**FINITE DIMENSIONAL DESCRIPTION OF
CONVECTIVE REACTION-DIFFUSION EQUATIONS**

By

Minkyu Kwak

IMA Preprint Series # 827

June 1991

FINITE DIMENSIONAL DESCRIPTION OF CONVECTIVE REACTION-DIFFUSION EQUATIONS*

MINKYU KWAK†

Abstract. We are concerned with the asymptotic dynamics of a certain type of semilinear parabolic equation, namely,

$$u_t = u_{xx} + (f(u))_x + g(u) + h(x)$$

on the interval $[0, L]$. Under the general condition we prove that this equation admits a dissipative dynamical system and it possesses the global attractor. But for large $L > 0$, we do not know whether there exists an inertial manifold or not.

Here we introduce a nonlinear change of variables so that we transform the above equation to reaction diffusion system which possess the exactly same asymptotic dynamics. We then prove the existence of inertial manifold for the transformed equation, thereby we find the ordinary differential equation which describe completely the long-time dynamics of the original equation.

Key words. attractors, dissipative, inertial manifold, ordinary differential equations, partial differential equations

1. INTRODUCTION.

During the last decade it has been shown that the long-time behavior of the solutions of a very large class of partial differential equations [PDEs] bears a striking resemblance to the behavior of solutions of ordinary differential equations [ODEs].

The first development for describing the dynamics was the discovery that dissipative partial differential equations have global attractors which have finite Hausdorff and fractal dimension (see Hale (1988), Temam (1988) and the references therein.)

Lately it has been shown that some of these PDEs have finite dimensional inertial manifolds which are invariant and will contain the global attractors. The concept of an inertial manifold allows us to go deeper into the study of dissipative systems. In fact when restricted to the inertial manifold, the long time dynamics of solutions of a PDE is completely determined by a finite dimensional ODE which is called an inertial form. There are many references, for example Chow and Lu (1988), Constantin, Foias, Nicolaenko, and Temam (1989), Foias, Sell, and Temam (1988), Luskin and Sell (1989), Mallet-Paret and Sell (1988), Miklavcic (1990), Sell and You (1990).

Up to now various theories have been developed on the existence of inertial manifolds and all the proofs have exploited the existence of suitably large spectral gaps for the principal linear partial differential operator. Therefore the existence of inertial manifolds

*This research was supported in part by grants from the National Science Foundation, the Applied and Computational Mathematics Program/DARPA, and the Army High Performance Computing Research Center.

†School of mathematics, University of Minnesota, Minneapolis, MN 55455.

is still unsolved for several important equations, for instance the Navier-Stokes equations in space dimension two.

In this paper we consider a scalar PDE of the form

$$(1.1) \quad u_t = u_{xx} + (f(u))_x + g(u) + h(x) \quad \text{on } I = [0, L]$$

with homogeneous Dirichlet boundary condition.

Under some general assumptions which we will clarify in section 2, the equation (1.1) is dissipative. But because of the gradient term in the nonlinearity, the spectral gap condition mentioned above is not satisfied for large $L > 0$. In fact it can be shown that there is a constant C_1 such that $C_1 = C_1(L) \rightarrow \infty$ as $L \rightarrow \infty$ and if the gap condition :

$$(1.2) \quad \lambda_{N+1} - \lambda_N \geq C_1(\lambda_{N+1}^{\frac{1}{2}} + \lambda_N^{\frac{1}{2}})$$

is satisfied, then the equation (1.1) has inertial manifold with dimension N . Here $\{\lambda_N = (\frac{2\pi N}{L})^2, N \geq 1\}$ denote the eigenvalues of the operator A induced by the Laplace operator with Dirichlet boundary condition on $[0, L]$, i.e.,

$$Au = -u_{xx}.$$

But because the gap condition is not satisfied for large L , we do not know whether there exists an inertial manifold or not.

In this paper we construct a new equation which possesses the exactly same long-time dynamics and which, best of all, has an inertial manifold for all $L > 0$. As a result we can prove that the dynamics near the global attractor is completely described by finite dimensional ODEs with no errors.

Let us explain our strategy. First of all we treat the solution of the equation (1.1) as a solution of a reaction diffusion system by introducing a nonlinear change of variables. Namely let u be a solution of (1.1) and let

$$J(u) = (u, u_x, f(u)),$$

then we find that $(u, v, w) = J(u)$ satisfies the equation

$$(1.3) \quad \begin{aligned} u_t &= u_{xx} + w_x + g(u) + h(x) \\ v_t &= v_{xx} + w_{xx} + g'(u)v + h'(x) \\ w_t &= w_{xx} - f''(u)v^2 + f'(u)^2v + f'(u)\{g(u) + h(x)\} \end{aligned}$$

where $'$ denotes the derivative of corresponding function. Conversely we see that if (u, v, w) is a solution of (1.3) with initial condition $(u, v, w)(0) = J(u_0)$, then from the uniqueness of solution $u(t)$ is a solution of (1.1) (see Proposition 2.5). In order to get to our results, we

need to modify the nonlinearity of (1.3) without losing the essential long-time dynamics. This process is explained in Section 2.

With a suitable modification on the nonlinearity we show that if $\mathcal{A}$ is global attractor of (1.1), then the image of $\mathcal{A}$ under the imbedding J is the global attractor of the modified reaction diffusion system (see Proposition 3.3). It follows that the study of the long-time behavior of solutions of (1.1) is transformed to the study of the modified equation of (1.3).

Next we apply inertial manifold theory to the modified equation of (1.3), for which the existence of inertial manifolds can be proved. As a matter of fact the linear differential operator of the modified equation of (1.3) generates a sectorial operator and we have removed the gradient term in nonlinearity by using the above nonlinear change of variables. Hence the new spectral gap condition becomes

$$(1.4) \quad \lambda_{N+1} - \lambda_N \geq C_2$$

for some constant C_2 such that $C_2 = C_2(L) \rightarrow \infty$ as $L \rightarrow \infty$. Thus we can find easily $N = N(L) > 0$ such that (1.4) is satisfied for all $L > 0$. And then we interpret the dynamics of the equation (1.1) by making use of the inertial form of the new equation. These two steps are described in Section 4.

ACKNOWLEDGEMENT.

The author wishes to express his gratitude to Prof. George R. Sell for the generous advices and stimulating discussions and he also thanks Dr. Mario Taboada for useful conversations.

2. PRELIMINARY RESULTS.

We consider a scalar partial differential equation of the form

$$(2.1) \quad u_t = u_{xx} + (f(u))_x + g(u) + h(x), \quad \text{on} \quad I = [0, L]$$

with homogeneous Dirichlet boundary condition, i.e.,

$$u(0, t) = 0, \quad u(L, t) = 0.$$

We assume that f is C^2 -function with $f(0) = f'(0) = 0$, g is a C^1 -function with $g(0) = 0$, and $h \in H_0^1(I)$. We also assume that f, g satisfy a certain growth conditions so that the equation (2.1) becomes a dissipative dynamical system. More precisely let us assume that there exist positive constants $\alpha_0 \geq 1, R$, and a positive integer m such that

$$(2.2) \quad |f'(u)| \leq \alpha_0(1 + |u|^m), \quad |f''(u)| \leq \alpha_0(1 + |u|^{m-1}), \quad |g'(u)| \leq \alpha_0(1 + |u|^m)$$

and

$$(2.3) \quad g(u)u \leq 0 \quad \text{for} \quad |u| \geq R.$$

We have local Burger's equation as a concrete example.

