varepsilon^2}{2} |\nabla S(x)|^2 + U(S(x)) \right) dx .$$

Note that the reversible measure of S_t^ε is the scaled Gibbs distribution, i.e., an image measure $\mu_\varepsilon = \mu \circ \sigma_\varepsilon^{-1}$ of μ introduced in Section 0 under the spatial scaling transformation σ_ε defined by $(\sigma_\varepsilon S)(x) = S(x/\varepsilon)$, $S \in C(R)$; or formally saying

$$(1.6) \quad \mu_\varepsilon(dS) = Z_\varepsilon^{-1} e^{-\mathcal{H}^\varepsilon(S)} dS .$$

A significant role will be played by the so-called local equilibrium distributions. To explain this concept we introduce a family Λ of chemical-potential profile functions: The class Λ consists of all $\lambda \in C^2(R)$ satisfying $\lambda' \in C_0(R)$ and accordingly

$\lambda = \text{constant}$ on two intervals $(-\infty, x_-]$ and $[x_+, \infty)$, respectively, with some $x_- < x_+$. With each $\lambda(\cdot) \in \Lambda$ we associate the local equilibrium distributions $\{\mu_{\lambda(\cdot), \varepsilon}\}_{\varepsilon > 0}$ formally defined by

$$(1.7) \quad \mu_{\lambda(\cdot), \varepsilon}(dS) = Z_{\lambda(\cdot), \varepsilon}^{-1} e^{-\mathcal{H}_{\lambda(\cdot)}^{\varepsilon}(S)} dS,$$

$$(1.8) \quad \mathcal{H}_{\lambda(\cdot)}^{\varepsilon}(S) = \mathcal{H}^{\varepsilon}(S) - \frac{1}{\varepsilon} \int \lambda(x) S(x) dx.$$

Note that $\mu_{\lambda(\cdot), \varepsilon} = \mu_{\lambda} \circ \sigma_{\varepsilon}^{-1}$ with μ_{λ} introduced in Section 0 when $\lambda(\cdot) = \lambda = \text{constant}$. An important property of $\{\mu_{\lambda(\cdot), \varepsilon}\}$ is the law of large numbers: As ε tends to 0, $\mu_{\lambda(\cdot), \varepsilon}$ converges in a proper sense to the δ -distribution concentrated on a configuration ρ defined by $\rho(x) = \bar{\rho}(\lambda(x))$ where $\bar{\rho}$ is the mean spin function, see Section 4.

As we have seen in Section 0 the main difficulty to analyze the asymptotic behavior of S_t^{ε} lies in the treatment of the nonlinear term $U'(S_t^{\varepsilon})$ appearing in the RHS of (0.10). To overcome this difficulty we notice a formula of integration by parts:

$$(1.9) \quad - \int \mathcal{G}^{\varepsilon} \Phi \cdot \Psi d\mu_{\varepsilon} = \int d\mu_{\varepsilon} \int \nabla \frac{\delta \Phi}{\delta S(x)} \cdot \nabla \frac{\delta \Psi}{\delta S(x)} dx,$$

which may hold for tame functions Φ and Ψ defined on the configuration space. We emphasize the nonlinear term disappears from the RHS of (1.9) which is nothing but the Dirichlet form of the process S_t^{ε} . If we can take $\Psi = \exp\{\frac{1}{\varepsilon} \int \lambda(x) S(x) dx\}$ and $T_t^{\varepsilon} \Phi \equiv e^{t\mathcal{G}^{\varepsilon}} \Phi$ instead of Φ in this formula, then we might obtain

$$(1.10) \quad \frac{\partial}{\partial t} \int T_t^{\varepsilon} \Phi d\mu_{\lambda(\cdot), \varepsilon} = \int \mathcal{G}^{\varepsilon} T_t^{\varepsilon} \Phi d\mu_{\lambda(\cdot), \varepsilon}$$

$$= \int d\mu_{\lambda(\cdot), \varepsilon} \int \Delta\lambda(x) \frac{\delta}{\delta S(x)} T_t^\varepsilon \Phi \, dx .$$

We of course need a proof of this formula since for example $T_t^\varepsilon \Phi$ is not a tame function, see Sections 6 - 8 for detail.

The compactness of two families $\{T_t^\varepsilon \Phi\}_{\varepsilon > 0}$ and $\{\frac{\delta}{\delta S(x)} T_t^\varepsilon \Phi\}_{\varepsilon > 0}$ as functions of t and configurations S will be shown by using energy inequalities, see Sections 9 - 11. We can therefore take subsequence ε' in such a way that $T_t^{\varepsilon'} \Phi$ and $\frac{\delta}{\delta S(x)} T_t^{\varepsilon'} \Phi$ converge to some functions Φ_t and $\frac{\delta}{\delta S(x)} \Phi_t$, respectively, in a proper sense as ε' tends to 0. Now with the help of the law of large numbers for $\{\mu_{\lambda(\cdot), \varepsilon}\}$ the equation (1.10) implies that the limit functions satisfy

$$(1.11) \quad \frac{\partial}{\partial t} \Phi_t(\rho) = \int \Delta\{\bar{\lambda}(\rho(x))\} \frac{\delta}{\delta S(x)} \Phi_t(\rho) \, dx ,$$

for every $\rho \in \bar{\rho}(\Lambda) = \{\bar{\rho}(\lambda(\cdot)) ; \lambda(\cdot) \in \Lambda\}$. However the equation (1.11) is just the Liouville equation corresponding to the nonlinear diffusion equation (0.11) and therefore we obtain $\Phi_t(\rho) = \Phi(\rho_t)$ with the solution ρ_t of (0.11) satisfying $\rho_0 = \rho$. Hence it holds $\lim_{\varepsilon \downarrow 0} T_t^\varepsilon \Phi(\rho) = \Phi(\rho_t)$ for each tame function Φ and this implies the conclusion of Theorem 1.1.

§2 Notations and preliminary facts

In this section we first introduce some notations which will be used throughout the paper and then summarize known properties and their simple consequences on the so-called Schrödinger operators on R .

2.1 Notations (i) Generally for a topological space X , $C_b(X)$ and $\mathcal{P}(X)$ stand for the space of all bounded continuous functions on X and the space of all Borel probability measures on X , respectively. We denote $\langle \mu, \Phi \rangle = \int_X \Phi d\mu$ for $\Phi \in C_b(X)$ and $\mu \in \mathcal{P}(X)$.

(ii) Configuration spaces and their dual spaces Let $\chi \in C^\infty(\mathbb{R})$ be a positive even function satisfying $\chi(x) = |x|$ for $|x| \geq 1$. The function χ will be fixed throughout the paper. Setting $\theta(x, r) = e^{-r\chi(x)}$, $r \in \mathbb{R}$, we introduce a family of Hilbert spaces $H_r = L^2(\mathbb{R}, \theta(x, r) dx)$, $r \in \mathbb{R}$, having norms defined by

$$(2.1) \quad \|S\|_r = \left(\int_{\mathbb{R}} S(x)^2 \theta(x, r) dx \right)^{1/2}, \quad S \in H_r.$$

The space H_{-r} can be identified with the dual space H_r^* of H_r . We set $H_e = \bigcap_{r>0} H_r$ and $H_e^* = \bigcup_{r>0} H_r^*$. The space H_e is a countably Hilbertian space with topology determined by the family of Hilbertian norms $\{\|\cdot\|_r\}_{r>0}$. We shall also consider a weak topology $\sigma(H_e, H_e^*)$ on this space. With this topology the space H_e will be written by $H_{e,w}$.

We denote by $\mathcal{C}$ the space $C(\mathbb{R})$ with the usual topology of uniform-convergence on each bounded set. Let $\mathcal{B}_I$ be a σ -field of the space $\mathcal{C}$ generated by $\{S(x); x \in I\}$, $S \in \mathcal{C}$, for every subset I of $\mathbb{R}$. We simply write $\mathcal{B}$ for $\mathcal{B}_{\mathbb{R}}$ and sometimes use the same notation $\mathcal{B}_I$ to denote the Borel field of the space $C(I)$ for each interval I of $\mathbb{R}$. Let $\mathcal{C}_r$, $r \in \mathbb{R}$, be the space of all $S \in \mathcal{C}$ satisfying $\|S\|_r = \sup_{x \in \mathbb{R}} |S(x)| \theta(x, r) < \infty$ and set $\mathcal{C}_e = \bigcap_{r>0} \mathcal{C}_r$ the countably normed space with topology determined by the norms $\{\|\cdot\|_r\}_{r>0}$. We also consider the space $\mathcal{C}_{-r}$, $r > 0$, of all $\varphi \in \mathcal{C}_{-r}$

satisfying

$$(2.2) \quad \lim_{|x| \rightarrow \infty} |\varphi(x)| \theta(x, -r) = 0.$$

The space $\mathcal{E}_{-r}$ is a Banach space with norm $\|\cdot\|_{-r}$.

(iii) The class of tame functions on the configuration space Let $\mathcal{D}$ be the class of all functions Ψ on the space $\mathcal{E}$ having the form:

$$(2.3) \quad \Psi(S) = \psi(\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle) \quad , \quad S \in \mathcal{E},$$

with $k = 1, 2, \dots$, $\psi = \psi(\alpha_1, \dots, \alpha_k) \in C_0^\infty(\mathbb{R}^k)$ and $\varphi_1, \dots, \varphi_k \in C_0^\infty(\mathbb{R})$, where $\langle S, \varphi \rangle = \int_{\mathbb{R}} S(x) \varphi(x) dx$. We also denote by $\mathcal{D}$ without

distinction the class of such functions on $\mathcal{E}_e$, H_e or H_r etc.

instead of $\mathcal{E}$ as introduced similarly as above. For $\ell > 0$, $\mathcal{D}_\ell$

is the class of all $\Psi \in \mathcal{D}$ of the form (2.3) with $\varphi_1, \dots, \varphi_k$ having supports in an open interval $(-\ell, \ell)$. We shall sometimes denote by

$\mathcal{D}_\ell$ again the class of functions Ψ on the space $C([- \ell, \ell])$ or $L^2([- \ell, \ell])$, which are defined in a similar manner.

2.2 Schrödinger operator Let $H_\lambda = -\frac{1}{2} \frac{d^2}{ds^2} + \{U(s) - \lambda s\}$, $\lambda \in \mathbb{R}$, be a self-adjoint operator on a Hilbert space $L^2(\mathbb{R}, ds)$. We assume the condition (1.1) on U . Then H_λ has purely discrete spectrum and its normalized eigenfunction Ω_λ corresponding to the minimum eigenvalue $\kappa(\lambda)$, which is simple, can be taken to be positive. The function Ω_λ decays exponentially fast, i.e., $\Omega_\lambda(s) \leq \text{const.} e^{-r|s|}$, $s \in \mathbb{R}$, for every $r > 0$. See Reed and Simon [32].

We can show that $\kappa(\lambda)$ is real-analytic in $\lambda \in \mathbb{R}$ and Ω_λ is strongly differentiable as an $L^2(\mathbb{R}, ds)$ -valued function of $\lambda \in \mathbb{R}$. These facts follow from Kato-Rellich theorem by proving that the family of operators $\{H_\lambda; \lambda \in \mathbb{C}\}$ on a complex Hilbert space $L^2(\mathbb{R} \rightarrow \mathbb{C}, ds)$

is an analytic family in the sense of Kato for some neighborhood $\emptyset$ of R in the complex plane $\mathbb{C}$, see Reed and Simon [32, vol IV, pp14-17]. Another consequence of this theorem is that $\inf_{\lambda \in I} \delta(\lambda) > 0$ for every bounded interval I of R , where $\delta(\lambda)$ is the gap between the second least eigenvalue of H_λ and $\kappa(\lambda)$.

Final remark is on the positivity of the diffusion coefficient $d(\rho)$ defined by (0.13). Indeed differentiating both sides of an equality $H_\lambda \Omega_\lambda = \kappa(\lambda) \Omega_\lambda$ in λ we obtain $(H_\lambda - \kappa(\lambda)) \frac{\partial \Omega_\lambda}{\partial \lambda} = (s + \kappa'(\lambda)) \Omega_\lambda$, which implies $\kappa'(\lambda) = -\bar{\rho}(\lambda)$ and therefore $\frac{\partial \Omega_\lambda}{\partial \lambda} = (H_\lambda - \kappa(\lambda))^{-1} \eta_\lambda$ since $\frac{\partial \Omega_\lambda}{\partial \lambda} \in L_0^2$. Here $(H_\lambda - \kappa(\lambda))^{-1}$ is a positive operator of $L_0^2 \rightarrow L_0^2$, $\eta_\lambda = (s - \bar{\rho}(\lambda)) \Omega_\lambda \in L_0^2$ and L_0^2 is the space of all $\eta \in L^2(R, ds)$ orthogonal to Ω_λ ; $(\eta, \Omega_\lambda) \equiv (\eta, \Omega_\lambda)_{L^2} = 0$. Hence we see $d(\rho) > 0$ from

$$\bar{\rho}'(\lambda) = 2(s \Omega_\lambda, \frac{\partial \Omega_\lambda}{\partial \lambda}) = 2(\eta_\lambda, (H_\lambda - \kappa(\lambda))^{-1} \eta_\lambda)$$

which is finite and positive.

§3 Spatially inhomogeneous Gibbs distributions

Throughout the paper we fix the function U and assume the condition (1.1).

3.1 Definition and construction Let $\mu_{x,s_1;y,s_2}$, $x < y$, $s_1, s_2 \in R$, be a probability distribution on the space $(C([x,y]), \mathcal{B}_{[x,y]})$ of the pinned Brownian motion $S = \{S(z); z \in [x,y]\}$ with time parameter z satisfying $S(x) = s_1$ and $S(y) = s_2$; more precisely, under this probability measure

$$S(z) = s_1 + B(z-x) + \frac{z-x}{y-x} \{s_2 - s_1 - B(y-x)\} \quad , \quad z \in [x,y],$$

with a standard Brownian motion $\{B(z); z \geq 0\}$. Let $\lambda(\cdot) \in \Lambda$ be a chemical-potential profile, see Section 1. The local specification is a probability measure on $C([x,y])$ defined by

$$\mu_{\lambda(\cdot)}^{x,y}(dS; s_1, s_2) = \Xi^{-1} \exp\{-\int_x^y U(z, S(z); \lambda(\cdot)) dz\} \mu_{x, s_1; y, s_2}(dS),$$

for each $x < y$ and $s_1, s_2 \in \mathbb{R}$, where $U(z, s; \lambda(\cdot)) = U(s) - \lambda(z)s$ and $\Xi = \Xi_{\lambda(\cdot)}^{x,y}(s_1, s_2)$ is a normalizing constant. We sometimes regard $\mu_{\lambda(\cdot)}^{x,y}(\cdot; s_1, s_2)$ as a probability measure on the space $(\mathcal{G}, \mathcal{B})$ by considering $S(z) = s_1$ for $z \leq x$ and $S(z) = s_2$ for $z \geq y$ under this probability distribution. A probability measure μ on $(\mathcal{G}, \mathcal{B})$ is called a $(U, \lambda(\cdot))$ -Gibbs distribution associated with the Ginzburg-Landau-Wilson free energy if and only if it satisfies the so-called DLR equation:

$$\mu(A | \mathcal{B}_{(x,y)^c}^c)(S) = \mu_{\lambda(\cdot)}^{x,y}(A; S(x), S(y)) \quad , \quad \mu\text{-a.e. } S,$$

for every $x < y$ and $A \in \mathcal{B}_{[x,y]}$.

The Gibbs distribution can be constructed in the following manner. Let two functions $\{\Omega^{(\pm)}(x, s)\}$ be the solutions of two diffusion equations:

$$(3.1) \quad \frac{\partial}{\partial x} \Omega^{(\pm)}(x, s) = \pm \bar{H}_x \Omega^{(\pm)}(x, s) \quad , \quad x \in \mathbb{R},$$

with initial conditions

$$(3.2) \quad \Omega^{(\pm)}(x_{\pm}, s) = \Omega(x_{\pm}, s) \quad ,$$

where $\bar{H}_x = H_{\lambda(x)} - \kappa(\lambda(x))$, $\Omega(x, s) = \Omega_{\lambda(x)}(s)$ and see Section 1 for $x_{\pm}$. The double signs $\pm$ are taken in same order in (3.1) and (3.2).

We define $\mu_{\lambda(\cdot)}(A)$ for $A \in \mathcal{B}_{[x,y]}$, $x < y$, by

$$(3.3) \quad \mu_{\lambda(\cdot)}(A) = Z^{-1} \int_{\mathbb{R}} ds_1 ds_2 \Omega^{(-)}(x, s_1) \Omega^{(+)}(y, s_2) p(y-x, s_1, s_2) \\ \times E^{\mu_{x, s_1; y, s_2}} [\exp(-\int_x^y \{U(z, S(z); \lambda(\cdot)) + \kappa(\lambda(z))\} dz); A] ,$$

and

$$(3.4) \quad Z = \int_{\mathbb{R}} \Omega^{(+)}(x, s) \Omega^{(-)}(x, s) ds ,$$

where $p(z, s_1, s_2)$ is a transition probability of the standard Brownian motion. Then Z is independent of x and the Feynman-Kac formula can be applied to prove that $\mu_{\lambda(\cdot)}$ is well-defined as a probability measure on $(\mathcal{G}, \mathcal{B})$. It is not difficult to see $\mu_{\lambda(\cdot)}$ constructed as above is a $(U, \lambda(\cdot))$ -Gibbs distribution. This is a spatially-inhomogeneous extension of the $P(\phi)_1$ -measure, see Simon [37]. Under the distribution $\mu_{\lambda(\cdot)}$, $\{S(x); x \in \mathbb{R}\}$ can be regarded as a temporally-inhomogeneous diffusion process with 1-dimensional marginal distribution $Z^{-1} \Omega^{(+)}(x, s) \Omega^{(-)}(x, s) ds$ and infinitesimal generator

$$G_x = \frac{1}{2} \frac{d^2}{ds^2} + \left\{ \frac{\partial}{\partial s} \log \Omega^{(+)}(x, s) \right\} \frac{d}{ds} , \quad x \in \mathbb{R}.$$

Although generally there exist another μ 's satisfying the DLR equation, in the following we mean by the $(U, \lambda(\cdot))$ -Gibbs distribution the probability measure $\mu_{\lambda(\cdot)}$ which has been constructed in this manner. We introduce spatial scalings σ_ε and τ_ε by $(\sigma_\varepsilon S)(x) = S(x/\varepsilon)$ and $(\tau_\varepsilon \phi)(x) = \phi(\varepsilon x)$, respectively. Define a scaled $(U, \lambda(\cdot))$ -Gibbs distribution by $\mu_{\lambda(\cdot), \varepsilon} = \mu_{\tau_\varepsilon \lambda(\cdot)} \circ \sigma_\varepsilon^{-1}$, $0 < \varepsilon < 1$, which is sometimes called local equilibrium distribution.

3.2 FKG inequality Here we prove the monotonicity property of $\mu_{\lambda}(\cdot)$ with respect to $\lambda(\cdot)$ and derive some uniform moment-estimates on $\{\mu_{\lambda}(\cdot), \varepsilon\}$ as its consequence.

Definition (see, e.g., Liggett [29] or Simon [36]) (i) For a Polish space X equipped with a partial order $\geq$, $\mathcal{M}(X)$ is a class of all $\Phi \in C_b(X)$ which are monotone-increasing in this order.

(ii) We say $\mu_1 \leq \mu_2$ for $\mu_1, \mu_2 \in \mathcal{P}(X)$ if $\langle \mu_1, \Phi \rangle \leq \langle \mu_2, \Phi \rangle$ holds for every $\Phi \in \mathcal{M}(X)$.

(iii) We say that $\mu \in \mathcal{P}(X)$ satisfies an FKG inequality on the space X if

$$(3.5) \quad \langle \mu, \Phi\Psi \rangle \geq \langle \mu, \Phi \rangle \langle \mu, \Psi \rangle$$

holds for every $\Phi, \Psi \in \mathcal{M}(X)$.

On the space $C(I)$ with interval I of $\mathbb{R}$ a usual partial order $S \leq \bar{S}$ is defined by the relation: $S(x) \leq \bar{S}(x)$ for every $x \in I$. For such $\nu_i \in \mathcal{P}(\mathbb{R})$, $i = 1, 2$, that $\langle \nu_i, \theta(\cdot, -r) \rangle < \infty$ for every $r > 0$ we define $\mu_{\lambda}^{x,y}(\cdot; \nu_1, \nu_2) \in \mathcal{P}(C([x, y]))$ by

$$\mu_{\lambda}^{x,y}(\cdot; \nu_1, \nu_2) = \int_{\mathbb{R}^2} \mu_{\lambda}^{x,y}(\cdot; s_1, s_2) \nu_1(ds_1) \nu_2(ds_2).$$

Lemma 3.1 The probability measures $\mu_{\lambda}^{x,y}(\cdot; s_1, s_2)$, $\mu_{\lambda}^{x,y}(\cdot; \nu_1, \nu_2)$ and $\mu_{\lambda}(\cdot)$ each satisfy the FKG inequality on the spaces on which these measures are defined.

Proof For each partition $\pi_n = \{x = x_0 < x_1 < \dots < x_n < x_{n+1} = y\}$ of the interval $[x, y]$ we introduce $\mu_n \in \mathcal{P}(\mathbb{R}^{n+2})$ by

$$(3.6) \quad \mu_n(\prod_{i=0}^{n+1} dS_i) = e^{-W} \nu_1(dS_0) \prod_{i=1}^n dS_i \nu_2(dS_{n+1}) / \text{normalization}$$

where for $\beta = 0$ or 1

$$(3.7) \quad W = \sum_{i=0}^n \{U(x_i, S_i; \lambda(\cdot))(x_{i+1} - x_i) + \frac{(S_{i+1} - S_i)^2}{2(x_{i+1} - x_i)}\} + \frac{\beta(S_{n+1} - S_0)^2}{2(x_{n+1} - x_0)}.$$

Since μ_n satisfies the FKG inequality on R^{n+2} (see Theorem VIII.16 in Simon [36]), by setting $\beta = 0$ in (3.7) and taking the limit $|\pi_n| = \sup_i |x_{i+1} - x_i| \rightarrow 0$, we can prove the inequality (3.5) for $\mu = \mu_{\lambda(\cdot)}^{x,y}(\cdot; \nu_1, \nu_2)$ and increasing functions Φ and Ψ depending only on finitely many $S(x)$'s, say $\{S(x); x \in \pi_m\}$ with some $m \geq 1$. We can then show (3.5) for general Φ and $\Psi \in \mathcal{M}(C([x, y]))$ by approximations and this proves the FKG inequality for

$$\mu_{\lambda(\cdot)}^{x,y}(\cdot; \nu_1, \nu_2).$$

If we take (i) $\nu_1 = \delta_{s_1}$, $\nu_2 = \delta_{s_2}$ (δ -measures) and $\beta = 0$ or (ii) $d\nu_1 = \Omega^{(-)}(x, s)ds$, $d\nu_2 = \Omega^{(+)}(y, s)ds$ and $\beta = 1$ in (3.6) and (3.7), then we obtain the FKG inequality for $\mu_{\lambda(\cdot)}^{x,y}(\cdot; s_1, s_2)$ in the case (i) or for $\mu_{\lambda(\cdot)}$ in the case (ii), respectively, by similar limiting procedure. $\square$

Lemma 3.2 For every $\mathcal{B}_{(x,y)}$ -measurable $\Phi \in C_b(\mathcal{E})$, $x < y$, and two profiles $\lambda_1(\cdot), \lambda_2(\cdot) \in \Lambda$, we have

$$\langle \mu_{\lambda_1(\cdot)}, \Phi \rangle = \lim_{\ell \rightarrow \infty} \langle \mu_{\lambda_2(\cdot)}, \Psi^\ell \rangle^{-1} \langle \mu_{\lambda_2(\cdot)}, \Phi \Psi^\ell \rangle$$

where

$$(3.8) \quad \Psi^\ell(S) = \exp \left\{ \int_{-\ell}^{\ell} (\lambda_1(z) - \lambda_2(z)) S(z) dz \right\}.$$

Proof Noting that we may assume two profiles are constants on two intervals $(-\infty, x]$ and $[y, \infty)$, respectively, the lemma can be shown by a similar method used in the proof of Theorem 6.9 in Simon [37]. $\square$

We define $\nu_\lambda \in \mathcal{P}(R)$, $\lambda \in R$, by

$$(3.9) \quad \nu_{\lambda}(ds) = \Omega_{\lambda}^2(s) ds.$$

Proposition 3.1 (i) For local specifications, we have

$$\mu_{\lambda_1(\cdot)}^{x,y}(\cdot; s_1, s_2) \geq \mu_{\lambda_2(\cdot)}^{x,y}(\cdot; s_1, s_2)$$

if $\lambda_1(\cdot), \lambda_2(\cdot) \in \Lambda$ satisfy $\lambda_1(\cdot) \geq \lambda_2(\cdot)$ in the space $\mathcal{E}$ and

$$\mu_{\lambda(\cdot)}^{x,y}(\cdot; s_1, s_2) \geq \mu_{\lambda(\cdot)}^{x,y}(\cdot; \bar{s}_1, \bar{s}_2)$$

if $s_1 \geq \bar{s}_1$ and $s_2 \geq \bar{s}_2$.

(ii) For Gibbs distributions we have

$$\mu_{\lambda_1(\cdot)} \geq \mu_{\lambda_2(\cdot)}$$

if $\lambda_1(\cdot), \lambda_2(\cdot) \in \Lambda$ satisfy $\lambda_1(\cdot) \geq \lambda_2(\cdot)$ in the space $\mathcal{E}$.

Especially we have $\nu_{\lambda_1} \geq \nu_{\lambda_2}$ in the space $\mathcal{P}(\mathbb{R})$ for two real

numbers λ_1, λ_2 such that $\lambda_1 \geq \lambda_2$.

(iii) For every $p \geq 1$ and $\lambda(\cdot) \in \Lambda$, we have

$$\sup_{x \in \mathbb{R}, 0 < \varepsilon < 1} \langle \mu_{\lambda(\cdot), \varepsilon}^p, |S(x)|^p \rangle < \infty.$$

Proof The first assertion of (i) is an easy consequence of the FKG inequality for $\mu_{\lambda_2(\cdot)}^{x,y}(\cdot; s_1, s_2)$ with the help of an equality:

$$\langle \mu_{\lambda_1(\cdot)}^{x,y}(\cdot; s_1, s_2), \Phi \rangle = \langle \mu_{\lambda_2(\cdot)}^{x,y}(\cdot; s_1, s_2), \Psi \rangle^{-1} \langle \mu_{\lambda_2(\cdot)}^{x,y}(\cdot; s_1, s_2), \Phi \Psi \rangle$$

for $\Phi \in \mathcal{M}(C([x, y]))$, where Ψ is an increasing function on $C([x, y])$

defined by the RHS of (3.8) with $-\ell$ and ℓ replaced by x and y

respectively. Similarly the first assertion of (ii) can be proved

by the FKG inequality for $\mu_{\lambda_2(\cdot)}$ and Lemma 3.2. We can show the

inequality $\nu_{\lambda_1} \geq \nu_{\lambda_2}$ immediately since ν_{λ} is the 1-dimensional

marginal distribution of $\mu_{\lambda(\cdot)}$ with $\lambda(\cdot) \equiv \lambda$. The assertion (iii)

follows from the properties of $\Omega_{\lambda_{\pm}}$: $\Omega_{\lambda_{\pm}}(s) \leq \text{const.} e^{-r|s|}$, $r > 0$, by

noting that (ii) shows $\mu_{\lambda_-} \leq \mu_{\lambda(\cdot)} \leq \mu_{\lambda_+}$, where $\lambda_- = \inf_x \lambda(x)$ and $\lambda_+ = \sup_x \lambda(x)$. Finally the second assertion of (i) can be proved as follows: Let $\nu_{s,v}$ be the Gaussian distribution on $\mathbb{R}$ with mean s and variance v . Then, since the function $\Psi(S) = (d\nu_{s_1,v}/d\nu_{s_1,v}^-)(S(x)) \cdot (d\nu_{s_2,v}/d\nu_{s_2,v}^-)(S(y))$ is increasing on the space $C([x,y])$, the FKG inequality for $\bar{\mu}_v^{x,y}$ proves an inequality $\langle \mu_v^{x,y}, \Phi \rangle \geq \langle \bar{\mu}_v^{x,y}, \Phi \rangle$, $\Phi \in \mathcal{M}(C([x,y]))$, where $\mu_v^{x,y} = \mu_{\lambda(\cdot)}^{x,y}(\cdot; \nu_{s_1,v}, \nu_{s_2,v})$ and $\bar{\mu}_v^{x,y} = \mu_{\lambda(\cdot)}^{x,y}(\cdot; \nu_{s_1,v}^-, \nu_{s_2,v}^-)$. Therefore the conclusion follows by taking the limit $v \downarrow 0$ of both sides of this inequality. $\square$

Remark 3.1 (i) Iwata [21] proved that $\mu_{\lambda(\cdot)}(\mathcal{E}_e) = 1$ when $\lambda(\cdot) = \text{constant}$. Therefore Proposition 3.1-(ii) shows that $\mu_{\lambda(\cdot)}(\mathcal{E}_e) = 1$ for general $\lambda(\cdot) \in \Lambda$.
(ii) Since the inclusion map of $\mathcal{E}_e$ into H_e or $H_{e,w}$ is continuous, we can regard $\mu_{\lambda(\cdot)} \in \mathcal{P}(H_e)$ or $\in \mathcal{P}(H_{e,w})$ by identifying it with its image measure under the inclusion map.

3.3 Thermodynamical limit The $(U, \lambda(\cdot))$ -Gibbs distribution $\mu_{\lambda(\cdot)}$ can be also obtained by taking the thermodynamical limit.

Proposition 3.2 The probability measure $\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)$ converges weakly to $\mu_{\lambda(\cdot)}$ as $\ell \rightarrow \infty$ on the space $\mathcal{E}_r$ for every $r > 0$.

Proof Assume $0 < \ell' < \ell$ and $-\ell' < x_- < x_+ < \ell'$. Then we have a representation:

$$(3.10) \quad \mu_{\lambda(\cdot)}^{-\ell, \ell}(S(-\ell') \in ds_1, S(\ell') \in ds_2; 0, 0)$$

$$= Z_{\ell}^{-1} f_{\ell}(s_1, s_2) e^{-(\ell-\ell')\bar{H}_-(0, s_1)} e^{-(\ell-\ell')\bar{H}_+(0, s_2)} ds_1 ds_2,$$

where

$$f_{\ell}(s_1, s_2) = e^{\ell'(\kappa(\lambda_-) + \kappa(\lambda_+))} \Xi_{\lambda(\cdot)}^{-\ell', \ell'}(s_1, s_2) p(2\ell', s_1, s_2)$$

$$Z_{\ell} = f_{\ell}(0, 0), \quad \lambda_{\pm} = \lambda(x_{\pm}), \quad \bar{H}_{\pm} = \bar{H}_{x_{\pm}},$$

and $e^{-\ell\bar{H}_{\pm}}(s, s')$ is the kernel function of the semigroup operator

$$e^{-\ell\bar{H}_{\pm}}:$$

$$(e^{-\ell\bar{H}_{\pm}}\varphi)(s) = \int_{\mathbb{R}} e^{-\ell\bar{H}_{\pm}}(s, s')\varphi(s')ds', \quad \varphi \in L^2(\mathbb{R}, ds), \quad \ell > 0.$$

Since it is easy to see that $e^{-\ell\bar{H}_{\pm}}(0, \cdot)$ converges respectively to $c_{\pm}\Omega(x_{\pm}, \cdot)$ in the space $L^2(\mathbb{R}, ds)$ as $\ell \rightarrow \infty$ with some positive constants $c_{\pm}$, we can prove using (3.10) and the Markov property of the local specifications that $\inf_{\ell \geq 1} Z_{\ell} > 0$ and

$$\lim_{\ell \rightarrow \infty} \mu_{\lambda(\cdot)}^{-\ell, \ell}(A; 0, 0) = \mu_{\lambda(\cdot)}(A)$$

for every $A \in \mathcal{B}_{(-\ell', \ell')}$, $\ell' > 0$. To conclude the proof it is

enough to show the tightness of $\{\mu^{\ell} = \mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)\}_{\ell \geq 1}$ on the

space $\mathcal{E}_r$ for every $r > 0$. To this end, we may show (i) the

tightness of $\{\mu^{\ell}\}_{\ell \geq 1}$ on the space $\mathcal{E}$ and (ii) an estimate:

$$\sup_{\ell \geq 1} \langle \mu^{\ell}, |||S|||_r \rangle < \infty \quad \text{for every } r > 0. \quad \text{Indeed (ii) can be proved by}$$

noting that Proposition 3.1-(i) implies

$$\langle \mu^{\ell}, \Phi \rangle \cdot \mu_{\lambda_-}(S(-\ell) > 0, S(\ell) > 0) \leq \langle \mu_{\lambda_+}, \Phi \rangle$$

for every non-negative and $\mathcal{B}_{(-\ell, \ell)}$ -measurable function $\Phi \in \mathcal{M}(\mathcal{E})$ and

similarly

$$\langle \mu_{\lambda_{\pm}}^{\ell}, \Phi' \rangle \cdot \mu_{\lambda_{\pm}}(S(-\ell) < 0, S(\ell) < 0) \leq \langle \mu_{\lambda_{\pm}}^{\ell}, \Phi' \rangle$$

for every non-negative and $\mathcal{B}_{(-\ell, \ell)}$ -measurable Φ' such that $-\Phi' \in \mathcal{H}(\mathcal{G})$. We note $\inf_{\ell \geq 1} \mu_{\lambda_{\pm}}(\pm S(-\ell) < 0, \pm S(\ell) < 0) > 0$ (see Iwata [22]) and $\langle \mu_{\lambda_{\pm}}^{\ell}, |||S|||_r \rangle < \infty$, $r > 0$, which follows easily from the stationarity of $\mu_{\lambda_{\pm}}$. Now to show (i) we may only prove

$$\langle \mu_{\lambda_{\pm}}^{\ell}, |S(x) - S(y)|^4 \rangle \leq \text{const. } |x - y|^2, \quad |x - y| < 1,$$

with some positive const. independent of ℓ . However simple calculation shows

$$\begin{aligned} & \langle \Xi_{\lambda(\cdot)}^{-\ell', \ell'}(s_1, s_2) \mu_{\lambda(\cdot)}^{-\ell', \ell'}(\cdot; s_1, s_2), |S(x) - S(y)|^4 \rangle \\ & \leq \text{const. } (1 + |s_1 - s_2|^4) |x - y|^2, \quad |x - y| < 1. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & \langle \mu_{\lambda_{\pm}}^{\ell}, |S(x) - S(y)|^4 \rangle \\ & \leq \text{const. } |x - y|^2 Z_{\ell}^{-1} \int_{\mathbb{R}^2} e^{\ell' \{ \kappa(\lambda_{-}) + \kappa(\lambda_{+}) \}} p(2\ell', s_1, s_2) \\ & \quad \times (1 + |s_1 - s_2|^4) e^{-(\ell - \ell') \bar{H}_{-}(0, s_1)} e^{-(\ell - \ell') \bar{H}_{+}(0, s_2)} ds_1 ds_2 \\ & \leq \text{const. } |x - y|^2 \pi \left[\int_{*=\pm} p(2\ell', s_1, s_2) (1 + |s_1 - s_2|^4) (e^{-(\ell - \ell') \bar{H}_{*}(0, s_1)})^2 \right. \\ & \quad \left. \times ds_1 ds_2 \right]^{1/2} \\ & \leq \text{const. } |x - y|^2, \end{aligned}$$

where const.'s are independent of ℓ . $\square$

§4 Law of large numbers for Gibbs distributions

The purpose of this section is to investigate the asymptotic behavior as $\varepsilon \downarrow 0$ of the scaled $(U, \lambda(\cdot))$ -Gibbs distributions $\mu_{\lambda(\cdot), \varepsilon}$ constructed in Section 3. We define a function $\rho \in H_e$ by $\rho(x) = \bar{\rho}(\lambda(x))$, $x \in R$, see (0.12) for the definition of the mean spin function $\bar{\rho}$. We shall prove the following theorem.