For the mathematical setting we consider Hilbert spaces $H = L^2(I)$ with standard norm and scalar product $|\cdot|, (\cdot, \cdot)$ respectively. We denote by A the sectorial operator in H , with domain $D(A)$, which is induced by Laplace operator with homogeneous boundary condition. We also denote by A^α the fractional power of A and its domain $V_\alpha \stackrel{\text{def}}{=} D(A^{\frac{\alpha}{2}})$. In particular we put $V = V_1 = D(A^{\frac{1}{2}})$. Notice that

$$D(A) = H^2(I) \cap H_0^1(I), \quad V = H_0^1(I),$$

see Henry (1981), Temam (1988). We define a nonlinear operator $B(\cdot)$ from V to H by

$$B(u)(x) = (f(u(x)))_x.$$

Thanks to the Sobolev imbedding $H_0^1(I) \subset L^\infty(I)$, the operator B is well-defined and has the following properties.

LEMMA 2.1. *For $u \in V$, we have*

$$(2.4) \quad |B(u)| \leq \alpha_0(1 + |u|_{L^\infty}^m) |A^{\frac{1}{2}} u|$$

$$(2.5) \quad (B(u), u^k) = 0 \quad \text{for all integer } k \geq 1.$$

Moreover for $u \in D(A)$, there exists constant C_1 such that

$$(2.6) \quad |(B(u), Au)| \leq C_1(L + |u|_{L^{2m}}^{2m})^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}} |Au|^{\frac{3}{2}}.$$

Proof. Let $u \in V$, then

$$(2.7) \quad \begin{aligned} |B(u)|^2 &= \int_0^L |f'(u)|^2 |u_x|^2 dx \\ &\leq \alpha_0^2 \int_0^L (1 + |u|^m)^2 |u_x|^2 dx \quad \text{from (2.2)} \\ &\leq \alpha_0^2 (1 + |u|_{L^\infty}^m)^2 |A^{\frac{1}{2}} u|^2, \end{aligned}$$

which yields (2.4). (2.5) is easily proved by integration by parts. In fact we have, for all $k \geq 1$,

$$(B(u), u^k) = \int_0^L f(u)_x u^k dx = -k \int_0^L f(u) u^{k-1} u_x dx = 0.$$

In order to get (2.6) we estimate (2.7) in different manner. In fact we obtain

$$|B(u)|^2 \leq \alpha_0^2 |u_x|_{L^\infty}^2 \int_0^L (1 + |u|^m)^2 dx.$$

By using the Agmon inequality

$$(2.8) \quad |u|_{L^\infty} \leq C_0 |u|^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}},$$

where C_0 is a positive constant, we obtain that

$$|B(u)|^2 \leq 2\alpha_0^2 C_0^2 |A^{\frac{1}{2}} u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} (L + |u|_{L^{2m}}^{2m}).$$

By taking $C_1 = \sqrt{2}\alpha_0 C_0$, we finally obtain (2.6) with the use of Schwartz inequality. $\square$

We will consider a modified equation of (2.1), i.e.,

$$(2.9) \quad u_t = -Au + B(u) - \alpha_0^2(1 - \phi_\gamma)(u + u^{2m+1}) + \phi_\gamma(g(u) + h(x)),$$

where

$$\phi_\gamma \stackrel{\text{def}}{=} \phi\left(\frac{|u|_{L^2}^2 + 2|u_x|_{L^2}^2 + |f(u)|_{L^2}^2}{\gamma^2}\right) \quad \text{for } 0 < \gamma \leq \infty,$$

and $\phi : \mathbf{R}_+ \rightarrow [0, 1]$ is a smooth monotone function such that $\phi(s) = 1$ for $0 \leq s \leq 1$, $\phi(s) = 0$ for $s \geq 2$, and $|\phi'(s)| \leq 2$. We note that ϕ_γ does not depend on space variable x . If $\gamma = \infty$, then $\phi_\gamma \equiv 1$ and we recover (2.1). Given the initial condition

$$(2.10) \quad u(x, 0) = u_0 \quad \text{in } H,$$

the classical theory ensures the existence of the unique solution of (2.9), (2.10) and there exists $0 < T \leq \infty$ which depends on u_0 such that the solution $u(t)$ satisfies $u \in C([0, T]; H) \cap L^2(0, T; V)$. Moreover if $u_0 \in V$, then $u \in ([0, T]; V) \cap L^2(0, T; D(A))$ (see, for instance, Friedman(1964), Lions(1969)). We first prove the following result.

PROPOSITION 2.2. *Under the above assumptions if $u(t)$ is a solution of (2.9) and (2.10), then there exist positive constants ρ_0, ρ_1 which do not depend on u_0 and γ such that*

$$(2.11) \quad \limsup_{t \rightarrow \infty} |u|^2 \leq \rho_0^2$$

$$(2.12) \quad \limsup_{t \rightarrow \infty} |A^{\frac{1}{2}} u|^2 \leq \rho_1^2.$$

Moreover if $u(t)$ is a solution of (2.1) and (2.6) with $u_0 \in V$, then there exists $\rho_2 > 0$ not depending on u_0 such that

$$(2.13) \quad \limsup_{t \rightarrow \infty} |Au|^2 \leq \rho_2^2.$$

Proof. Let $u(t)$ be the solution of (2.9) and (2.10) and without loss of generality we can assume that $u(t)$ is sufficiently smooth so that all the estimates in the proof are justified. In general case we can use the usual approximation arguments to obtain Proposition.

If we take the scalar product of (2.9) with u^{2p-1} , $p \geq 1$, then we obtain from (2.5)

$$(2.14) \quad \begin{aligned} \frac{1}{2p} \frac{d}{dt} \int_0^L u^{2p} dx &= -(2p-1) \int_0^L u^{2p-2} u_x^2 dx - \alpha_0^2 (1 - \phi_\gamma) \int_0^L (u^{2p} + u^{2(m+p)}) dx \\ &\quad + \phi_\gamma \int_0^L (g(u) u^{2p-1} + h(x) u^{2p-1}) dx. \end{aligned}$$

By using (2.3) we obtain

$$(2.15) \quad \int_0^L g(u) u^{2p-1} dx \leq \alpha_p \stackrel{\text{def}}{=} L \sup_{|u| \leq R} |g(u) u^{2p-1}|.$$

One has from Hölder inequality that

$$(2.16) \quad \int_0^L h(x) u^{2p-1} dx \leq \frac{1}{2p} \left(\frac{\lambda_1}{p} \right)^{1-2p} \int_0^L h^{2p} dx + \frac{(2p-1)\lambda_1}{2p^2} \int_0^L u^{2p} dx,$$

where $\lambda_1 > 0$ is the first eigenvalue of A , i.e., $\lambda_1 = (\frac{2\pi}{L})^2$. We recall the inequality

$$(2.17) \quad |A^{\frac{1}{2}} u|^2 \geq \lambda_1 |u|^2 \quad \text{for all } u \in V.$$