Theorem 4.1 The probability measure $\mu_{\lambda(\cdot), \varepsilon}$ converges weakly to the δ -distribution δ_ρ on the space $H_{e,w}$ as $\varepsilon \downarrow 0$; namely

$$\lim_{\varepsilon \downarrow 0} \langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle = \Psi(\rho)$$

for every $\Psi \in C_b(H_{e,w})$.

The proof of the theorem will be divided into three parts.

4.1 Convergence of 1-dimensional distribution Let $\Omega_\varepsilon^{(\pm)}(x)$ be two solutions of the diffusion equation (3.1) with $\bar{H}_x$ replaced by $\frac{1}{\varepsilon} \bar{H}_x$ having the same initial conditions (3.2). These functions play same roles for $\mu_{\lambda(\cdot), \varepsilon}$ as $\Omega^{(\pm)}(x)$ do for $\mu_{\lambda(\cdot)}$. For example, the distribution of $S(x)$ under $\mu_{\lambda(\cdot), \varepsilon}$ is given by $Z_\varepsilon^{-1} \Omega_\varepsilon^{(+)}(x, s) \Omega_\varepsilon^{(-)}(x, s) ds$, where Z_ε is a constant defined by the RHS of (3.4) with $\Omega^{(\pm)}$ replaced by $\Omega_\varepsilon^{(\pm)}$, i.e., $Z_\varepsilon = (\Omega_\varepsilon^{(+)}(x), \Omega_\varepsilon^{(-)}(x))$.

First we analyze the asymptotic behavior of the solution $\eta(y, s) = \eta_\varepsilon(y, s)$, $0 < \varepsilon < 1$, of the diffusion equation:

$$(4.1) \quad \begin{cases} \frac{\partial \eta}{\partial y}(y, s) = -\frac{1}{\varepsilon} \bar{H}_y \eta(y, s) & , \quad y > x, \\ \eta(x, s) = \bar{\eta}(s) & , \end{cases}$$

with a given initial value $\bar{\eta} \in L^2(R, ds)$. It is easy to know the

existence and uniqueness of solutions of (4.1) in the space $L^2(R, ds)$. Put $c_\varepsilon(y) \equiv c_\varepsilon(y; x, \bar{\eta}) = (\eta_\varepsilon(y), \Omega(y))$, where $\Omega(y) = \Omega_\lambda(y)$ is the function which was introduced in Section 3. We shall denote the norm of the space $L^2(R, ds)$ simply by $\|\cdot\|$ in this section.

Lemma 4.1 (i) We have two estimates:

$$|c_\varepsilon(y)| \leq \|\bar{\eta}\|, \quad \left| \frac{\partial}{\partial y} c_\varepsilon(y) \right| \leq \|\bar{\eta}\| \left\| \frac{\partial \Omega}{\partial y}(y) \right\|, \quad y \geq x, \varepsilon > 0.$$

$$(ii) \quad \lim_{\varepsilon \downarrow 0} \|\eta_\varepsilon(y) - c_\varepsilon(y)\Omega(y)\| = 0, \quad y > x.$$

Proof Since the bound $\|\eta_\varepsilon(y)\| \leq \|\bar{\eta}\|$, $y \geq x$, follows from

$$\frac{\partial}{\partial y} \|\eta_\varepsilon(y)\|^2 = -\frac{2}{\varepsilon} (\bar{H}_y \eta_\varepsilon, \eta_\varepsilon) \leq 0,$$

we obtain the first estimate in (i) by noting $\|\Omega(y)\| = 1$. The second estimate can be shown since $\bar{H}_y \Omega(y) = 0$ implies

$$\left| \frac{\partial}{\partial y} c_\varepsilon(y) \right| = |(\eta_\varepsilon(y), \frac{\partial \Omega}{\partial y}(y))| \leq \|\bar{\eta}\| \left\| \frac{\partial \Omega}{\partial y}(y) \right\|.$$

Note the differentiability of $\Omega(y)$ in y , see Section 2. To prove the assertion (ii), we derive an equality

$$(4.2) \quad \frac{\partial}{\partial y} \|\xi_\varepsilon(y)\|^2 = -\frac{2}{\varepsilon} (\bar{H}_y \xi_\varepsilon, \xi_\varepsilon) - 2 \left(\frac{\partial}{\partial y} (c_\varepsilon \Omega), \xi_\varepsilon \right),$$

for $\xi_\varepsilon(y) = \eta_\varepsilon(y) - c_\varepsilon(y)\Omega(y)$. Since $(\xi_\varepsilon(y), \Omega(y)) = 0$, the first term in the RHS of (4.2) can be bounded from above by $-2\delta \|\xi_\varepsilon\|^2/\varepsilon$,

where $\delta = \inf_{x \in [x_-, x_+]} \delta_x > 0$ and δ_x is the second least eigenvalue of

$\bar{H}_x$, see Section 2. On the other hand, since (i) implies $K =$

$\sup_{y \geq x, \varepsilon > 0} \left\| \frac{\partial}{\partial y} (c_\varepsilon \Omega) \right\| < \infty$, the second term is bounded from above by

$2K \|\xi_\varepsilon\|$ and therefore by $K(1 + \|\xi_\varepsilon\|^2)$. These estimates are now summed up into

$$\frac{\partial}{\partial y} \|\xi_\varepsilon(y)\|^2 \leq -\left(\frac{2\delta}{\varepsilon} - K\right) \|\xi_\varepsilon(y)\|^2 + K,$$

which shows

$$\|\xi_\varepsilon(y)\|^2 \leq \|\bar{\eta} - c_\varepsilon(x)\Omega(x)\|^2 \exp\left\{-\left(\frac{2\delta}{\varepsilon} - K\right)(y-x)\right\} + K\left(\frac{2\delta}{\varepsilon} - K\right)^{-1}.$$

This completes the proof of the assertion (ii). $\square$

Put $\bar{c}_\varepsilon^{(\pm)}(x) = (\Omega_\varepsilon^{(\pm)}(x), \Omega(x))$. Then the following is an immediate consequence of Lemma 4.1. Consider in the reverse direction for $\Omega_\varepsilon^{(+)}$.

Corollary 4.1 (i) We have for every $x \in R$ and $\varepsilon > 0$:

$$|\bar{c}_\varepsilon^{(\pm)}(x)| \leq 1, \quad \left| \frac{\partial}{\partial x} \bar{c}_\varepsilon^{(\pm)}(x) \right| \leq \left\| \frac{\partial \Omega}{\partial x}(x) \right\|.$$

$$(ii) \quad \lim_{\varepsilon \downarrow 0} \|\Omega_\varepsilon^{(\pm)}(x) - \bar{c}_\varepsilon^{(\pm)}(x)\Omega(x)\| = 0, \quad x \in R.$$

Now we introduce the following additional condition on the profile $\lambda(\cdot) \in \Lambda$:

$$(4.3) \quad \beta = (x_+ - x_-) \sup_{x \in R} \left\| \frac{\partial \Omega}{\partial x}(x) \right\| < 1.$$

Corollary 4.2 Assume the condition (4.3). Then we have

$$(4.4) \quad \inf_{0 < \varepsilon < 1} Z_\varepsilon > 0$$

and

$$(4.5) \quad \lim_{\varepsilon \downarrow 0} \|Z_\varepsilon^{-1} \Omega_\varepsilon^{(+)}(x) \Omega_\varepsilon^{(-)}(x) - \Omega^2(x)\|_{L^1(R, ds)} = 0, \quad x \in R.$$

Proof Since $\bar{c}_\varepsilon^{(\pm)}(x_\pm) = 1$, Corollary 4.1-(i) combined with the condition (4.3) proves that $\bar{c}_\varepsilon^{(\pm)}(x) \geq 1 - \beta$, $\varepsilon > 0$, $x \in [x_-, x_+]$. We therefore see (4.4) with the help of Corollary 4.1-(ii). Now the assertion (4.5) follows also from Corollary 4.1-(ii) noting (4.4). $\square$

4.2 Asymptotic independence We recall (3.9) for the definition of $\nu_\lambda \in \mathcal{P}(R)$.

Lemma 4.2 Under the condition (4.3) we have

$$\lim_{\varepsilon \downarrow 0} |\langle \mu_{\lambda(\cdot), \varepsilon}, \xi(S(x))\Phi \rangle - \langle \nu_{\lambda(x), \varepsilon}, \xi \rangle \langle \mu_{\lambda(\cdot), \varepsilon}, \Phi \rangle| = 0 ,$$

for every $\xi \in C_b(R)$ and $\mathcal{B}_{(y,z)}$ -measurable bounded function Φ on the space $\mathcal{E}$, where we assume (i) $x < y < z$ and $y \leq x_+$ or (ii) $y < z < x$ and $x_- \leq z$.

Proof We may only discuss the case (i) because of the symmetry.

Let $\tilde{\eta}_\varepsilon = \tilde{\eta}_\varepsilon(y, s; x, \xi)$ be a solution of the equation (4.1) with initial value $\bar{\eta}(s) = \xi(s) \Omega_\varepsilon^{(-)}(x, s)$ for given $\xi \in C_b(R)$. Then we have the following representation for the conditional distribution:

$$E^{\mu_{\lambda(\cdot), \varepsilon}}[\xi(S(x)) | \mathcal{B}_{(y)}](S(y)=s) = \Omega_\varepsilon^{(-)}(y, s)^{-1} \tilde{\eta}_\varepsilon(y, s; x, \xi), x < y,$$

and therefore

$$(4.6) \quad \begin{aligned} & \langle \mu_{\lambda(\cdot), \varepsilon}, \xi(S(x))\Phi \rangle \\ &= Z_\varepsilon^{-1} \int \tilde{\eta}_\varepsilon(y, s) \Omega_\varepsilon^{(+)}(y, s) E^{\mu_{\lambda(\cdot), \varepsilon}}[\Phi | \mathcal{B}_{(y)}](S(y)=s) ds . \end{aligned}$$

Note that $\tilde{\eta}_\varepsilon = \Omega_\varepsilon^{(-)}$ in the RHS of (4.6) when $\xi \equiv 1$. Hence, putting $\tilde{c}_\varepsilon(y) \equiv \tilde{c}_\varepsilon(y; x, \xi) = (\tilde{\eta}_\varepsilon(y), \Omega(y))$, we obtain

$$\begin{aligned} I_\varepsilon &\equiv |\langle \mu_{\lambda(\cdot), \varepsilon}, \xi(S(x))\Phi \rangle - \{\bar{c}_\varepsilon^{(-)}(y)\}^{-1} \tilde{c}_\varepsilon(y) \langle \mu_{\lambda(\cdot), \varepsilon}, \Phi \rangle| \\ &\leq Z_\varepsilon^{-1} \|\Omega_\varepsilon^{(+)}(y)\| \|\Phi\|_\infty I'_\varepsilon . \end{aligned}$$

The term I'_ε is defined and estimated as follows:

$$\begin{aligned} I'_\varepsilon &\equiv \|\tilde{\eta}_\varepsilon(y) - \{\bar{c}_\varepsilon^{(-)}(y)\}^{-1} \tilde{c}_\varepsilon(y) \Omega_\varepsilon^{(-)}(y)\| \\ &\leq \|\tilde{\eta}_\varepsilon(y) - \tilde{c}_\varepsilon(y) \Omega(y)\| + \{\bar{c}_\varepsilon^{(-)}(y)\}^{-1} \tilde{c}_\varepsilon(y) \|\Omega_\varepsilon^{(-)}(y) - \bar{c}_\varepsilon^{(-)}(y) \Omega(y)\| . \end{aligned}$$

Here Lemma 4.1-(ii) shows that the first term in the RHS converges to

0 as $\varepsilon \downarrow 0$. On the other hand, noting that $\bar{c}_\varepsilon^{(-)}(y) \geq 1-\beta$, $y \leq x_+$, and $|\tilde{c}_\varepsilon(y)| \leq \|\xi \Omega_\varepsilon^{(-)}(y)\| \leq \|\xi\|_\infty$, Corollary 4.1-(ii) proves that the second term also converges to 0. Therefore we get $\lim_{\varepsilon \downarrow 0} I_\varepsilon = 0$ since $\|\Omega_\varepsilon^{(+)}(y)\| \leq 1$ and $\sup_{0 < \varepsilon < 1} Z_\varepsilon^{-1} < \infty$. We especially obtain by taking $\Phi \equiv 1$:

$$(4.7) \quad \lim_{\varepsilon \downarrow 0} |\langle \mu_{\lambda(\cdot), \varepsilon}, \xi(S(x)) \rangle - \{\bar{c}_\varepsilon^{(-)}(y)\}^{-1} \tilde{c}_\varepsilon(y; x, \xi)| = 0.$$

Now the conclusion of the lemma follows from the fact $\lim_{\varepsilon \downarrow 0} I_\varepsilon = 0$ and (4.7), since (4.5) shows that $\langle \mu_{\lambda(\cdot), \varepsilon}, \xi(S(x)) \rangle$ converges to $\langle \nu_{\lambda(x), \xi} \rangle$ as $\varepsilon \downarrow 0$. $\square$

Use an inductive method to prove the following.

Corollary 4.3 Assume the condition (4.3). Then we have

$$(4.8) \quad \lim_{\varepsilon \downarrow 0} \langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^n \xi_i(S(x_i)) \rangle = \prod_{i=1}^n \langle \nu_{\lambda(x_i), \xi_i} \rangle$$

for every $x_1 < x_2 < \dots < x_n$ and $\xi_i \in C_b(R)$, $1 \leq i \leq n$, $n = 1, 2, \dots$.

Let $\mu_{\lambda(\cdot), \varepsilon}^{x, y}(\cdot; s_1, s_2) \equiv \mu_{\tau_\varepsilon \lambda(\cdot)}^{x/\varepsilon, y/\varepsilon}(\cdot; s_1, s_2) \cdot \sigma_\varepsilon^{-1}$ be the local specifications corresponding to the scaled Gibbs distribution $\mu_{\lambda(\cdot), \varepsilon}$.

Lemma 4.3 Let $\lambda(\cdot) \in \Lambda$ be given and assume that we can find another profile $\tilde{\lambda}(\cdot) \in \Lambda$ which satisfies (4.3) and coincides with $\lambda(\cdot)$ on an interval $[x, y]$. Then the convergence (4.8) holds with $\mu_{\lambda(\cdot), \varepsilon}^{x, y}(\cdot; s_1, s_2)$ instead of $\mu_{\lambda(\cdot), \varepsilon}$ for every $s_1, s_2 \in R$, $x < x_1 < \dots < x_n < y$ and $\xi_i \in C_b(R)$. This convergence is uniform in (s_1, s_2) on each bounded subset of R^2 .

Proof To complete the proof we may assume that the functions $\xi_i \in$

$C_b(R)$, $1 \leq i \leq n$, are non-negative and monotone increasing. Then Proposition 3.1-(i) proves that

$$f_\varepsilon(s_1, s_2) := \langle \mu_{\lambda(\cdot), \varepsilon}^{x, y}(\cdot; s_1, s_2), \prod_{i=1}^n \xi_i(S(x_i)) \rangle$$

is monotone in $(s_1, s_2) \in R^2$. Put $c = \prod_{i=1}^n \langle \nu_{\lambda(x_i)}, \xi_i \rangle$ and denote the 2-dimensional marginal distribution of $(S(x), S(y))$ under $\mu_{\lambda(\cdot), \varepsilon}^{(2)}$ by $\nu_{\lambda(\cdot), \varepsilon}^{(2)} \in \mathcal{P}(R^2)$. Then Corollary 4.3 shows that the measures $f_\varepsilon(s_1, s_2) \nu_{\lambda(\cdot), \varepsilon}^{(2)}(ds_1 ds_2)$ and $\nu_{\lambda(\cdot), \varepsilon}^{(2)}(ds_1 ds_2)$ on R^2 converge weakly to $c \nu_{\lambda(x)} \otimes \nu_{\lambda(y)}$ and $\nu_{\lambda(x)} \otimes \nu_{\lambda(y)}$, respectively, as $\varepsilon \downarrow 0$. Since the limiting measure $\nu_{\lambda(x)} \otimes \nu_{\lambda(y)}$ is absolutely continuous with respect to the Lebesgue measure on R^2 , every interval $I = [s_1, s'_1] \times [s_2, s'_2]$ in R^2 is its continuity set and therefore (see, e.g., Billingsley [3]) we get

$$\lim_{\varepsilon \downarrow 0} \int_I f_\varepsilon d\nu_{\lambda(\cdot), \varepsilon}^{(2)} = c \nu_{\lambda(x)} \otimes \nu_{\lambda(y)}(I)$$

and

$$\lim_{\varepsilon \downarrow 0} \nu_{\lambda(\cdot), \varepsilon}^{(2)}(I) = \nu_{\lambda(x)} \otimes \nu_{\lambda(y)}(I).$$

Now the monotonicity of the function f_ε implies

$$\overline{\lim}_{\varepsilon \downarrow 0} f_\varepsilon(s_1, s_2) \leq c \leq \underline{\lim}_{\varepsilon \downarrow 0} f_\varepsilon(s'_1, s'_2),$$

and this proves

$$\lim_{\varepsilon \downarrow 0} f_\varepsilon(s_1, s_2) = c$$

for every $(s_1, s_2) \in R^2$. The uniformity of this convergence in (s_1, s_2) on every bounded set follows from Dini's theorem. $\square$

Lemma 4.4 The conclusion of Corollary 4.3 still holds without assuming the condition (4.3).

Proof Let $x_1 < \dots < x_n$ and $\xi_1, \dots, \xi_n \in C_b(R)$ be given. We can then find a sequence $y_0 < y_1 < \dots < y_m$ in such a manner that $y_0 < x_1 < x_n < y_m$, $x_i \neq y_k$ for every i, k and there exists a profile $\tilde{\lambda}_k(\cdot) \in \Lambda$ for each k ; $1 \leq k \leq m$, which satisfies (4.3) and coincides with $\lambda(\cdot)$ on the interval $[y_{k-1}, y_k]$. We define a sequence $0 \leq j_1 \leq \dots \leq j_m = n$ by the condition $x_{j_k} < y_k < x_{j_{k+1}}$, $1 \leq k \leq m-1$ (put $x_0 = y_0$ for convenience) and set

$$f_{\varepsilon}^{(k)}(S_{k-1}, S_k) = \langle \mu_{\lambda(\cdot), \varepsilon}^{y_{k-1}, y_k}(\cdot; S_{k-1}, S_k), \prod_{i=j_{k-1}+1}^{j_k} \xi_i(S(x_i)) \rangle.$$

Here we interpret the integrand of the RHS is equal to 1 when $j_{k-1} = j_k$. Then Lemma 4.3 proves that $f_{\varepsilon}^{(k)}$ converges to

$$\prod_{i=j_{k-1}+1}^{j_k} \langle \nu_{\lambda(x_i)}^{y_{k-1}, y_k}, \xi_i \rangle \text{ as } \varepsilon \downarrow 0 \text{ uniformly on each compact set of } R^2.$$

Since Proposition 3.1-(iii) proves that the family of marginal distributions $\{\nu_{\lambda(\cdot), \varepsilon}^{(m)}(dS_0 \dots dS_m) = \mu_{\lambda(\cdot), \varepsilon}(S(y_0) \in dS_0, \dots, S(y_m) \in dS_m) ; 0 < \varepsilon < 1\}$ is tight in $\mathcal{P}(R^{m+1})$, the following equality completes the proof of the lemma:

$$\langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^n \xi_i(S(x_i)) \rangle = \langle \nu_{\lambda(\cdot), \varepsilon}^{(m)}, \prod_{k=1}^m f_{\varepsilon}^{(k)}(S_{k-1}, S_k) \rangle. \quad \square$$

4.3 The proof of Theorem 4.1

Lemma 4.5 For every $\Psi \in \mathcal{D}$, $\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle$ converges to $\Psi(\rho)$ as $\varepsilon \downarrow 0$.

Proof Noting Proposition 3.1-(iii) we may take $\xi_i(s) = s$, $1 \leq i \leq n$, in (4.8) although these functions are not bounded and therefore we obtain

$$(4.9) \quad \lim_{\varepsilon \downarrow 0} \langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^n S(x_i) \rangle = \prod_{i=1}^n \rho(x_i), \quad x_1 < x_2 < \dots < x_n.$$

First we consider such Ψ that the function ψ defining Ψ is a monomial, i.e., $\Psi(S) = \psi(\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle)$ with $\varphi_i \in C_0^\infty(R)$ and $\psi(\alpha_1, \dots, \alpha_k) = \prod_{i=1}^k \alpha_i^{m_i}$, $m_i \in \mathbb{N}$. We may assume $m_i = 1$, $1 \leq i \leq k$, by making k large if necessary. For Ψ of this form, we have

$$\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle = \int_R \prod_{i=1}^k \varphi_i(x_i) \langle \mu_{\lambda(\cdot), \varepsilon}, \prod_{i=1}^k S(x_i) \rangle \prod_{i=1}^k dx_i.$$

Since Proposition 3.1-(iii) implies that the integrand of the RHS is uniformly bounded in $(\varepsilon, x_1, x_2, \dots, x_k)$, Lebesgue's dominated convergence theorem proves with the help of (4.9) that $\langle \mu_{\lambda(\cdot), \varepsilon}, \Psi \rangle$ tends to $\Psi(\rho)$ as $\varepsilon \downarrow 0$. It is now standard to show this convergence for every $\Psi \in \mathcal{D}$. $\square$

Now we are ready to give the proof of Theorem 4.1. First we note that the locally convex space $H_{e,w}$ is completely regular and a Radon space (Schwartz [35]). Its balls $B(\{b_r\}) = \{S \in H_e ; |S|_r \leq b_r \text{ for every } r > 0\}$ are compact in this space for all sequences $\{b_r > 0\}_{r>0}$ (Dunford and Schwartz [8, p423]). Conversely, every compact subset B of $H_{e,w}$ is a closed set which is contained in some ball. Moreover, each ball is metrizable. From these observations, we see that Prokhorov's theorem holds on the space $H_{e,w}$ (Smolyanov and Fomin [38]).

Proposition 3.1-(iii) proves the tightness of the family $\{\mu_{\lambda(\cdot), \varepsilon}\}_{0 < \varepsilon < 1}$ in $\mathcal{P}(H_{e,w})$; for every $\delta > 0$, there exists a ball $B = B(\{b_r\})$ such that $\inf_{0 < \varepsilon < 1} \mu_{\lambda(\cdot), \varepsilon}(B) > 1 - \delta$. Hence from every subsequence $\{\mu_{\lambda(\cdot), \varepsilon'}\}$ of $\{\mu_{\lambda(\cdot), \varepsilon}\}$ we can choose further subsequence $\{\mu_{\lambda(\cdot), \varepsilon''}\}$ and $\mu \in \mathcal{P}(H_{e,w})$ such that $\mu_{\lambda(\cdot), \varepsilon''}$ converges weakly to μ in $\mathcal{P}(H_{e,w})$ as $\varepsilon'' \downarrow 0$. However Lemma 4.5

shows that $\langle \mu, \Psi \rangle = \langle \delta_\rho, \Psi \rangle$ for every $\Psi \in \mathcal{D}$ and this proves $\mu = \delta_\rho$ since $\mathcal{D}$ is a determining class for the space $\mathcal{P}(H_{e,w})$. Therefore the sequence $\mu_{\lambda(\cdot), \varepsilon}$ itself converges weakly to δ_ρ as $\varepsilon \downarrow 0$, which completes the proof of Theorem 4.1.

§5 Existence and uniqueness theorem for the TDGL equation

The TDGL equation of conservative type is described by the SPDE:

$$(0.5) \quad dS_t(x) = -\Delta^2 S_t(x)dt + \Delta\{U'(S_t(x))\}dt + \sqrt{2}\nabla dw_t(x), \quad t > 0, \quad x \in \mathbb{R}.$$

Here the cylindrical Brownian motion $w_t(x)$ on a Hilbert space $L^2(\mathbb{R}, dx)$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ with reference family $\{\mathcal{F}_t\}$, see Yor [42] or Funaki [15] for example. We assume that w_t is $\{\mathcal{F}_t\}$ -adapted and its increment $w_t - w_u$ is independent of $\mathcal{F}_u$ for every $0 \leq u \leq t$.

From the assumption (1.1) on U the SPDE (0.5) can be rewritten at least formally into

$$(5.1) \quad dS_t(x) = -AS_t(x)dt + \Delta\{V'(S_t(x))\}dt + \sqrt{2}\nabla dw_t(x), \quad t > 0, \quad x \in \mathbb{R}.$$

where $A = \Delta^2 - \gamma\Delta$. The fundamental solution $q = q(t, x)$ of the parabolic operator $\frac{\partial}{\partial t} + A$ has the following estimates for $T > 0$:

$$(5.2) \quad \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x) \right| \leq K_1 t^{-j - \frac{1+k}{4}} \exp\{-L_1(\frac{x^4}{t})^{1/3}\},$$

$$0 < t \leq T, \quad x \in \mathbb{R}; \quad j=0,1, \quad k=0,1,2,3,$$

with positive constants K_1 and L_1 , which depend on T . See Eidel'man [9].

Now, similarly to Dawson [5], Marcus [30] and others, we rewrite the equation (5.1) again formally into an integral equation:

$$(5.3) \quad S_t(x) = \int_{\mathbb{R}} q(t, x, y) S_0(y) dy - \sqrt{2} \int_0^t \int_{\mathbb{R}} q_y(t-u, x, y) dw_u(y) dy \\ + \int_0^t \int_{\mathbb{R}} q_{yy}(t-u, x, y) V'(S_u(y)) du dy, \quad t \geq 0, x \in \mathbb{R},$$

where $q(t, x, y) = q(t, x-y)$ and the subscripts to q mean derivatives with respect to those variables; e.g., $q_y = \partial q / \partial y$, $q_t = \partial q / \partial t$ etc. The initial data S_0 of the SPDE (0.5) accordingly of the integral equation (5.3) is always taken from the space H_e .

Let $\mathcal{T}$ be the class of all stochastic processes $S_t = \{S_t(x; \omega); x \in \mathbb{R}\}$, $t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\})$, which are $\{\mathcal{F}_t\}$ -adapted and jointly measurable in $(t, x, \omega) \in [0, \infty) \times \mathbb{R} \times \Omega$. We call S_t a solution of the SPDE (0.5) if S_t belongs to the class $\mathcal{T}$ and satisfies the integral equation (5.3) with probability one. We denote by $\mathcal{T}'$ the subclass of $\mathcal{T}$ consisting of all S_t such that $S_t \in C((0, \infty), \mathcal{E}_e)$ (a.s.) and $\sup_{0 < t \leq T} t^{1/8} \|S_t\|_r < \infty$ (a.s.) for every $T > 0$ and $r > 0$. The existence and uniqueness result for the SPDE (0.5) can be formulated as follows.

- Theorem 5.1** (i) There exists a solution S_t of the SPDE (0.5). Every solution belongs to the class $\mathcal{T}'$.
- (ii) Let S_t and S'_t be two solutions of the SPDE (0.5). If $S_0 = S'_0$, then we have $S_t = S'_t$, $t \geq 0$, with probability one.
- (iii) Suppose $S_0 \in \mathcal{E}_e$, then we have $S_t \in C([0, \infty), \mathcal{E}_e)$ (a.s.).

For given $S_0 \in H_e$ and $S_t \in \mathcal{T}$, we denote

$$S_{t,1}(x) \equiv S_{t,1}(x; S_0) = \int_R q(t, x, y) S_0(y) dy,$$

$$S_{t,3}(x) \equiv S_{t,3}(x; S_0) = \int_0^t \int_R q_{yy}(t-u, x, y) V'(S_u(y)) du dy,$$

and set

$$S_{t,2}(x) = \int_0^t \int_R q_y(t-u, x, y) dw_u(y) dy, \quad t \geq 0, \quad x \in R.$$

Lemma 5.1 For every $T > 0$ and $0 < \alpha < 1$, there exists a positive constant C such that

$$E[|S_{t,2}(x) - S_{t',2}(x')|^2] \leq C(|t-t'|^{1/4} + |x-x'|^\alpha), \\ 0 \leq t, t' \leq T, \quad x, x' \in R.$$

Proof First consider the case of $t = t'$ and $x < x'$:

$$\begin{aligned} I_1 &\equiv E[|S_{t,2}(x) - S_{t,2}(x')|^2] \\ &= \int_0^t du \int_R \{q_x(t-u, x, y) - q_x(t-u, x', y)\}^2 dy \\ &= \int_0^t \{q_{xy}(2u, x, x) - 2q_{xy}(2u, x, x') + q_{xy}(2u, x', x')\} du. \end{aligned}$$

Here we divide the integrand into two terms and each term can be estimated by using (5.2) and Hölder's inequality. In fact, for example, we have

$$\begin{aligned} |q_{xy}(2u, x, x) - q_{xy}(2u, x, x')| &= \left| \int_x^{x'} q_{xyy}(2u, x, y) dy \right| \\ &\leq K_1 (2u)^{-1} \int_x^{x'} \exp(-L_1 (\frac{|x-y|^4}{2u})^{1/3}) dy \\ &\leq \text{const.} u^{-1+1/4p'} (x'-x)^{1/p}, \quad 1 < p, p' < \infty; 1/p + 1/p' = 1. \end{aligned}$$

We therefore obtain

$$(5.4) \quad I_1 \leq \text{const.} (x'-x)^{1/p}, \quad t \in [0, T].$$

Next we consider the case of $x = x'$ and $0 \leq t < t'$:

$$\begin{aligned}
I_2 &\equiv E[|S_{t,2}(x) - S_{t',2}(x)|^2] \\
&= \int_t^{t'} du \int_R q_x(t'-u, x, y)^2 dy \\
&\quad + \int_0^t du \int_R \{q_x(t'-u, x, y) - q_x(t-u, x, y)\}^2 dy \\
&= \int_0^\tau q_{xy}(2u, x, x) du \\
&\quad + \int_0^t \{q_{xy}(2u+2\tau, x, x) - 2q_{xy}(2u+\tau, x, x) + q_{xy}(2u, x, x)\} du \\
&\equiv I_{2,1} + I_{2,2},
\end{aligned}$$

where $\tau = t' - t > 0$. The first term $I_{2,1}$ can be estimated by using (5.2) as follows:

$$I_{2,1} \leq \text{const.} \int_0^\tau u^{-3/4} du \leq \text{const.} \tau^{1/4}.$$

The second term $I_{2,2}$ has also a same bound as $I_{2,1}$, since we have

$$\begin{aligned}
&|q_{xy}(2u+\tau, x, x) - q_{xy}(2u, x, x)| \\
&= \left| \int_{2u}^{2u+\tau} q_{xyt}(v, x, x) dv \right| \leq \text{const.} \{(2u)^{-3/4} - (2u+\tau)^{-3/4}\},
\end{aligned}$$

and a similar estimate on $|q_{xy}(2u+2\tau, x, x) - q_{xy}(2u+\tau, x, x)|$. We have therefore proved that

$$(5.5) \quad I_2 \leq \text{const.} \tau^{1/4}.$$

Now the conclusion follows from two estimates (5.4) and (5.5). $\square$

Lemma 5.2 (i) $\sup\left\{\left|\frac{\partial S_{t,3}}{\partial x}(x; S_\cdot)\right|; 0 \leq t \leq T, x \in R, S_\cdot \in \mathcal{F}\right\} < \infty, T > 0$.

(ii) For every $T > 0$, there exists a positive constant C such that

$$|S_{t,3}(x) - S_{t',3}(x)| \leq C|t-t'|^{1/2}, \quad 0 \leq t, t' \leq T, x \in R.$$

Proof Since V' is bounded, the uniform estimate (i) follows from (5.2) easily. The assertion (ii) also follows by using (5.2), since we have

$$\begin{aligned}
|S_{t,3}(x) - S_{t',3}(x)| &\leq \|V'\|_{\infty} \int_t^{t'} du \int_R |q_{xx}(t'-u, x, y)| dy \\
&\quad + \|V'\|_{\infty} \int_0^t du \int_R |\int_{t-u}^{t'-u} q_{xx}(v, x, y) dv| dy \\
&\leq \text{const. } (t'-t)^{1/2}, \quad 0 \leq t \leq t' \leq T. \quad \square
\end{aligned}$$

Corollary 5.1 The processes $S_{t,2}$ and $S_{t,3} \in C([0, \infty), \mathcal{G}_e)$ (a.s.).

Proof It is an immediate consequence of Lemma 5.2 that $S_{t,3} \in C([0, \infty), \mathcal{G}_e)$. On the other hand, since $(S_{t,2}(x); t \geq 0, x \in \mathbb{R})$ is a Gaussian system, Lemma 5.1 shows

$$\begin{aligned}
E[|S_{t,2}(x) - S_{t',2}(x')|^{2p}] &\leq C(|t-t'|^{p/4} + |x-x'|^{\alpha p}), \\
0 \leq t, t' \leq T, \quad x, x' \in \mathbb{R},
\end{aligned}$$

for every $p \geq 1$ with positive constant $C = C_{T,p}$. Hence, taking $p > 8$, Kolmogorov-Totoki's theorem (see, e.g., Walsh [41], Kunita [27]) implies that there exists a continuous version of $(S_{t,2}(x))$ such that for every small $\delta > 0$

$$|S_{t,2}(x) - S_{t',2}(x')| \leq \eta_i (|t-t'| + |x-x'|)^{1/8-1/p-\delta},$$

$$t, t' \in [0, T], \quad x, x' \in [i-1, i+1], \quad i \in \mathbb{Z},$$

holds with some sequence $(\eta_i)_{i \in \mathbb{Z}}$ of positive random variables satisfying $\sup_{i \in \mathbb{Z}} E[\eta_i^{2p}] < \infty$. This proves the conclusion for $S_{t,2}$. $\square$

We shall use frequently an estimate which follows from (5.2):

$$(5.6) \quad \int \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x, y) \right| \theta(y, r) dy \leq K t^{-j-k/4} \theta(x, r),$$

$$0 < t \leq T, \quad x \in \mathbb{R}; \quad j=0,1, \quad k=0,1,2,3,$$

for every $r \in \mathbb{R}$ with a positive constant K depending only on r

and T .

Lemma 5.3 (i) If $S_0 \in \mathcal{G}_e$, then $S_{t,1} = S_{t,1}(\cdot; S_0) \in C([0, \infty), \mathcal{G}_e)$.
(ii) If $S_0 \in H_e$, then $S_{t,1} \in C((0, \infty), \mathcal{G}_e)$ and $\sup_{0 < t \leq T} t^{1/8} \|S_{t,1}\|_r < \infty$
for every $T > 0$ and $r > 0$.

Proof First for general $S_0 \in H_e$, we have using (5.2), (5.6) and Schwarz's inequality:

$$\begin{aligned} |S_{t,1}(x)| &\leq |S_0|_{2r} \left(\int_R |q(t, x, y)|^2 \theta(y, -2r) dy \right)^{1/2} \\ &\leq |S_0|_{2r} \{ \text{const. } t^{-1/4} \theta(x, -2r) \}^{1/2}, \quad 0 < t \leq T, \end{aligned}$$

which implies

$$\sup_{0 < t \leq T} t^{1/8} \|S_{t,1}\|_r < \infty, \quad T > 0, r > 0.$$

Especially we see that $S_{t,1} \in \mathcal{G}_e$ for each $t > 0$ because $S_{t,1} \in C(R)$ is easy to be shown. Therefore, if the assertion (i) is proved, then the semigroup property of $S_{t,1}$:

$$S_{t,1}(x; S_{u,1}(\cdot; S_0)) = S_{t+u,1}(x; S_0), \quad t, u \geq 0,$$

completes the proof of (ii).

From now on we assume $S_0 \in \mathcal{G}_e$. Then we have

$$\begin{aligned} |S_{t,1}(x) - S_{t',1}(x)| &\leq \|S_0\|_r \int_t^{t'} du \int_R |q_t(u, x, y)| \theta(y, -r) dy \\ &\leq K \|S_0\|_r \theta(x, -r) \int_t^{t'} u^{-1} du, \quad 0 < t \leq t' \leq T. \end{aligned}$$

Hence

$$\|S_{t,1} - S_{t',1}\|_r \leq \text{const. } \|S_0\|_r (\log t' - \log t), \quad r > 0,$$

and this proves that $S_{t,1} \in C((0, \infty), \mathcal{G}_e)$. The final task is to prove the continuity of $S_{t,1} \in \mathcal{G}_e$ at $t = 0$. To this end, take a function $\psi \in C_0^\infty(R)$, $0 \leq \psi \leq 1$, satisfying $\psi \equiv 1$ on the interval

$[-1,1]$ and put $\psi_\varepsilon(x) = \psi(\varepsilon x)$, $\varepsilon > 0$. Then $S_{t,1}$ can be represented as a sum:

$$\begin{aligned} S_{t,1}(x) &= S_{t,1}^{\varepsilon,1}(x) + S_{t,1}^{\varepsilon,2}(x) \\ &\equiv S_{t,1}(x; \psi_\varepsilon \cdot S_0) + S_{t,1}(x; (1-\psi_\varepsilon)S_0). \end{aligned}$$

From the estimation for $r > 0$

$$\begin{aligned} |S_{t,1}^{\varepsilon,2}(x)| &\leq \|S_0\|_{r/2} \theta(\varepsilon^{-1}, r/2) \int_R |q(t, x, y)| \theta(y, -r) dy \\ &\leq K \|S_0\|_{r/2} \theta(\varepsilon^{-1}, r/2) \theta(x, -r), \end{aligned}$$

we get

$$(5.7) \quad \|S_{t,1}^{\varepsilon,2}\|_r \leq K \|S_0\|_{r/2} \theta(\varepsilon^{-1}, r/2), \quad r > 0, \quad 0 < t \leq T.$$

On the other hand, since it is known that $\int_R q(t, x, y) \varphi(y) dy$ converges to $\varphi(x)$ as $t \downarrow 0$ uniformly on each bounded interval if $\varphi \in C_b(R)$ (see Arima [2]), we see

$$(5.8) \quad \lim_{t \downarrow 0} \sup_{|x| \leq \ell} |S_{t,1}^{\varepsilon,1}(x) - (\psi_\varepsilon \cdot S_0)(x)| \theta(x, r) = 0$$

for every ℓ , $\varepsilon > 0$. However, since (5.6) can be used to show for $r \in R$:

$$(5.9) \quad \|S_{t,1}\|_r \leq K \|S_0\|_r, \quad 0 \leq t \leq T,$$

we obtain for $r > 0$

$$\begin{aligned} (5.10) \quad &\sup_{|x| \geq \ell} |S_{t,1}^{\varepsilon,1}(x) - (\psi_\varepsilon \cdot S_0)(x)| \theta(x, r) \\ &\leq \sup_{|x| \geq \ell} (\text{const.} \|S_0\|_{r/2} \theta(x, -r/2) \theta(x, r)) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty. \end{aligned}$$

Therefore two estimates (5.8) and (5.10) are combined to prove

$$(5.11) \quad \lim_{t \downarrow 0} \|S_{t,1}^{\varepsilon,1} - \psi_\varepsilon S_0\|_r = 0$$

for every $\varepsilon > 0$. Hence we get the continuity of $S_{t,1}$ at $t = 0$

from (5.7) and (5.11) by noting a trivial inequality:

$$\|S_{t,1} - S_0\|_r \leq \|S_{t,1}^{\varepsilon,1} - \psi_\varepsilon \cdot S_0\|_r + \|S_{t,1}^{\varepsilon,2}\|_r + \|(1 - \psi_\varepsilon) S_0\|_r. \quad \square$$

Lemma 5.4 For every $2 < p < 8$, $T > 0$ and $r > 0$, there exists a positive constant C such that

$$\|S_{t,3}(\cdot; S) - S_{t,3}(\cdot; \bar{S})\|_r^p \leq C \int_0^t \|S_u - \bar{S}_u\|_r^p du, \quad 0 \leq t \leq T, \quad S, \bar{S} \in \mathcal{T}'.$$

Proof We obtain the conclusion from the following calculation:

$$\begin{aligned} & |S_{t,3}(x; S) - S_{t,3}(x; \bar{S})| \\ & \leq \gamma_0 \int_0^t \|S_u - \bar{S}_u\|_r du \int_R |q_{xx}(t-u, x, y)| \theta(y, -r) dy \\ & \leq \gamma_0 K \theta(x, -r) \int_0^t (t-u)^{-1/2} \|S_u - \bar{S}_u\|_r du \\ & \leq \gamma_0 K \theta(x, -r) \left(\int_0^t (t-u)^{-p'/2} du \right)^{1/p'} \left(\int_0^t \|S_u - \bar{S}_u\|_r^p du \right)^{1/p}, \\ & \quad 0 \leq t \leq T, \end{aligned}$$

where $1 < p' < 2 < p < \infty$ such that $1/p + 1/p' = 1$. Note that the RHS is finite if $p < 8$. $\square$

Proof of Theorem 5.1 Corollary 5.1 and Lemma 5.3 prove that $S_t \in \mathcal{T}'$ for every solution S_t of the SPDE (0.5) if exists. Therefore the uniqueness of solutions may be discussed in the class $\mathcal{T}'$. However this follows from Lemma 5.4 immediately. Lemma 5.4 can be also applied to construct a solution of the SPDE (0.5) by using the usual method of successive approximation. The assertion (iii) follows from Lemma 5.3-(i). $\square$

Remark 5.1 (i) Mathematical theory on the TDGL equation of non-conservative type has been developed by Marcus [30], Funaki [15], Faris and Jona-Lasinio [11], Iwata [20], Cassandro, Olivieri and

Picco [4] and others both in finite and infinite volume cases. See Funaki [17] for a non-scalar TDGL equation.

(ii) We did not discuss the equivalence between the integral equation (5.3) and the SPDE (0.5) or (5.1). See Iwata [20] for such problem.

§6 Approximation theorems for the TDGL equation

In order to develop an infinite-dimensional analysis on the stochastic process S_t , we need to approximate it by finite-dimensional processes. Here we shall discuss two types of approximation theorems, namely, finite volume approximation and its further approximation using the so-called Galerkin method. The result will be applied in Section 8.

6.1 Finite volume approximation Consider the following SPDE on a finite interval $[-l, l]$, $l \in \mathbb{N}$:

$$(6.1) \quad dS_t^l(x) = -A^l S_t^l(x) dt + \Delta \{V'(S_t^l(x))\} dt + \sqrt{2} \nabla dw_t(x), \quad t > 0, \quad x \in (-l, l),$$

with an initial condition:

$$(6.2) \quad S_0^l = S^l \quad \text{on} \quad [-l, l].$$

Here $S^l \in L^2_l \equiv L^2([-l, l], dx)$ and $A^l = (-\Delta)^2 - \gamma \Delta$ should be understood as an operator defined as a function of the self-adjoint operator $-\Delta$ on the space L^2_l having the Dirichlet 0-boundary condition at $\pm l$. The precise mathematical meaning of the equation (6.1) is given similarly to the equation (5.1) by rewriting it into an integral equation:

$$(6.3) \quad S_t^\ell(x) = \int_{-\ell}^{\ell} q^\ell(t, x, y) S^\ell(y) dy - \sqrt{2} \int_0^t \int_{-\ell}^{\ell} q_y^\ell(t-u, x, y) dw_u(y) dy \\ + \int_0^t \int_{-\ell}^{\ell} q_{yy}^\ell(t-u, x, y) V(S_u^\ell(y)) du dy, \quad t \geq 0, \quad x \in [-\ell, \ell],$$

where q^ℓ is the fundamental solution of the operator $\frac{\partial}{\partial t} + A^\ell$.

The second term in the RHS of (6.3) should be understood as

$$- \sqrt{2} \int_0^t \int_{\mathbb{R}^1} 1_{[-\ell, \ell]}(y) q_y^\ell(t-u, x, y) dw_u(y) dy$$

or being defined with the cylindrical Brownian motion on the Hilbert space L_ℓ^2 . We note the relation:

$$(6.4) \quad q^\ell(t, x, y) = \sum_{n=-\infty}^{\infty} \{q(t, x-y+4n\ell) - q(t, x+y+2(2n+1)\ell)\}, \\ x, y \in [-\ell, \ell], \quad t > 0,$$

where $q = q(t, x)$ is the function introduced in Section 5. In fact, this follows by seeing that both sides satisfy the same boundary conditions $q(\pm\ell) = q''(\pm\ell) = 0$ as functions of x . The existence and uniqueness theorem for the equation (6.1) can be formulated as follows.

Proposition 6.1 (i) There exists a unique solution S_t^ℓ of (6.1) satisfying $S_t^\ell \in C([0, \infty), L_\ell^2)$ (a.s.).
(ii) If the initial data $S^\ell \in C([-\ell, \ell])$ and $S^\ell(\pm\ell) = 0$, then the solution S_t^ℓ satisfies $S_t^\ell(\pm\ell) = 0$ and $S_t^\ell \in C([0, \infty), C([-\ell, \ell]))$ (a.s.).

The meaning of "uniqueness" in the statement of this proposition is the same as in Theorem 5.1-(ii). The first assertion is shown by using the Galerkin method (see Theorem 6.2 below) and the second one is proved by a similar method used in Section 5 noting the relation (6.4). We therefore omit the proof of the proposition.

Let $S, S^\ell \in \mathcal{E}_e$, $\ell \in \mathbb{N}$, be given and satisfy that $S^\ell(\pm\ell) = 0$ and $S^\ell \rightarrow S$ as $\ell \rightarrow \infty$ in the space $\mathcal{E}_r$ with some $r > 0$. The positive number r will be fixed throughout this paragraph. In the following we sometimes regard $S_t^\ell \in \mathcal{E}$ by setting $S_t^\ell(x) = 0$ for $x \in \mathbb{R} \setminus [-\ell, \ell]$. Denote by P and P^ℓ the distributions on the space $C([0, \infty), \mathcal{E})$ of the solution S_t of the SPDE (0.5) with an initial data S respectively of the solution S_t^ℓ of (6.1) satisfying (6.2). The purpose of this paragraph is to prove the following theorem.

Theorem 6.1 The probability distribution P^ℓ converges weakly to P as $\ell \rightarrow \infty$ on the space $C((0, \infty), \mathcal{E})$.

Similarly to the definitions of $S_{t,1}$, $S_{t,2}$ and $S_{t,3}$ given in Section 5, we denote three terms in the RHS of (6.3) in due order by $S_{t,1}^\ell$, $S_{t,2}^\ell$ and $S_{t,3}^\ell$, respectively (we neglect the factor $-\sqrt{2}$ for the definition of $S_{t,2}^\ell$). We regard $S_{t,1}^\ell$, $S_{t,2}^\ell$ and $S_{t,3}^\ell \in \mathcal{E}$ by setting $= 0$ outside $[-\ell, \ell]$ similarly to S_t^ℓ . The proof of Theorem 6.1 will be divided into four steps.

6.1.1 Convergence of $S_{t,1}^\ell$ — We shall prove that $S_{t,1}^\ell$ converges to $S_{t,1}$ in the space $C((0, \infty), \mathcal{E})$ as $\ell \rightarrow \infty$. The following estimate is an easy consequence of (6.4):

$$(6.5) \quad \int_{-\ell}^{\ell} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q^\ell(t, x, y) \right| \theta(y, -r) dy \leq e^{r \int_R} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x, y) \right| \theta(y, -r) dy$$

$$t > 0, x \in [-\ell, \ell] ; j = 0, 1, k = 0, 1, 2, 3,$$

for every $r \geq 0$. Although this bound holds only for $r \geq 0$, we can derive the following supplementary estimate for $r > 0$:

$$(6.6) \quad \int_{-\ell}^{\ell} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q^{\ell}(t, x, y) \right| \theta(y, r) dy \leq K' t^{-j-k/4} \theta(x, r),$$

$$0 < t \leq T, \quad x \in [-\ell, \ell], \quad \ell \geq 1; \quad j=0,1, \quad k=0,1,2,3,$$

with a positive constant K' depending only on r and T . Indeed, noting that x is an even function and using (6.4) and (5.6), the LHS of (6.6) can be bounded by

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \int_{\mathbb{R}} \left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial x^k} q(t, x+2m\ell, y) \right| \theta(y, r) dy \\ & \leq K t^{-j-k/4} \sum_{m=-\infty}^{\infty} \theta(x+2m\ell, r), \quad 0 < t \leq T. \end{aligned}$$

However, it is not difficult to see

$$\sum_{m=-\infty}^{\infty} \theta(x+2m\ell, r) \leq \text{const. } \theta(x, r), \quad x \in [-\ell, \ell], \quad r > 0,$$

with a const. independent of x and $\ell \geq 1$.

Lemma 6.1 The family of functions $\{S_{t,1}^{\ell}\}_{\ell \in \mathbb{N}}$ is relatively compact in the space $C((0, \infty), \mathcal{E}) = C((0, \infty) \times \mathbb{R})$.

Proof Since the assumption on $\{S^{\ell}\}_{\ell \in \mathbb{N}}$ implies $\sup_{\ell} \|S^{\ell}\|_r < \infty$, the following three bounds which follow from (5.6) and (6.5) indicate the conclusion:

$$\|S_{t,1}^{\ell}\|_r \leq K e^r \|S^{\ell}\|_r$$

$$\left\| \frac{\partial}{\partial x} S_{t,1}^{\ell} \right\|_r \leq K e^r t^{-1/4} \|S^{\ell}\|_r$$

and

$$\left\| \frac{\partial}{\partial t} S_{t,1}^{\ell} \right\|_r \leq K e^r t^{-1} \|S^{\ell}\|_r, \quad 0 < t \leq T, \quad T > 0. \quad \square$$

Let $\mathcal{E}_{-r,\ell}$, $r > 0$, be the space $C([-\ell, \ell])$ with norm $\|\varphi\|_{-r} =$

$\sup_{x \in [-\ell, \ell]} |\varphi(x)| \theta(x, -r)$ and let $\hat{\mathcal{E}}_{-r, \ell}$ be its subspace consisting of all φ such that $\varphi(\pm \ell) = 0$. We denote by e^{-tA} an integral operator with the kernel $q(t, x, y)$, which is defined on the space $\mathcal{E}_{-r}$; i.e., $e^{-tA} \varphi = S_{t, 1}(\cdot; \varphi)$, $\varphi \in \mathcal{E}_{-r}$. Similarly e^{-tA^ℓ} can be defined as an operator on the space $\mathcal{E}_{-r, \ell}$. Let $\hat{\mathcal{E}}_{-r}$ be the space introduced in Section 2.

Lemma 6.2 (i) $\{e^{-tA}\}_{t \geq 0}$ and $\{e^{-tA^\ell}\}_{t \geq 0}$ are strongly continuous semigroups on the spaces $\hat{\mathcal{E}}_{-r}$ and $\hat{\mathcal{E}}_{-r, \ell}$, respectively. (ii) With some constants $M \geq 1$ and $\delta > 0$ which are independent of ℓ and t , operator norms of e^{-tA^ℓ} on $\hat{\mathcal{E}}_{-r, \ell}$ and of e^{-tA} on $\hat{\mathcal{E}}_{-r}$ can be estimated as follows:

$$\|e^{-tA^\ell}\| \leq M e^{\delta t}, \quad \|e^{-tA}\| \leq M e^{\delta t}, \quad \ell \in \mathbb{N}, \quad t > 0.$$

Proof (i) The estimate (5.9) shows that $\varphi \in \mathcal{E}_{-r}$ implies $e^{-tA} \varphi \in \mathcal{E}_{-r}$, $t \geq 0$. However, since the condition (2.2) on $\varphi \in \hat{\mathcal{E}}_{-r}$ proves with the help of (5.6) that

$$(6.7) \quad \lim_{\ell \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{|x| \geq \ell} |e^{-tA} \varphi(x)| \theta(x, -r) = 0, \quad T > 0,$$

we see that $\varphi \in \hat{\mathcal{E}}_{-r}$ implies $e^{-tA} \varphi \in \hat{\mathcal{E}}_{-r}$ and therefore e^{-tA} is a semigroup on the space $\hat{\mathcal{E}}_{-r}$. The strong-continuity of this semigroup follows from (6.7) and the fact that $e^{-tA} \varphi$ converges to φ as $t \downarrow 0$ uniformly on each bounded interval of R if $\varphi \in C_b(R)$ (see Arima [2]). For the assertion on e^{-tA^ℓ} , see Arima [2] noting that A^ℓ is an operator with boundary conditions: $\varphi(\pm \ell) = \varphi''(\pm \ell) = 0$. We notice that the uniform convergence of $e^{-tA^\ell} \varphi$ to φ on $[-\ell, \ell]$ as $t \downarrow 0$ can be shown by using (5.2), (6.4) and the property $\varphi(\pm \ell)$

= 0 of $\varphi \in \hat{\mathcal{E}}_{-r, \ell}$.

(ii) Since the estimate (6.6) proves

$$\|e^{-tA} \varphi\|_{-r} \leq K' \|\varphi\|_{-r}, \quad 0 \leq t \leq T,$$

the assertion on $\{e^{-tA}\}$ follows from their semigroup property, see Tanabe [40, Theorem 3.1.1]. The assertion on e^{-tA} follows from (5.9). $\square$

Let π_ℓ , $\ell \in \mathbb{N}$, be a linear operator from $\hat{\mathcal{E}}_{-r}$ into $\hat{\mathcal{E}}_{-r, \ell}$ defined as follows:

$$\pi_\ell \varphi(x) = \begin{cases} \varphi(x) & , \quad |x| \leq \frac{\ell}{2}, \\ \varphi(\ell/2)(2-2x/\ell)\theta(\frac{\ell}{2}, -r)\theta(x, r) & , \quad \frac{\ell}{2} \leq x \leq \ell, \\ \varphi(-\ell/2)(2+2x/\ell)\theta(\frac{\ell}{2}, -r)\theta(x, r) & , \quad -\ell \leq x \leq -\frac{\ell}{2}. \end{cases}$$

Then, the operator norm $\|\pi_\ell\|$ of π_ℓ is equal to 1. We also define a mapping π_ℓ^{-1} from $\hat{\mathcal{E}}_{-r, \ell}$ into $\hat{\mathcal{E}}_{-r}$ by setting $\pi_\ell^{-1} \varphi(x) = \varphi(x)$, $x \in [-\ell, \ell]$, and $= 0$, $x \in \mathbb{R} \setminus [-\ell, \ell]$, for $\varphi \in \hat{\mathcal{E}}_{-r, \ell}$. If $\varphi \in C_0(\mathbb{R})$ satisfies $\text{supp } \varphi \subset [-\ell/2, \ell/2]$, then $\pi_\ell^{-1} \pi_\ell \varphi = \varphi$.

Lemma 6.3 For every $r, T > 0$ and $\varphi \in \hat{\mathcal{E}}_{-r}$, we have

$$\lim_{\ell \rightarrow \infty} \sup_{t \in [0, T]} \|\pi_\ell^{-1} e^{-tA} \pi_\ell \varphi - e^{-tA} \varphi\|_{-r} = 0.$$

Proof We use the results in the book of Ethier and Kurtz [10]. Let D be the space of all $\varphi \in C^\infty(\mathbb{R})$ such that its k -th derivative $\varphi^{(k)}$ belongs to $\hat{\mathcal{E}}_{-r}$ for every $k = 0, 1, 2, \dots$. Then D is a core for the generator $-A$ of the semigroup e^{-tA} on the space $\hat{\mathcal{E}}_{-r}$. Indeed, this can be shown with the help of Proposition 3.3 of [10, p17], if we can prove (a) $C_0^\infty(\mathbb{R})$ is dense in $\hat{\mathcal{E}}_{-r}$ and (b) $\varphi \in C_0^\infty(\mathbb{R})$ implies $e^{-tA} \varphi \in D$ for all $t \geq 0$. However (a) is standard and (b)

follows from Lemma 6.2-(i) and the relation:

$$\frac{\partial^k}{\partial x^k} e^{-tA} \varphi(x) = e^{-tA} \varphi^{(k)}(x) \quad , \quad \varphi \in C_0^\infty(R).$$

Therefore, noting Lemma 6.2, Theorem 6.1 of [10, p28] (see also Remark 1.3 of [10, p7]) implies

$$\lim_{\ell \rightarrow \infty} \sup_{t \in [0, T]} \|e^{-tA} \pi_\ell \varphi - \pi_\ell e^{-tA} \varphi\|_{-r} = 0 \quad , \quad T > 0,$$

for all $\varphi \in \mathcal{D}_{-r}$, because for every $\varphi \in D$ we can find a sequence $\{\varphi_\ell \in C_0^\infty(R)\}_{\ell=1}^\infty$ such that $\text{supp } \varphi_\ell \subset [-\ell/2, \ell/2]$ and

$$\lim_{\ell \rightarrow \infty} \sum_{k=0}^4 \|\varphi_\ell^{(k)} - \varphi^{(k)}\|_{-r} = 0.$$

This completes the proof of the lemma with the help of (6.7) and

$$\lim_{\ell \rightarrow \infty} \sup_{t \in [0, T]} \sup_{\ell/2 \leq |x| \leq \ell} |e^{-tA} \pi_\ell \varphi(x)| \theta(x, -r) = 0$$

for all $\varphi \in \mathcal{D}_{-r}$ and $T > 0$, which can be shown similarly to (6.7). $\square$

Lemma 6.4 For every $t > 0$, $S_{t,1}^\ell$ converges to $S_{t,1}$ as $\ell \rightarrow \infty$ in the following sense: For all $\varphi \in C_0(R)$

$$\lim_{\ell \rightarrow \infty} \langle S_{t,1}^\ell, \varphi \rangle = \langle S_{t,1}, \varphi \rangle.$$

Proof The symmetry of the fundamental solutions $q^\ell(t, x, y)$ and $q(t, x, y)$ in the variables (x, y) implies

$$\langle S_{t,1}^\ell, \varphi \rangle = \langle S^\ell, \pi_\ell^{-1} e^{-tA} \pi_\ell \varphi \rangle, \quad \langle S_{t,1}, \varphi \rangle = \langle S, e^{-tA} \varphi \rangle,$$

for every $\varphi \in C_0(R)$ such that $\text{supp } \varphi \subset [-\ell/2, \ell/2]$. We therefore obtain the conclusion from an estimation:

$$\begin{aligned} & |\langle S_{t,1}^\ell - S_{t,1}, \varphi \rangle| \\ & \leq |\langle S^\ell - S, \pi_\ell^{-1} e^{-tA} \pi_\ell \varphi \rangle| + |\langle S, \pi_\ell^{-1} e^{-tA} \pi_\ell \varphi - e^{-tA} \varphi \rangle| \end{aligned}$$

$$\leq \{ \|\pi_\ell^{-1} e^{-tA} \pi_\ell \varphi\|_{-r-\varepsilon} \|S^\ell - S\|_r + \|\pi_\ell^{-1} e^{-tA} \pi_\ell \varphi - e^{-tA} \varphi\|_{-r} \|S\|_{r-\varepsilon} \} \\ \times \int_R \theta(x, \varepsilon) dx, \quad \varepsilon > 0,$$

which converges to 0 as $\ell \rightarrow \infty$ by Lemma 6.3. $\square$

Now we can prove the following:

Proposition 6.2 In the space $C((0, \infty), \mathcal{E})$, $S_{t,1}^\ell$ converges to $S_{t,1}$ as $\ell \rightarrow \infty$.

Proof Lemma 6.1 shows that from every subsequence $\{S_{\cdot,1}^{\ell'}\}$ of $\{S_{\cdot,1}^\ell\}$ we can take further subsequence $\{S_{\cdot,1}^{\ell''}\}$ in such a way that $S_{\cdot,1}^{\ell''}$ converges to some $\tilde{S}_\cdot \in C((0, \infty), \mathcal{E})$ as $\ell'' \rightarrow \infty$ in the space $C((0, \infty), \mathcal{E})$. However Lemma 6.4 proves $\tilde{S}_t = S_{t,1}$ for every $t > 0$ and this completes the proof. $\square$

6.1.2 Convergence of $S_{t,2}^\ell$ Let $\kappa_n \equiv \kappa_n^\ell = (\frac{n\pi}{2\ell})^2$ and $e_n(x) \equiv e_n^\ell(x) = \sqrt{1/\ell} \sin(\sqrt{\kappa_n}(x+\ell))$, $n \in \mathbb{N}$, be eigenvalues and their corresponding normalized eigenfunctions, respectively, of the operator $-\Delta$ with Dirichlet boundary conditions defined on the space L_ℓ^2 . We denote by $\langle \cdot, \cdot \rangle_\ell$ the usual inner product of L_ℓ^2 . Let $\{w_t^n\}_{n \in \mathbb{N}}$ be a system of mutually independent 1-dimensional standard Brownian motions defined by

$$w_t^n = \langle w_t(\cdot), \sqrt{1/\ell} \cos(\sqrt{\kappa_n}(\cdot + \ell)) \rangle_\ell.$$

See [15] for the RHS of this expression. Then $S_{t,2}^\ell$ has the following representation:

$$S_{t,2}^\ell(x) = \sum_{n=1}^{\infty} e_n^\ell(x) \int_0^t (\kappa_n^\ell)^{1/2} \exp[-(t-u)((\kappa_n^\ell)^2 + \gamma \kappa_n^\ell)] dw_u^n.$$

Lemma 6.5 For every $T > 0$ and $0 < \alpha < 1$, there exists a

positive constant C independent of ℓ such that

$$E[|S_{t,2}^{\ell}(x) - S_{t',2}^{\ell}(x')|^2] \leq C(|t-t'|^{1/4} + |x-x'|^{\alpha}),$$

$$0 \leq t, t' \leq T, \quad x, x' \in [-\ell, \ell], \quad \ell \in \mathbb{N}.$$

Proof First we consider the case of $t = t'$ and $x < x'$:

$$I_1 \equiv E[|S_{t,2}^{\ell}(x) - S_{t,2}^{\ell}(x')|^2]$$

$$= \sum_{n=1}^{\infty} f_t(\kappa_n) \{e_n(x) - e_n(x')\}^2,$$

where the function f_t is defined by

$$f_t(\kappa) = (\kappa + \gamma)^{-1} (1 - e^{-t(\kappa^2 + \gamma\kappa)}) \quad , \quad \kappa > 0.$$

Since it is easy to see

$$|e_n(x) - e_n(x')| \leq \sqrt{1/\ell} \min(\sqrt{\kappa_n} |x-x'|, 2)$$

$$\leq \sqrt{1/\ell} 2^{1-\beta} (\sqrt{\kappa_n} |x-x'|)^{\beta}$$

for every $0 < \beta < 1$, we obtain

$$(6.8) \quad I_1 \leq M |x-x'|^{2\beta}, \quad 0 < \beta < 1/2,$$

where M is a positive constant independent of ℓ , t , x , x' and given by

$$M = \sup_{\ell} M_{\beta}^{\ell}$$

and

$$M_{\beta}^{\ell} = 2^{2-2\beta} \frac{1}{\ell} \sum_{n=1}^{\infty} (\kappa_n^{\ell} + \gamma)^{-1} (\kappa_n^{\ell})^{\beta}.$$

Note that $0 < \beta < 1/2$ implies $M < \infty$. Next we consider the case of $x = x'$ and $0 \leq t < t'$:

$$I_2 \equiv E[|S_{t,2}^{\ell}(x) - S_{t',2}^{\ell}(x)|^2]$$

$$= \sum_{n=1}^{\infty} \{e_n(x)\}^2 \{2 + e^{-(t+\tau)(\kappa_n^2 + \gamma\kappa_n)} - e^{-t(\kappa_n^2 + \gamma\kappa_n)}\} f_{\tau}(\kappa_n)$$

$$\leq \frac{3}{\ell} \sum_{n=1}^{\infty} f_{\tau}(\kappa_n),$$

where $\tau = (t'-t)/2 > 0$. The function $f_{\tau}(\kappa)$ is increasing in κ if $0 < \kappa < \kappa^*$ and decreasing if $\kappa > \kappa^*$. Here $\kappa^* = \kappa^*(\tau)$ is a unique positive solution of a quadratic equation $(2\kappa+\gamma)(\kappa+\gamma) = 1/\tau$ when $0 < \tau < \gamma^{-2}$ and $\kappa^* = \kappa^*(\tau) = \infty$ when $\tau > \gamma^{-2}$. Hence for every $T > 0$,

$$(6.9) \quad I_2 \leq \int_0^{\infty} f_{\tau}(\pi^2 \lambda^2 / 4) d\lambda + \frac{2}{\ell} f_{\tau}(\kappa^*(\tau)) \\ \leq \text{const.} \tau^{1/4}, \quad \tau \leq T, \quad \ell \in \mathbb{N},$$

with some positive const. independent of ℓ . We have the conclusion combining two estimates (6.8) and (6.9). $\square$

Proposition 6.3 The probability distribution of $S_{t,2}^{\ell}$ on the space $C([0,\infty),\mathcal{E})$ converges weakly to that of $S_{t,2}$ as $\ell \rightarrow \infty$.

Proof Since $\{S_{t,2}^{\ell}(x); t \geq 0, x \in \mathbb{R}\}$ is a Gaussian system and $S_{t,2}^{\ell}(\pm\ell) = 0$, Lemma 6.5 with the help of Kolmogorov-Totoki's theorem proves that $S_{\cdot,2}^{\ell} \in C([0,\infty),\mathcal{E})$ (a.s.) and the family of distributions on the space $C([0,\infty),\mathcal{E})$ of $\{S_{\cdot,2}^{\ell}\}_{\ell \in \mathbb{N}}$ is tight. Therefore the conclusion follows from an observation that for every $t > 0$ and $\varphi \in C_0^{\infty}(\mathbb{R})$ satisfying $\text{supp } \varphi \subset [-\ell/2, \ell/2]$ we have

$$E[\langle S_{t,2}^{\ell} - S_{t,2}, \varphi \rangle^2] = \int_0^t \|\pi_{\ell}^{-1} e^{-(t-u)A} \pi_{\ell} \nabla \varphi - e^{-(t-u)A} \nabla \varphi\|_{L^2(\mathbb{R})}^2 du$$

which converges to 0 as $\ell \rightarrow \infty$, see Lemma 6.3. $\square$

6.1.3 Convergence of $S_{t,3}^{\ell}$ As in Section 5, for given $S_{\cdot} = \{S_t(x)\} \in C([0,\infty),\mathcal{E})$ we define

$$S_{t,3}^{\ell}(x) \equiv S_{t,3}^{\ell}(x;S_{\cdot}) = \int_0^t \int_{-\ell}^{\ell} q_{yy}^{\ell}(t-u, x, y) V'(S_u(y)) dy du,$$

and

$$S_{t,3}(x) \equiv S_{t,3}(x;S_{\cdot}) = \int_0^t \int_R q_{yy}(t-u, x, y) V'(S_u(y)) dy du.$$

Lemma 6.6 For every $T > 0$ there exists a positive constant C depending only on T such that

$$|S_{t,3}^{\ell}(x) - S_{t',3}^{\ell}(x')| \leq C(|t-t'|^{1/2} + |x-x'|), \\ 0 \leq t, t' \leq T, \quad x, x' \in [-\ell, \ell], \quad \ell \in \mathbb{N}.$$

Proof The proof can be completed quite similarly to that of Lemma 5.2. We may use the estimate (6.5) with $r = 0$. $\square$

Proposition 6.4 Let functions $\{S_{\cdot}^{\ell}\}_{\ell \in \mathbb{N}}$ and $S_{\cdot} \in C([0, \infty), \mathcal{E})$ be given and assume that $S_{\cdot}^{\ell}$ converges to $S_{\cdot}$ as $\ell \rightarrow \infty$ in this space. Then $S_{t,3}^{\ell}(\cdot; S_{\cdot}^{\ell})$ converges to $S_t(\cdot; S_{\cdot})$ in the space $C([0, \infty), \mathcal{E})$.

Proof First we note that Lemma 6.6 shows the relative-compactness of the family of functions $\{S_{t,3}^{\ell}(\cdot; S_{\cdot}^{\ell})\}_{\ell \in \mathbb{N}}$ in the space $C([0, \infty), \mathcal{E})$. Therefore the conclusion follows by seeing that for $\varphi \in C_0^{\infty}(R)$ such that $\text{supp } \varphi \subset [-\ell/2, \ell/2]$,

$$\begin{aligned} & | \langle S_{t,3}^{\ell} - S_{t,3}, \varphi \rangle | \\ & \leq \left| \int_0^t \langle V'(S_u^{\ell}), \pi_{\ell}^{-1} e^{-(t-u)A} \pi_{\ell} \Delta \varphi - e^{-(t-u)A} \Delta \varphi \rangle du \right| \\ & \quad + \left| \int_0^t \langle V'(S_u^{\ell}) - V'(S_u), e^{-(t-u)A} \Delta \varphi \rangle du \right| \\ & \leq t \|V'\|_{\infty} \sup_{0 \leq u \leq t} \| \pi_{\ell}^{-1} e^{-uA} \pi_{\ell} \Delta \varphi - e^{-uA} \Delta \varphi \|_{-1} \int_R \theta(x, 1) dx \\ & \quad + \sup_{0 \leq u \leq t} \| e^{-uA} \Delta \varphi \|_{-1} \int_0^t du \int_R |V'(S_u^{\ell}(x)) - V'(S_u(x))| \theta(x, 1) dx, \end{aligned}$$

which converges to 0 as $\ell \rightarrow \infty$ for every $t > 0$ and $\varphi \in C_0^{\infty}(R)$,

see Lemma 6.3. $\square$

6.1.4 Proof of Theorem 6.1 Now we give the proof of Theorem 6.1.

Propositions 6.2, 6.3 and 6.4 are combined to show that the family of joint probability distributions $\{\tilde{P}^{\ell}\}_{\ell \in \mathbb{N}}$ of $\{(S_t^{\ell}, S_{t,1}^{\ell}, S_{t,2}^{\ell})\}_{\ell \in \mathbb{N}}$ on the space $C((0, \infty), \mathcal{G}^3)$ is tight. Therefore every subsequence $\{\ell'\}$ of $\{\ell\}$ has further subsequence $\{\ell''\}$ in such a way that $\tilde{P}^{\ell''}$ converges weakly to some $\tilde{P} \in \mathcal{P}(C((0, \infty), \mathcal{G}^3))$ as $\ell'' \rightarrow \infty$. We can now apply Skorohod's representation theorem to construct $C((0, \infty), \mathcal{G}^3)$ -valued random variables $\bar{W}_t^{\ell''} = (\bar{S}_t^{\ell''}, \bar{S}_{t,1}^{\ell''}, \bar{S}_{t,2}^{\ell''})$ and $\bar{W}_t = (\bar{S}_t, \bar{S}_{t,1}, \bar{S}_{t,2})$ on a proper probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ in the following manner: (i) Under $\bar{P}$, the distributions of $\bar{W}_t^{\ell''}$ and $\bar{W}_t$ are $\tilde{P}^{\ell''}$ and $\tilde{P}$, respectively, and (ii) $\bar{W}_t^{\ell''}$ converges almost surely to $\bar{W}_t$ in the space $C((0, \infty), \mathcal{G}^3)$ as $\ell'' \rightarrow \infty$. Define $\bar{S}_{t,3}^{\ell''} = S_{t,3}^{\ell''}(\cdot; \bar{S}^{\ell''})$ and $\bar{S}_{t,3} = S_{t,3}(\cdot; \bar{S})$, then an equality $\bar{S}_t^{\ell''} = \bar{S}_{t,1}^{\ell''} - \sqrt{2} \bar{S}_{t,2}^{\ell''} + \bar{S}_{t,3}^{\ell''}$ holds and Proposition 6.4 proves that $\bar{S}_{t,3}^{\ell''}$ converges almost surely to $\bar{S}_{t,3}$ in the space $C((0, \infty), \mathcal{G})$. Hence we obtain $\bar{S}_t = \bar{S}_{t,1} - \sqrt{2} \bar{S}_{t,2} + \bar{S}_{t,3}$. However Proposition 6.3 implies that $\bar{S}_{t,2}$ has a representation:

$$\bar{S}_{t,2}(x) = \int_0^t \int_{\mathbb{R}} q_y(t-u, x, y) d\bar{W}_u(y) dy,$$

with a cylindrical Brownian motion $\bar{W}_t$ which is defined on the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ or its proper extension if necessary.