From this one obtains

$$(2.18) \quad \lambda_1 \int_0^L u^{2p} dx \leq p^2 \int_0^L u^{2p-2} u_x^2 dx.$$

Since $\phi_\gamma \leq 1$, we obtain from (2.15), (2.16), and (2.18) that

$$\frac{1}{2p} \frac{d}{dt} \int_0^L u^{2p} dx \leq -\frac{2p-1}{2} \int_0^L u^{2p-2} u_x^2 dx + \alpha_p + \frac{1}{2p} \left(\frac{\lambda_1}{p} \right)^{1-2p} |h|_{L^{2p}}^{2p}.$$

By multiplying $2p$ to both sides, we get

$$(2.19) \quad \frac{d}{dt} \int_0^L u^{2p} dx \leq -p(2p-1) \int_0^L u^{2p-2} u_x^2 dx + \beta_p,$$

where

$$\beta_p \stackrel{\text{def}}{=} 2p\alpha_p + \left(\frac{\lambda_1}{p} \right)^{1-2p} |h|_{L^{2p}}^{2p}.$$

First let $p = 1$ and integrate (2.19) in time to obtain

$$(2.20) \quad \int_t^{t+1} |A^{\frac{1}{2}} u|^2 ds \leq |u(t)|_{L^2}^2 + \beta_1.$$

Using (2.18) we also have

$$(2.21) \quad \frac{d}{dt} \int_0^L u^{2p} dx \leq -\frac{2p-1}{p} \lambda_1 \int_0^L u^{2p} dx + \beta_p.$$

The classical Gronwall inequality yields

$$(2.22) \quad \limsup_{t \rightarrow \infty} |u|_{L^{2p}}^{2p} \leq \beta_p \quad \text{for any } p \geq 1.$$

In particular we define $\rho_0 = \beta_1$ to get (2.11).

We now take the scalar product of (2.9) with Au and by using (2.6) we obtain

$$(2.23) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |A^{\frac{1}{2}} u|^2 &\leq -|Au|^2 + C_1(L + |u|_{L^{2m}}^{2m})^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}} |Au|^{\frac{3}{2}} \\ &\quad + |(g(u), Au)| + |(h, Au)|. \end{aligned}$$

We have

$$\begin{aligned} |(g(u), Au)| &= \left| \int_0^L g'(u) u_x^2 dx \right| \\ &\leq \alpha_0 \int_0^L (1 + |u|^m) u_x^2 dx \quad \text{from (2.2)} \\ &\leq \alpha_0 |A^{\frac{1}{2}} u|^2 + \alpha_0 |u_x|_{L^\infty}^2 \int_0^L |u|^m dx. \end{aligned}$$

By using (2.8), we see that the last term is bounded by

$$\alpha_0 |A^{\frac{1}{2}} u|^2 + \alpha_0 |A^{\frac{1}{2}} u| |Au| |u|_{L^m}^m,$$

which is in turn bounded, with an use of Young inequality, by

$$(2.24) \quad \alpha_0 |A^{\frac{1}{2}} u|^2 + \alpha_0^2 C_0^4 |u|_{L^m}^{2m} |A^{\frac{1}{2}} u|^2 + \frac{1}{4} |Au|^2.$$

Another uses of Young inequality yield

$$(2.25) \quad C_1(L + |u|_{L^{2m}}^{2m})^{\frac{1}{2}} |A^{\frac{1}{2}} u|^{\frac{1}{2}} |Au|^{\frac{3}{2}} \leq 27C_1^4(L + |u|_{L^{2m}}^{2m})^2 |A^{\frac{1}{2}} u|^2 + \frac{1}{4} |Au|^2$$

and

$$(2.26) \quad |(h, Au)| \leq |h|^2 + \frac{1}{4} |Au|^2.$$

By using (2.24), (2.25), and (2.26) we obtain

$$(2.27) \quad \frac{d}{dt} |A^{\frac{1}{2}} u|^2 \leq -\frac{1}{2} |Au|^2 + \psi_1(u) |A^{\frac{1}{2}} u|^2 + 2|h|^2,$$

where

$$\psi_1(u) \stackrel{\text{def}}{=} 2\alpha_0 + 2\alpha_0^2 C_0^4 |u|_{L^m}^{2m} + 54C_1^4 (L + |u|_{L^{2m}}^{2m})^2.$$

From (2.22) there exists $T_0 > 0$ such that

$$\psi_1(u) \leq C_2 \quad \text{for } t \geq T_0,$$

where

$$C_2 \stackrel{\text{def}}{=} 2\{2\alpha_0 + 2\alpha_0^2 C_0^4 \beta_m^{2m} + 54C_1^4 (L + \beta_m^{2m})^2\}.$$

Hence we can rewrite (2.27) as

$$(2.28) \quad \frac{d}{dt} |A^{\frac{1}{2}} u|^2 \leq -\frac{1}{2} |Au|^2 + C_2 |A^{\frac{1}{2}} u|^2 + 2|h|^2.$$

Now (2.20) and the Uniform Gronwall inequality (see Temam (1988), page 89) implies (2.12).

Finally by taking scalar product of (2.1) with $A^2 u$, we obtain

$$(2.29) \quad \frac{1}{2} \frac{d}{dt} |Au|^2 \leq -|A^{\frac{3}{2}} u|^2 + |A^{\frac{1}{2}} B(u)| |A^{\frac{3}{2}} u| + |A^{\frac{1}{2}} g(u)| |A^{\frac{3}{2}} u| + |A^{\frac{1}{2}} h| |A^{\frac{3}{2}} u|.$$

Firstly one has

$$\begin{aligned} |A^{\frac{1}{2}} B(u)|^2 &= \int_0^L (f'(u) u_{xx} + f''(u) u_x^2)^2 dx \\ &\leq 2 \int_0^L f'(u)^2 u_{xx}^2 dx + 2 \int_0^L f''(u)^2 u_x^2 dx \end{aligned}$$

and with the use of (2.2),

$$|A^{\frac{1}{2}} B(u)|^2 \leq 2\alpha_0^2 (1 + |u|_{L^\infty}^m)^2 |Au|^2 + 2\alpha_0^2 (1 + |u|_{L^\infty}^{m-1})^2 |A^{\frac{1}{2}} u|^2.$$

Secondly

$$|A^{\frac{1}{2}} g(u)|^2 \leq \int_0^L g'(u)^2 u_x^2 dx \leq \alpha_0^2 (1 + |u|_{L^\infty}^m)^2 |A^{\frac{1}{2}} u|^2.$$