Therefore $\bar{S}_t$ is a solution of the integral equation (5.3) with $w_t = \bar{W}_t$. This shows that the distribution of $\bar{S}_t$ is just given by $\tilde{P}$ appearing in the statement of Theorem 6.1, which completes the proof of the theorem.

We conclude this paragraph by showing the following lemma which

will be used in Section 8.

Lemma 6.7 $\sup_{\ell; \ell \geq \bar{\ell}} E[\|S_t^\ell\|_{L_{\frac{2}{\ell}}}^2] < \infty, \quad \bar{\ell} \in \mathbb{N}, t > 0.$

Proof The assertion can be proved easily by the following three

facts. (i) Proposition 6.2 shows that $S_{t,1}^\ell(x)$ converges to

$S_{t,1}(x)$ uniformly in $x \in [-\bar{\ell}, \bar{\ell}]$ as $\ell \rightarrow \infty$ and therefore

$\sup_{\ell \geq \bar{\ell}} \|S_{t,1}^\ell\|_{L_{\frac{2}{\ell}}} < \infty.$ (ii) In the proof of Lemma 6.5 (take $t=0$ in the

definition of I_2), we have proved that

$$E[|S_{t,2}^\ell(x)|^2] \leq \text{const. } t^{1/4}, \quad 0 \leq t \leq T, x \in [-\ell, \ell], \ell \in \mathbb{N},$$

with a const. independent of ℓ and x . (iii) Lemma 6.6 shows

$$|S_{t,3}^\ell(x)| \leq C t^{1/2}, \quad 0 \leq t \leq T, x \in [-\ell, \ell], \ell \in \mathbb{N}. \quad \square$$

6.2 Galerkin approximation In this paragraph we fix $\ell \in \mathbb{N}$ and

discuss a finite-dimensional approximation to the solution S_t^ℓ of

the SPDE (6.1) having initial data $S^\ell \in L_\ell^2$. We shall denote for

simplicity by S_t and S instead of S_t^ℓ and S^ℓ , respectively.

The norm of the space L_ℓ^2 will be denoted simply by $\|\cdot\|$. Define a

sequence of L_ℓ^2 -valued processes $S_t^{(N)} = \sum_{n=1}^N a_n^{(N)}(t) e_n$, $N \in \mathbb{N}$, by

solving the following finite-dimensional stochastic differential equation(SDE):

$$(6.10) \quad \begin{cases} da_n^{(N)}(t) = -\theta_n a_n^{(N)}(t) dt - \kappa_n \langle V'(S_t^{(N)}(\cdot)), e_n \rangle_\ell dt - \sqrt{2\kappa_n} dw_t^n, \\ a_n^{(N)}(0) = \langle S, e_n \rangle_\ell, \quad n = 1, 2, \dots, N, \end{cases}$$

where $\theta_n \equiv \theta_n^\ell = \kappa_n^2 + \gamma \kappa_n$ and κ_n, e_n, w_t^n are the same as those

introduced in the paragraph 6.1.2. We shall prove the following.

Theorem 6.2 (i) There exists a unique solution $S_t = S_t^\ell$ of the SPDE (6.1) satisfying $S_t \in C([0, \infty), L_\ell^2)$ (a.s.).

(ii) We have a uniform estimate:

$$(6.11) \quad E[\|S_t^{(N)}\|^2] \leq \text{const.} + \|S\|^2,$$

with a positive const. independent of N, t, S and

$$(6.12) \quad \lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} E[\|S_t^{(N)} - S_t\|^2] = 0, \quad T > 0.$$

Lemma 6.8 There exists a positive constant C independent of N, n, t, S such that

$$E[\{a_n^{(N)}(t)\}^2] \leq C\kappa_n^{-1} + \langle S, e_n \rangle^2, \quad 1 \leq n \leq N.$$

Proof Using Itô's formula it is easy to show

$$\frac{d}{dt} E[\{a_n^{(N)}(t)\}^2] \leq -(2\theta_n - \kappa_n^2) E[\{a_n^{(N)}(t)\}^2] + (2\kappa_n + 2\|V'\|_\infty^2),$$

$$1 \leq n \leq N,$$

which gives the desired estimate. $\square$

The estimate (6.11) in Theorem 6.2 is an immediate consequence of this lemma. Let π_N be an orthogonal projection of L_ℓ^2 onto its subspace spanned by $\{e_n\}_{n=1}^N$. Remind that $\gamma_0 = \|V''\|_\infty$.

Lemma 6.9 The following two estimates hold for every $1 \leq N \leq N'$ and $t > 0$:

$$(6.13) \quad E[\|\pi_N(S_t^{(N)} - S_t^{(N')})\|^2] \leq \gamma_0^2 \int_0^t E[\|S_u^{(N)} - S_u^{(N')}\|^2] du,$$

$$(6.14) \quad E[\|S_t^{(N)} - S_t^{(N')}\|^2] \leq 2 e^{2\gamma_0^2 t} \sum_{n=N+1}^{N'} \{C\kappa_n^{-1} + \langle S, e_n \rangle^2\}.$$

Proof For $1 \leq n \leq N$ we have

$$\begin{aligned} & \frac{d}{dt} \{a_n^{(N)}(t) - a_n^{(N')}(t)\}^2 \\ & \leq -(2\theta_n - \kappa_n^2) \{a_n^{(N)}(t) - a_n^{(N')}(t)\}^2 + \langle V'(S_t^{(N)}) - V'(S_t^{(N')}), e_n \rangle_{\ell}^2 \end{aligned}$$

and therefore

$$\frac{d}{dt} \|\pi_N(S_t^{(N)} - S_t^{(N')})\|^2 \leq \gamma_0^2 \|S_t^{(N)} - S_t^{(N')}\|^2.$$

This proves (6.13). On the other hand, Lemma 6.8 and (6.13) show

$$\begin{aligned} & E[\|S_t^{(N)} - S_t^{(N')}\|^2] \\ & \leq 2\gamma_0^2 \int_0^t E[\|S_u^{(N)} - S_u^{(N')}\|^2] du + 2 \sum_{n=N+1}^{N'} \{C\kappa_n^{-1} + \langle S, e_n \rangle_{\ell}^2\}. \end{aligned}$$

We get (6.14) by using Gronwall's lemma. $\square$

The estimate (6.14) shows that for each $t > 0$ $\{S_t^{(N)}\}_{N \in \mathbb{N}}$ is a Cauchy sequence in the space $L^2(\Omega \rightarrow L_{\ell}^2)$ and therefore it has a limit $S_t^* \in L^2(\Omega \rightarrow L_{\ell}^2)$ as $N \rightarrow \infty$. The random variable S_t^* is $\mathcal{F}_t$ -measurable and satisfies

$$(6.15) \quad \sup_{t \geq 0} E[\|S_t^*\|^2] < \infty.$$

Indeed this follows from (6.11) by using Fatou's lemma. Now we define $\{a_n(t)\}_{n \in \mathbb{N}}$ by solving the following family of 1-dimensional SDE's:

$$(6.16) \quad \begin{cases} da_n(t) = -\theta_n a_n(t) dt - \kappa_n \langle V'(S_t^*(\cdot)), e_n \rangle_{\ell} dt - \sqrt{2\kappa_n} dw_t^n, \\ a_n(0) = \langle S, e_n \rangle_{\ell}, \end{cases}$$

and set $S_t = \sum_{n=1}^{\infty} a_n(t) e_n$; or we can say that S_t is given by

$$(6.17) \quad S_t(x) = S_{t,1}^{\ell}(x) - \sqrt{2} S_{t,2}^{\ell}(x) + S_{t,3}^{\ell}(x; S_t^*).$$

Each term in the RHS of (6.17) is the same as that introduced in the paragraph 6.1.

Lemma 6.10 (i) With the same constant C as in Lemma 6.8, we have

$$E[\{a_n(t)\}^2] \leq C\kappa_n^{-1} + \langle S, e_n \rangle^2, \quad n \in \mathbb{N}.$$

(ii) The following two estimates hold for every $t > 0$ and $N \in \mathbb{N}$:

$$(6.18) \quad E[\|\pi_N(S_t^{(N)} - S_t^*)\|^2] \leq \gamma_0^2 \int_0^t E[\|S_u^{(N)} - S_u^*\|^2] du,$$

$$(6.19) \quad E[\|S_t^{(N)} - S_t^*\|^2] \leq 4 e^{2\gamma_0^2 t} \sum_{n=N+1}^{\infty} (C\kappa_n^{-1} + \langle S, e_n \rangle^2).$$

Proof The estimates (i) and (6.18) can be shown in a similar manner to the proof of Lemma 6.8 and (6.13), respectively. To prove (6.19) we first show

$$(6.20) \quad \lim_{N \rightarrow \infty} E[\|S_t^{(N)} - S_t^*\|^2] = 0, \quad t \geq 0.$$

Indeed using (i) and (6.18) we have

$$(6.21) \quad E[\|S_t^{(N)} - S_t^*\|^2] \leq 2\gamma_0^2 \int_0^t E[\|S_u^{(N)} - S_u^*\|^2] du \\ + 2 \sum_{n=N+1}^{\infty} (C\kappa_n^{-1} + \langle S, e_n \rangle^2).$$

Here the second term in the RHS of (6.21) tends to 0 as $N \rightarrow \infty$.

On the other hand, the integrand of the first term converges to 0 for each $u \in [0, t]$. Therefore, noting two estimates (6.11) and (6.15), we can apply the Lebesgue's dominated convergence theorem to show that the first term in the RHS of (6.21) also tends to 0 as $N \rightarrow \infty$. This proves (6.20). Now the estimate (6.19) follows from (6.14) and (6.21) by noting a trivial inequality:

$$E[\|S_t^{(N)} - S_t\|^2] \leq 2E[\|S_t^{(N)} - S_t^{(N')}\|^2] + 2E[\|S_t^{(N')} - S_t\|^2], \quad N' \geq N. \quad \square$$

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2 The bound (6.19) implies that $S_t^* = S_t$ (a.s.) for every $t \geq 0$ and therefore $P(S_t^* = S_t \text{ for a.e. } t \geq 0) = 1$. Hence from (6.17) we see that S_t is a solution of the SPDE (6.1).

Proposition 6.3 and Lemma 6.6 show that $S_{t,2}^\ell$ and $S_{t,3}^\ell(\cdot; S_\cdot^*) \in C([0, \infty), \mathcal{E})$ (a.s.) and accordingly $\in C([0, \infty), L_\ell^2)$ (a.s.). However, since it is easy to see that $S \in L_\ell^2$ implies $S_{t,1}^\ell = e^{-tA} S \in C([0, \infty), L_\ell^2)$ (a.s.), we obtain that $S_t \in C([0, \infty), L_\ell^2)$ (a.s.). The convergence (6.12) follows from (6.19).

Finally the uniqueness of solutions of the SPDE (6.1) is shown as follows: Let S_t and $\bar{S}_t$ be two solutions with same initial value satisfying $S_t, \bar{S}_t \in C([0, \infty), L_\ell^2)$ (a.s.). Then we have

$$(6.22) \quad \begin{aligned} \|S_t - \bar{S}_t\| &= \|S_{t,3}^\ell(\cdot; S_\cdot) - S_{t,3}^\ell(\cdot; \bar{S}_\cdot)\| \\ &\leq C_T \int_0^t (t-u)^{-1/2} \|S_u - \bar{S}_u\| du, \quad 0 \leq t \leq T, \end{aligned}$$

with some constant C_T depending on T . Here we have used the estimates (5.6) and (6.5) on the fundamental solution q^ℓ . A recursive usage of (6.22) implies

$$(6.23) \quad \|S_t - \bar{S}_t\| \leq (C_T \pi t)^{n/2} \Gamma^{-1}(\frac{n}{2}+1) \sup_{0 \leq u \leq T} \|S_u - \bar{S}_u\|, \quad 0 \leq t \leq T,$$

for every $n \in \mathbb{N}$. We therefore obtain $S_t = \bar{S}_t$, since the RHS of (6.23) tends to 0 as $n \rightarrow \infty$. $\square$

§7 Integral equations

In this section we analyze parabolic equations with measurable coefficients as a preparation for Sections 8 and 9.

Let $\mathcal{B}([0, \infty) \times \mathbb{R})$ be a class of all measurable functions on $[0, \infty) \times \mathbb{R}$ and let $\mathcal{B}_b = \mathcal{B}_b([0, \infty) \times \mathbb{R})$ be its subclass consisting of bounded functions. We associate with each $c = c(u, x) \in \mathcal{B}_b$ an operator $\mathcal{L}_u \equiv \mathcal{L}_{u, x} = -A + c(u, x)\Delta_x$, $x \in \mathbb{R}$, and consider a backward equation for $Z_{u, t} = Z_{u, t}(x)$, $0 \leq u \leq t < \infty$:

$$(7.1) \quad \begin{cases} \frac{\partial}{\partial u} Z_{u, t} = -\mathcal{L}_u Z_{u, t}, & u \in [0, t], \\ Z_{t, t} = \varphi \in C_0^\infty(\mathbb{R}). \end{cases}$$

More precisely, we consider the corresponding integral equation:

$$(7.2) \quad Z_{u, t} = e^{-A(t-u)}\varphi + \int_u^t e^{-A(v-u)} \{c(v, \cdot)\Delta Z_{v, t}\} dv.$$

This equation will be studied on the space $\mathcal{G}_{-r}^2$, $r > 0$, consisting of all functions $Z \in \mathcal{G}_{-r}$ such that ∇Z and $\Delta Z \in \mathcal{G}_{-r}$. The space $\mathcal{G}_{-r}^2$

has a norm defined by $\|Z\|_{2, -r} = \sum_{i=0}^2 \|Z^{(i)}\|_{-r}$. The solution

$Z_{u, t}$ of (7.2) will be sometimes denoted by $Z_{u, t}(x, \varphi)$ or

$Z_{u, t}(x, \varphi; c)$ to make its dependence on the terminal condition φ and the coefficient c clear. The purpose of this section is to prove

(i) the existence and uniqueness results for (7.2) and also for the corresponding forward integral equation and (ii) the existence of its fundamental solution, which depends continuously on the coefficient c .

The fundamental solutions of second order parabolic equations with measurable coefficients were constructed by Nash and Aronson.

Eidel'man discussed higher order equations with smooth coefficients. Although the technique employed here is quite standard, we add this section since the result seems not to be available in this form in any other literature.

7.1 Existence and uniqueness results On the space $\mathcal{E}_{-r}$, $r > 0$, consider the following auxiliary integral equation for $Y_{u,t} = Y_{u,t}(x, \varphi; c)$, $0 \leq u \leq t < \infty$, $\varphi \in C_0^\infty(R)$:

$$(7.3) \quad Y_{u,t} = e^{-A(t-u)}(\Delta\varphi) + \int_u^t \Delta e^{-A(v-u)} \{c(v, \cdot) Y_{v,t}\} dv.$$

We put $D_T = \{(u, t) \in R^2; 0 \leq u \leq t \leq T\}$ and $\dot{D}_T = \{(u, t) \in D_T; u \neq t\}$, $T > 0$.

Lemma 7.1 (i) For every $r > 0$ and $T > 0$, there exists a solution of (7.3) satisfying $Y_{u,t} \in C(D_T, \mathcal{E}_{-r})$.

(ii) The uniqueness of solutions of (7.3) holds in the class of measurable functions $Y_{u,t}$ satisfying

$$(7.4) \quad \sup_{0 \leq u \leq t \leq T} \|Y_{u,t}\|_{-r} (t-u)^{1-\varepsilon} < \infty, \quad T > 0,$$

with some $0 < \varepsilon < 1$.

Proof (i) Denote by $Q_{u,t} = Q_{u,t}(Y)$ the second term in the RHS of (7.3) for given $Y = Y_{u,t} \in C(D_T, \mathcal{E}_{-r})$. The first task is to prove $Q_{u,t} \in C(D_T, \mathcal{E}_{-r})$. To this end, we estimate for $0 \leq u_1 < u_2 \leq t \leq T$:

$$\|Q_{u_1,t} - Q_{u_2,t}\|_{-r} \leq I_1 + I_2,$$

where

$$I_1 = \left\| \int_{u_1}^{u_2} \Delta e^{-A(v-u_1)} \{c(v, \cdot) Y_{v,t}\} dv \right\|_{-r},$$

$$I_2 = \left\| \int_{u_2}^t \Delta (e^{-A(v-u_1)} - e^{-A(v-u_2)}) \{c(v, \cdot) Y_{v,t}\} dv \right\|_{-r}.$$

The first term I_1 is bounded by $\text{const.}(u_2-u_1)^{1/2} \sup_{0 \leq v \leq t \leq T} \|Y_{v,t}\|_{-r}$.

Indeed the estimate (5.6) can be applied to derive an estimation on the integrand of $Q_{u,t}$:

$$\begin{aligned} & |\int q_{xx}(v-u, x, y) c(v, y) Y_{v,t}(y) dy| \\ & \leq \|c\|_{\infty} K \|Y_{v,t}\|_{-r} (v-u)^{-1/2} \theta(x, r), \end{aligned}$$

and from this we obtain

$$(7.5) \quad \|\Delta e^{-A(v-u)} \{c(v, \cdot) Y_{v,t}\}\|_{-r} \leq C_T (v-u)^{-1/2} \|Y_{v,t}\|_{-r}, \quad 0 \leq u \leq v \leq t \leq T,$$

with $C_T = K \|c\|_{\infty}$. As for the second term I_2 , the function inside the norm is bounded using (5.6) again as follows:

$$\begin{aligned} & |\int_{u_2}^t dv \int_R dy \int_{v-u_2}^{v-u_1} dv' q_{txx}(v', x, y) c(v, y) Y_{v,t}(y) dy| \\ & \leq \text{const.} \sup_{0 \leq v \leq t \leq T} \|Y_{v,t}\|_{-r} \theta(x, r) \int_{u_2}^t dv \int_{v-u_2}^{v-u_1} (v')^{-3/2} dv'. \end{aligned}$$

Hence we see that I_2 is also bounded by $\text{const.}(u_2-u_1)^{1/2} \times$

$\sup_{0 \leq v \leq t \leq T} \|Y_{v,t}\|_{-r}$. Two estimates on I_1 and I_2 are summed up to

show that $Q_{u,t}$ is continuous in $u \in [0, t]$ as a $\mathcal{G}_{-r}$ -valued function. Now the continuity of $Q_{u,t}$ in t and therefore the assertion $Q_{u,t} \in C(D_T, \mathcal{G}_{-r})$ follow by seeing that

$$\begin{aligned} (7.6) \quad & \|Q_{u,t_1} - Q_{u,t_2}\|_{-r} \leq C_T \int_{t_1}^{t_2} (v-u)^{-1/2} \|Y_{v,t_2}\|_{-r} dv \\ & + C_T \int_u^{t_1} (v-u)^{-1/2} \|Y_{v,t_1} - Y_{v,t_2}\|_{-r} dv, \quad 0 \leq u \leq t_1 < t_2 \leq T, \end{aligned}$$

and the RHS of (7.6) converges to 0 as $|t_1 - t_2| \rightarrow 0$.

The next task is to construct a solution of (7.3) satisfying $Y_{u,t} \in C(D_T, \mathcal{G}_{-r})$. We use a usual method of iteration; namely, we

define a sequence $\{Y_{u,t}^n \in C(D_T, \mathcal{G}_{-1})\}_{n=1}^\infty$ inductively by

$$\begin{cases} Y_{u,t}^{n+1} = e^{-A(t-u)}(\Delta\varphi) + Q_{u,t}(Y_{u,t}^n), & n \in \mathbb{N}, \\ Y_{u,t}^1 = e^{-A(t-u)}(\Delta\varphi). \end{cases}$$

Note that Lemma 6.2 proves $e^{-A(t-u)}(\Delta\varphi) \in C(D_T, \mathcal{G}_{-1})$. Then (7.5) shows

$$\|Y_{u,t}^{n+1} - Y_{u,t}^n\|_{-r} \leq C_T \int_u^t (v-u)^{-1/2} \|Y_{v,t}^n - Y_{v,t}^{n-1}\|_{-r} dv,$$

which can be used recursively to prove

$$\|Y_{u,t}^{n+1} - Y_{u,t}^n\|_{-r} \leq \{C_T^2 \pi(t-u)\}^{n/2} \sup_{0 \leq u' \leq t \leq T} \|Y_{u',t}^1\|_{-r} / \Gamma(1 + \frac{n}{2}).$$

Therefore $\{Y_{u,t}^n\}$ is a Cauchy sequence in the space $C(D_T, \mathcal{G}_{-1})$ and its limit gives a solution of (7.3).

(ii) Let $Y_{u,t}$ and $\bar{Y}_{u,t}$ be two solutions of (7.3) satisfying the condition (7.4). Then, since (7.5) gives

$$\|Y_{u,t} - \bar{Y}_{u,t}\|_{-r} \leq C_T \int_u^t (v-u)^{-1/2} \|Y_{v,t} - \bar{Y}_{v,t}\|_{-r} dv,$$

we obtain for every $n \in \mathbb{N}$:

$$(7.7) \quad \|Y_{u,t} - \bar{Y}_{u,t}\|_{-r} \leq C_T' (C_T^2 \pi)^{n/2} (t-u)^{\varepsilon-1+n/2} \Gamma(\varepsilon) / \Gamma(\varepsilon-1+n/2),$$

with a constant C_T' defined by

$$C_T' = \sup_{0 \leq u \leq t \leq T} (\|Y_{u,t}\|_{-r} + \|\bar{Y}_{u,t}\|_{-r}) (t-u)^{1-\varepsilon} < \infty.$$

We can therefore conclude $Y_{u,t} = \bar{Y}_{u,t}$ by making $n \rightarrow \infty$ in the RHS of (7.7). $\square$

Now we return to the equation (7.2).

Lemma 7.2 (i) There exists a solution of (7.2) such that $Z_{u,t}$

$\in C(D_T, \mathcal{G}_{-r}^2)$, $r > 0$, $T > 0$. The uniqueness of solutions holds in the class of functions $Z_{u,t}$ satisfying

$$(7.8) \quad \sup_{0 \leq u \leq t \leq T} \|Z_{u,t}\|_{2,-r} (t-u)^{1-\varepsilon} < \infty, \quad T > 0,$$

with some $0 < \varepsilon < 1$.

(ii) The solutions $Z_{u,t}$ of (7.2) and $Y_{u,t}$ of (7.3) are tied up by the relation: $\Delta Z_{u,t} = Y_{u,t}$.

Proof Denote by $P_{u,t} = P_{u,t}(Z)$ the second term in the RHS of (7.2) for given $Z_{u,t} \in C(D_T, \mathcal{G}_{-r}^2)$, $T > 0$. Then similar calculations for showing (7.5) prove the following estimations on the integrand of $P_{u,t}$:

$$\left\| \frac{\partial^i}{\partial x^i} e^{-A(v-u)} \{c(v, \cdot) \Delta Z_{v,t}\} \right\|_{-r} \leq C_T (v-u)^{-i/4} \|\Delta Z_{v,t}\|_{-r},$$

$$i = 0, 1, 2, \quad 0 \leq u \leq v \leq t \leq T.$$

We therefore obtain

$$(7.9) \quad \|e^{-A(v-u)} \{c(v, \cdot) \Delta Z_{v,t}\}\|_{2,-r} \leq C_T' (v-u)^{-1/2} \|Z_{v,t}\|_{2,-r},$$

with another positive constant C_T' . Based on this estimate we can complete the proof of the assertion (i) by similar arguments developed in the proof of Lemma 7.1 noting that $\varphi \in C_0^\infty(R)$ implies $e^{-A(t-u)} \varphi \in C(D_T, \mathcal{G}_{-r}^2)$. To prove the assertion (ii) we set

$$\bar{Z}_{u,t} = e^{-A(t-u)} \varphi + \int_u^t e^{-A(v-u)} \{c(v, \cdot) Y_{v,t}\} dv,$$

in which $Y_{v,t}$ is the solution of (7.3). Then we can show $\bar{Z}_{u,t} \in C(D_T, \mathcal{G}_{-r}^2)$, $T > 0$, and $\Delta \bar{Z}_{u,t} = Y_{u,t}$, since we have $\Delta e^{-A(t-u)} \varphi = e^{-A(t-u)} \Delta \varphi$. Therefore $\bar{Z}_{u,t}$ is a solution of (7.2) and this implies the conclusion from the uniqueness of solutions of (7.2). $\square$

We also consider the following integral equation for $\tilde{Z}_t = \tilde{Z}_t(x; \eta, c)$, $0 \leq t < \infty$, $\eta \in H_e$, $c \in \mathcal{B}_b$:

$$(7.10) \quad \tilde{Z}_t = e^{-tA} \eta + \int_0^t \Delta e^{-A(t-u)} \{c(u, \cdot) \tilde{Z}_u\} du, \quad t \geq 0.$$

Let $\mathcal{T}''$ be the class of all $\tilde{Z}_t(x)$ such that $\tilde{Z}_t \in C((0, \infty), \mathcal{B}_e)$ and $\sup_{0 \leq t \leq T} t^{1/8} \|\tilde{Z}_t\|_r < \infty$ for every $T, r > 0$.

Lemma 7.3 There exists a unique solution of (7.10) in the class $\mathcal{T}''$.

Proof It is a consequence of Lemma 5.3 and its proof that $e^{-tA} \eta \in C((0, \infty), \mathcal{B}_e)$ and

$$(7.11) \quad \|e^{-tA} \eta\|_r \leq \sqrt{KK_1} t^{-1/8} \|\eta\|_{2r}, \quad 0 < t \leq T.$$

Denote by $\tilde{Q}_t(\tilde{Z}_\cdot) \equiv \tilde{Q}_t(x; \tilde{Z}_\cdot)$ the second term in the RHS of (7.10) for given $\tilde{Z}_t \in \mathcal{T}''$. Then the estimate (5.6) proves for $0 \leq t_1 < t_2 \leq T$:

$$(7.12) \quad \|\tilde{Q}_{t_1}(\tilde{Z}_\cdot) - \tilde{Q}_{t_2}(\tilde{Z}_\cdot)\|_r \leq K \|c\|_\infty \sup_{0 \leq t \leq T} t^{1/8} \|\tilde{Z}_t\|_r \\ \times \left[\int_{t_1}^{t_2} u^{-1/8} (t_2 - u)^{-1/2} du + \int_0^{t_1} 2u^{-1/8} \{ (t_1 - u)^{-1/2} - (t_2 - u)^{-1/2} \} du \right],$$

from which we see $\tilde{Q}_t(\tilde{Z}_\cdot) \in C([0, \infty), \mathcal{B}_r)$. Now, for $\tilde{Z}_\cdot$ and $\tilde{Z}'_\cdot \in \mathcal{T}''$, we derive an estimate:

$$(7.13) \quad \|\tilde{Q}_t(\tilde{Z}_\cdot) - \tilde{Q}_t(\tilde{Z}'_\cdot)\|_r \\ \leq K \|c\|_\infty \int_0^t (t-u)^{-1/2} \|\tilde{Z}_u - \tilde{Z}'_u\|_r du, \quad 0 \leq t \leq T.$$

From these bounds, we can complete the proof similarly but more easily than that of Theorem 5.1, see also Lemma 5.4. $\square$

Lemma 7.4 (i) For every T, r and $C > 0$, we have

$$(7.14) \quad \sup_{0 \leq t \leq T} \sup_{\eta \in H_e; |\eta|_{2r} \leq C} \sup_{c \in \mathcal{B}_b; \|c\|_\infty \leq C} t^{1/8} \|\tilde{Z}_t(\cdot; \eta, c)\|_r < \infty.$$

(ii) For each $\eta \in H_e$ and $r, C > 0$, the family $(\tilde{Z}_t(\cdot; \eta, c); \|c\|_\infty \leq C)$ is relatively compact in the space $C((0, \infty), \mathcal{G}_r)$.

Proof From the bounds (7.11) and (7.13) with $\tilde{Z}_t' \equiv 0$, we obtain

$$z(t) \leq \sqrt{KK_1} |\eta|_{2r} + K\|c\|_\infty t^{1/8} \int_0^t (t-u)^{-1/2} u^{-1/8} z(u) du, \quad 0 \leq t \leq T,$$

for $z(t) = t^{1/8} \|\tilde{Z}_t(\cdot; \eta, c)\|_r$. This estimate implies (7.14). To prove the assertion (ii), we see from (5.6):

$$\begin{aligned} & \left\| \frac{\partial}{\partial x} \tilde{Q}_t(\cdot; \tilde{Z}_t) \right\|_r \\ & \leq K\|c\|_\infty \left\{ \sup_{0 \leq u \leq T} u^{1/8} \|\tilde{Z}_u\|_r \right\} \int_0^t u^{-1/8} (t-u)^{-3/4} du, \quad 0 \leq t \leq T, \end{aligned}$$

which proves with the help of (7.14) the equicontinuity of the family $(\tilde{Q}_t(x; \tilde{Z}_t(\cdot; \eta, c)); \|c\|_\infty \leq C)$ in x . The equicontinuity of this family in $t \in (0, \infty)$ follows from (7.12) and (7.14). We therefore see that the family $(\tilde{Z}_t(\cdot; \eta, c); \|c\|_\infty \leq C)$ is relatively compact in the space $C((0, \infty), \mathcal{G})$, since $e^{-tA}\eta \in C((0, \infty), \mathcal{G}_e)$. However, it is now easy to obtain the conclusion by noting the uniform estimate (7.14) which holds for every $r > 0$. $\square$

Finally in this paragraph, we notice the continuity of $\tilde{Z}_t(\cdot; \eta, c)$ in $\eta \in H_e$ and $c \in \mathcal{B}_b$. For this purpose, we define for $c, \bar{c} \in \mathcal{B}_b$:

$$(7.15) \quad d_1(c, \bar{c}) = \sup_{0 \leq t \leq T, x \in R} |c(t, x) - \bar{c}(t, x)| (1 + |x|)^{-1}, \quad T > 0.$$

Lemma 7.5 For every T, r and $C > 0$, there exists a positive constant M such that

$$(7.16) \quad \sup_{0 \leq t \leq T} t^{1/8} \|\tilde{Z}_t(\cdot; \eta, c) - \tilde{Z}_t(\cdot; \bar{\eta}, c)\|_r \leq M |\eta - \bar{\eta}|_{2r},$$

$$\eta, \bar{\eta} \in H_e, \quad c \in \mathcal{B}_b: \|c\|_\infty \leq C,$$

and

$$(7.17) \quad \sup_{0 \leq t \leq T} \|\tilde{Z}_t(\cdot; \eta, c) - \tilde{Z}_t(\cdot; \eta, \bar{c})\|_r \leq M d_1(c, \bar{c}),$$

$$\eta \in H_e: |\eta|_{2r} \leq C, \quad c, \bar{c} \in \mathcal{B}_b: \|c\|_\infty, \|\bar{c}\|_\infty \leq C.$$

Proof The estimate (7.16) follows from

$$\begin{aligned} \|\tilde{Z}_t(\cdot; \eta, c) - \tilde{Z}_t(\cdot; \bar{\eta}, c)\|_r &\leq \sqrt{KK_1} t^{-1/8} |\eta - \bar{\eta}|_{2r} \\ &+ K \|c\|_\infty \int_0^t (t-u)^{-1/2} \|\tilde{Z}_u(\cdot; \eta, c) - \tilde{Z}_u(\cdot; \bar{\eta}, c)\|_r du, \quad 0 \leq t \leq T. \end{aligned}$$

To prove (7.17), assume $\|c\|_\infty, \|\bar{c}\|_\infty \leq C$. Then (5.6) can be used to show

$$(7.18) \quad \begin{aligned} &\|\tilde{Z}_t(\cdot; \eta, c) - \tilde{Z}_t(\cdot; \eta, \bar{c})\|_r \\ &\leq K \int_0^t (t-u)^{-1/2} \{C \|\tilde{Z}_u(\cdot; \eta, c) - \tilde{Z}_u(\cdot; \eta, \bar{c})\|_r \\ &\quad + a d_1(c, \bar{c}) \|\tilde{Z}_u(\cdot; \eta, \bar{c})\|_r\} du, \end{aligned}$$

where $0 < r' < r$ and $a = \sup_{y \in R} \{1 + |y|\} \theta(y, r-r') < \infty$. The recursive

usage of (7.18) gives (7.17) with the help of (7.14). $\square$

7.2 Construction of fundamental solutions A function $Z_{u,t}(x, y) \equiv Z_{u,t}(x, y; c)$ is called a fundamental solution of the integral equation (7.2) if the unique solution $Z_{u,t}(x, \varphi)$ of (7.2) can be represented as follows:

$$(7.19) \quad Z_{u,t}(x, \varphi) = \int_R \varphi(y) Z_{u,t}(x, y) dy.$$

In this paragraph we first construct the fundamental solution by using E.E. Levi's method (see Eidel'man [9]) which is fairly simple in our situation and then prove its continuity in the coefficient c .

We define $\{Z_{u,t}^{(n)}(x,y) \equiv Z_{u,t}^{(n)}(x,y;c)\}_{n=0}^{\infty}$ inductively by

$$(7.20) \quad \begin{cases} Z_{u,t}^{(0)}(x,y) = q(t-u, y-x), \\ Z_{u,t}^{(n+1)}(x,y) = \int_u^t dv \int q(v-u, z-x) c(v,z) \Delta_z Z_{v,t}^{(n)}(z,y) dz, \\ n = 0, 1, 2, \dots \end{cases}$$

Here q is the function introduced in Section 5. Take a positive number L^* which is strictly less than the constant L_1 appearing in (5.2).

Lemma 7.6 An absolutely-converging series

$$(7.21) \quad Z_{u,t}(x,y) \equiv Z_{u,t}(x,y;c) = \sum_{n=0}^{\infty} Z_{u,t}^{(n)}(x,y)$$

gives a fundamental solution of (7.2) and has the following bounds:

$$(7.22) \quad \left| \frac{\partial^k}{\partial x^k} Z_{u,t}(x,y;c) \right| \leq K^*(t-u)^{-\frac{1+k}{4}} e^{-L^* \rho},$$

$$k = 0, 1, 2, 3, \quad 0 \leq u \leq t \leq T, \quad \|c\|_{\infty} \leq C,$$

for every $T, C > 0$, where $\rho = \left\{ \frac{|x-y|^4}{t-u} \right\}^{1/3}$ and K^* is a constant depending only on T and C .

Proof First we prove by using inductive method:

$$(7.23) \quad \left| \frac{\partial^k}{\partial x^k} Z_{u,t}^{(n)}(x,y) \right| \leq \frac{a_{1,k} a_2^n}{\Gamma\left(\frac{3}{4} - \frac{k}{4} + \frac{n}{2}\right)} (t-u)^{-\frac{1+k}{4} + \frac{n}{2}} e^{-L^* \rho},$$

$$0 \leq u \leq t \leq T,$$

for $(n,k) = (0,2)$ and for $n \in \mathbb{N}$, $k = 0, 1, 2, 3$. Here

$$a_{1,k} = K_1 \Gamma(1/4) \Gamma(1-k/4) / \Gamma(1/2),$$

$$a_2 = K_1 \Gamma(1/2) \|c\|_{\infty} F,$$

$$F = \int_R e^{-(L_1 - L^*)|z|^{4/3}} dz,$$

and K_1 is the constant appearing in (5.2). It is easy to see that (7.23) holds for $(n,k) = (0,2)$. Therefore the induction can be completed if we can prove (7.23) for $n+1$ and $k = 0,1,2,3$ assuming that it is true for $n \in \mathbb{N}$ and $k = 2$. Indeed from this assumption we have

$$\begin{aligned}
 (7.24) \quad & \left| \frac{\partial^k}{\partial x^k} Z_{u,t}^{(n+1)}(x,y) \right| \\
 & \leq \frac{K_1 a_{1,2} a_2^n \|c\|_\infty}{\Gamma(\frac{1}{4} + \frac{n}{2})} \int_u^t dv (v-u)^{-\frac{1+k}{4}} (t-v)^{-\frac{3}{4} + \frac{n}{2}} \\
 & \quad \times \int \exp \left\{ -L_1 \left(\frac{|x-z|^4}{v-u} \right)^{1/3} - L^* \left(\frac{|z-y|^4}{t-v} \right)^{1/3} \right\} dz \\
 & \leq \frac{K_1 a_{1,2} a_2^n \|c\|_\infty}{\Gamma(\frac{1}{4} + \frac{n}{2})} F e^{-L^* \rho} \int_u^t dv (v-u)^{-\frac{k}{4}} (t-v)^{-\frac{3}{4} + \frac{n}{2}} dv,
 \end{aligned}$$

which coincides with the RHS of (7.23) with n replaced by $n+1$.

We have used the following estimate in Eidel'man [9] to derive

(7.24):

$$(7.25) \quad \left(\frac{|x-z|^4}{v-u} \right)^{1/3} + \left(\frac{|z-y|^4}{t-v} \right)^{1/3} \geq \rho.$$

Therefore we obtain (7.23), from which we see that the series (7.21) itself and its derivatives in x up to the third order converge absolutely and the function $Z_{u,t}(x,y)$ has the bounds (7.22). Now it is easy to see that $Z_{u,t}(x,y)$ constructed in this manner is the fundamental solution of (7.2). Note the estimate (7.22) and use the uniqueness result in Lemma 7.2 (take $\varepsilon = 1/2$). $\square$

Next we prove the continuity property of $Z_{u,t}(x,y;c)$ in c .

Lemma 7.7 The following estimates hold:

$$(7.26) \quad \left| \frac{\partial^k}{\partial x^k} Z_{u,t}(x,y;c) - \frac{\partial^k}{\partial x^k} Z_{u,t}(x,y;\bar{c}) \right| \\ \leq K^{**} d_1(c, \bar{c}) \{1+|x|+|y|\} (t-u)^{-\frac{1+k}{4}} e^{-L^* \rho}, \\ k = 0, 1, 2, 3, \quad 0 \leq u \leq t \leq T; \quad \|c\|_\infty, \|\bar{c}\|_\infty \leq C,$$

where the constant K^{**} depends only on T and C .

Proof These estimates follow immediately if we can show

$$(7.27) \quad \left| \frac{\partial^k}{\partial x^k} Z_{u,t}^{(n)}(x,y;c) - \frac{\partial^k}{\partial x^k} Z_{u,t}^{(n)}(x,y;\bar{c}) \right| \\ \leq \frac{d_1(c, \bar{c}) a_{1,k} \tilde{a}_2^n C^{n-1}}{\Gamma(\frac{3}{4} - \frac{k}{4} + \frac{n}{2})} \{1+|x|+|y|\} (t-u)^{-\frac{1+k}{4} + \frac{n}{2}} e^{-L^* \rho},$$

for $n \in \mathbb{N}$ and $k = 0, 1, 2, 3$. Here $\tilde{a}_2 = 2 K_1 \Gamma(1/2) F_T$ and

$$F_T = F + T^{1/4} \int |z| \exp\{-(L_1 - L^*)|z|^{4/3}\} dz.$$

To this end, we see

$$\text{the LHS of (7.27)} \leq I_1^{(n)} + I_2^{(n)}, \quad n \in \mathbb{N},$$

where

$$I_1^{(n)} = \left| \int_u^t dv \int \frac{\partial^k}{\partial x^k} q(v-u, z-x) \{c(v,z) - \bar{c}(v,z)\} \Delta_z Z_{v,t}^{(n-1)}(z,y;\bar{c}) dz \right|, \\ I_2^{(n)} = \left| \int_u^t dv \int \frac{\partial^k}{\partial x^k} q(v-u, z-x) c(v,z) \Delta_z \{Z_{v,t}^{(n-1)}(z,y;c) - Z_{v,t}^{(n-1)}(z,y;\bar{c})\} dz \right|.$$

Using (7.23) and (7.25), the first term $I_1^{(n)}$ can be bounded by a half of the RHS of (7.27). Especially we obtain (7.27) with $n = 1$ since $I_2^{(1)} = 0$. Now we can prove (7.27) by the induction. Indeed assuming that (7.27) is true for $n-1$, we can show that $I_2^{(n)}$ is also bounded by a half of the RHS of (7.27) and this completes the

proof of (7.27). $\square$

Remark 7.1 Assume that the coefficient c is smooth enough. Then $Z_{u,t}(x,\varphi)$ actually solves the backward equation (7.1). In addition, the function $Z_{u,t}(x,y)$ is also a fundamental solution of the corresponding forward equation:

$$(7.28) \quad \begin{cases} \frac{\partial}{\partial t} Z_{u,t} = \mathcal{L}_{t,y}^* Z_{u,t}, & t \in [u, \infty), \\ Z_{u,u} = \varphi \in C_0^\infty(R), \end{cases}$$

where $\mathcal{L}_{t,y}^* = -A + \Delta_y \{c(t,y) \cdot\}$. Especially the function

$$(7.29) \quad Z_{u,t}(\varphi, y) = \int \varphi(x) Z_{u,t}(x, y) dx$$

gives the solution of (7.28). See Eidel'man [9].

§8 A formula of integration by parts

The solution S_t of the SPDE (0.5) determines a Markov process on the space H_e . The (formal) generator $\mathcal{G}$ of this process is given by

$$(8.1) \quad \begin{aligned} \mathcal{G}\Psi(S) = & \sum_{i=1}^k \frac{\partial \Psi}{\partial \alpha_i} (\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle) \{ -\langle S, \Delta^2 \varphi_i \rangle + \langle U'(S(\cdot)), \Delta \varphi_i \rangle \} \\ & + \sum_{i,j=1}^k \frac{\partial^2 \Psi}{\partial \alpha_i \partial \alpha_j} (\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle) \langle -\Delta \varphi_i, \varphi_j \rangle, \quad S \in H_e, \end{aligned}$$

for $\Psi \in \mathcal{D}$ having the form (2.3); i.e., $\mathcal{G}$ is the generator in the sense that

$$(8.2) \quad \Psi(S_t) - \int_0^t \mathcal{G}\Psi(S_u) du$$

is a martingale for every $\Psi \in \mathcal{D}$. We denote $E_S[\Psi(S_t)]$, $S \in H_e$, $\Psi \in$

$\mathcal{D}$, by $T_t\Psi(S)$ or simply by $\Psi_t(S)$, where $E_S[\cdot]$ stands for an expectation with respect to the probability distribution of the process S_t starting from S . Assume a chemical-potential profile function $\lambda(\cdot) \in \Lambda$ is given and let $\mu_{\lambda}(\cdot)$ be the $(U, \lambda(\cdot))$ -Gibbs distribution being constructed in Section 3.

Definition (see Hille and Phillips [18]) A real valued function Φ on the space H_e is called Fréchet differentiable at $S \in H_e$ if $\Phi(S+\delta\eta)$ is differentiable at $\delta = 0$ for every $\eta \in H_e$ and the derivative has an expression:

$$\left. \frac{d}{d\delta} \Phi(S+\delta\eta) \right|_{\delta=0} = \langle D\Phi(\cdot, S), \eta \rangle, \quad \eta \in H_e,$$

with some $D\Phi(\cdot, S) \in H_e^*$. We simply call Φ Fréchet differentiable on H_e if it is Fréchet differentiable at every $S \in H_e$.

The purpose of the present section is to prove the following formula.

Theorem 8.1 (i) For every $t > 0$ and $\Psi \in \mathcal{D}$, Ψ_t is Fréchet differentiable on H_e and the following formula holds:

$$(8.3) \quad E^{\mu_{\lambda}(\cdot)}[\mathcal{G}\Psi(S_t)] = E^{\mu_{\lambda}(\cdot)}[\langle \Delta\lambda(\cdot), D\Psi_t(\cdot, S) \rangle],$$

where the LHS is an expectation with respect to the distribution of the process S_t having initial distribution $\mu_{\lambda}(\cdot)$ and the RHS is an integration with respect to $\mu_{\lambda}(\cdot)(dS)$ over the space H_e .

(ii) The Fréchet derivative $D\Psi_t(x, S)$ of Ψ_t can be expressed explicitly as follows:

$$(8.4) \quad D\Psi_t(x, S) = \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i} (\langle S_t, \varphi_1 \rangle, \dots, \langle S_t, \varphi_k \rangle) Z_{0,t}(x, \varphi_i; S.) \right].$$

Here we denote by $Z_{0,t}(x, \varphi; S_.)$ the solution $Z_{0,t}(x, \varphi; V''(S_.))$ of the integral equation (7.2) with $c(u, x) = V''(S_u(x))$.

The proof of the theorem will be completed after the following three main steps: The first step is to prove the corresponding formula for the finite-dimensional process $S_t^{(N)} \in L_{\ell}^2 = L^2([- \ell, \ell])$ which was constructed in Section 6. Then it is derived the formula for the solution S_t^{ℓ} of the SPDE (6.1) by taking the limit of $N \rightarrow \infty$ and finally we take the limit of $\ell \rightarrow \infty$.

8.1 A formula for $S_t^{(N)}$ The N -dimensional process $S_t^{(N)} = \sum_{n=1}^N a_n^{(N)}(t) \times e_n \in L_{\ell}^2$ was constructed by solving the SDE (6.10). We shall fix $N \in \mathbb{N}$ throughout this paragraph. The generator of the diffusion process $a_t = (a_n^{(N)}(t))_{n=1}^N \in \mathbb{R}^N$ is given by

$$\mathcal{A} \equiv \mathcal{A}^{(N)} = \sum_{n=1}^N \left[\kappa_n \frac{\partial^2}{\partial a_n^2} - \left(\kappa_n^2 a_n + \kappa_n \langle U' \left(\sum_{m=1}^N a_m e_m(\cdot) \right), e_n(\cdot) \rangle_{\ell} \right) \frac{\partial}{\partial a_n} \right].$$

Define a finite measure $\tilde{\mu}^{(N)}$ on $\mathbb{R}^N$ by

$$\tilde{\mu}^{(N)}(da) = \exp \left(- \int_{-\ell}^{\ell} U \left(\sum_{n=1}^N a_n e_n(x) \right) dx - \frac{1}{2} \sum_{n=1}^N \kappa_n a_n^2 \right) da, \quad da = \prod_{n=1}^N da_n,$$

then $\tilde{\mu}^{(N)}$ is a reversible measure for the process a_t :

$$\begin{aligned} (8.5) \quad \int \mathcal{A} f(a) g(a) \tilde{\mu}^{(N)}(da) &= \int f(a) \mathcal{A} g(a) \tilde{\mu}^{(N)}(da) \\ &= - \sum_{n=1}^N \kappa_n \int \frac{\partial f}{\partial a_n} \frac{\partial g}{\partial a_n} \tilde{\mu}^{(N)}(da), \end{aligned}$$

for every $f, g \in C_b^2(\mathbb{R}^N)$. This formula is easily shown by using integration by parts. We set

$$f_t(a) \equiv \tilde{T}_t^{(N)} f(a) = E_a[f(a_t)], \quad a \in \mathbb{R}^N, \quad t \geq 0, \quad f \in C_b(\mathbb{R}^N),$$

the expectation with respect to the distribution of a_t starting

from a and

$$\tilde{\mu}_{\lambda(\cdot)}^{(N)}(da) = \exp\left\{\sum_{n=1}^N \langle \lambda, e_n \rangle_{\ell} a_n\right\} \tilde{\mu}^{(N)}(da),$$

where $\lambda(\cdot) \in \Lambda$ is an already given profile function.

Lemma 8.1
$$E_{\tilde{\mu}_{\lambda(\cdot)}^{(N)}}[\mathcal{A}^{(N)} f(a_t)] = E_{\tilde{\mu}_{\lambda(\cdot)}^{(N)}}\left[\sum_{n=1}^N \langle \lambda, \Delta e_n \rangle_{\ell} \frac{\partial f_t}{\partial a_n}(a)\right],$$

$$f \in C_0^2(\mathbb{R}^N).$$

Proof We note that $\tilde{T}_t^{(N)}$ is a strongly continuous Feller semigroup on a Banach space $C_{\infty}(\mathbb{R}^N)$, the space consisting of all $f \in C(\mathbb{R}^N)$ satisfying $\lim_{|x| \rightarrow \infty} f(x) = 0$, with the usual supremum-norm and the domain of its generator contains $C_0^2(\mathbb{R}^N)$. Therefore we have $\tilde{T}_t^{(N)} \mathcal{A}^{(N)} f = \mathcal{A}^{(N)} \tilde{T}_t^{(N)} f$, $f \in C_0^2(\mathbb{R}^N)$, see Ethier and Kurtz [10] or Tanabe [40] for example. The conclusion now follows by taking $f = f_t$ and $g = \exp\left\{\sum_{n=1}^N \langle \lambda, e_n \rangle_{\ell} a_n\right\}$ in (8.5). $\square$

Define a mapping $I_N: \mathbb{R}^N \rightarrow \pi_N L_{\ell}^2$ by $I_N a = \sum_{n=1}^N a_n e_n$, $a = \{a_n\}_{n=1}^N$, and set $({}_N I \Psi)(a) = \Psi(I_N a)$, $a \in \mathbb{R}^N$, for functions Ψ on $\pi_N L_{\ell}^2$ (or L_{ℓ}^2). Here π_N is the orthogonal projection of the space L_{ℓ}^2 defined in the paragraph 6.2. Let $\mathcal{D}_{\ell}$ be the class of tame functions on L_{ℓ}^2 , which was introduced in Section 2. We introduce an operator $\mathcal{G}^{(N)}$ by

$$\begin{aligned} \mathcal{G}^{(N)} \Psi(S) &= \sum_{i=1}^k \frac{\partial \Psi}{\partial \alpha_i}(\langle S, \varphi_1 \rangle_{\ell}, \dots, \langle S, \varphi_k \rangle_{\ell}) \\ &\quad \times \{-\langle S, \Delta^2 \pi_N \varphi_i \rangle_{\ell} + \langle U'(S(\cdot)), \Delta \pi_N \varphi_i \rangle_{\ell}\} \\ &+ \sum_{i,j=1}^k \frac{\partial^2 \Psi}{\partial \alpha_i \partial \alpha_j}(\langle S, \varphi_1 \rangle_{\ell}, \dots, \langle S, \varphi_k \rangle_{\ell}) \langle -\Delta \pi_N \varphi_i, \varphi_j \rangle_{\ell}, \quad S \in L_{\ell}^2, \end{aligned}$$

for $\Psi \in \mathcal{D}_{\ell}$ having the form (2.3). Note that the supports of φ_i ,

$i = 1, 2, \dots, k$, are included in the interval $(-\ell, \ell)$. The operators $\mathcal{G}^{(N)}$ and $\mathcal{A}^{(N)}$ are linked by the following relation:

$$(8.6) \quad {}_N I \mathcal{G}^{(N)} \Psi = \mathcal{A}^{(N)} {}_N I \Psi, \quad \Psi \in \mathcal{D}_\ell.$$

Let $\mu_{\lambda(\cdot)}^{(N)} = \tilde{\mu}_{\lambda(\cdot)}^{(N)} \circ I_N^{-1}$ be an image measure on $\pi_N L_\ell^2$ of $\tilde{\mu}_{\lambda(\cdot)}^{(N)}$ under the mapping I_N . The derivative $D^{(N)}\Phi(x, S)$, $x \in [-\ell, \ell]$, $S \in \pi_N L_\ell^2$, of a function $\Phi = \Phi(S)$ on the space $\pi_N L_\ell^2$ (or L_ℓ^2) is defined by

$$D^{(N)}\Phi(x, S) = \sum_{n=1}^N e_n(x) \frac{\partial({}_N I \Phi)}{\partial a_n} (I_N^{-1} S),$$

when the RHS exists, where $I_N^{-1}: \pi_N L_\ell^2 \rightarrow \mathbb{R}^N$ is an inverse mapping of I_N . We set

$$\Psi_t^{(N)}(S) \equiv T_t^{(N)} \Psi(S) = E_S[\Psi(S_t^{(N)})], \quad S \in \pi_N L_\ell^2, \quad \Psi \in \mathcal{D}_\ell,$$

which will be sometimes considered as a function on the space L_ℓ^2 by putting $\pi_N S$ instead of S in the RHS. Then we have the following proposition as an immediate consequence of Lemma 8.1 and (8.6).

Proposition 8.1 For every $\Psi \in \mathcal{D}_\ell$,

$$(8.7) \quad E^{\mu_{\lambda(\cdot)}^{(N)}}[\mathcal{G}^{(N)} \Psi(S_t^{(N)})] = E^{\mu_{\lambda(\cdot)}^{(N)}}[\langle \lambda(\cdot), \Delta D^{(N)} \Psi_t^{(N)}(\cdot, S) \rangle_\ell].$$

Before concluding this paragraph we give useful representations of $\mu_{\lambda(\cdot)}^{(N)}$ and $\Delta D^{(N)} \Psi_t^{(N)}(x, S)$, respectively.

Lemma 8.2 With some positive constant c_N , we have

$$\mu_{\lambda(\cdot)}^{(N)}(dS) = c_N \exp\{\langle \lambda, S \rangle_\ell - \langle U(S), 1 \rangle_\ell\} (\mu \circ \pi_N^{-1})(dS), \quad S \in \pi_N L_\ell^2,$$

where $\mu = \mu_{-\ell, 0; \ell, 0}$ is the probability distribution of the pinned Brownian motion as in Section 3.

Proof The conclusion follows from the Wiener's representation of the Brownian motion: μ is the distribution of $\tilde{S}(x; \{a_n\}) = \sum_{n=1}^{\infty} a_n e_n(x)$, $x \in [-l, l]$, which is realized on a probability space

$$(R^{\infty}, \prod_{n=1}^{\infty} \sqrt{\kappa_n/2\pi} \exp\{-\kappa_n a_n^2/2\} da_n). \quad \square$$

Consider the following backward integral equation for $Y_{u,t}^{(N)} = Y_{u,t}^{(N)}(x, \varphi; S_.)$, $0 \leq u \leq t < \infty$, with given $\varphi \in C_0^{\infty}(-l, l)$ and $S_. \in C([0, \infty), L_{\ell}^2)$:

$$(8.8) \quad Y_{u,t}^{(N)} = e^{-A^{\ell}(t-u)} \pi_N(\Delta\varphi) + \int_u^t \Delta e^{-A^{\ell}(v-u)} \pi_N\{V''(S_v) Y_{v,t}^{(N)}\} dv,$$

where A^{ℓ} is the operator on L_{ℓ}^2 introduced in Section 6.

Lemma 8.3 (i) The integral equation (8.8) has a unique solution satisfying $Y_{.,t}^{(N)} \in C([0, t], \pi_N L_{\ell}^2)$ for each $t \geq 0$.

(ii) For every $\Psi \in \mathcal{D}_{\ell}$,

$$(8.9) \quad \begin{aligned} \Delta D^{(N)} \Psi_t^{(N)}(x, S) \\ = \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i} (\langle S_t^{(N)}, \varphi_1 \rangle_{\ell}, \dots, \langle S_t^{(N)}, \varphi_k \rangle_{\ell}) Y_{0,t}^{(N)}(x, \varphi_i; S_.)^{(N)} \right]. \end{aligned}$$

Proof (i) Consider the following system of linear temporally inhomogeneous backward ODE with continuous coefficients for $\{Y_{u,t;n}^{(N)}\}_{n=1}^N \in R^N$ and each $t > 0$:

$$(8.10) \quad \begin{cases} \frac{\partial}{\partial u} Y_{u,t;n}^{(N)} = \theta_n Y_{u,t;n}^{(N)} + \kappa_n \sum_{m=1}^N Y_{u,t;m}^{(N)} \langle V''(S_u) e_m, e_n \rangle_{\ell}, & 0 \leq u \leq t, \\ Y_{t,t;n}^{(N)} = \langle \Delta\varphi, e_n \rangle_{\ell}, & 1 \leq n \leq N. \end{cases}$$

Then the equation (8.8) is equivalent to (8.10) through the relation:

$$Y_{u,t}^{(N)} = \sum_{n=1}^N Y_{u,t;n}^{(N)} e_n. \quad \text{However it is easy to see the existence and}$$

uniqueness of solutions of (8.10).

(ii) Let us denote the solution of SDE (6.10) starting at time $u \geq 0$ (instead of time 0) from the point $a \in \mathbb{R}^N$ by $a_{u,t}(a) = \{a_{u,t;n}(a)\} \equiv \{a_{u,t;n}^{(N)}(a)\}_{n=1}^N$, $0 \leq u \leq t < \infty$. Then its derivatives $\{b_{u,t;n}^m = \partial a_{u,t;n}(a) / \partial a_m\}_{1 \leq n, m \leq N}$ in a satisfy

$$\begin{cases} \frac{\partial}{\partial t} b_{u,t;n}^m = -\theta_n b_{u,t;n}^m - \sum_{n'=1}^N \kappa_{n'} \langle V''(\sum_{j=1}^N a_{u,t;j} e_j) e_{n'}, e_n \rangle b_{u,t;n'}^m, \\ b_{u,u;n}^m = \delta_{n,m}, \end{cases}$$

and therefore the corresponding backward equation is

$$\begin{cases} \frac{\partial}{\partial u} b_{u,t;n}^m = \theta_m b_{u,t;n}^m + \sum_{m'=1}^N \kappa_{m'} \langle V''(\sum_{j=1}^N a_{u,t;j} e_j) e_m, e_{m'} \rangle b_{u,t;n}^{m'}, \\ b_{t,t;n}^m = \delta_{n,m}. \end{cases}$$

Put $\bar{Y}_{u,t;n}(\varphi) \equiv \bar{Y}_{u,t;n}(a; \varphi) = -\kappa_n \sum_{n'=1}^N b_{u,t;n'}^n \langle e_{n'}, \varphi \rangle$, then $\bar{Y}$ satisfies (8.10). Hence the uniqueness of solutions of (8.10) implies $\bar{Y}_{u,t;n} = Y_{u,t;n}^{(N)}$ with $S_{\cdot} = S_{\cdot}^{(N)}$. This proves (8.9) since it is easy to see

$$\begin{aligned} & \Delta D^{(N)} \Psi_t^{(N)}(x, S) \\ &= \sum_{n=1}^N e_n(x) \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i} (\langle S_t^{(N)}, \varphi_1 \rangle_{\ell}, \dots, \langle S_t^{(N)}, \varphi_k \rangle_{\ell}) \bar{Y}_{0,t;n} (I_N^{-1} S; \varphi_i) \right]. \end{aligned}$$

□

8.2 A formula for S_t^{ℓ} Let $\varphi \in C_0^{\infty}(-\ell, \ell)$ and $S_{\cdot} = S_t(x) \in \mathcal{B}^{\ell} =$

$\mathcal{B}([0, \infty) \times [-\ell, \ell])$, the class of all measurable functions on

$[0, \infty) \times [-\ell, \ell]$, be given and consider the following backward integral

equation for $Y_{u,t}^{\ell} = Y_{u,t}^{\ell}(x, \varphi; S_{\cdot})$, $0 \leq u \leq t < \infty$, in the space $\mathcal{E}_{-r, \ell}$,

$r > 0$:

$$(8.11) \quad Y_{u,t}^\ell = e^{-A^\ell(t-u)}(\Delta\varphi) + \int_u^t \Delta e^{-A^\ell(v-u)} \{V''(S_v)Y_{v,t}^\ell\} dv, \quad 0 \leq u \leq t$$

See Lemma 8.6 below for the existence and uniqueness result for this integral equation. We define $\Delta\Psi_t^\ell(x, S)$, $x \in [-\ell, \ell]$, $S \in L_2^\ell$, for $\Psi \in \mathcal{D}_\ell$ by

$$(8.12) \quad \Delta\Psi_t^\ell(x, S) = \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i} (\langle S_t^\ell, \varphi_1 \rangle_\ell, \dots, \langle S_t^\ell, \varphi_k \rangle_\ell) Y_{0,t}^\ell(x, \varphi_i; S_\cdot^\ell) \right],$$

the expectation with respect to the distribution of the process $S_\cdot^\ell = \{S_t^\ell; t \geq 0\}$ which is the solution of (6.1) with an initial condition $S_0^\ell = S$. We shall prove the following proposition.

Proposition 8.2 For every $\Psi \in \mathcal{D}_\ell$, we have

$$(8.13) \quad E_{\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)} [\mathcal{G}\Psi(S_t^\ell)] = E_{\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)} [\langle \lambda(\cdot), \Delta\Psi_t^\ell(\cdot, S) \rangle_\ell].$$

The LHS of (8.13) is the expectation with respect to the distribution of S_t^ℓ having initial distribution $\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)$, the local specification introduced in Section 3, and $\mathcal{G}\Psi(S)$ is defined by replacing $\langle \cdot, \cdot \rangle$ with $\langle \cdot, \cdot \rangle_\ell$ in (8.1). The proof of the proposition will be completed by taking the limit $N \rightarrow \infty$ of the both sides of (8.7). In this paragraph we denote the norm of the space L_ℓ^2 simply by $\|\cdot\|$.

Lemma 8.4 We have the following estimates for each $\Psi \in \mathcal{D}_\ell$ with a sequence $\{\beta_N = \beta_N(\Psi) \downarrow 0\}_{N=1}^\infty$ and a positive constant $C = C(\Psi)$:

$$(8.14) \quad |\mathcal{G}^{(N)}\Psi(S) - \mathcal{G}\Psi(S)| \leq \beta_N(1 + \|S\|), \quad S \in L_\ell^2,$$

$$(8.15) \quad |\mathcal{G}\Psi(S)| \leq C(1 + \|S\|), \quad S \in L_\ell^2,$$

$$(8.16) \quad |\mathcal{G}\Psi(S_1) - \mathcal{G}\Psi(S_2)| \leq C(1 + \|S_1\| + \|S_2\|)\|S_1 - S_2\|, \quad S_1, S_2 \in L_\ell^2.$$

This lemma can be shown by simple calculations so that we omit the proof. Now we can prove the convergence of the LHS of (8.7).

Lemma 8.5 For every $\Psi \in \mathcal{D}_\ell$, we have

$$(8.17) \quad \lim_{N \rightarrow \infty} c_N^{-1} E^{\mu_{\lambda(\cdot)}^{(N)}} [\mathcal{G}^{(N)} \Psi(S_t^{(N)})] = \Xi^\ell E^{\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)} [\mathcal{G} \Psi(S_t^\ell)],$$

where Ξ^ℓ is defined by

$$\Xi^\ell = E^\mu [\exp\{-\int_{-\ell}^\ell U(z, S(z); \lambda(\cdot)) dz\}] , \quad \mu = \mu_{-\ell, 0; \ell, 0} .$$

Proof First we note that Theorem 6.2 combined with Lemma 8.4 proves

$$(8.18) \quad \lim_{N \rightarrow \infty} E_{\pi_N S} [\mathcal{G}^{(N)} \Psi(S_t^{(N)})] = E_S [\mathcal{G} \Psi(S_t^\ell)] .$$

for every $S \in L_\ell^2$ and $\Psi \in \mathcal{D}_\ell$. Therefore we can complete the proof, since Lemma 8.2 shows

$$\begin{aligned} & c_N^{-1} E^{\mu_{\lambda(\cdot)}^{(N)}} [\mathcal{G}^{(N)} \Psi(S_t^{(N)})] \\ &= \int_{\pi_N L_\ell^2} E_S [\mathcal{G}^{(N)} \Psi(S_t^{(N)})] \exp\{-\int_{-\ell}^\ell U(z, S(z); \lambda(\cdot)) dz\} \mu \circ \pi_N^{-1}(dS) \\ &= \int_{L_\ell^2} E_{\pi_N S} [\mathcal{G}^{(N)} \Psi(S_t^{(N)})] \exp\{-\int_{-\ell}^\ell U(z, \pi_N S(z); \lambda(\cdot)) dz\} \mu(dS), \end{aligned}$$

and this converges to the RHS of (8.17) by using Lebesgue's dominated convergence theorem with the help of (8.18). $\square$

The second task is to show the convergence of the RHS of (8.7). We discuss the integral equation (8.11).

Lemma 8.6 (i) For every $r > 0$ and $T > 0$, there exists a solution of (8.11) satisfying $y_{u, t}^\ell \in C(D_T, \mathcal{G}_{-r, \ell})$.

(ii) The uniqueness of solutions of (8.11) holds in the class of measurable functions $y_{u, t}^\ell$ satisfying

$$\sup_{0 \leq u \leq t \leq T} \|Y_{u,t}^\ell\|_{-r} (t-u)^{1-\varepsilon} < \infty, \quad T > 0,$$

with some $0 < \varepsilon < 1$.

(iii) The following uniform bound holds for every $\varphi \in C_0^\infty(-\bar{\ell}, \bar{\ell})$, $\bar{\ell} \in \mathbb{N}$:

$$(8.19) \quad \sup\{\|Y_{u,t}^\ell(\cdot, \varphi; S)\|_{-r}; \ell \geq \bar{\ell}, (u,t) \in D_T, S \in \mathcal{S}^\ell\} < \infty.$$

Here $\|\cdot\|_{-r}$ denotes the norm of the space $\mathcal{B}_{-r,\ell}$, see Section 6.

Proof To prove (8.19) we notice a uniform estimate:

$$(8.20) \quad \|\Delta e^{-A^\ell(v-u)} \{V''(S_v) Y_{v,t}^\ell\}\|_{-r} \leq C_T (v-u)^{-1/2} \|Y_{v,t}^\ell\|_{-r}, \quad 0 \leq v \leq t \leq T,$$

with a constant C_T which is independent of ℓ . Indeed this estimate can be derived similarly to (7.5) using (6.6). Therefore we get from Lemma 6.2-(ii) (see also its proof):

$$\|Y_{u,t}^\ell\|_{-r} \leq M e^{\delta T} \|\Delta \varphi\|_{-r} + C_T \int_u^t (v-u)^{-1/2} \|Y_{v,t}^\ell\|_{-r} dv, \quad 0 \leq u \leq t \leq T.$$

This inequality proves (8.19). The assertions (i) and (ii) are shown similarly to the proof of Lemma 7.1. Actually we can use, for example, (8.20) instead of (7.5) and other estimations also can be done in a quite parallel manner. $\square$

Every solution of (8.11) satisfies $Y_{u,t}^\ell(\pm \ell) = 0$ and therefore $Y_{u,t}^\ell \in C(D_T, \mathcal{B}_{-r,\ell})$. We prepare the following estimates on $\{Y_{u,t}^\ell\}$:

Lemma 8.7 For every $T > 0$, there exists a positive constant C_T such that

$$(8.21) \quad \|Y_{u,t}^\ell\| \leq C_T \|\Delta \varphi\|,$$

$$(8.22) \quad \|Y_{u,t}^\ell\|_\infty \leq C_T (1 + (t-u)^{-1/8}) \|\Delta \varphi\|, \quad 0 \leq u \leq t \leq T.$$

Proof Since the operator norms of $e^{-A^\ell t}$ and $\Delta e^{-A^\ell t}$ on the space L_ℓ^2 are bounded by 1 and $\sup_{\kappa > 0} \kappa \exp\{-t(\kappa^2 + \gamma \kappa)\} (\leq 1/\sqrt{2t})$

respectively, from the equation (8.11) we have

$$\|Y_{u,t}^\ell\| \leq \|\Delta\varphi\| + \text{const.} \int_u^t (v-u)^{-1/2} \|Y_{v,t}^\ell\| dv ,$$

which can be used recursively to obtain

$$\|Y_{u,t}^\ell\| \leq \|\Delta\varphi\| \sum_{i=0}^{\infty} \{(\text{const.})^2 \pi(t-u)\}^{i/2} \Gamma^{-1}(1+i/2) .$$

This proves (8.21). To show (8.22), we note the following two

bounds which can be derived by using the Fourier series expansions of $e^{-A^\ell t} Y$ and $\Delta e^{-A^\ell t} Y$, respectively:

$$\|e^{-A^\ell t} Y\|_\infty \leq \text{const.} \ t^{-1/8} \|Y\| ,$$

$$\|\Delta e^{-A^\ell t} Y\|_\infty \leq \text{const.} \ t^{-5/8} \|Y\| , \quad Y \in L_\ell^2 ,$$

with const. independent of t and $\ell \in \mathbb{N}$. Due to these estimates and using the equation (8.11), we see that $\|Y_{u,t}^\ell\|_\infty$ is bounded from above by

$$\text{const.} (t-u)^{-1/8} \|\Delta\varphi\| + \text{const.} \int_u^t (v-u)^{-5/8} \gamma_0 \|Y_{v,t}^\ell\| dv .$$

This implies (8.22) with the help of (8.21). $\square$

To make initial values clear, we shall denote by $S_t^{(N)}(S)$ and $S_t^\ell(S)$ the processes determined by (6.10) respectively (6.1) such that $S_0^{(N)}(S) = S$ and $S_0^\ell(S) = S$.