Because of (2.12) there exists $T_1 \geq T_0$ such that

$$|A^{\frac{1}{2}} u| \leq \rho_1 \quad \text{for all } t \geq T_1.$$

From (2.8) one also has

$$|u|_{L^\infty}^2 \leq C_0^2 \rho_0 \rho_1 \quad \text{for } t \geq T_1.$$

Applying Young inequality to (2.29) and using above estimates we obtain

$$(2.30) \quad \frac{d}{dt} |Au|^2 \leq -\frac{1}{2} |A^{\frac{3}{2}} u|^2 + C_3 |Au|^2 + C_4$$

where

$$\begin{aligned} C_3 &= 4\alpha_0^2 (1 + C_0^m \rho_0^{\frac{m}{2}} \rho_1^{\frac{m}{2}})^2 \\ C_4 &= 2\alpha_0^2 \{2(1 + C_0^{m-1} \rho_0^{\frac{m-1}{2}} \rho_1^{\frac{m-1}{2}})^2 + (1 + C_0^m \rho_0^{\frac{m}{2}} \rho_1^{\frac{m}{2}})^2\} \rho_1^2 + 2|A^{\frac{1}{2}} h|^2. \end{aligned}$$

Another use of Uniform Gronwall inequality yields (2.22). $\square$

As a corollary we have

COROLLARY 2.3. *Under the same assumption as in Proposition 2.2, there exists the global attractor $\mathcal{A}_\gamma \subset V$ of (2.9) for all $0 < \gamma \leq \infty$ and we have for $u \in \mathcal{A}_\gamma$,*

$$(2.31) \quad |u|^2 \leq \rho_0^2 \quad \text{and} \quad |A^{\frac{1}{2}} u|^2 \leq \rho_1^2.$$

In the case $\phi_\gamma \equiv 1$, i.e., $\gamma = \infty$, we have $\mathcal{A}_\infty \subset D(A)$ and

$$(2.32) \quad |Au|^2 \leq \rho_2^2 \quad \text{for } u \in \mathcal{A}_\infty.$$

Proof. Proposition 2.2 implies that the ball $\mathcal{B}_{\rho_1}$ of V centered at 0 of radius ρ_1 is absorbing in V and if u_0 belongs to the ball $B(0, R)$ of H centered at 0 of radius R , then $u(t)$ enters $\mathcal{B}_{\rho_1}$ at a time $t \leq T_1$ and remains in it for $t \geq T_1$ for some $T_1 > 0$. This provides the uniform compactness of the nonlinear semigroup $S(t)$ defined by the solution of (2.9) since V is compactly imbedded in H . Hence Theorem 1.1.1, Temam (1988) implies the existence of global attractor and the inequalities (2.31) hold obviously. A similar argument shows the second statement. $\square$

REMARK 2.4. *Here we remark that there exists $\rho_3 > 0$ not depending on γ such that*

$$|f(u)|_{L^2}^2 \leq \rho_3^2$$

and thus

$$|u|_{L^2}^2 + 2|u_x|_{L^2}^2 + |f(u)|_{L^2}^2 \leq \rho^2 \quad \text{for } u \in \mathcal{A}_\gamma,$$

where

$$\rho^2 = \rho_0^2 + 2\rho_1^2 + \rho_3^2.$$

Therefore we find that if $\gamma \geq 2\rho$, then

$$\phi_\gamma = 1 \quad \text{for } u \in \mathcal{A}_\gamma$$

and (2.9) has the exactly same long-time dynamics as (2.1).

We now describe the nonlinear change of variables. First of all, we notice that if $u_0 \in V$ and $h \in H_0^1(I)$, then the solution $u(t)$ of (2.9) and (2.10) satisfies

$$u \in C((0, T]; D(A)) \cap L^2(0, T; D(A^{\frac{3}{2}})).$$

In particular $u \in D(A^{\frac{3}{2}})$ almost everywhere on $(0, T)$. Hence we can differentiate the equation (2.9) and we obtain

$$\begin{aligned} (u_x)_t &= u_{xxx} + (B(u))_x - \alpha_0^2(1 - \phi_\gamma)(u_x + (2m + 1)u^{2m}u_x) \\ &\quad + \phi_\gamma(g'(u)u_x + h'(x)). \end{aligned}$$

Let B be the unbounded operator defined by

$$Bu = -u_{xx} + \delta_0 u \quad \text{for } u \in D(B),$$

where $\delta_0 > 0$ and

$$D(B) = \{u \in H^2(I) : u_x(0) = u_x(L) = 0\}.$$

Let $v = u_x$, then we see that $v \in D(B)$ and v satisfies

$$v_t = -Bv + \delta_0 v + (f(u))_{xx} + \eta_2,$$

where

$$\eta_2 \stackrel{\text{def}}{=} -\alpha_0^2(1 - \phi_\gamma)(v + (2m + 1)u^{2m}v) + \phi_\gamma(g'(u)v + h'(x)).$$

We are not satisfied yet because of the nonlinearity $(f(u))_{xx}$. Hence we need to introduce another variable, namely,

$$w = f(u).$$

Then w satisfies the followings:

$$\begin{aligned} w_t &= f'(u)u_t \\ w_{xx} &= f'(u)u_{xx} + f''(u)u_x^2. \end{aligned}$$

Moreover if $u \in D(A)$, then $w = f(u) \in D(B)$ and

$$w_t = -Bw + \delta_0 w - f''(u)u_x^2 + f'(u)\{B(u) + \eta_1\},$$

where

$$\eta_1 \stackrel{\text{def}}{=} -10(1 - \phi_\gamma)(u + u^{2m+1}) + \phi_\gamma(g(u) + h(x)).$$

Finally by using (2.33), we obtain that $\phi_\gamma = \phi(\frac{|u|_{L^2}^2 + |u_x|_{L^2}^2 + |v|_{L^2}^2 + |w|_{L^2}^2}{\gamma^2})$ and

$$\begin{aligned} (2.34) \quad u_t &= -Au + w_x + \eta_1 \\ v_t &= -Bv + \delta_0 v + w_{xx} + \eta_2 \\ w_t &= -Bw + \delta_0 w - f''(u)v^2 + f'(u)^2 v + f'(u)\eta_1. \end{aligned}$$

We would like to add additional terms $(\xi_1, \xi_2$ and the terms involving $k_1, k_2)$ to the equation (2.34) so that we obtain

$$\begin{aligned} (2.35) \quad u_t &= -Au + w_x + \eta_1 + \xi_1 \\ v_t &= -Bv + \delta_0 v + w_{xx} + \eta_2 + \xi_2 - k_1(v - u_x) \\ w_t &= -Bw + \delta_0 w - f''(u)v^2 + f'(u)^2 v + f'(u)(\eta_1 + \xi_1) - (k_2 + 2f'(u)^2)(w - f(u)). \end{aligned}$$

Notice that the additional terms vanish if $v = u_x$ and $w = f(u)$. Let

$$J(u) = (u, u_x, f(u)) \quad \text{for } u \in V,$$

then these additional terms will give us the exponential convergence of the solution of (2.35) to the set $J(V)$.

So far we have seen that if $u(t)$ is a solution of (2.9) with initial condition $u_0 \in V$, then $J(u(t))$ is a solution of the reaction diffusion system (2.35). The converse statement is also true by the uniqueness of the solutions (the existence and uniqueness of (2.35) will be clarified soon). Hence the solutions of the equations (2.9) and (2.35) have the same dynamics on the set $J(V)$. To summarize we state the following Proposition.