Lemma 8.8 For every $0 \leq u \leq t < \infty$, $\varphi \in C_0^\infty(-\ell, \ell)$ and $S \in L_\ell^2$, $Y_{u,t}^{(N)} = Y_{u,t}^{(N)}(\cdot, \varphi; S^{(N)}(\pi_N S))$ converges to $Y_{u,t}^\ell = Y_{u,t}^\ell(\cdot, \varphi; S^\ell(S))$ as $N \rightarrow \infty$ in the following sense:

$$(8.23) \quad \lim_{N \rightarrow \infty} E[\|Y_{u,t}^{(N)} - Y_{u,t}^\ell\|^2] = 0 .$$

Proof By the equations (8.8) and (8.11) we have

$$\begin{aligned}
& \|Y_{u,t}^{(N)} - Y_{u,t}^{\ell}\| \leq \|(\pi_N - I)\Delta\varphi\| \\
& + \int_u^t \|\Delta e^{-A^{\ell}(v-u)} \pi_N \{V''(S_v^{(N)}) (Y_{v,t}^{(N)} - Y_{v,t}^{\ell})\}\| dv \\
& + \int_u^t \|\Delta e^{-A^{\ell}(v-u)} \pi_N [\{V''(S_v^{(N)}) - V''(S_v^{\ell})\} Y_{v,t}^{\ell}]\| dv \\
& + \int_u^t \|\Delta e^{-A^{\ell}(v-u)} (-\Delta)^{\beta} (-\Delta)^{-\beta} (I - \pi_N) \{V''(S_v^{\ell}) Y_{v,t}^{\ell}\}\| dv \\
& \equiv I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

for every $0 < \beta < 1$. The RHS can be estimated further as follows:

$$I_2 \leq \text{const.} \int_u^t (v-u)^{-1/2} \|Y_{v,t}^{(N)} - Y_{v,t}^{\ell}\| dv,$$

$$I_3 \leq \text{const.} \|\Delta\varphi\| \int_u^t (v-u)^{-1/2} \{1+(t-v)^{-1/8}\} \|S_v^{(N)} - S_v^{\ell}\| dv,$$

and

$$I_4 \leq \text{const.} \kappa_N^{-\beta} \|\Delta\varphi\| \int_u^t (v-u)^{-(1+\beta)/2} dv, \quad 0 \leq u \leq t \leq T.$$

We have used (8.22) to derive the bound on I_3 . The bound on I_4 follows from (8.21) with an observation that the operator norms of $(-\Delta)^{1+\beta} e^{-A^{\ell}t}$ and $(-\Delta)^{-\beta} (I - \pi_N): L_{\ell}^2 \rightarrow L_{\ell}^2$ are bounded by $\text{const.} t^{-(1+\beta)/2}$ and $\kappa_N^{-\beta}$, respectively. Using Schwarz's inequality we have

$$\{I_2\}^2 \leq (\text{const.})^2 2\sqrt{t-u} \int_u^t (v-u)^{-1/2} \|Y_{v,t}^{(N)} - Y_{v,t}^{\ell}\|^2 dv,$$

and also

$$\begin{aligned}
(8.24) \quad E[\{I_3\}^2] & \leq (\text{const.})^2 \|\Delta\varphi\|^2 \int_u^t (v-u)^{-1/2} \{1+(t-v)^{-1/8}\} dv \\
& \times \int_u^t (v-u)^{-1/2} \{1+(t-v)^{-1/8}\} E[\|S_v^{(N)} - S_v^{\ell}\|^2] dv.
\end{aligned}$$

Note that the RHS of (8.24) converges to 0 as $N \rightarrow \infty$ by (6.12) using Lebesgue's dominated convergence theorem. Now the above

calculations can be combined into

$$E[\|Y_{u,t}^{(N)} - Y_{u,t}^\ell\|^2] \leq \delta^{(N)} + C \int_u^t (v-u)^{-1/2} E[\|Y_{v,t}^{(N)} - Y_{v,t}^\ell\|^2] dv$$

with some $\delta^{(N)}, \delta^{(N)} \downarrow 0$ as $N \rightarrow \infty$, and C which depend only on T , ℓ and φ . From this inequality, it is easy to complete the proof of the lemma. $\square$

Lemma 8.9 For every $\Psi \in \mathcal{D}_\ell$, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} c_N^{-1} E^{\mu_{\lambda(\cdot)}^{(N)}} [\langle \lambda(\cdot), \Delta D^{(N)} \Psi_t^{(N)}(\cdot, S) \rangle_\ell] \\ = E^\ell E^{\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)} [\langle \lambda(\cdot), \Delta D \Psi_t^\ell(\cdot, S) \rangle_\ell] . \end{aligned}$$

Proof Recall the expressions (8.9) and (8.12). Then, using (6.12), (8.21) and (8.23), we can prove that $\Delta D^{(N)} \Psi_t^{(N)}(\cdot, \pi_N S)$ converges to $\Delta D \Psi_t^\ell(\cdot, S)$ as $N \rightarrow \infty$ in the space L_ℓ^2 for every fixed $t > 0$ and $S \in L_\ell^2$. However, since we have from Lemma 8.2:

$$\begin{aligned} c_N^{-1} E^{\mu_{\lambda(\cdot)}^{(N)}} [\langle \lambda(\cdot), \Delta D^{(N)} \Psi_t^{(N)}(\cdot, S) \rangle_\ell] \\ = \int_{L_\ell^2} \langle \lambda(\cdot), \Delta D^{(N)} \Psi_t^{(N)}(\cdot, \pi_N S) \rangle_\ell \exp\{-\int_{-\ell}^\ell U(z, \pi_N S(z); \lambda(\cdot)) dz\} d\mu(S) , \end{aligned}$$

the Lebesgue's dominated convergence theorem completes the proof.

Note that a uniform bound:

$$\sup_{N, S} \|\Delta D^{(N)} \Psi_t^{(N)}(\cdot, \pi_N S)\| < \infty$$

can be obtained by deriving a uniform estimate on $\|Y_{u,t}^{(N)}\|$, which is similar to (8.21). $\square$

Now Proposition 8.2 follows as a combination of Proposition 8.1 with Lemmas 8.5 and 8.9.

8.3 Convergence of $Y_{u,t}^\ell$ Before completing the proof of Theorem

8.1, we need to examine the convergence of the solution $Y_{u,t}^\ell$ of the integral equation (8.11) as $\ell \rightarrow \infty$. We regard $Y_{u,t}^\ell \in \mathcal{E}$ by setting $Y_{u,t}^\ell(x) = 0$ for $x \in \mathbb{R} \setminus [-\ell, \ell]$ as before. Note that $Y_{u,t}^\ell(\pm\ell) = 0$.

Lemma 8.10 For every $T > 0$, $r > 0$ and $\varphi \in C_0^\infty(-\bar{\ell}, \bar{\ell})$, $\bar{\ell} \in \mathbb{N}$, the family of functions $\{Y_{u,t}^\ell(\cdot, \varphi; S_\cdot) ; \ell \geq \bar{\ell}, S_\cdot \in \mathcal{S}^\ell\}$ is relatively compact in the space $C(\hat{D}_T, \mathcal{E}_{-r})$ equipped with the usual topology of uniform convergence on each compact subset of $\hat{D}_T$.

Proof First we shall show that $\{Y_{u,t}^\ell\}$ are differentiable in x and their derivatives are uniformly bounded. Indeed the derivative of the first term in the RHS of (8.11) is bounded as follows:

$$|\int_{-\ell}^\ell q_{xx}^\ell(t-u, x, y) \Delta\varphi(y) dy| \leq K (t-u)^{-1/4} \|\Delta\varphi\|_\infty.$$

Use the estimates (5.6) and (6.5). As for the second term $Q_{u,t}^\ell$ in the RHS of (8.11), we can derive the following estimate from (6.6):

$$\begin{aligned} |\frac{\partial}{\partial x} Q_{u,t}^\ell(x)| &= |\int_u^t dv \int_{-\ell}^\ell q_{xxx}^\ell(v-u, x, y) V''(S_v(y)) Y_{v,t}^\ell(y) dy| \\ &\leq \text{const.} \sup_{(v,t) \in \hat{D}_T} \|Y_{v,t}^\ell\|_{-r} \theta(x, r) \int_u^t (v-u)^{-3/4} dv. \end{aligned}$$

Here the const. is independent of $S_\cdot$ and ℓ . These two estimates with the help of (8.19) prove that

$$\sup\{|\frac{\partial}{\partial x} Y_{u,t}^\ell(x)| ; x \in [-\ell, \ell], \ell \geq \bar{\ell}, (u, t) \in \hat{D}_T, |u-t| > \varepsilon, S_\cdot \in \mathcal{S}^\ell\} < \infty,$$

for every $\varepsilon > 0$. Especially, $\{Y_{u,t}^\ell(x)\}$ are equicontinuous in $x \in \mathbb{R}$ for $(u, t) \in \hat{D}_T$.

Next we show the equicontinuity of $\{Y_{u,t}^\ell(x)\}$ in $(u, t) \in \hat{D}_T$. Actually the first term $e^{-A^\ell(t-u)} \Delta\varphi$ in the RHS of (8.11) is equicontinuous in (u, t) , since we see

$$(8.25) \quad |e^{-A^\ell t'} \Delta \varphi(x) - e^{-A^\ell t} \Delta \varphi(x)| = \left| \int_t^{t'} du \int_{-\ell}^\ell q_t^\ell(u, x, y) \Delta \varphi(y) dy \right| \\ \leq K' \|\Delta \varphi\|_{-r} \theta(x, r) \int_t^{t'} u^{-1} du ,$$

which converges to 0 as $t' \downarrow t$ if $t > 0$. For the second term $Q_{u,t}^\ell$, using (6.6), similar calculations in the proof of Lemma 7.1 verify the following two estimates:

$$(8.26) \quad \|Q_{u_1,t}^\ell - Q_{u_2,t}^\ell\|_{-r} \leq C_T (u_2 - u_1)^{1/2} \sup_{(v,t) \in D_T} \|Y_{v,t}^\ell\|_{-r} , \\ 0 \leq u_1 \leq u_2 \leq t \leq T,$$

and

$$(8.27) \quad \|Q_{u,t_1}^\ell - Q_{u,t_2}^\ell\|_{-r} \leq C_T \int_{t_1}^{t_2} (v-u)^{-1/2} \|Y_{v,t_2}^\ell\|_{-r} dv \\ + C_T \int_u^{t_1} (v-u)^{-1/2} \|Y_{v,t_1}^\ell - Y_{v,t_2}^\ell\|_{-r} dv, \quad 0 \leq u \leq t_1 \leq t_2 \leq T,$$

with a constant C_T independent of S and ℓ . The equicontinuity of $\{Q_{u,t}^\ell\}$ in u follows immediately from (8.19) and (8.26). On the other hand, the equicontinuity in t follows by using (8.27) recursively noting (8.19) and (8.25), see the proof of Lemma 8.7.

Now we have shown that the family $\{Y_{u,t}^\ell\}$ is relatively compact in the space $C(\hat{D}_T, \mathcal{E})$, since (8.19) proves the uniform-boundedness of $\{Y_{u,t}^\ell\}$. The proof of the lemma can be completed by noting the following fact: Generally if functions $\bar{Y}_{u,t}^n$ converge to $\bar{Y}_{u,t}$ as $n \rightarrow \infty$ in the space $C(\hat{D}_T, \mathcal{E})$ and a uniform estimate

$$\sup_n \sup_{(u,t) \in D_T} \|\bar{Y}_{u,t}^n\|_{-r'} < \infty$$

holds for $r' > 0$ and $T > 0$, then this convergence also holds in the space $C(\hat{D}_T, \mathcal{E}_{-r})$, $0 < r < r'$. $\square$

Lemma 8.11 Let $\{S_t^\ell \in \mathcal{B}\}_{\ell=1}^\infty$ and $S_t \in \mathcal{B}([0, \infty) \times R)$ be

functions satisfying that $(\pi_{\ell}^{-1} S_t^{\ell})(x)$ converges to $S_t(x)$ a.e. $-(t, x) \in [0, \infty) \times R$ as $\ell \rightarrow \infty$, where π_{ℓ}^{-1} is a mapping of $\mathcal{B}^{\ell}$ into $\mathcal{B}$ defined similarly to in Section 6. Then $Y_{u,t}^{\ell} = Y_{u,t}^{\ell}(x, \varphi; S_{\cdot}^{\ell})$ converges to the solution $Y_{u,t} = Y_{u,t}(x, \varphi; S_{\cdot}) \equiv Y_{u,t}(x, \varphi; V''(S_{\cdot}))$ of (7.3) with $c(u, x) = V''(S_u(x))$ in the space $C(\dot{D}_T, \mathcal{B}_{-r})$, $r > 0$, as $\ell \rightarrow \infty$ for every $\varphi \in C_0^{\infty}(R)$.

Proof Since Lemma 8.10 proves the relative compactness of $\{Y_{u,t}^{\ell}\}$ in the space $C(\dot{D}_T, \mathcal{B}_{-r})$, $r > 0$, we can find from every subsequence $\{\ell' \uparrow \infty\}$ of $\{\ell\}$ further subsequence $\{\ell'' \uparrow \infty\}$ and $\bar{Y}_{u,t} \in C(\dot{D}_T, \mathcal{B}_{-r})$ in such a way that $Y_{u,t}^{\ell''}$ converges to $\bar{Y}_{u,t}$ in the space $C(\dot{D}_T, \mathcal{B}_{-r})$. Note that $\sup \{\|\bar{Y}_{u,t}\|_{-r} ; (u,t) \in D_T\} < \infty$, $T > 0$, follows from (8.19). We shall prove that $\bar{Y}_{u,t}$ is a solution of the integral equation (7.3) with $c(u, x) = V''(S_u(x))$ and this completes the proof of this lemma because of the uniqueness of its solutions. To this end, we may only show the following three equalities for every test function $\eta \in C_0^{\infty}(R)$:

$$(8.28) \quad \lim_{\ell'' \uparrow \infty} \langle Y_{u,t}^{\ell''}, \eta \rangle = \langle \bar{Y}_{u,t}, \eta \rangle,$$

$$(8.29) \quad \lim_{\ell'' \uparrow \infty} \langle \pi_{\ell''}^{-1} e^{-A^{\ell''}(t-u)}(\Delta \varphi), \eta \rangle = \langle e^{-A(t-u)}(\Delta \varphi), \eta \rangle,$$

and

$$(8.30) \quad \begin{aligned} \lim_{\ell'' \uparrow \infty} \int_u^t \langle V''(\pi_{\ell''}^{-1} S_v^{\ell''}) Y_{v,t}^{\ell''}, \pi_{\ell''}^{-1} e^{-A^{\ell''}(v-u)} \Delta \eta \rangle dv \\ = \int_u^t \langle V''(S_v) \bar{Y}_{v,t}, e^{-A(v-u)} \Delta \eta \rangle dv. \end{aligned}$$

However (8.28) is trivially shown and (8.29) follows by using Lemma 6.3. Finally, (8.30) follows from Lemma 6.3 and (8.19), since $\pi_{\ell''}^{-1} S_v^{\ell''}(x)$ converges to $S_v(x)$ a.e. $-(v, x) \in [u, t] \times R$ and $Y_{v,t}^{\ell''}$ converges to $\bar{Y}_{v,t}$ in the space $C(\dot{D}_T, \mathcal{B}_{-r})$ as $\ell'' \rightarrow \infty$. $\square$

8.4 The proof of Theorem 8.1 For $\Psi \in \mathcal{D}$ having the form (2.3), we define $\Delta D\Psi_t(x, S)$, $x \in \mathbb{R}$, $S \in \mathcal{G}_e$, by

$$\Delta D\Psi_t(x, S) = \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i}(\langle S_t, \varphi_1 \rangle, \dots, \langle S_t, \varphi_k \rangle) Y_{0,t}(x, \varphi_i; S_\cdot) \right],$$

where $S_\cdot = \{S_t\}$ is the solution of the SPDE (0.5) with initial value S and $Y_{0,t}$ is the solution of the integral equation (7.3) with $c(u, x) = V''(S_u(x))$. First we show the following result:

Proposition 8.3 For every $\Psi \in \mathcal{D}$, we have

$$(8.31) \quad E^{\mu_{\lambda(\cdot)}}[\Psi(S_t)] = E^{\mu_{\lambda(\cdot)}}[\langle \lambda(\cdot), \Delta D\Psi_t(\cdot, S) \rangle].$$

Proof The proof will be completed by taking the limit $\ell \rightarrow \infty$ of the both sides of (8.13). First we remind that Proposition 3.2 proves $\mu_{\lambda(\cdot)}^{-\ell, \ell}(\cdot; 0, 0)$ converges weakly to $\mu_{\lambda(\cdot)}$ on the space $\mathcal{G}_{r_0}$, $r_0 > 0$, and Theorem 6.1 asserts the distribution $P_{S^\ell}^\ell$ of the solution S_t^ℓ of (6.1) converges weakly to the distribution P_S of the solution S_t of (0.5) on the space $C((0, \infty), \mathcal{G})$ if the initial value S^ℓ converges to S in the space $\mathcal{G}_r$, $r > 0$. Therefore Skorohod's representation theorem can be applied to construct stochastic processes $\{\bar{S}_t^\ell\}$ and $\bar{S}_t$ on a proper probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ in the following manner: (i) Under $\bar{P}$, the probability distributions on the space $C([0, \infty), \mathcal{G})$ of $\bar{S}_t^\ell$ and $\bar{S}_t$ are $\int P_S^\ell(\cdot) \mu_{\lambda(\cdot)}^{-\ell, \ell}(dS; 0, 0)$ and $\int P_S(\cdot) \mu_{\lambda(\cdot)}(dS)$, respectively. (ii) $\bar{S}_t^\ell$ converges almost surely to $\bar{S}_t$ in the space $C((0, \infty), \mathcal{G})$ as $\ell \rightarrow \infty$. (iii) $\bar{S}_0^\ell$ converges almost surely to $\bar{S}_0$ in the space $\mathcal{G}_{r_0}$. Now we may assume $\Psi \in \mathcal{D}_{\bar{\ell}}$ with some $\bar{\ell} \in \mathbb{N}$, because $\mathcal{D} = \bigcup_{\ell \in \mathbb{N}} \mathcal{D}_\ell$. Then the LHS

of (8.13), which can be expressed as $E^{\bar{P}}[\mathcal{G}\Psi(\bar{S}_t^\ell)]$, converges to $E^{\bar{P}}[\mathcal{G}\Psi(\bar{S}_t)]$ as $\ell \rightarrow \infty$. Here we should note that $\Psi \in \mathcal{D}_{\bar{\ell}}$ with the help of (8.16) implies $\mathcal{G}\Psi \in C(\mathcal{E})$ and (8.15) shows

$$|\mathcal{G}\Psi(\bar{S}_t^\ell)| \leq C(1 + \|\bar{S}_t^\ell\|_{L_{\frac{2}{\ell}}}) \quad , \quad \ell \geq \bar{\ell},$$

which is uniformly integrable with respect to $\bar{P}$ because of Lemma 6.7. On the other hand, the convergence of the RHS of (8.13) is shown by noting Lemma 8.11 and the uniform bound (8.19). $\square$

In order to prove the Fréchet differentiability of the function $\Psi_t(S)$, we denote the solution of the SPDE (0.5) with the initial value $S \in H_e$ by $S_t(S) \equiv S_t(x; S)$. Set

$$D^\delta S_t(x) \equiv D^\delta S_t(x; S, \eta) = \frac{1}{\delta} (S_t(x; S + \delta\eta) - S_t(x; S))$$

for $0 < \delta < 1$ and $\eta \in H_e$. Then we see that $D^\delta S_t$ is a solution of the integral equation (7.10) with $c(u, x) = V''(X_u^\delta(x))$, where X^δ is some random element of $\mathcal{B}([0, \infty) \times R)$. Since $D^\delta S_t \in \mathcal{T}''$ (a.s.), the uniqueness result for (7.10) implies $D^\delta S_t(x; S, \eta) = \hat{Z}_t(x; \eta, V''(X_t^\delta))$.

Lemma 8.12 The function $D^\delta S_t$ converges to $\hat{Z}_t(\cdot; \eta, V''(S_t(S)))$ as $\delta \downarrow 0$ in the space $C((0, \infty), \mathcal{E}_r)$, $r > 0$, with probability one.

Proof Lemma 7.4 proves the relative compactness of the family $\{D^\delta S_t\}_{0 < \delta < 1}$ so that from every subsequence $\{\delta' \downarrow 0\}$ of $\{\delta\}$ we can find further subsequence $\{\delta'' \downarrow 0\}$ such that $D^{\delta''} S_t$ converges to some $\bar{Z}_t$ in the space $C((0, \infty), \mathcal{E}_r)$. Note that the uniform estimate (7.14) verifies that $\bar{Z}_t \in \mathcal{T}''$. However, since $D^\delta S_t$ satisfies

$$D^\delta S_t = e^{-tA} \eta + \int_0^t \Delta e^{-A(t-u)} \{V''(S_u(\cdot; S)) D^\delta S_u(\cdot)\} du$$

$$+ \frac{\delta}{2} \int_0^t \Delta e^{-A(t-u)} [V'''(\tilde{X}_u(\cdot)) (D^\delta S_u(\cdot))^2] du,$$

with some $\tilde{X}_u(y) \in \mathcal{B}([0, \infty) \times R)$, it is easy to see that $\bar{Z}_t = \tilde{Z}_t(\cdot; \eta, V''(S_\cdot(S)))$ by taking the limit $\delta \downarrow 0$. This gives the conclusion. $\square$

This lemma with the help of the estimate (7.14) implies that

$$(8.32) \quad \frac{d}{d\delta} \Psi_t(S+\delta\eta)|_{\delta=0} = E \left[\sum_{i=1}^k \frac{\partial \psi}{\partial \alpha_i} (\langle S_t(S), \varphi_1 \rangle, \dots, \langle S_t(S), \varphi_k \rangle) \times \langle \tilde{Z}_t(\cdot; \eta, V''(S_\cdot(S))), \varphi_i \rangle \right],$$

because for $0 < \delta < 1$

$$\begin{aligned} & \frac{1}{\delta} (\Psi_t(S+\delta\eta) - \Psi_t(S)) \\ &= E \left[\sum_{i=1}^k \frac{\partial \psi}{\partial \alpha_i} (\langle S_t(S), \varphi_1 \rangle, \dots, \langle S_t(S), \varphi_k \rangle) \langle D^\delta S_t(\cdot; S, \eta), \varphi_i \rangle \right] + R_\delta, \end{aligned}$$

and the remainder term R_δ has an estimate:

$$|R_\delta| \leq \frac{\delta}{2} \sum_{i,j=1}^k \left\| \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} \right\|_\infty E[|\langle D^\delta S_t(\cdot; S, \eta), \varphi_i \rangle \langle D^\delta S_t(\cdot; S, \eta), \varphi_j \rangle|].$$

Therefore the following lemma shows that $\Psi_t(S)$ is Fréchet differentiable on H_e and the equality (8.4) holds. Note that the RHS of (8.4) belongs to the space H_e^* , see Section 7.

Lemma 8.13 For every $\eta \in H_e$ and $c \in C_b([0, \infty) \times R)$, we have

$$(8.33) \quad Z_{0,t}(\eta, x; c) = \tilde{Z}_t(x; \eta, c).$$

Proof First we prove (8.33) for $\eta \in C_0^\infty(R)$ and $c \in C^\infty(R) \cap \mathcal{B}_b$.

In this case, both sides of (8.33) solve a PDE:

$$(8.34) \quad \begin{cases} \frac{\partial}{\partial t} Z_t(x) = -AZ_t(x) + \Delta\{c(t, x)Z_t(x)\} \\ Z_0 = \eta \end{cases}$$

and satisfy for every $T, r > 0$

$$(8.35) \quad \sup_{0 \leq t \leq T} \|Z_t\|_r < \infty,$$

see Remark 7.1 for $Z_{0,t}$. However, it is known the uniqueness of solutions of (8.34) satisfying the condition (8.35), see Eidel'man [9]. Therefore we obtain the conclusion for smooth η and c . Now it is not difficult to get the conclusion for general η and c , since we can prove the continuity of the both sides of (8.33) in $\eta \in H_e$ and $c \in C_b([0, \infty) \times \mathbb{R})$, see Lemmas 7.5 and 7.7. $\square$

We have an equality:

$$\langle \lambda(\cdot), Y_{0,t} \rangle = \langle \Delta \lambda(\cdot), Z_{0,t} \rangle$$

by using integration by parts, since $\lambda(\cdot) \in \Lambda$, $Z_{0,t} \in \mathcal{G}_{-r}^2$ and $\Delta Z_{0,t} = Y_{0,t}$, see Lemma 7.2. Therefore the formula (8.3) follows from (8.4) and Proposition 8.3. This completes the proof of Theorem 8.1.

We conclude this section by noticing that a similar formula holds for the solution S_t^ε , $\varepsilon > 0$, of the scaled TDGL equation (0.10), for which the existence and uniqueness result follows immediately from Theorem 5.1. The (formal) generator $\mathcal{G}^\varepsilon$ of the process S_t^ε is given by

$$\begin{aligned} \mathcal{G}^\varepsilon \Psi(S) = & \sum_{i=1}^k \frac{\partial \Psi}{\partial \alpha_i} (\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle) \{-\varepsilon^2 \langle S, \Delta^2 \varphi_i \rangle + \langle U'(S(\cdot)), \Delta \varphi_i \rangle\} \\ & + \varepsilon \sum_{i,j=1}^k \frac{\partial^2 \Psi}{\partial \alpha_i \partial \alpha_j} (\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle) \langle -\Delta \varphi_i, \varphi_j \rangle, \quad S \in H_e, \end{aligned}$$

for $\Psi \in \mathcal{D}$ having the form (2.3). We set

$$(8.36) \quad \Psi_t^\varepsilon(S) = E_S[\Psi(S_t^\varepsilon)], \quad S \in H_e, \quad \Psi \in \mathcal{D}.$$

Let $\mu_{\lambda(\cdot), \varepsilon}$ be the scaled $(U, \lambda(\cdot))$ -Gibbs distribution being constructed in Section 3. For every $S_\cdot \in \mathcal{B}([0, \infty) \times R)$ and $\varphi \in C_0^\infty(R)$, $Z_{u, t}^\varepsilon(x, \varphi; S_\cdot) = Z_{u, t}^\varepsilon(x, \varphi; V''(S_\cdot))$ denotes a solution of the integral equation (7.2) with $c(u, x) = V''(S_u(x))$ and with A replaced by an operator $A^\varepsilon = \varepsilon^2 \Delta^2 - \gamma \Delta$. Then we can prove the following formula as an easy consequence of Theorem 8.1.

Corollary 8.1 (i) For every $t > 0$ and $\Psi \in \mathcal{D}$, Ψ_t^ε is Fréchet differentiable on H_e and the following formula holds:

$$E^{\mu_{\lambda(\cdot), \varepsilon}}[\Psi_t^\varepsilon(S_t^\varepsilon)] = E^{\mu_{\lambda(\cdot), \varepsilon}}[\langle \Delta \lambda(\cdot), D\Psi_t^\varepsilon(\cdot, S) \rangle].$$

(ii) The Fréchet derivative $D\Psi_t^\varepsilon(x, S)$ of Ψ_t^ε can be expressed explicitly as follows:

$$(8.37) \quad D\Psi_t^\varepsilon(x, S) = \sum_{i=1}^k E_S \left[\frac{\partial \Psi}{\partial \alpha_i}(\langle S_t^\varepsilon, \varphi_1 \rangle, \dots, \langle S_t^\varepsilon, \varphi_k \rangle) Z_{0, t}^\varepsilon(x, \varphi_i; S_\cdot) \right].$$

§9 Basic estimates

The purpose of this section is twofold. We derive energy estimates for solutions of parabolic equations and then give L^p -estimates on the fundamental solutions, see Fritz [13], [14].

9.1 Energy inequalities Let $q^\varepsilon(t, x-y) \equiv q^\varepsilon(t, x, y)$, $\varepsilon > 0$, be the fundamental solution of the parabolic operator $\frac{\partial}{\partial t} + A^\varepsilon$, $A^\varepsilon = \varepsilon^2 \Delta^2 - \gamma \Delta$. For $\eta \in H_r$, $r > 0$, we define a function $\omega = \omega(x; \eta) \in \bigcap_{r' > r} H_{r'}$, by $\omega(0) = 0$ and $\nabla \omega = \eta$. For given $\eta_0 \in H_e$ and $f = f_t(x) \in$

$C_b([0, \infty) \times \mathbb{R})$, we define two functions $\eta_t^\varepsilon(x)$ and $\xi_t^\varepsilon(x)$, respectively, by

$$(9.1) \quad \eta_t^\varepsilon(x) = \int_{\mathbb{R}} \eta_0(y) q^\varepsilon(t, x, y) dy + \int_0^t du \int_{\mathbb{R}} q_{yy}^\varepsilon(t-u, x, y) f_u(y) dy$$

and

$$(9.2) \quad \xi_t^\varepsilon(x) = \int_{\mathbb{R}} \omega(y; \eta_0) q^\varepsilon(t, x, y) dy - \int_0^t du \int_{\mathbb{R}} q_y^\varepsilon(t-u, x, y) f_u(y) dy.$$

Then the following lemma is easily shown so that we omit the proof.

Lemma 9.1 (i) For every $t \geq 0$, we have $\eta_t^\varepsilon, \xi_t^\varepsilon \in H_e$ and $\nabla \xi_t^\varepsilon = \eta_t^\varepsilon$.

(ii) Assume $f \in C_b^\infty([0, \infty) \times \mathbb{R})$. Then we have $\xi_t^\varepsilon \in C^\infty(\mathbb{R})$ and $\frac{d^k}{dx^k} \xi_t^\varepsilon \in H_e$ for every $t > 0$ and $k = 0, 1, 2, \dots$.

Now we give the first energy inequality. The constant γ_0 is the same that was introduced in (1.2) and we assume $\gamma_0 < \gamma$.

Lemma 9.2 We suppose the condition:

$$(9.3) \quad |f_t(x)| \leq \gamma_0 |\eta_t^\varepsilon(x)|, \quad t \geq 0, x \in \mathbb{R}, 0 < \varepsilon < 1.$$

Then there exist positive constants r_0, C_1 and C_2 , which are independent of ε, r and t , such that

$$\int_0^t |\eta_u^\varepsilon|_r^2 du \leq C_2 |\omega(\cdot; \eta_0)|_r^2 e^{C_1 r t}, \quad 0 < \varepsilon < 1, 0 < r \leq r_0, t \geq 0.$$

Proof We consider the case of $f \in C_b^\infty([0, \infty) \times \mathbb{R})$, on which we do not assume the condition (9.3) at first. In this case, ξ_t^ε is a solution of the following parabolic equation:

$$\frac{\partial}{\partial t} \xi_t^\varepsilon = -A^\varepsilon \xi_t^\varepsilon + \nabla f, \quad t > 0.$$

Therefore, using this equation, simple calculations show

$$e^{C_1 r t} \frac{d}{dt} (e^{-C_1 r t} |\xi_t^\varepsilon|_r^2) \\ \leq -(2\gamma - 4Mr) |\eta_t^\varepsilon|_r^2 + Mr |f_t|_r^2 + 2 \int_R |\eta_t^\varepsilon(x)| |f_t(x)| \theta(x, r) dx, \\ 0 < \varepsilon < 1, \quad 0 < r < 1, \quad t > 0,$$

where $C_1 = M(2 + \gamma)$. Here we have applied integration by parts in the variable x with the help of Lemma 9.1 and used the following estimates:

$$2 |\xi_t^\varepsilon(x)| |f_t(x)| \leq |\xi_t^\varepsilon(x)|^2 + |f_t(x)|^2$$

and

$$(9.4) \quad \left| \frac{\partial^k}{\partial x^k} \theta(x, r) \right| \leq Mr \theta(x, r), \quad k = 1, 2, 3, 4, \quad 0 < r < 1,$$

with some $M > 0$. Hence we obtain

$$(9.5) \quad \int_0^t \{ (2\gamma - 4Mr) |\eta_u^\varepsilon|_r^2 - Mr |f_u|_r^2 - 2 \int_R |\eta_u^\varepsilon(x)| |f_u(x)| \theta(x, r) dx \} e^{-C_1 r u} du \\ \leq |\omega(\cdot; \eta_0)|_r^2.$$

Now we can remove the assumption $f \in C_b^\infty([0, \infty) \times R)$ to derive (9.5).

Use the usual approximation method. Therefore the conclusion follows from the condition (9.3) by taking r_0 ; $0 < r_0 < 1$ in such a way that $C_2^{-1} = 2\gamma - 2\gamma_0 - 4Mr_0 - Mr_0 \gamma_0^2 > 0$. $\square$

The second task is to give estimates on the fundamental solution $Z_{u,t}^\varepsilon(x, y) = Z_{u,t}^\varepsilon(x, y; c)$, $\varepsilon > 0$, of the equation (7.1) with $\mathcal{L}_u$ replaced by $\mathcal{L}_{u,c}^\varepsilon = -A^\varepsilon + c(u, x) \Delta_x$ for given $c = c(u, x) \in C_b \equiv C_b([0, \infty) \times R)$. This fundamental solution can be constructed in a similar manner to the case of $\varepsilon = 1$, see Section 7. We define $Z_{u,t}^\varepsilon(\eta, y)$ by

$$(9.6) \quad Z_{u,t}^\varepsilon(\eta, y) = \int_R \eta(x) Z_{u,t}^\varepsilon(x, y) dx, \quad \eta \in H_r, \quad r > 0.$$

Lemma 9.3 There exist positive constants r_0, C_1 and C_2 , which are independent of ε, r, t, c and η , such that the following three estimates hold for every $0 < \varepsilon < 1, 0 < r \leq r_0, t \geq 0$ and $c \in C_b$; $\|c\|_\infty \leq \gamma_0$:

$$(9.7) \quad \int_0^t |Z_{0,u}^\varepsilon(\eta, \cdot; c)|_r^2 du \leq C_2 e^{C_1 r t} |\omega(\cdot; \eta)|_r^2, \quad \eta \in H_r,$$

$$(9.8) \quad \int_0^t |Z_{0,u}^\varepsilon(\nabla \eta, \cdot; c)|_r^2 du \leq C_2 e^{C_1 r t} |\eta|_r^2, \quad \eta \in C_b^\infty(R),$$

and

$$(9.9) \quad \int_0^t |Z_{0,u}^\varepsilon(x, \cdot; c)|_r^2 du \leq C_2 r^{-1} e^{C_1 r t} \theta(x, r), \quad x \in R.$$

Proof To complete the proof, we may assume $c \in C_0^\infty([0, \infty) \times R)$.

Indeed, after proving the concluding estimates for such smooth c 's, we can generalize them for $c \in C_b$ by using approximation method with the help of Lemma 7.7. Remind that $\eta_t^\varepsilon \equiv Z_{0,t}^\varepsilon(\eta, \cdot; c)$ solves the forward equation (7.28) with $\mathcal{L}_{t,y}^*$ replaced by $\mathcal{L}_{t,y}^{\varepsilon*} = -A^\varepsilon + \Delta_y\{c(t, y) \cdot\}$ if the coefficient c is smooth. Consider the following equation:

$$(9.10) \quad \begin{cases} \frac{\partial}{\partial t} \xi_t^\varepsilon = -A^\varepsilon \xi_t^\varepsilon + \nabla(c(t, y) \nabla \xi_t^\varepsilon), & t > 0, \\ \xi_0^\varepsilon = \omega(\cdot; \eta) + a, & a \in R. \end{cases}$$

Then we have $\nabla \xi_t^\varepsilon = \eta_t^\varepsilon$ and simple calculations show

$$\begin{aligned} \frac{d}{dt} |\xi_t^\varepsilon|_r^2 &\leq -(2\gamma - 2\gamma_0 - 4Mr - Mr\gamma_0) |\eta_t^\varepsilon|_r^2 + C_1 r |\xi_t^\varepsilon|_r^2, \\ 0 &< \varepsilon < 1, \quad 0 < r < 1, \quad t > 0, \end{aligned}$$

with $C_1 = M(1 + \gamma + \gamma_0)$, where M is the same constant appearing in

(9.4). Therefore, taking r_0 ; $0 < r_0 < 1$ in such a way that $C_2^{-1} = 2\gamma - 2\gamma_0 - 4Mr_0 - Mr_0\gamma_0 > 0$, we obtain

$$(9.11) \quad \int_0^t |\eta_u^\varepsilon|_r^2 du \leq C_2 e^{C_1 r t} |\omega(\cdot; \eta) + a|_r^2, \quad 0 < r \leq r_0.$$

Now the estimate (9.7) follows by taking $a = 0$, while (9.8) follows by taking $\nabla \eta$ instead of η and $a = \eta(0)$ in (9.11), since $\omega(\cdot; \nabla \eta) + \eta(0) = \eta$. The third estimate (9.9) is a consequence of (9.8); we may take an approximating sequence $\{\eta_n \in C_b^\infty(R)\}_{n=1}^\infty$ such that $\nabla \eta_n$ converges to δ_x in a proper sense. $\square$

Lemma 9.4 There exist positive constants r_0 , C_1 and C_2 , which are independent of ε , r , t , c , $\bar{c}$ and η , such that

$$\begin{aligned} & \int_0^t |Z_{0,u}^\varepsilon(\eta, \cdot; c) - Z_{0,u}^\varepsilon(\eta, \cdot; \bar{c})|_r^2 du \\ & \leq C_2 e^{C_1 r t} \int_0^t |(c - \bar{c})(u) Z_{0,u}^\varepsilon(\eta, \cdot; c)|_r^2 du, \end{aligned}$$

for every $0 < \varepsilon < 1$, $0 < r \leq r_0$, $t \geq 0$, $c, \bar{c} \in C_b$; $\|c\|_\infty, \|\bar{c}\|_\infty \leq \gamma_0$ and $\eta \in H_r$.

Proof As in the proof of Lemma 9.3, we may assume $c, \bar{c} \in C_0^\infty([0, \infty) \times R)$. We define ξ_t^ε as in the proof of Lemma 9.3 and also introduce $\bar{\xi}_t^\varepsilon$, the solution of (9.10) with c replaced by $\bar{c}$. We assume the initial values are same; $\xi_0^\varepsilon = \bar{\xi}_0^\varepsilon = \omega(\cdot; \eta)$. Set $\eta_t^\varepsilon = Z_{0,t}^\varepsilon(\eta, \cdot; c)$ and $\bar{\eta}_t^\varepsilon = Z_{0,t}^\varepsilon(\eta, \cdot; \bar{c})$. Then we have

$$\begin{aligned} \frac{d}{dt} |\xi_t^\varepsilon - \bar{\xi}_t^\varepsilon|_r^2 & \leq -(2\gamma - 2\gamma_0 - 4Mr - Mr\gamma_0) |\eta_t^\varepsilon - \bar{\eta}_t^\varepsilon|_r^2 \\ & + rM(1 + \gamma + \gamma_0) |\xi_t^\varepsilon - \bar{\xi}_t^\varepsilon|_r^2 + I, \quad 0 < \varepsilon < 1, \quad 0 < r < 1, \quad t > 0, \end{aligned}$$

where

$$I = -2 \int (c_t - \bar{c}_t) \eta_t^\varepsilon \{ (\eta_t^\varepsilon - \bar{\eta}_t^\varepsilon) \theta(x, r) + (\xi_t^\varepsilon - \bar{\xi}_t^\varepsilon) \nabla \theta(x, r) \} dx.$$

We get the conclusion by estimating each term in I as follows:

$$2|(c_t - \bar{c}_t)\eta_t^\varepsilon(\eta_t^\varepsilon - \bar{\eta}_t^\varepsilon)| \leq (\gamma - \gamma_0)|\eta_t^\varepsilon - \bar{\eta}_t^\varepsilon|^2 + (\gamma - \gamma_0)^{-1}|(c_t - \bar{c}_t)\eta_t^\varepsilon|^2,$$

$$2|(c_t - \bar{c}_t)\eta_t^\varepsilon(\xi_t^\varepsilon - \bar{\xi}_t^\varepsilon)| \leq |\xi_t^\varepsilon - \bar{\xi}_t^\varepsilon|^2 + |(c_t - \bar{c}_t)\eta_t^\varepsilon|^2.$$

This time we take $C_1 = M(2 + \gamma + \gamma_0)$ and r_0 ; $0 < r_0 < 1$ such that $C_2 = (Mr_0 + (\gamma - \gamma_0)^{-1})(\gamma - \gamma_0 - 4Mr_0 - Mr_0\gamma_0)^{-1} > 0$. $\square$

We may assume the constants r_0 , C_1 and C_2 appearing in Lemmas 9.2 - 9.4 are common.

9.2 L^p -estimates on the fundamental solutions Here we shall show the following type of estimate:

Lemma 9.5 For every $1 < p < 7/3$, there exist positive constants $\gamma_1^{(p)}$ and $C_2^{(p)}$ such that, if $\gamma_0 < \gamma_1^{(p)} \wedge \gamma$, then we have

$$(9.12) \quad \int_0^t du \int_R |Z_{0,u}^\varepsilon(x, y; c)|^p \theta(y, r) dy$$

$$\leq C_2^{(p)} r^{-(p-1)/2} t^{(7-3p)/4} e^{C_1 r t} \theta(x, r),$$

for every $0 < \varepsilon < 1$, $0 < r \leq (p-1)r_0/2$, $t \geq 0$, $x \in R$ and $c \in C_b$; $\|c\|_\infty \leq \gamma_0$. Here r_0 and C_1 are the constants appearing in Lemma 9.3.

To give the proof of this lemma, we regard $Z_{u,t}^\varepsilon(x, y; c)$ as a perturbation from $q^\varepsilon(t-u, x-y) = Z_{u,t}^\varepsilon(x, y; 0)$. We may consider that $Z_{u,t}^\varepsilon(x, y; c)$ is defined for every $-\infty < u < t < \infty$, $x, y \in R$, by extending $c \in C_b([0, \infty) \times R)$ to $c \in C_b((-\infty, \infty) \times R)$ properly. Define an operator Q_c^ε on the space $L^p = L^p(R \times R)$, $p > 1$, by

$$(Q_c^\varepsilon h)(u, x) = \int_u^\infty dt \int_R Z_{u,t}^\varepsilon(x, y; c) h(t, y) dy, \quad (u, x) \in R \times R, h \in L^p,$$

and set $\bar{Q}^\varepsilon = Q_0^\varepsilon$ (Q_c^ε with $c \equiv 0$) and $R_c^\varepsilon = c \Delta \bar{Q}^\varepsilon (= (\mathcal{L}_{u,c}^\varepsilon - \mathcal{L}_{u,0}^\varepsilon) \bar{Q}^\varepsilon)$.

Then we have the following.

Lemma 9.6 (i) There exist positive constants $C_{1,p}$, $1 < p \leq 3$, and $C_{2,p}$, $p > 1$, which depend only on p , such that

$$(9.13) \quad \int_0^t du \int_R \{q^\varepsilon(u, x)\}^p dx \leq C_{1,p} t^{(3-p)/2}, \quad 1 < p \leq 3,$$

$$(9.14) \quad \|\Delta \bar{Q}^\varepsilon h\|_{L^p} \leq C_{2,p} \|h\|_{L^p}, \quad p > 1,$$

for every $0 < \varepsilon < 1$ and $t \geq 0$.

(ii) Assume $\gamma_0 C_{2,p} < 1$ and $\|c\|_\infty \leq \gamma_0$. Then $(1 - R_c^\varepsilon)^{-1} = \sum_{n=0}^{\infty} \{R_c^\varepsilon\}^n$ exists as an operator on L^p and

$$\|(1 - R_c^\varepsilon)^{-1} h\|_{L^p} \leq (1 - \gamma_0 C_{2,p})^{-1} \|h\|_{L^p}.$$

(iii) In addition, if $h \in L^p$ satisfies $h(t, y) = 0$, $t \geq T$, $y \in R$, with some $T \in R$, then we have $(1 - R_c^\varepsilon)^{-1} h(t, y) = 0$ for $t \geq T$, $y \in R$.

Proof To prove (9.13), we notice that q^ε is given by a convolution; $q^\varepsilon(t, x) = \{q_{\varepsilon^2 t}^{(1)} * q_t^{(2)}\}(x)$, where $q_t^{(1)}$ and $q_t^{(2)}$ are fundamental solutions of $\frac{\partial}{\partial t} + \Delta^2$ and $\frac{\partial}{\partial t} - \gamma \Delta$, respectively.

Therefore (9.13) follows by using Hausdorff-Young's inequality and by the facts; $\|q_t^{(2)}\|_{L^p(R)}^p = p^{-1/2} (2\pi t \gamma)^{(1-p)/2}$ and $\sup_{t>0} \|q_t^{(1)}\|_{L^1(R)} < \infty$, which is a consequence of the scaling property: $q_t^{(1)}(x) = t^{-1/4} q_1^{(1)}(t^{-1/4} x)$.

For the proof of (9.14), we need to give estimates on singular integrals. However, these can be derived by similar argument in the proof of Lemma 5 of Fritz [14]. The assertions (ii) and (iii) follow by using Neumann series expansion. $\square$

Proof of Lemma 9.5 Assume $\gamma_0 C_{2,\tilde{p}} < 1$ with $\tilde{p} = 2(p-1)^{-1}$. Then, for every $h \in L^{\tilde{p}}$ such that $h(u) = 0$ for $u \geq t$, we have

$$\begin{aligned}
(9.15) \quad |Q_c^\varepsilon h(0, x)| &= |\bar{Q}^\varepsilon (1 - R_c^\varepsilon)^{-1} h(0, x)| \\
&= \left| \int_0^t du \int_R q^\varepsilon(u, x-y) \{ (1 - R_c^\varepsilon)^{-1} h \}(u, y) dy \right| \\
&\leq (C_{1,p'})^{1/p'} t^{(7-3p)/4} (1 - \gamma_0 C_{2,\tilde{p}})^{-1} \|h\|_{L^{\tilde{p}}},
\end{aligned}$$

with $p' = 2(3-p)^{-1}$, i.e., $1/\tilde{p} + 1/p' = 1$. Here we have used Lemma 9.6 and Hölder's inequality. The first equality in (9.15) is a consequence of

$$\bar{Q}^\varepsilon \{R_c^\varepsilon\}^n h(u, x) = \int_u^T dt \int_R Z_{u,t}^{\varepsilon, (n)}(x, y) h(t, y) dy,$$

$$n = 0, 1, 2, \dots, \quad (u, x) \in \mathbb{R} \times \mathbb{R},$$

where $\{Z_{u,t}^{\varepsilon, (n)}\}_{n=0}^\infty$ are the functions defined by (7.20) with q replaced by q^ε . Now consider functions $h_{t,x}^\varepsilon \in L^{\tilde{p}}$, $t > 0$, $x \in \mathbb{R}$, defined by

$$h_{t,x}^\varepsilon(u, y) = \begin{cases} \text{sign}\{Z_{0,u}^\varepsilon(x, y; c)\} |Z_{0,u}^\varepsilon(x, y; c)|^{p-1} \theta(y, r), & 0 \leq u \leq t, \\ 0, & \text{otherwise,} \end{cases}$$

then we see

$$Q_c^\varepsilon h_{t,x}^\varepsilon(0, x) = \text{the LHS of (9.12)}$$

and

$$\begin{aligned}
\|h_{t,x}^\varepsilon\|_{L^{\tilde{p}}} &= \left(\int_0^t |Z_{0,u}^\varepsilon(x, \cdot; c)|_{r\tilde{p}}^2 du \right)^{1/\tilde{p}} \\
&\leq (C_2/r\tilde{p})^{1/\tilde{p}} e^{C_1 r t} \theta(x, r),
\end{aligned}$$

if $0 < r\tilde{p} \leq r_0$. Therefore we obtain the conclusion from (9.15) by taking $\gamma_1^{(p)} = C_{2,\tilde{p}}^{-1}$ and $C_2^{(p)} = (C_{1,p'})^{1/p'} (1 - \gamma_0 C_{2,\tilde{p}})^{-1} (C_2/\tilde{p})^{1/\tilde{p}}$. $\square$

§10 Compactness argument I

We need to investigate, for every $\Psi \in \mathcal{D}$, the compactness of the families $\{\Psi_t^\varepsilon(S)\}_{0 < \varepsilon < 1}$ defined by (8.36) and their Fréchet derivatives $\{D\Psi_t^\varepsilon(\cdot, S)\}_{0 < \varepsilon < 1}$ as weak-continuous functions of $S \in H_{e,w}$. For this purpose, however, it is more convenient to treat their Laplace transforms defined by

$$(10.1) \quad R_a \Psi^\varepsilon(S) = \int_0^\infty e^{-at} \Psi_t^\varepsilon(S) dt,$$

and

$$(10.2) \quad R_a D\Psi^\varepsilon(x, S) = \int_0^\infty e^{-at} D\Psi_t^\varepsilon(x, S) dt, \quad a > 0.$$

We shall denote by $S_t^\varepsilon(\cdot; S)$ the solution of the scaled TDGL equation (0.10) to make its initial value $S \in H_e$ clear as before. We assume the conditions (1.1) and (1.2) on the potential function U taking $\gamma_1 \equiv \sup_{2 < p < 7/3} \gamma_1^{(p)}$, where $\gamma_1^{(p)}$ is the constant appearing in Lemma 9.5.

10.1 Compactness of $\{R_a \Psi^\varepsilon; 0 < \varepsilon < 1\}$ The following lemma is an immediate consequence of Lemma 9.2 in which we take $f_t(x) = v'(S_t^\varepsilon(x; S)) - v'(S_t^\varepsilon(x; \bar{S}))$, $\eta_t^\varepsilon(x) = S_t^\varepsilon(x; S) - S_t^\varepsilon(x; \bar{S})$ and $\eta_0 = S - \bar{S}$ for $S, \bar{S} \in H_e$.

Lemma 10.1 For every $0 < \varepsilon < 1$, $0 < r \leq r_0$, $t \geq 0$ and $S, \bar{S} \in H_e$, we have

$$\int_0^t |S_u^\varepsilon(\cdot; S) - S_u^\varepsilon(\cdot; \bar{S})|_r^2 du \leq C_2 |\omega(\cdot; S - \bar{S})|_r^2 e^{C_1 r t},$$

with the same constants r_0 , C_1 and C_2 appearing in Lemma 9.2.

Lemma 10.2 For every $a_0 > 0$, there exist positive constants $C = C(a_0, \Psi)$ and $\tilde{r} = \tilde{r}(a_0)$ such that

$$|R_a \Psi^\varepsilon(S) - R_a \Psi^\varepsilon(\bar{S})| \leq C |\omega(\cdot; S - \bar{S})|_{\tilde{r}},$$

for every $0 < \varepsilon < 1$, $a \geq a_0$ and $S, \bar{S} \in H_e$.

Proof Since the function Ψ of the form (2.3) has a bound:

$$(10.3) \quad |\Psi(S_1) - \Psi(S_2)| \leq C_r(\Psi) |S_1 - S_2|_r, \quad r > 0,$$

with

$$(10.4) \quad C_r(\Psi) = \sum_{i=1}^k \left\| \frac{\partial \Psi}{\partial \alpha_i} \right\|_{\infty} |\varphi_i|_{-r} < \infty,$$

we obtain

$$\begin{aligned} & |R_a \Psi^{\varepsilon}(S) - R_a \Psi^{\varepsilon}(\bar{S})| \\ & \leq C_r(\Psi) E[a \int_0^{\infty} e^{-at} dt \int_0^t |S_u^{\varepsilon}(\cdot; S) - S_u^{\varepsilon}(\cdot; \bar{S})|_r du] \\ & \leq C_r(\Psi) C(a, r) |\omega(\cdot; S - \bar{S})|_r, \end{aligned}$$

where

$$(10.5) \quad C(a, r) = a \sqrt{C_2} \int_0^{\infty} \sqrt{t} \exp\{-t(a - C_1 r/2)\} dt, \quad a > 0, \quad 0 < r < 2a/C_1.$$

We have used Schwarz's inequality and Lemma 10.1. Therefore the conclusion follows by taking $\tilde{r}$; $0 < \tilde{r} < r_0 \wedge 2a_0/C_1$ and $C = C_{\tilde{r}}(\Psi) \sup\{C(a, \tilde{r}); a \geq a_0\} < \infty$. $\square$

We regard $R_a \Psi^{\varepsilon}(S)$ as real-valued functions of $(a, S) \in (0, \infty) \times H_{e,w}$. Remind that the set $B(\{b_r\}) = \{S \in H_{e,w}; |S|_r \leq b_r, r > 0\}$ is (weakly) compact in $H_{e,w}$ for every sequence $\{b_r > 0; r > 0\}$.

Proposition 10.1 For every $0 < a_0 < a_1 < \infty$ and $\{b_r > 0; r > 0\}$, the family of functions $\{R_a \Psi^{\varepsilon}(S); 0 < \varepsilon < 1\}$ restricted on $[a_0, a_1] \times B(\{b_r\})$ is relatively compact in the space $C([a_0, a_1] \times B(\{b_r\}))$ having the usual uniform-convergence topology.

Proof Since the uniform boundedness of the family $\{R_a \Psi^{\varepsilon}(S)\}$ follows from $|R_a \Psi^{\varepsilon}(S)| \leq \|\Psi\|_{\infty}/a$ and the equicontinuity in a

follows from $|\frac{\partial}{\partial a} R_a \Psi^\varepsilon(S)| \leq \|\psi\|_\infty / a^2$, the proof can be completed if one can show the equicontinuity in S of this family. For this purpose, note that the fundamental system of neighborhoods of $0 \in H_{e,w}$ consists of all subsets U of $H_{e,w}$ having the forms:

$$(10.6) \quad U \equiv U_\alpha(\eta_1, \dots, \eta_n) = \{S \in H_{e,w}; |\langle S, \eta_i \rangle| < \alpha, i=1, 2, \dots, n\},$$

with $n \in \mathbb{N}$, $\alpha > 0$ and $\eta_i \in H_e^*$, $i = 1, 2, \dots, n$. Therefore, with the help of Lemma 10.2, we may only prove that for every $\delta > 0$ there exists a set U of the form (10.6) such that

$$\sup\{|\omega(\cdot; S)|_{\tilde{r}}; S \in U \cap B(\{2b_r\})\} < \delta.$$

However, this is not difficult. $\square$

10.2 Compactness of $\{R_a D\Psi^\varepsilon; 0 < \varepsilon < 1\}$ The first task is to show the relative-compactness in the space $H_{-\bar{r}}$ of the family $\{R_a D\Psi^\varepsilon(\cdot, S); 0 < \varepsilon < 1, a \geq a_0, S \in H_e\}$, $a_0 > 0$, with some $\bar{r} = \bar{r}(a_0) > 0$. The following lemma gives a criterion for the relative-compactness of a subset in the Hilbert space H_{-r} , $r > 0$. The proof is easy and omitted.

Lemma 10.3 A subset δ is relatively compact in the space H_{-r} if it satisfies the following two conditions:

- (a) $\sup\{|\xi|_{-r'}; \xi \in \delta\} < \infty$ with some $r' > r$.
- (b) For every bounded interval I of R , the family of restrictions $\{\xi|_I; \xi \in \delta\}$ is relatively compact in the space $L^2(I, dx)$.

Lemma 10.4 For every $a_0 > 0$, there exists a positive constant $\bar{r} = \bar{r}(a_0)$ such that $\{R_a D\Psi^\varepsilon(\cdot, S); 0 < \varepsilon < 1, a \geq a_0, S \in H_e\}$ is relatively compact in the space $H_{-\bar{r}}$ (equipped with the usual strong topology).

Proof We show that two conditions (a) and (b) of Lemma 10.3 hold for

$\mathcal{E} = \{R_a D\Psi^\mathcal{E}(\cdot, S); 0 < \mathcal{E} < 1, a \geq a_0, S \in H_e\}$. First, using Corollary 8.1-(ii) and then (9.7), we obtain for every $\eta \in H_r$:

$$\begin{aligned}
 (10.7) \quad & |\langle R_a D\Psi^\mathcal{E}(\cdot, S), \eta \rangle| \\
 & \leq C_r(\Psi) \int_0^\infty e^{-at} dt E_S[|Z_{0,t}^\mathcal{E}(\eta, \cdot; S^\mathcal{E})|_r] \\
 & = C_r(\Psi) \int_0^\infty a e^{-at} dt E_S[\int_0^t |Z_{0,u}^\mathcal{E}(\eta, \cdot; S^\mathcal{E})|_r du] \\
 & \leq C_r(\Psi) C(a, r) |\omega(\cdot; \eta)|_r, \quad 0 < \mathcal{E} < 1, 0 < r < r_0 \wedge 2a_0/C_1, a \geq a_0,
 \end{aligned}$$

where $C_r(\Psi)$ and $C(a, r)$ are the constants defined by (10.4) and (10.5), respectively. The RHS of (10.7) can be estimated further by noting that if $0 < r' < r$, then

$$(10.8) \quad |\omega(\cdot; \eta)|_r \leq C_{r,r'} |\eta|_{r'},$$

with some constant $C_{r,r'}$ depending on r and r' . Therefore we obtain for every $0 < r' < r_0 \wedge 2a_0/C_1$:

$$(10.9) \quad |R_a D\Psi^\mathcal{E}(\cdot, S)|_{-r'} \leq \inf_{r; r > r'} \{C_r(\Psi) C(a, r) C_{r,r'}\}, \quad 0 < \mathcal{E} < 1, a \geq a_0, S \in H_e.$$

Now take $\bar{r} = \bar{r}(a_0)$ such that $0 < \bar{r} < r_0 \wedge 2a_0/C_1$. Then, since $\sup\{C(a, r); a \geq a_0\} < \infty$ for $r < 2a_0/C_1$, we get the condition (a) with $r = \bar{r}$ from (10.9) by taking $r': \bar{r} < r' < r_0 \wedge 2a_0/C_1$.

Secondly to show the condition (b), we see for every $\eta \in C_0^\infty(R)$:

$$(10.10) \quad |\langle R_a D\Psi^\mathcal{E}(\cdot, S), \nabla \eta \rangle| \leq C_r(\Psi) C(a, r) |\eta|_r,$$

holds for $0 < \mathcal{E} < 1, 0 < r < r_0 \wedge 2a_0/C_1, a \geq a_0$ and $S \in H_e$. This follows by a similar calculation as above using (9.8). Since it holds $|\eta|_r \leq \|\eta\|_{L^2(I)}$, $r > 0$, for η satisfying $\text{supp } \eta \subset I$, (10.9) and (10.10)

imply

$$\sup \left\{ \sum_{k=0}^1 \|\nabla^k R_a D\Psi^\varepsilon(\cdot, S)\|_{L^2(I)} ; 0 < \varepsilon < 1, a \geq a_0, S \in H_e \right\} < \infty,$$

where ∇ is the derivative in the distribution's sense. This proves the condition (b) with the help of Rellich's theorem (see, e.g., Adams [11]). $\square$

The second task in this paragraph is proving the equicontinuity of $\{R_a D\Psi^\varepsilon(\cdot, S); 0 < \varepsilon < 1\}$ as $H_{-\bar{r}}$ -valued functions of (a, S) . The assumption $\gamma_0 < \gamma_1$ will be used to show the following lemma.

Lemma 10.5 For every $a_0 > 0$, there exist positive constants $\bar{r} = \bar{r}(a_0)$, $\tilde{r} = \tilde{r}(a_0)$, $C = C(a_0, \Psi)$ and $0 < \alpha < 1$ such that

$$|R_a D\Psi^\varepsilon(\cdot, S) - R_a D\Psi^\varepsilon(\cdot, \bar{S})|_{-\bar{r}} \leq C(|\omega(\cdot; S - \bar{S})|_{\tilde{r}}^\alpha + |\omega(\cdot; S - \bar{S})|_{\tilde{r}}),$$

holds for every $0 < \varepsilon < 1$, $a \geq a_0$ and $S, \bar{S} \in H_e$.

Proof Positive constants $\bar{r}$ and $\tilde{r}$ will be chosen later.

Corollary 8.1-(ii) shows for every $\eta \in H_{\bar{r}}$:

$$\begin{aligned} & |\langle R_a D\Psi^\varepsilon(\cdot, S) - R_a D\Psi^\varepsilon(\cdot, \bar{S}), \eta \rangle| \\ &= \left| \sum_{i=1}^k \int_0^\infty a e^{-at} dt \int_0^t \{I_{1,i}^\varepsilon(u) + I_{2,i}^\varepsilon(u)\} du \right|, \end{aligned}$$

where

$$I_{1,i}^\varepsilon(t) = E[\langle \Psi_i(S_t^\varepsilon) - \Psi_i(\bar{S}_t^\varepsilon) \rangle \langle Z_{0,t}^\varepsilon(\eta, \cdot; S_t^\varepsilon), \varphi_i \rangle],$$

$$I_{2,i}^\varepsilon(t) = E[\langle \Psi_i(\bar{S}_t^\varepsilon) \rangle \langle Z_{0,t}^\varepsilon(\eta, \cdot; S_t^\varepsilon) - Z_{0,t}^\varepsilon(\eta, \cdot; \bar{S}_t^\varepsilon), \varphi_i \rangle],$$

$$\Psi_i(S) = \frac{\partial \psi}{\partial \alpha_i}(\langle S, \varphi_1 \rangle, \dots, \langle S, \varphi_k \rangle), \quad i = 1, 2, \dots, k,$$

and

$$S_t^\varepsilon = S_t^\varepsilon(\cdot; S), \quad \bar{S}_t^\varepsilon = S_t^\varepsilon(\cdot; \bar{S}).$$

Estimation on a term including $I_{1,i}^\varepsilon$ goes as follows:

$$\begin{aligned}
& |\int_0^t I_{1,i}^\varepsilon(u) du| \\
& \leq C_{\tilde{r}}(\Psi_i) \text{EI} \{ \int_0^t |S_u^\varepsilon - \bar{S}_u^\varepsilon|_{\tilde{r}}^2 du \}^{1/2} \{ \int_0^t \langle Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon), \varphi_i \rangle^2 du \}^{1/2} \\
& \leq C_{\tilde{r}}(\Psi_i) C_2 C_{\tilde{r}', \tilde{r}} |\varphi_i|_{-\tilde{r}} |\omega(\cdot; S - \bar{S})|_{\tilde{r}} |\eta|_{\tilde{r}} \exp(C_1 t(\tilde{r} + \tilde{r}')/2), \\
& \quad 0 < \varepsilon < 1, \quad 0 < \tilde{r} \leq r_0, \quad 0 < \tilde{r}' < \tilde{r} \leq r_0.
\end{aligned}$$

Here we have used Schwarz's inequality and (10.3) for the second line and then (9.7), (10.8) and Lemma 10.1 for the third line. To estimate the term including $I_{2,i}^\varepsilon$, we fix $2 < p < 7/3$ in such a way that $\gamma_0 < \gamma_1^{(p)}$ holds. Then we have

$$\begin{aligned}
(10.11) \quad & |\int_0^t I_{2,i}^\varepsilon(u) du| \\
& \leq \|\frac{\partial \psi}{\partial \alpha_i}\|_\infty |\varphi_i|_{-\tilde{r}} \int_0^t \text{EI} [|Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon) - Z_{0,u}^\varepsilon(\eta, \cdot; \bar{S}_\cdot^\varepsilon)|_{\tilde{r}}] du \\
& \leq \|\frac{\partial \psi}{\partial \alpha_i}\|_\infty |\varphi_i|_{-\tilde{r}} \sqrt{t} C_2 e^{C_1 \tilde{r} t/2} \\
& \quad \times \text{EI} \{ \int_0^t | \{ V''(S_u^\varepsilon(\cdot)) - V''(\bar{S}_u^\varepsilon(\cdot)) \} Z_{0,u}^\varepsilon(\eta, \cdot; S_\cdot^\varepsilon) |_{\tilde{r}}^2 du \}^{1/2} \\
& \leq \|\frac{\partial \psi}{\partial \alpha_i}\|_\infty |\varphi_i|_{-\tilde{r}} \sqrt{t} C_2 e^{C_1 \tilde{r} t/2} \text{EI} \{ I_{3,i}^\varepsilon(t) \}^{1/q} \{ I_{4,i}^\varepsilon(t) \}^{1/p}, \\
& \quad 0 < \varepsilon < 1, \quad 0 < \tilde{r} \leq r_0,
\end{aligned}$$

with $q > 2$ such that $1/p + 1/q = 1/2$, where

$$I_{3,i}^\varepsilon(t) = \int_0^t du \int_R \{ V''(S_u^\varepsilon(x)) - V''(\bar{S}_u^\varepsilon(x)) \}^q \theta(x, \tilde{r}) dx,$$

and

$$I_{4,i}^\varepsilon(t) = \int_0^t du \int_R |Z_{0,u}^\varepsilon(\eta, y; S_\cdot^\varepsilon)|^p \theta(y, \tilde{r}) dy.$$

We have used Schwarz's inequality and then applied Lemma 9.4 for the second inequality in (10.11). We estimate further as follows using Lemma 10.1:

$$I_{3,i}^\varepsilon(t) \leq (2\gamma_0)^{q-2} \gamma_*^2 \int_0^t |S_u^\varepsilon - \bar{S}_u^\varepsilon|_{\tilde{r}}^2 du$$

$$\leq (2\gamma_0)^{q-2} \gamma_*^2 C_2 |\omega(\cdot; S-\bar{S})|_{\tilde{r}}^2 e^{C_1 \tilde{r} t},$$

where $\gamma_* = \|V'''\|_\infty$. On the other hand,

$$\begin{aligned} I_{4,i}^\varepsilon(t) &\leq |\eta| \frac{p}{\bar{r}} \int_0^t du \int_R |Z_{0,u}^\varepsilon(\cdot, y; S_\cdot^\varepsilon)|_{-\bar{r}}^p \theta(y, \tilde{r}) dy \\ &\leq |\eta| \frac{p}{\bar{r}} \{M(\frac{p}{p-2}(\bar{r}''-r))\}^{-1+p/2} \\ &\quad \times \int_R \theta(x, -p\bar{r}''/2) dx \int_0^t du \int_R |Z_{0,u}^\varepsilon(x, y; S_\cdot^\varepsilon)|^p \theta(y, \tilde{r}) dy \\ &\leq |\eta| \frac{p}{\bar{r}} \{M(\frac{p}{p-2}(\bar{r}''-r))\}^{-1+p/2} \\ &\quad \times M(\tilde{r}-p\bar{r}''/2) C_2^{(p)} \tilde{r}^{-(p-1)/2} t^{(7-3p)/4} e^{C_1 t \tilde{r}}, \\ &\quad 0 < \tilde{r} \leq (p-1)r_0/2, \quad 0 < \bar{r} < \bar{r}'' < 2\tilde{r}/p, \end{aligned}$$

where

$$M(r) = \int \theta(x, r) dx < \infty \quad \text{if } r > 0.$$

We have used Hölder's inequality for the second inequality and Lemma 9.5 for the third inequality. Now the combination of these estimates leads us to the conclusion by taking $\alpha = 2/q$, choosing $\tilde{r}$ and $\bar{r}$ in such a way that $0 < p\bar{r}/2 < \tilde{r} < \{a_0/C_1 \wedge (p-1)r_0/2\}$, and noting an equality:

$$|R_a D\Psi^\varepsilon(\cdot, S) - R_a D\Psi^\varepsilon(\cdot, \bar{S})|_{-\bar{r}} = \sup_{|\eta|_{\bar{r}}=1} |\langle R_a D\Psi^\varepsilon(\cdot, S) - R_a D\Psi^\varepsilon(\cdot, \bar{S}), \eta \rangle|. \quad \square$$

We may assume that two $\bar{r}(a_0)$'s appearing in Lemmas 10.4 and 10.5 are common.

Proposition 10.2 For every $0 < a_0 < a_1 < \infty$ and $\{b_r > 0; r > 0\}$, the family of $H_{-\bar{r}(a_0)}$ -valued functions $\{R_a D\Psi^\varepsilon(\cdot, S); 0 < \varepsilon < 1\}$ restricted on $[a_0, a_1] \times B(\{b_r\})$ is relatively compact in the space $C([a_0, a_1] \times B(\{b_r\}), H_{-\bar{r}(a_0)})$ equipped with the uniform-convergence

topology.

Proof We apply Ascoli-Arzela's theorem using Lemmas 10.4 and 10.5.

We note that the equicontinuity in a follows from

$$\sup \left\{ \left| \frac{\partial}{\partial a} R_a D\Psi^\varepsilon(\cdot, S) \right|_{-\overline{r}(a)}; a \in [a_0, a_1], 0 < \varepsilon < 1, S \in H_e \right\} < \infty,$$

which can be shown similarly to (10.9). Therefore the proof can be completed in a similar manner to that of Proposition 10.1. $\square$

§11 Compactness argument II

Here in this section we shall prove the compactness of the families $\{\Psi_t^\varepsilon(S)\}_{0 < \varepsilon < 1}$ and $\{D\Psi_t^\varepsilon(\cdot, S)\}_{0 < \varepsilon < 1}$ regarding as functions of t . We begin with preparing the following uniform estimate in ε on the solution S_t^ε of the SPDE (0.10).

Lemma 11.1 For every $\varphi \in C_0^\infty(R)$, there exist positive constants C_1 and $C_2 = C_2(\varphi)$ such that

$$E_S[\langle S_t^\varepsilon, \varphi \rangle^2] \leq C_2 (1 + |S|_r^2) \{ e^{C_1 r t} + t^2 \}$$

holds for every $0 < \varepsilon < 1$, $0 < r < 1$, $t \geq 0$ and $S \in H_e$.

Proof Let $\{S_{t,i}^\varepsilon\}_{i=1}^3$ be three functions defined similarly to $\{S_{t,i}\}_{i=1}^3$, respectively, which were introduced in Section 5; we replace q by q^ε and, in addition, S_0 by S for $S_{t,1}^\varepsilon$ and $S_u(y)$ by $S_u^\varepsilon(y)$ for $S_{t,3}^\varepsilon$. Then an equality $S_t^\varepsilon = S_{t,1}^\varepsilon - \sqrt{2}S_{t,2}^\varepsilon + S_{t,3}^\varepsilon$ holds. Each term can be estimated as follows:

$$\begin{aligned} |\langle S_{t,1}^\varepsilon, \varphi \rangle| &= |\langle S, q^\varepsilon(t, \cdot) * \varphi \rangle| \\ &\leq |S|_r \|\varphi\|_{1,-r} \|q_{\varepsilon^2 t}^{(1)}\|_{1,-r} \|q_t^{(2)}\|_{1,-r}, \end{aligned}$$

by using Hausdorff-Young's inequality, where $\|\cdot\|_{1,-r}$ stands for the norm of the space $L^1(R, \theta(x, -r)dx)$. However, it is not difficult to show the following estimates with some $K, L > 0$:

$$\|q_{\varepsilon^2 t}^{(1)}\|_{1,-r} \leq K e^{Lrt}, \quad \|q_t^{(2)}\|_{1,-r} \leq K e^{Lrt}, \quad \text{for every}$$

$0 < \varepsilon < 1, 0 \leq r < 1$ and $t > 0$. Therefore we obtain

$$(11.1) \quad |\langle S_{t,1}^\varepsilon, \varphi \rangle| \leq |S|_r \|\varphi\|_{1,-r} K^2 e^{2Lrt}.$$

For the second term, we have

$$\begin{aligned} (11.2) \quad E[\langle S_{t,2}^\varepsilon, \varphi \rangle^2] &= \int_0^t \|q_u^\varepsilon * \nabla \varphi\|_{L^2(R, dx)}^2 du \\ &\leq \|\nabla \varphi\|_{L^2(R, dx)}^2 \int_0^t \|q_u^{(1)}\|_{L^1(R, dx)}^2 \|q_u^{(2)}\|_{L^1(R, dx)}^2 du \\ &\leq K^4 \|\nabla \varphi\|_{L^2(R, dx)}^2 t. \end{aligned}$$

Finally, for the third term:

$$\begin{aligned} (11.3) \quad |\langle S_{t,3}^\varepsilon, \varphi \rangle| &\leq \|V'\|_\infty \int_0^t \|q_u^\varepsilon * \Delta \varphi\|_{L^1(R, dx)} du \\ &\leq K \|V'\|_\infty \|\Delta \varphi\|_{L^1(R, dx)} t. \end{aligned}$$

Now the combination of (11.1)-(11.3) leads us to the conclusion. $\square$

Lemma 11.2 (i) $\sup\{|\Psi_t^\varepsilon(S)|; 0 < \varepsilon < 1, t \geq 0, S \in H_e\} < \infty$

(ii) As functions of t , a family $\{\Psi_t^\varepsilon(S); 0 < \varepsilon < 1, S \in H_e: |S|_r \leq b\}$ is relatively compact in the space $C([0, \infty))$ equipped with the usual compact-open topology for every $0 < r < 1$ and $b > 0$.

Proof The assertion (i) is trivially shown since $|\Psi_t^\varepsilon(S)| \leq \|\psi\|_\infty$.