PROPOSITION 2.5. *If $u(t)$ is a solution of (2.9) with initial condition $u_0 \in V$, then $(u, v, w)^t(t) = J(u(t))$ is a solution of (2.35). Conversely if $(u, v, w)^t(t)$ is a solution of (2.35) with initial condition $J(u_0)$ with $u_0 \in V$, then $u(t)$ is a solution of (2.9).*

We are now led to the study of reaction diffusion system (2.35). Let $\mathbf{U} = (u, v, w)^t$ and we define an operator on the space $D(A) \times D(B) \times D(B)$ by

$$\mathbf{AU} = \begin{pmatrix} Au - w_x \\ Bv - \delta_0 w - k_1 u_x - w_{xx} \\ Bw - \delta_0 w \end{pmatrix}.$$

We also define

$$\tilde{\mathbf{F}}(\mathbf{U}) = (F_1, F_2, F_3)^t,$$

where

$$\begin{aligned} F_1 &= \eta_1 + \xi_1, & F_2 &= \eta_2 + \xi_2 - k_1 v, \\ F_3 &= -f''(u)v^2 + f'(u)\eta_1 - (k_2 + 2f'(u)^2)(w - f(u)). \end{aligned}$$

Then the equation (2.35) can be rewritten in the form

$$(2.36) \quad \frac{d}{dt} \mathbf{U} = -\mathbf{A}\mathbf{U} + \tilde{\mathbf{F}}(\mathbf{U}).$$

The operator $\mathbf{A}$ is not self-adjoint but it is sectorial, i.e., $-\mathbf{A}$ generates an analytic semi-group on $\mathbf{H} \stackrel{\text{def}}{=} L^2(I)^3$ (see Henry (1981) and Pazy (1983) for definition). This follows from the next Lemma.

LEMMA 2.6. *The operator $\mathbf{A}$ is a sectorial operator on $\mathbf{H}$.*

Proof. We first recall that since A, B are sectorial operators, there exist $0 < \theta < \frac{\pi}{2}$ and $M \geq 1$ such that

$$(2.37) \quad \rho(A) \supset \Sigma = \{\lambda : \theta \leq |\arg \lambda| \leq \pi, \lambda \neq 0\}, \quad \rho(B) \supset \Sigma$$

and

$$(2.38) \quad \|(\lambda - A)^{-1}\|_{op} \leq \frac{M}{|\lambda|}, \quad \|(\lambda - B)^{-1}\|_{op} \leq \frac{M}{|\lambda|} \quad \text{for } \lambda \in \Sigma.$$

We now suppose that

$$(\mathbf{A} + 2\delta_0 - \lambda)\mathbf{U} = F,$$

for $\lambda \in \Sigma$ and $F = (f_1, f_2, f_3)^t \in \mathbf{H}$, then w -equation reads

$$(2.39) \quad Bw + \delta_0 w - \lambda w = f_3$$

and since $\lambda - \delta_0 \in \Sigma$ we see that w is uniquely determined by f_3 and that from (2.38),

$$(2.40) \quad |w|_{L^2} \leq \frac{M}{|\lambda - \delta_0|} |f_3|_{L^2}.$$

Taking the scalar product of (2.39) with w , we obtain

$$(2.41) \quad |w_x|_{L^2}^2 \leq |\lambda - \delta_0| |w|_{L^2}^2 + |f_3| |w|_{L^2} \leq \frac{M^2 + M}{|\lambda - \delta_0|} |f_3|_{L^2}^2.$$

One also has

$$(2.42) \quad |w_{xx}|_{L^2} \leq |\lambda - \delta_0| |w|_{L^2} + |f_3|_{L^2} \leq (M + 1) |f_3|_{L^2}.$$

Now u -equation reads

$$Au + 2\delta_0 u - w_x - \lambda u = f_1.$$

Since $\lambda - 2\delta_0 \in \Sigma$, u is uniquely solvable and

$$|u|_{L^2} \leq \frac{M}{|\lambda - 2\delta_0|} (|f_1|_{L^2} + |w_x|_{L^2}).$$

We obtain from (2.41) that

$$\begin{aligned} (2.43) \quad |u|_{L^2} &\leq \frac{M}{|\lambda - 2\delta_0|} \{ |f_1|_{L^2} + (\frac{M^2 + M}{|\lambda - \delta_0|})^{\frac{1}{2}} |f_3|_{L^2} \} \\ &\leq \frac{M_1}{|\lambda|} (|f_1|_{L^2} + |f_3|_{L^2}), \end{aligned}$$

for some constant $M_1 \geq 1$. One also has

$$(2.44) \quad |A^{\frac{1}{2}} u|^2 \leq (|\lambda| + 1) |u|_{L^2}^2 + |w_x|_{L^2}^2 + |f_1|_{L^2}^2.$$

Finally v -equation becomes

$$Bv + \delta_0 v + k_1 u_x - w_{xx} - \lambda v = f_2.$$

From (2.38), v is uniquely found and

$$|v|_{L^2} \leq \frac{M}{|\lambda - \delta_0|} \{ k_1 |u_x|_{L^2} + |w_{xx}|_{L^2} + |f_2|_{L^2} \}.$$

By using (2.41), (2.42), (2.43), and (2.44), we finally obtain

$$(2.45) \quad |v|_{L^2} \leq \frac{M_2}{|\lambda|} \{ |f_1|_{L^2} + |f_2|_{L^2} + |f_3|_{L^2} \},$$

for some constant $M_2 \geq 1$. We see easily from (2.40), (2.43), and (2.45) that $(A - \lambda I)^{-1}$ exists for $\lambda \in \Sigma$ and that

$$\|(A - \lambda I)^{-1}\|_{op} \leq \frac{M_3}{|\lambda|}$$

for some constant $M_3 \geq 1$, which proves that A is sectorial. It follows also from Theorem 5.2, Pazy (1983) that $-A$ generates an analytic semigroup. $\square$

Since A is sectorial, one can define the fractional powers of A and it is easy to see that the nonlinearity $\tilde{F} : D(A^{\frac{1}{2}}) \rightarrow H$ is locally Lipschitz continuous. Hence if $U(0) = U_0 \in D(A^{\frac{1}{2}})$, then there exists a unique (strong) solution $U(t)$ of (2.36) such that

$$U \in C([0, T]; A^{\frac{1}{2}}),$$

see Henry (1981) and Pazy (1983).

3. MAIN RESULTS.

In this section we prove that for large k_1, k_2 , the transformed equation (2.35) has the global attractor $\mathcal{A}_{RD}$ and that the attractor is, in fact, the image of $\mathcal{A}_\gamma$ under the imbedding J , i.e., $J(\mathcal{A}_\gamma) = \mathcal{A}_{RD}$.