For the assertion (ii), we may only prove the equicontinuity in t of this family. Indeed this follows by showing

$$(11.4) \quad \sup\{|\mathcal{E}_S[\Psi_t^\varepsilon(S)]|; 0 < \varepsilon < 1, 0 \leq t \leq T, |S|_r \leq b\} < \infty,$$

for every $T > 0$, since we have

$$(11.5) \quad \frac{\partial}{\partial t} \Psi_t^\varepsilon(S) = E_S[\mathcal{G}^\varepsilon \Psi(S_t^\varepsilon)] , \quad t > 0.$$

However, the uniform estimate (11.4) can be proved by using Lemma 11.1 and by noting that

$$|\mathcal{G}^\varepsilon \Psi(S)| \leq C \sum_{i=1}^k \{1 + |\langle S, \Delta^2 \varphi_i \rangle| + |\langle S, \Delta \varphi_i \rangle|\} , \quad 0 < \varepsilon < 1,$$

with a constant C which is independent of ε . $\square$

Before discussing the compactness of $\{D\Psi_t^\varepsilon(\cdot, S)\}$, we prepare the following lemma in which we put

$$Z_{0,t}^\varepsilon(\eta, \varphi; c) \equiv \langle Z_{0,t}^\varepsilon(\eta, \cdot; c), \varphi \rangle = \langle Z_{0,t}^\varepsilon(\cdot, \varphi; c), \eta \rangle .$$

Lemma 11.3 For every $\varphi \in C_0^\infty(R)$ and $0 < r < r' \leq r_0$, there exists a positive constant $C_2 = C_2(\varphi, r, r')$ such that

$$(11.6) \quad |Z_{0,t}^\varepsilon(\nabla^k \eta, \varphi; c) - Z_{0,t'}^\varepsilon(\nabla^k \eta, \varphi; c)| \leq C_2 |t' - t|^{1/2} e^{C_1 r' t'/2} |\eta|_r$$

holds for every $k = 0, 1$, $0 < \varepsilon < 1$, $0 < t < t' < \infty$, $\eta \in H_r$ and $c \in C_b$; $\|c\|_\infty \leq \gamma_0$.

Here two constants r_0 and C_1 are the same as in Lemma 9.3.

Proof Noting Lemma 7.7, we may assume $c \in C_0^\infty([0, \infty) \times R)$. Then we have

$$\frac{\partial}{\partial t} Z_{0,t}^\varepsilon(\eta, \varphi; c) = -\langle Z_{0,t}^\varepsilon(\eta, \cdot; c), A^\varepsilon \varphi \rangle + \langle c(t) Z_{0,t}^\varepsilon(\eta, \cdot; c), \Delta \varphi \rangle, \quad t > 0.$$

Therefore, using (9.7) and (10.8), the LHS of (11.6) with $k = 0$ can be bounded from above by

$$\begin{aligned} & (\gamma_0 |\Delta \varphi|_{-r} + |A^\varepsilon \varphi|_{-r}) \int_t^{t'} |Z_{0,u}^\varepsilon(\eta, \cdot; c)|_r du \\ & \leq ((\gamma_0 + \gamma) |\Delta \varphi|_{-r} + |\Delta^2 \varphi|_{-r}) (t' - t)^{1/2} \sqrt{C_2} e^{C_1 r' t'/2} C_{r', r} |\eta|_r . \end{aligned}$$

The case of $k = 1$ can be also discussed in a quite parallel manner

by using (9.8) instead of (9.7). $\square$

Lemma 11.4 A family $\{D\Psi_t^\varepsilon(\cdot, S); 0 < \varepsilon < 1, |S|_{\tilde{r}} \leq b\}$ is relatively compact in the space $C([0, \infty), H_{-\bar{r}})$ for every $0 < \bar{r} < r_0$; $0 < \tilde{r} < 1$ and $b > 0$ and has the following uniform bound with some $C_3 = C_3(\Psi, \bar{r})$:

$$(11.7) \quad |D\Psi_t^\varepsilon(\cdot, S)|_{-\bar{r}} \leq C_3(1+\sqrt{t}) e^{C_1 \bar{r} t/2}$$

for every $0 < \varepsilon < 1$, $0 \leq t < \infty$ and $S \in H_e$.

Proof The estimate (11.6) with $t = 0$ proves

$$(11.8) \quad |Z_{0,t}^\varepsilon(\cdot, \varphi; c)|_{-r} \leq |\varphi|_{-r} + C_2(\varphi, r, r') \sqrt{t} e^{C_1 r' t/2}$$

and

$$(11.9) \quad |\nabla Z_{0,t}^\varepsilon(\cdot, \varphi; c)|_{-r} \leq |\nabla \varphi|_{-r} + C_2(\varphi, r, r') \sqrt{t} e^{C_1 r' t/2}$$

for every $0 < \varepsilon < 1$, $0 < t < \infty$ and $c; \|c\|_\infty \leq \gamma_0$, where $\nabla Z_{0,t}^\varepsilon$ is the derivative in the distribution's sense. Now the uniform estimate (11.7) follows from (11.8) combined with Corollary 8.1-(ii), while the relative compactness of the family $\{D\Psi_t^\varepsilon(\cdot, S); 0 < \varepsilon < 1, S \in H_e, 0 \leq t \leq T\}$, $T > 0$, in the space $H_{-\bar{r}}$, $0 < \bar{r} < r$, follows from (11.8), (11.9) and Lemma 10.3 by using similar arguments developed in the proof of Lemma 10.4. Therefore, to complete the proof of the lemma, we may only show the equicontinuity of the family $\{D\Psi_t^\varepsilon(\cdot, S)\}$ in t . For this purpose, we notice

$$(11.10) \quad |D\Psi_t^\varepsilon(\cdot, S) - D\Psi_{t'}^\varepsilon(\cdot, S)|_{-\bar{r}} = \sup_{|\eta|_{\bar{r}}=1} \left| \sum_{i=1}^k \{I_{1,i} + I_{2,i}\} \right|, \quad 0 \leq t < t',$$

where

$$I_{1,i} = I_{1,i}(\eta, S) = E_S[\Psi_i(S_t^\varepsilon)(Z_{0,t}^\varepsilon(\eta, \varphi_i; S_t^\varepsilon) - Z_{0,t}^\varepsilon(\eta, \varphi_i; S_{t'}^\varepsilon))],$$

$$I_{2,i} = I_{2,i}(\eta, S) = E_S[(\Psi_i(S_t^\varepsilon) - \Psi_i(S_{t'}^\varepsilon))Z_{0,t}^\varepsilon(\eta, \varphi_i; S_t^\varepsilon)],$$

and Ψ_i is the function introduced in the proof of Lemma 10.5.

Lemma 11.3 can be applied to derive an estimate on $I_{1,i}$:

$$(11.11) \quad |I_{1,i}| \leq \left\| \frac{\partial \Psi}{\partial \alpha_i} \right\|_{\infty} C_2 (t' - t)^{1/2} e^{C_1 r' t' / 2} |\eta|_{\bar{r}}, \quad \bar{r} < r' \leq r_0.$$

As for the $I_{2,i}$, we see

$$|\Psi_i(S_t^{\varepsilon}) - \Psi_i(S_t^{\varepsilon})| \leq \sum_{j=1}^k \left\| \frac{\partial^2 \Psi}{\partial \alpha_i \partial \alpha_j} \right\|_{\infty} |\langle S_t^{\varepsilon}, -S_t^{\varepsilon}, \varphi_j \rangle|$$

and

$$\begin{aligned} E[|\langle S_t^{\varepsilon}, -S_t^{\varepsilon}, \varphi \rangle|] &\leq \int_t^{t'} E[|\langle S_u^{\varepsilon}, A^{\varepsilon} \varphi \rangle| + |\langle V'(S_u^{\varepsilon}), \Delta \varphi \rangle|] du \\ &\quad + \sqrt{2\varepsilon} E[|\langle w_t, -w_t, \nabla \varphi \rangle|] \\ &\leq \text{const.} (t' - t)^{1/2} \end{aligned}$$

for every $0 < \varepsilon < 1$, $0 < t < t' \leq T$, $|S|_{\bar{r}} \leq b$ with a const. depending only on (φ, T, b) ; we have used Lemma 11.1 to estimate the term $E[|\langle S_u^{\varepsilon}, A^{\varepsilon} \varphi \rangle|]$. Therefore, we can complete the work giving a bound on $I_{2,i}$ using (11.8). Now the conclusion follows by combining this with (11.11). $\square$

Corollary 11.1 For every $S, \eta \in H_e$, a family of functions $\{\frac{1}{\delta}(\Psi_t^{\varepsilon}(S + \delta \eta) - \Psi_t^{\varepsilon}(S)) ; 0 < \varepsilon, \delta < 1\}$ is relatively compact in the space $C([0, \infty))$.

Proof We can show the uniform-boundedness and the equicontinuity of this family from an equality:

$$\frac{1}{\delta}(\Psi_t^{\varepsilon}(S + \delta \eta) - \Psi_t^{\varepsilon}(S)) = \frac{1}{\delta} \int_0^{\delta} D\Psi_t^{\varepsilon}(\eta, S + \delta' \eta) d\delta' . \quad \square$$

We conclude this section by making the definition of an operator D more restrictive. With fixed $\bar{r} > 0$, we define an operator $(D, \mathcal{D}(D))$ on the locally convex space $C(H_{e,w})$ as follows: The domain $\mathcal{D}(D)$ consists of all $\Psi \in C(H_{e,w})$ which are Fréchet

differentiable on H_e and satisfy $D\Psi \in C(H_{e,w}, H_{-\bar{r}})$. For $\Psi \in \mathcal{D}(D)$, we set $D\Psi(\cdot, S)$ = the Fréchet derivative of Ψ at S . The following lemma is not difficult to be proved so that we omit the proof.

Lemma 11.5 The operator D defined as above is "closed" in the following sense: Let $(\Psi^\varepsilon)_{0 < \varepsilon < 1}$, $\Psi \in C(H_{e,w})$ and $\Phi \in C(H_{e,w}, H_{-\bar{r}})$ be given and satisfy that Ψ^ε and $D\Psi^\varepsilon$ converge as $\varepsilon \downarrow 0$ to Ψ and Φ , respectively, uniformly on each weakly compact ball $B(\{b_r\})$ of $H_{e,w}$. Then we have $\Psi \in \mathcal{D}(D)$ and $D\Psi = \Phi$.

§12 The proof of main theorem

We conclude the proof of Theorem 1.1 dividing it into three steps.

Step 1: Convergence of Ψ^ε and $D\Psi^\varepsilon$ We fix an increasing sequence of weakly compact balls $(B_n = B(\{b_r^{(n)}\}))_{n=1}^\infty$ in $H_{e,w}$ satisfying $\bigcup_{n=1}^\infty B_n = H_{e,w}$. Take any $a_0 \in (0, \infty)$ and subsequence $\{\varepsilon' \downarrow 0\}$ of $\{\varepsilon\}$. We set again $\bar{r}$ the smaller number of $\bar{r}(a_0)$ and $\bar{r}$ appearing in Proposition 10.2 and Lemma 11.4, respectively.

Lemma 12.1 (i) There exist $\hat{\Psi}_a^{(1)}(S) \in C_b([a_0, \infty) \times H_{e,w})$, $\hat{\Psi}_a^{(2)}(\cdot, S) \in C_b([a_0, \infty) \times H_{e,w}, H_{-\bar{r}})$ and a subsequence $\{\varepsilon'' \downarrow 0\}$ of $\{\varepsilon'\}$ such that $R_a \Psi^{\varepsilon''}(S)$ and $R_a D\Psi^{\varepsilon''}(\cdot, S)$ converge to $\hat{\Psi}_a^{(1)}(S)$ and $\hat{\Psi}_a^{(2)}(\cdot, S)$, respectively, uniformly on $[a_0, a_1] \times B_n$ for every a_1 ; $a_1 > a_0$ and $n \in \mathbb{N}$ as $\varepsilon'' \downarrow 0$.

(ii) We have $\hat{\Psi}_a^{(1)} \in \mathcal{D}(D)$ and $D\hat{\Psi}_a^{(1)}(\cdot, S) = \hat{\Psi}_a^{(2)}(\cdot, S)$ for every

$a \in [a_0, \infty)$.

Proof To prove the assertion (i), we may use diagonal method noting Propositions 10.1 and 10.2. The boundedness of the limiting functions $\hat{\Psi}_a^{(1)}(S)$ and $\hat{\Psi}_a^{(2)}(\cdot, S)$ follows from

$$|\hat{\Psi}_a^{(1)}(S)| = \lim_{\varepsilon'' \downarrow 0} |R_a \Psi^{\varepsilon''}(S)| \leq \|\psi\|_{\infty} / a$$

and (10.9), respectively. The assertion (ii) is a consequence of Lemma 11.5. $\square$

Lemma 12.2 For every $\lambda(\cdot) \in \Lambda$ and $a \in [a_0, \infty)$, we have

$$(12.1) \quad \lim_{\varepsilon'' \downarrow 0} E^{\mu_{\lambda(\cdot), \varepsilon''}} [R_a \Psi^{\varepsilon''}(S)] = \hat{\Psi}_a^{(1)}(\rho),$$

$$(12.2) \quad \lim_{\varepsilon'' \downarrow 0} E^{\mu_{\lambda(\cdot), \varepsilon''}} [\langle \Delta \lambda(\cdot), R_a D \Psi^{\varepsilon''}(\cdot, S) \rangle] = \langle \Delta \lambda(\cdot), \hat{\Psi}_a^{(2)}(\cdot, \rho) \rangle,$$

where $\rho \equiv \rho(\cdot) = \bar{\rho}(\lambda(\cdot))$ and $\bar{\rho}$ is the mean spin function.

Proof For the proof of (12.1), we see

$$\begin{aligned} & |E^{\mu_{\lambda(\cdot), \varepsilon''}} [R_a \Psi^{\varepsilon''}(S)] - \hat{\Psi}_a^{(1)}(\rho)| \\ & \leq |E^{\mu_{\lambda(\cdot), \varepsilon''}} [(R_a \Psi^{\varepsilon''} - \hat{\Psi}_a^{(1)})(S)]| + |E^{\mu_{\lambda(\cdot), \varepsilon''}} [\hat{\Psi}_a^{(1)}(S)] - \hat{\Psi}_a^{(1)}(\rho)| \\ & \equiv I_1^{\varepsilon''} + I_2^{\varepsilon''}. \end{aligned}$$

Since $\hat{\Psi}_a^{(1)} \in C_b(H_{e,w})$ for every $a \geq a_0$, Theorem 4.1 implies

$\lim_{\varepsilon'' \downarrow 0} I_2^{\varepsilon''} = 0$. On the other hand, we have

$$\begin{aligned} I_1^{\varepsilon''} & \leq \sup_{S \in B_n} |(R_a \Psi^{\varepsilon''} - \hat{\Psi}_a^{(1)})(S)| + 2\|\psi\|_{\infty} \mu_{\lambda(\cdot), \varepsilon''}(B_n^c) / a \\ & \equiv I_{3,n}^{\varepsilon''} + I_{4,n}^{\varepsilon''}, \end{aligned}$$

for every $n \in \mathbb{N}$. Lemma 12.1 shows $\lim_{\varepsilon'' \downarrow 0} I_{3,n}^{\varepsilon''} = 0$, $n \in \mathbb{N}$, while

Chebyshev's inequality can be applied to see

$$I_{4,n}^{\varepsilon''} \leq 2\|\psi\|_{\infty} \{b_r^{(n)}\}^{-2} E^{\mu_{\lambda}(\cdot), \varepsilon''} [|S|_r^2] / a ,$$

which converges to 0 as $n \rightarrow \infty$ uniformly in ε'' ; see Proposition 3.1-(iii). Therefore we obtain (12.1). The convergence (12.2) can be shown similarly. $\square$

Lemma 12.3 (i) For every $S \in H_{e,w}$, there exist $\Psi_t^{(1)}(S) \in C_b([0, \infty))$ and $\Psi_t^{(2)}(\cdot, S) \in C([0, \infty), H_{-\bar{r}})$ such that $\Psi_t^{\varepsilon''}(S)$ and $D\Psi_t^{\varepsilon''}(\cdot, S)$ converge to $\Psi_t^{(1)}(S)$ and $\Psi_t^{(2)}(\cdot, S)$, respectively, uniformly on each bounded interval of $[0, \infty)$ as $\varepsilon'' \downarrow 0$.

(ii) We have an estimate:

$$(12.3) \quad |\Psi_t^{(2)}(\cdot, S)|_{-\bar{r}} \leq C_3(1+\sqrt{t}) e^{C_1 r t / 2}, \quad 0 < r < \bar{r},$$

with constants C_1 and $C_3 = C_3(\Psi, r)$ appearing in Lemma 11.4.

(iii) We have $\hat{\Psi}_a^{(1)}(S) = R_a \Psi^{(1)}(S)$ and $\hat{\Psi}_a^{(2)}(\cdot, S) = R_a \{\Psi^{(2)}(\cdot, S)\}$.

Proof Using Lemma 11.2, we are able to find from every subsequence $\{\varepsilon''' \downarrow 0\}$ of $\{\varepsilon''\}$ further subsequence $\{\varepsilon'''' \downarrow 0\}$ such that $\Psi_t^{\varepsilon''''}(S)$ converges to some $\Psi_t^{(1)}(S) \in C_b([0, \infty))$ as $\varepsilon'''' \downarrow 0$ in this space. However, since we have

$$R_a \Psi^{(1)}(S) = \lim_{\varepsilon'''' \downarrow 0} R_a \Psi^{\varepsilon''''}(S) = \hat{\Psi}_a^{(1)}(S), \quad a \in [a_0, \infty),$$

the uniqueness of the Laplace transform proves that the limit $\Psi_t^{(1)}(S)$ does not depend on $\{\varepsilon''''\}$ and therefore $\Psi_t^{\varepsilon''}(S)$ itself converges to $\Psi_t^{(1)}(S)$ as $\varepsilon'' \downarrow 0$ in $C_b([0, \infty))$. The assertions on $\{D\Psi_t^{\varepsilon''}(\cdot, S)\}$ can be shown similarly by using Lemma 11.4. $\square$

Lemma 12.4 The function $\Psi_t^{(1)}(S)$, $t > 0$, is Fréchet differentiable on H_e and an equality $D\Psi_t^{(1)}(\cdot, S) = \Psi_t^{(2)}(\cdot, S)$ holds. We especially have $D R_a \Psi^{(1)}(\cdot, S) = R_a \{D\Psi^{(1)}(\cdot, S)\}$.

Proof From the relation:

$$(12.4) \quad (DR_a \Psi^{(1)})(\cdot, S) = (D\hat{\Psi}_a^{(1)})(\cdot, S) = \hat{\Psi}_a^{(2)}(\cdot, S) = R_a \Psi^{(2)}(\cdot, S),$$

we have for every $S, \eta \in H_e$ and $a \geq a_0$:

$$\lim_{\delta \downarrow 0} R_a \left[\frac{1}{\delta} \{ \Psi^{(1)}(S + \delta \eta) - \Psi^{(1)}(S) \} \right] = R_a \langle \Psi^{(2)}(\cdot, S), \eta \rangle.$$

Therefore the continuity theorem for the Laplace transform (Feller [12]) implies that $\frac{1}{\delta} \{ \Psi_t^{(1)}(S + \delta \eta) - \Psi_t^{(1)}(S) \} dt$ converges to $\langle \Psi_t^{(2)}(\cdot, S), \eta \rangle dt$ as $\delta \downarrow 0$ as measures on $[0, \infty)$. On the other hand, Corollary 11.1 combined with Lemma 12.3 proves that the family $\{ \frac{1}{\delta} \{ \Psi_t^{(1)}(S + \delta \eta) - \Psi_t^{(1)}(S) \} \}_{0 < \delta < 1}$ is relatively compact in the space $C([0, \infty))$. Therefore a convergence:

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \{ \Psi_t^{(1)}(S + \delta \eta) - \Psi_t^{(1)}(S) \} = \langle \Psi_t^{(2)}(\cdot, S), \eta \rangle$$

holds in the space $C([0, \infty))$, which implies that $\Psi_t^{(1)}(S)$ is Fréchet differentiable at S and $D\Psi_t^{(1)}(\cdot, S) = \Psi_t^{(2)}(\cdot, S)$. The equality (12.4) combined with this conclusion implies $DR_a \Psi^{(1)}(\cdot, S) = R_a \{ D\Psi^{(1)}(\cdot, S) \}$. $\square$

Step 2: Derivation of the Liouville equation

Lemma 12.5 For every $a > 0$, $0 < \varepsilon < 1$ and $\lambda(\cdot) \in \Lambda$,

$${}_a E^{\mu_{\lambda(\cdot), \varepsilon}} [R_a \Psi^\varepsilon(S)] = E^{\mu_{\lambda(\cdot), \varepsilon}} [\Psi(S)] + E^{\mu_{\lambda(\cdot), \varepsilon}} [\langle \Delta \lambda(\cdot), DR_a \Psi^\varepsilon(\cdot, S) \rangle].$$

Proof Since $\Psi(S_t^\varepsilon) - \int_0^t \mathcal{G}^\varepsilon \Psi(S_u^\varepsilon) du$ is a martingale, we have,

$$\begin{aligned} {}_a R_a \Psi^\varepsilon(S) &= a \int_0^\infty e^{-at} \Psi(S) dt + a \int_0^\infty e^{-at} dt \int_0^t E_S [\mathcal{G}^\varepsilon \Psi(S_u^\varepsilon)] du \\ &= \Psi(S) + \int_0^\infty e^{-at} E_S [\mathcal{G}^\varepsilon \Psi(S_t^\varepsilon)] dt. \end{aligned}$$

In this calculation, we have used integration by parts by noting the

result of Lemma 11.1 (see also the estimation on $|\mathcal{G}^\varepsilon \Psi(S)|$ in the proof of Lemma 11.2). Therefore the conclusion follows from Corollary 8.1 and an equality: $R_a \{D\Psi^\varepsilon(\cdot, S)\} = DR_a \Psi^\varepsilon(\cdot, S)$. $\square$

The following is an immediate consequence of Lemmas 12.2-12.5.

Corollary 12.1 For every $a \geq a_0$ and $\rho = \rho(\cdot) \in \bar{\rho}(\Lambda)$, we have

$$(12.5) \quad aR_a \Psi^{(1)}(\rho) = \Psi(\rho) + \langle \Delta\{\bar{\lambda}(\rho(\cdot))\}, DR_a \Psi^{(1)}(\cdot, \rho) \rangle,$$

where $\bar{\lambda}$ is an inverse function of the mean spin function $\bar{\rho}$ and $\bar{\rho}(\Lambda)$ is a class of functions $\rho(\cdot)$ having the forms $\rho(\cdot) = \bar{\rho}(\lambda(\cdot))$ with $\lambda(\cdot) \in \Lambda$.

Now we prepare the following lemma.

Lemma 12.6 $\lim_{\lambda \rightarrow \pm\infty} \bar{\rho}(\lambda) = \pm\infty$.

Proof Consider self-adjoint operators $\tilde{H}_0 = -\frac{1}{2} \frac{d^2}{ds^2} + \frac{\gamma}{2} s^2$ and $\tilde{H}_W = \tilde{H}_0 + W$ defined on the space $L^2(R, ds)$ for a bounded function W and let $\tilde{\Omega}_W$ be a positive and normalized eigenfunction of $\tilde{H}_W$ corresponding to its minimal eigenvalue $\tilde{\kappa}(W)$. Then Rayleigh-Ritz principle (Simon [37, p 199]) proves

$$|\tilde{\kappa}(W) - \tilde{\kappa}(0)| \leq \|W\|_\infty.$$

Using this estimation, similar argument employed by Reed and Simon [32, IV p 251] shows

$$\begin{aligned} \tilde{\Omega}_W(s) &= e^{t\{\tilde{\kappa}(W) + \|W\|_\infty\}} e^{-t\{\tilde{H}_W + \|W\|_\infty\}} \tilde{\Omega}_W(s) \\ &\leq e^{t\{\tilde{\kappa}(0) + 2\|W\|_\infty\}} e^{-t\tilde{H}_0} \tilde{\Omega}_W(s) \end{aligned}$$

The RHS can be estimated further by noting $\|\tilde{\Omega}_W\|_{L^2} = 1$ and using Mehler's formula and finally, by taking $t = 1$, we obtain

$$(12.6) \quad \tilde{\Omega}_W(s) \leq C_2 \exp\{2\|W\|_\infty - C_1 s^2\}, \quad s \in \mathbb{R},$$

with some positive constants C_1 and C_2 which are independent of W and s .

Now we look at the mean spin function $\bar{\rho}(\lambda)$. Since the ground state Ω_λ of the operator H_λ can be represented as $\Omega_\lambda(s) = \tilde{\Omega}_{V(\cdot + \lambda/\gamma)}(s - \lambda/\gamma)$, we have from (12.6):

$$\begin{aligned} |\bar{\rho}(\lambda) - \frac{\lambda}{\gamma}| &= \left| \int_{\mathbb{R}} s \tilde{\Omega}_{V(\cdot + \lambda/\gamma)}^2(s) ds \right| \\ &\leq C_2^2 e^{4\|V\|_\infty} \int_{\mathbb{R}} |s| e^{-2C_1 s^2} ds < \infty, \end{aligned}$$

and this implies the conclusion. $\square$

Lemma 12.7 The equality (12.5) holds for every $a \geq a_0$ and $\rho \in C_b^2(\mathbb{R})$.

Proof First we note that Lemma 12.6 proves $\bar{\rho}(\Lambda) = \Lambda$, since the function $\bar{\rho} = \bar{\rho}(\lambda)$ is real analytic and strictly increasing in λ . For every $\rho \in C_b^2(\mathbb{R})$, we can take a sequence $\{\rho_n \in \Lambda\}_{n=1}^\infty$ in such a manner that (a) ρ_n converges to ρ in the space $H_{e,w}$ (i.e., for every neighborhood U of ρ , $\rho_n \in U$ for all sufficiently large n) and (b) $\Delta\{\bar{\lambda}(\rho_n(\cdot))\}$ converges to $\Delta\{\bar{\lambda}(\rho(\cdot))\}$ in the space H_{-r} . Therefore we obtain the conclusion from Corollary 12.1 because $R_a \Psi^{(1)} \in C(H_{e,w})$ and $DR_a \Psi^{(1)}(\cdot, \cdot) \in C_b(H_{e,w}, H_{-r})$. $\square$

Lemma 12.8 For every $\rho \in C_b^2(\mathbb{R})$, we have

$$(12.7) \quad \Psi_t^{(1)}(\rho) = \Psi(\rho) + \int_0^t \langle \Delta\{\bar{\lambda}(\rho(\cdot))\}, D\Psi_u^{(1)}(\cdot, \rho) \rangle du, \quad t \geq 0.$$

Proof Noting the estimate (12.3) with r sufficiently small, we can apply integration by parts to see:

$$\text{the RHS of (12.5)} = a \int_0^\infty e^{-at} dt \text{ (the RHS of (12.7)) ,}$$

for every $\rho \in C_b^2(\mathbb{R})$. Therefore Lemma 12.7 implies with the help of the uniqueness of the Laplace transform that (12.7) holds for a.e. t and consequently for every $t \geq 0$, since the both sides are continuous in t . $\square$

Step 3: Identification of the limit See the book of Ladyzenskaya, Solonnikov and Ural'ceva [28, Theorem 8.1] for the following result on the nonlinear diffusion equation (0.11).

Theorem Assume the initial value $\rho_0 \in C_b^{2+\beta}(\mathbb{R})$, $0 < \beta < 1$.

(i) There exists a classical solution ρ_t of (0.11) belonging to the class $H^{2+\beta, 1+\beta/2}(\mathbb{R} \times [0, T])$ for every $T > 0$, see [28, p 7] for the definition of this class.

(ii) The classical solution of (0.11) satisfying the following condition is unique:

$$\sup \left\{ \sum_{k=0}^2 |\nabla^k \rho_t(x)| ; t \in [0, T], x \in \mathbb{R} \right\} < \infty, \quad T > 0.$$

We denote by $\rho_t(\rho) = \rho_t(\cdot; \rho)$ the unique solution of (0.11) with initial value $\rho \in C_b^{2+\beta}(\mathbb{R})$.

Lemma 12.9 $\Psi_t^{(1)}(\rho) = \Psi(\rho_t(\rho))$ holds for every $t \geq 0$ and $\rho \in C_b^{2+\beta}(\mathbb{R})$.

Proof For every $t > 0$, consider a sequence of decompositions $\pi_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$ of the interval $[0, t]$, $n \in \mathbb{N}$,

satisfying $|\pi_n| = \sup_{1 \leq k \leq n} |t_k - t_{k-1}| \rightarrow 0$ as $n \rightarrow \infty$. Then, for $\rho \in C_b^{2+\beta}(\mathbb{R})$, Lemma 12.8 shows

$$\begin{aligned} \Psi_t^{(1)}(\rho) - \Psi_0^{(1)}(\rho_t) &= \sum_{k=1}^n (\Psi_{t_k}^{(1)}(\rho_{t-t_k}) - \Psi_{t_{k-1}}^{(1)}(\rho_{t-t_{k-1}})) \\ &= \sum_{k=1}^n (\Psi_{t_{k-1}}^{(1)}(\rho_{t-t_k}) - \Psi_{t_{k-1}}^{(1)}(\rho_{t-t_{k-1}}) \\ &\quad + \int_{t_{k-1}}^{t_k} \langle \Delta\{\bar{\lambda}(\rho_{t-t_k}(\cdot))\}, D\Psi_u^{(1)}(\cdot, \rho_{t-t_k}) \rangle du), \end{aligned}$$

where we denote simply $\rho_t = \rho_t(\rho)$. On the other hand, since $\Psi_t^{(1)}$ is Fréchet differentiable (see Lemma 12.4), we have

$$\begin{aligned} \Psi_{t_{k-1}}^{(1)}(\rho_{t-t_k}) - \Psi_{t_{k-1}}^{(1)}(\rho_{t-t_{k-1}}) \\ = - \langle D\Psi_{t_{k-1}}^{(1)}(\cdot, (1-\delta_k)\rho_{t-t_k} + \delta_k\rho_{t-t_{k-1}}), \rho_{t-t_{k-1}} - \rho_{t-t_k} \rangle \end{aligned}$$

with some $\delta_k \in (0,1)$. Therefore, noting that $\Psi_0^{(1)} = \Psi$ and ρ_t is a solution of (0.11), we obtain

$$\begin{aligned} &\Psi_t^{(1)}(\rho) - \Psi(\rho_t) \\ &= \int_0^t \sum_{k=1}^n 1_{[t_{k-1}, t_k]}(u) [\langle \Delta\{\bar{\lambda}(\rho_{t-t_k}(\cdot))\}, D\Psi_u^{(1)}(\cdot, \rho_{t-t_k}) \rangle \\ &\quad - \langle \Delta\{\bar{\lambda}(\rho_{t-u}(\cdot))\}, D\Psi_{t_{k-1}}^{(1)}(\cdot, (1-\delta_k)\rho_{t-t_k} + \delta_k\rho_{t-t_{k-1}}) \rangle] du. \end{aligned}$$

However, the integrand is uniformly bounded in n because of (12.3) and $\sup \{|\Delta\{\bar{\lambda}(\rho_u(x))\}|; u \in [0, t], x \in \mathbb{R}\} < \infty$. Therefore, Lebesgue's dominated convergence theorem verifies that the RHS of this equality tends to 0 as $n \rightarrow \infty$, since we see that $\Delta\{\bar{\lambda}(\rho_t(\cdot))\} \in C([0, \infty), H_R)$, $D\Psi_t^{(1)}(\cdot, \rho) \in C([0, \infty) \times H_{e,w}, H_{-r})$ and $\rho_t \in C([0, \infty), H_{e,w})$. $\square$

Theorem 12.1 (i) Assume $S \in C_b^{2+\beta}(\mathbb{R})$ with some $0 < \beta < 1$.

Then $E_S[\Psi(S_t^\varepsilon)]$ converges to $\Psi(\rho_t(S))$ uniformly in $t \in [0, T]$, $T > 0$, as $\varepsilon \downarrow 0$ for every $\Psi \in \mathcal{D}$.

(ii) Assume $\lambda(\cdot) \in \Lambda \cap C_b^{2+\beta}(\mathbb{R})$ with some $0 < \beta < 1$. Then $E^{\mu_{\lambda(\cdot)}, \varepsilon}[\Psi(S_t^\varepsilon)]$ converges to $\Psi(\rho_t)$ uniformly in $t \in [0, T]$, $T > 0$, as $\varepsilon \downarrow 0$ for every $\Psi \in \mathcal{D}$, where $\rho_t = \rho_t(\bar{\rho}(\lambda(\cdot)))$.

Proof (i) Lemmas 12.3 and 12.9 prove that $\Psi_t^{\varepsilon''}(S) \equiv E_S[\Psi(S_t^{\varepsilon''})]$ converges to $\Psi(\rho_t(S))$ uniformly in $t \in [0, T]$ as $\varepsilon'' \downarrow 0$. However the limit function $\Psi(\rho_t(S))$ is independent of the subsequence $\{\varepsilon''\}$. Therefore $\{\Psi_t^\varepsilon(S)\}$ itself converges to $\Psi(\rho_t(S))$ as $\varepsilon \downarrow 0$.

(ii) Similarly we see from (12.1), Lemma 12.3-(iii) and Lemma 12.9:

$$\lim_{\varepsilon \downarrow 0} R_a E^{\mu_{\lambda(\cdot)}, \varepsilon}[\Psi(S_t^\varepsilon)] = R_a \Psi(\rho_t) \quad , \quad a \geq a_0.$$

Therefore we have the assertion (ii), since we can show the family $\{E^{\mu_{\lambda(\cdot)}, \varepsilon}[\Psi(S_t^\varepsilon)] ; 0 < \varepsilon < 1\}$ is relatively compact in the space $C([0, \infty))$, see Proposition 3.1 and Lemma 11.2. $\square$

It is not difficult to see that the assertions of Theorem 1.1 and Theorem 12.1-(i) are mutually equivalent and therefore we complete the proof of Theorem 1.1.

Concluding remark The method employed in this paper seems to work for another types of TDGL equations:

$$dS_t(x) = -(-\Delta)^{\alpha+1} S_t(x) dt - (-\Delta)^\alpha \{U'(S_t(x))\} dt + \sqrt{2}(-\Delta)^{\alpha/2} dw_t(x)$$

with $\alpha = 0, 2, 3, \dots$, which are given by taking $A = (-\Delta)^\alpha$ in the equation (0.4). The appropriate scaling for this equation is

$$t \longrightarrow t/\varepsilon^{2\alpha} \quad , \quad x \longrightarrow x/\varepsilon$$

and we obtain after the scaling:

$$dS_t^\varepsilon(x) = -\varepsilon^2(-\Delta)^{\alpha+1}S_t^\varepsilon(x)dt - (-\Delta)^\alpha\{U'(S_t^\varepsilon(x))\}dt + \sqrt{2\varepsilon}(-\Delta)^{\alpha/2}dw_t(x)$$

Especially when $\alpha = 0$, no scaling is involved in the time variable. This is because of the non-conservative nature of the dynamics and the limiting hydrodynamical equation might be an ODE:

$$\frac{\partial}{\partial t} \rho_t(x) = -\bar{\lambda}(\rho_t(x)) \quad , \quad t > 0,$$

for each $x \in \mathbb{R}$. It is easy to see that

$$\lim_{t \rightarrow \infty} \rho_t(x) = \rho_* \quad ,$$

where $\rho_* \in \mathbb{R}$ is a unique point satisfying $\bar{\lambda}(\rho_*) = 0$. This contrasts highly with the conservative case where the limiting equation (0.11) has a family of stationary solutions.

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