Let $\mathbf{U} = (u, v, w)^t$ and we recall $\mathbf{H} = (L^2(I))^3$ and define $\mathbf{V} = V \times H^1(I) \times H^1(I)$. If $\mathbf{U}$ is a sufficiently smooth solution of (2.35), then u -equation reads

$$(3.1) \quad u_t = -Au + w_x + \eta_1 + \xi_1.$$

By differentiating (3.1) with respect to x , one obtains

$$(u_x)_t = -Bu_x + \delta_0 u_x + w_{xx} + \eta_{1x} + \xi_{1x}.$$

Subtracting this equation from v -equation of (2.35), one gets

$$(3.2) \quad (v - u_x)_t = -B(v - u_x) + \delta_0(v - u_x) + \eta_2 - \eta_{1x} + \xi_2 - \xi_{1x} - k_1(v - u_x).$$

Let $\tilde{w} = f(u)$, then one easily sees that, as before, $\tilde{w}$ satisfies

$$\tilde{w}_t = -B\tilde{w} + \delta_0\tilde{w} - f''(u)^2 u_x^2 + f'(u)w_x + f'(u)(\eta_1 + \xi_1).$$

Subtracting this equation from w -equation of (2.35), we obtain

$$(3.3) \quad \begin{aligned} (w - \tilde{w})_t = & -B(w - \tilde{w}) + \delta_0(w - \tilde{w}) - f''(u)^2(v^2 - u_x^2) \\ & + f'(u)(f'(u)v - w_x) - (k_2 + 2f'(u)^2)(w - \tilde{w}). \end{aligned}$$

We first obtain a uniform bound for the norm

$$(3.4) \quad |u|_{L^2}^2 + |u_x|_{L^2}^2 + |v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2$$

and with the use of Minkowsky inequality we obtain the uniform bound of the norm $|\mathbf{U}|_{\mathbf{H}}$. In fact we prove the next proposition

PROPOSITION 3.1. *Let $k_1 > 2m\alpha_0^2, k_2 > \frac{3}{2}$ and $\mathbf{U}$ be a solution of (2.35) with initial condition $\mathbf{U}(0) \in \mathbf{V}$, then there exists $\rho_4 \geq \sqrt{2}\gamma$ such that*

$$(3.5) \quad \limsup_{t \rightarrow \infty} |\mathbf{U}|_{\mathbf{H}}^2 \leq \rho_4^2.$$

Proof. Let $\mathbf{U} = (u, v, w)^t$ be a solution of (2.35), then the estimates (3.4) is obtained by taking the scalar product of (3.1), (3.2), (3.3) with $u, Au, v - u_x, w - f(u)$ respectively.

Firstly we have

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} |u|_{L^2}^2 = -|A^{\frac{1}{2}} u|^2 + (w_x, u) + (\eta_1 + \xi_1, u).$$

One sees that

$$\begin{aligned} (w_x, u) &= -(w, u_x) \quad \text{by integrating by parts} \\ &= -(w - f(u), u_x) - (f(u), u_x) \\ &= -(w - f(u), u_x). \end{aligned}$$

Applying Schwartz inequality and Young inequality, we obtain

$$(3.7) \quad |(w_x, u)| \leq \frac{1}{2} |w - f(u)|^2 + \frac{1}{2} |A^{\frac{1}{2}} u|^2.$$

Secondly

$$\begin{aligned} (\eta_1 + \xi_1, u) &= -\alpha_0^2 (1 - \phi_\gamma) \int_0^L (u^2 + u^{2m+2}) dx \\ &\quad + \phi_\gamma \int_0^L (ug(u) + uh(x)) dx + \int_0^L u f'(u) (w - f(u)) dx. \end{aligned}$$

It follows from the assumption (2.2) and Schwartz inequality that

$$(\eta_1 + \xi_1, u) \leq -\alpha_0^2 (1 - \phi_\gamma) I_1 + \phi_\gamma |u|_{L^2} |h|_{L^2} + 2\alpha_0 I_1^{\frac{1}{2}} |w - f(u)|_{L^2},$$

where

$$I_1 = \int_0^L (u^2 + u^{2m+2}) dx.$$

The Young inequality implies that

$$(3.8) \quad (\eta_1 + \xi_1, u) \leq \alpha_0^2 \phi_\gamma I_1 + |w - \tilde{w}|_{L^2}^2 + \phi_\gamma |u|_{L^2} |h|_{L^2}.$$

By using (3.7) and (3.8), one sees that (3.6) becomes

$$(3.9) \quad \frac{d}{dt} |u|_{L^2}^2 \leq -|A^{\frac{1}{2}} u|^2 + 3|w - \tilde{w}|_{L^2}^2 + 2\alpha_0^2 \phi_\gamma I_1 + 2\phi_\gamma |u|_{L^2} |h|_{L^2}.$$

We next take the scalar product of (3.1) with Au to obtain

$$(3.10) \quad \frac{1}{2} \frac{d}{dt} |A^{\frac{1}{2}} u|^2 = -|Au|^2 + (w_x, Au) + (\eta_1 + \xi_1, Au).$$

First of all,

$$\begin{aligned}
|(w_x, Au)| &= |((w - f(u))_x + f(u)_x, Au)| \\
&\leq (\text{with Young inequality}) \\
&\leq \frac{1}{2} |(w - \tilde{w})_x|_{L^2}^2 + \int_0^L f'(u)^2 u_x^2 dx + |Au|^2.
\end{aligned}$$

From (2.2), one has

$$(3.11) \quad |(w_x, Au)| \leq \frac{1}{2} |(w - \tilde{w})_x|_{L^2}^2 + \alpha_0^2 I_2 + |Au|^2,$$

where

$$I_2 = \int_0^L (u_x^2 + u^{2m} u_x^2) dx.$$

One also has

$$\begin{aligned}
(3.12) \quad (\eta_1 + \xi_1, Au) &= (\eta_{1x} + \xi_{1x}, u_x) \\
&= -\alpha_0^2 (1 - \phi_\gamma) \int_0^L (u_x^2 + (2m+1) u^{2m} u_x^2) dx + \phi_\gamma (g(u) + h(x), Au).
\end{aligned}$$

It follows from (3.10), (3.11), (3.12) that

$$(3.13) \quad \frac{d}{dt} |A^{\frac{1}{2}} u|^2 \leq |(w - \tilde{w})_x|_{L^2}^2 + 2\alpha_0^2 \phi_\gamma I_2 + 2\phi_\gamma (g(u) + h(x), Au).$$

Now we turn to (3.2). By taking the scalar product with $v - u_x$, one obtains

$$(3.14) \quad \frac{1}{2} \frac{d}{dt} |v - u_x|_{L^2}^2 = -|(v - u_x)_x|_{L^2}^2 + (\eta_2 - \eta_{1x}, v - u_x) + (\xi_2 - \xi_{1x}, v - u_x) - k_1 |v - u_x|_{L^2}^2.$$

Firstly $(\eta_2 - \eta_{1x}, v - u_x)$ becomes

$$(3.15) \quad -\alpha_0^2 (1 - \phi_\gamma) ((2m+1) I_3 - 2m |v - u_x|_{L^2}^2) + \phi_\gamma \int_0^L g'(u) (v - u_x)^2 dx,$$

where

$$I_3 = \int_0^L \{(v - u_x)^2 + u^{2m} (v - u_x)^2\} dx.$$

Secondly $(\xi_2 - \xi_{1x}, v - u_x)$ becomes

$$-I_4 - \int_0^L f'(u) (w - f(u))_x (v - u_x) dx,$$

which is in turn majorized by

$$-I_4 + |f'(u)(v - u_x)|_{L^2} |(w - \tilde{w})_x|_{L^2},$$

where

$$I_4 = \int_0^L f''(u)(v^2 - u_x^2)(w - f(u))dx.$$

From (2.2) and Young inequality, we see that

$$(3.16) \quad (\xi_2 - \xi_{1x}, v - u_x) \leq -I_4 + 2\alpha_0^2 I_3 + \frac{1}{4} |(w - \tilde{w})_x|_{L^2}^2.$$

We obtain from (3.14), (3.15), (3.16) that

$$(3.17) \quad \begin{aligned} \frac{d}{dt} |v - u_x|_{L^2}^2 &\leq -2 |(v - u_x)_x|_{L^2}^2 + (4m\alpha_0^2 - 2k_1) |v - u_x|_{L^2}^2 + \frac{1}{2} |(w - \tilde{w})_x|_{L^2}^2 \\ &\quad - 2I_4 - (4m - 2)\alpha_0^2 I_3 + (4m + 2)\alpha_0^2 \phi_\gamma I_3 + 2\phi_\gamma \int_0^L g'(u)(v - u_x)^2 dx. \end{aligned}$$

Finally we obtain from (3.3) that

$$\frac{1}{2} \frac{d}{dt} |w - \tilde{w}|_{L^2}^2 = - |(w - \tilde{w})_x|_{L^2}^2 + I_4 - \int_0^L (k_2 + 2f'(u)^2)(w - \tilde{w})^2 dx + I_5,$$

where

$$I_5 = \int_0^L f'(u)(f'(u)v - w_x)(w - \tilde{w})dx.$$

Notice that

$$I_5 = \int_0^L f'(u)^2(v - u_x)(w - \tilde{w})dx + \int_0^L f'(u)(f(u) - w)_x(w - \tilde{w})dx.$$

From Hölder and Young inequality we have

$$\begin{aligned} I_5 &\leq |f'(u)(v - u_x)|_{L^2} (|f'(u)(w - \tilde{w})|_{L^2} + |(w - \tilde{w})_x|_{L^2}) \\ &\leq \alpha_0^2 I_3 + \frac{1}{4} |(w - \tilde{w})_x|_{L^2}^2 + \frac{3}{2} |f'(u)(w - \tilde{w})|_{L^2}^2 \end{aligned}$$

and it follows that

$$(3.18) \quad \frac{d}{dt} |w - \tilde{w}|_{L^2}^2 \leq -\frac{3}{2} |(w - \tilde{w})_x|_{L^2}^2 + 2I_4 - 2k_2 |w - \tilde{w}|_{L^2}^2 + 2\alpha_0^2 I_3.$$

By adding up (3.9), (3.13), (3.17), and (3.18), we obtain

$$\begin{aligned}
(3.19) \quad & \frac{d}{dt} \{ |u|_{L^2}^2 + |u_x|_{L^2}^2 + |v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2 \} \\
& \leq -|A^{\frac{1}{2}}u|^2 + (4m - \alpha_0^2 - 2k_1)|v - u_x|_{L^2}^2 + (3 - 2k_2)|w - \tilde{w}|_{L^2}^2 \\
& \quad + 2\alpha_0^2\phi_\gamma(I_1 + I_2 + (2m + 1)I_3) \\
& \quad + 2\phi_\gamma\{|u|_{L^2}|h|_{L^2} + (g(u) + h, Au) + \int_0^L g'(u)(v - u_x)^2 dx\}.
\end{aligned}$$

Without loss of generality we may assume that $|u|_{L^2}^2 + |u_x|_{L^2}^2 + |v|_{L^2}^2 + |w|_{L^2}^2 \geq 2\gamma^2$ so that $\phi_\gamma = 0$. It follows from (3.19) that

$$\begin{aligned}
& \frac{d}{dt} \{ |u|_{L^2}^2 + |u_x|_{L^2}^2 + |v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2 \} \\
& \leq -|A^{\frac{1}{2}}u|^2 + (4m\alpha_0^2 - 2k_1)|v - u_x|_{L^2}^2 + (3 - 2k_2)|w - \tilde{w}|_{L^2}^2,
\end{aligned}$$

which implies that if $k_1 > 2m\alpha_0^2$ and $k_2 > \frac{3}{2}$, then the norm $|u|_{L^2}^2 + |u_x|_{L^2}^2 + |v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2$ decays at the exponential rate and

$$\limsup_{t \rightarrow \infty} \{ |u|_{L^2}^2 + |u_x|_{L^2}^2 + |v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2 \}$$

is bounded. Since

$$|u|_{L^2}^2 + |u_x|_{L^2}^2 + |v|_{L^2}^2 + |w|_{L^2}^2 \leq |u|_{L^2}^2 + 3|u_x|_{L^2}^2 + 2|v - u_x|_{L^2}^2 + 2|w - \tilde{w}|_{L^2}^2 + 2|\tilde{w}|_{L^2}^2$$

and with the use of (2.2),

$$|\tilde{w}|_{L^2}^2 = \int_0^L \left(\int_0^{u(x)} f'(s) ds \right)^2 dx \leq 2\alpha_0^2 \int_0^L (|u| + |u|^{2m+1}) dx,$$

we can conclude that there exists $\rho_4 \geq \sqrt{2}\gamma$ such that

$$\limsup_{t \rightarrow \infty} \{ |u|_{L^2}^2 + |u_x|_{L^2}^2 + |v|_{L^2}^2 + |w|_{L^2}^2 \} \leq \rho_4^2. \quad \square$$

As a corollary we can prove that every solution of (2.35) converges to the set $J(V)$ at exponential rate.

COROLLARY 3.2. *Under the same assumption as in Proposition 3.1, if $\mathbf{U}(t)$ is a solution of (2.35), then there exists $T_1, K_1 > 0$ such that for $t \geq T_1$,*

$$(3.20) \quad \frac{d}{dt}(|v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2) \leq (K_1 - 2k_1)|v - u_x|_{L^2}^2 - 2k_2|w - \tilde{w}|_{L^2}^2.$$

Proof. By adding (3.17) and (3.18), one obtains

$$(3.21) \quad \begin{aligned} & \frac{d}{dt}(|v - u_x|_{L^2}^2 + |w - \tilde{w}|_{L^2}^2) \\ & - 2|(v - u_x)_x|_{L^2}^2 + (4m\alpha_0^2 - 2k_1)|v - u_x|_{L^2}^2 - |(w - \tilde{w})_x|_{L^2}^2 \\ & - 2k_2|w - \tilde{w}|_{L^2}^2 + (4m + 2)\alpha_0^2\phi_\gamma I_3 + 2\phi_\gamma \int_0^L g'(u)(v - u_x)^2 dx. \end{aligned}$$

Firstly one has

$$I_3 \leq (1 + |u|_{L^\infty}^{2m})|v - u_x|_{L^2}^2$$

and from (2.2)

$$2\phi_\gamma \int_0^L g'(u)(v - u_x)^2 dx \leq 2\alpha_0(1 + |u|_{L^\infty}^m)|v - u_x|_{L^2}^2.$$

Secondly from Proposition 3.1 and Sobolev imbedding

$$(3.22) \quad H_0^1(I) \subset L^\infty(I),$$

there exists $T_1 > 0$ such that

$$|u|_{L^\infty} \leq C_1 \quad \text{for } t \geq T_1.$$

Hence we obtain (3.20) by taking

$$K_1 = 4m\alpha_0^2 + (4m + 2)(1 + C_1^{2m}) + 2\alpha_0(1 + C_1^m). \quad \square$$

We now prove the uniform boundedness in the space $\mathbf{V}$ and the existence of global attractor, $\mathcal{A}_{RD}$ of (2.35).

PROPOSITION 3.3. *Let $k_1 > \frac{K_1}{2}, k_2 > \frac{3}{2}$, then the solution of (2.35) is uniformly bounded in $\mathbf{V}$ and there exists the global attractor for (2.35).*

Proof. We easily see from Proposition 3.1, Corollary 3.2, and (3.22) that there exists $T_2 > 0$ such that for $t \geq T_2$,

$$|F_1|_{L^2} \leq C_2, \quad |F_2 + k_1 u_x|_{L^2} \leq C_3, \quad |F_3|_{L^2} \leq C_4|v|_{L^4}^2 + C_5.$$

By multiplying u, v, w to (2.35), integrating over I , and using Hölder inequality, we obtain

$$\begin{aligned}
(3.23) \quad & \frac{1}{2} \frac{d}{dt} |u|_{L^2}^2 \leq -|A^{\frac{1}{2}} u|^2 + |w_x|_{L^2} |u|_{L^2} + C_2 |u|_{L^2}, \\
& \frac{1}{2} \frac{d}{dt} |v|_{L^2}^2 \leq -|v_x|_{L^2}^2 + |w_x|_{L^2} |v_x|_{L^2} + C_3 |v|_{L^2}, \\
& \frac{1}{2} \frac{d}{dt} |w|_{L^2}^2 \leq -|w_x|_{L^2}^2 + (C_4 |v|_{L^4}^2 + C_5) |w|_{L^2}.
\end{aligned}$$

By using (3.22) one has

$$(3.24) \quad |v|_{L^4}^2 = \left(\int_0^L v^4 dx \right)^{\frac{1}{2}} \leq |v|_{L^\infty} |v|_{L^2} \leq C_6 |v_x|_{L^2} |v|_{L^2}$$

and hence one obtains together with Young inequality that for $t \geq T_2$,

$$(3.25) \quad \frac{d}{dt} (|u|_{L^2}^2 + |v|_{L^2}^2 + |w|_{L^2}^2) \leq -\frac{1}{2} (|A^{\frac{1}{2}} u|^2 |v_x|_{L^2}^2 + |w_x|_{L^2}^2) + C_7.$$

We now multiply $Au, -v_{xx}, -w_{xx}$ to (2.35), integrate over I , and use Hölder inequality to obtain

$$\begin{aligned}
(3.26) \quad & \frac{1}{2} \frac{d}{dt} |A^{\frac{1}{2}} u|_{L^2}^2 \leq -|Au|^2 + |w_x|_{L^2} |Au| + C_2 |Au|, \\
& \frac{1}{2} \frac{d}{dt} |v_x|_{L^2}^2 \leq -|v_{xx}|_{L^2}^2 + |w_{xx}|_{L^2} |v_{xx}|_{L^2} + C_3 |v_{xx}|_{L^2}, \\
& \frac{1}{2} \frac{d}{dt} |w_x|_{L^2}^2 \leq -|w_{xx}|_{L^2}^2 + (C_4 |v|_{L^4}^2 + C_5) |w_{xx}|_{L^2}.
\end{aligned}$$

Using Young inequality and (3.24), we obtain that

$$\begin{aligned}
(3.27) \quad & \frac{d}{dt} (|A^{\frac{1}{2}} u|^2 + |v_x|_{L^2}^2 + |w_x|_{L^2}^2) \leq |w_x|_{L^2}^2 + C_2^2 + C_3^2 + (C_4 |v|_{L^4}^2 + C_5)^2 \\
& \leq |w_x|_{L^2}^2 + C_8 |v_x|_{L^2}^2 + C_9.
\end{aligned}$$

The inequalities (3.25), (3.27) and the Uniform Gronwall inequality (Temam (1988), page 89) imply the uniform boundedness. The existence of a global attractor follows directly from Theorem 1.1, Temam (1988) (see also Hale (1988)). $\square$

By using Corollary 3.2, we can prove an useful lemma.

LEMMA 3.4. *Let $k_1 > \frac{K_1}{2}, k_2 > \frac{3}{2}$, then $\mathcal{A}_{RD}$ is contained in the set $J(V)$.*

Proof. We first note that there exists a constant ρ_5 such that

$$|v - u_x|_{L^2}^2 + |w - f(u)|_{L^2}^2 < \rho_5^2, \quad \text{for all } (u, v, w) \in \mathcal{A}_{RD}.$$

We now let $U(U_0, t) = (u(t), v(t), w(t))$ be a solution of (2.35) with initial condition $U_0 \in \mathcal{A}_{RD}$, then from Corollary 3.2,

$$\begin{aligned}
(3.28) \quad & |v(t) - u_x(t)|_{L^2}^2 + |w(t) - f(u(t))|_{L^2}^2 \leq e^{-\nu t} |v(T_1) - u_x(T_1)|_{L^2}^2 + |w(T_1) - f(u(T_1))|_{L^2}^2 \\
& \leq (\text{since } U(U_0, T_1) \in \mathcal{A}_{RD}) \\
& \leq \rho_5^2 e^{-\nu t},
\end{aligned}$$

for all $t \geq T_1$, where

$$\nu = \min(2k_1 - K_1, 2k_2).$$

Moreover by the invariance property of the global attractor, for every $U_\infty \in \mathcal{A}_{RD}$ and every $t \geq T_1$, there exists U_0 such that

$$U(U_0, t) = U_\infty.$$

Hence (3.28) yields

$$\text{dist}(U_\infty, J(V)) \leq \rho_5^2 e^{-\nu t}$$

for all $t \geq T_1$. Therefore

$$\text{dist}(\mathcal{A}_{RD}, J(V)) = \sup_{U \in \mathcal{A}_{RD}} \text{dist}(U, J(V)) \leq \rho_5^2 e^{-\nu t}$$

for all $t \geq T_1$. This implies $\text{dist}(\mathcal{A}_{RD}, J(V)) = 0$.

If we can prove that $J(V)$ is closed in $L^2(I)^3$, we easily obtain that $\mathcal{A}_{RD} \subset J(V)$. In fact, if $(u_n, (u_n)_x, f(u_n))$ converges to (u, v, w) in $L^2(I)^3$, then v is a weak derivative of u and $v \in H$ because

$$|v|_{L^2} \leq |v - (u_n)_x|_{L^2} + |(u_n)_x|_{L^2} < \infty.$$

Now

$$|w - f(u)|_{L^2} \leq |w - f(u_n)|_{L^2} + |f(u_n) - f(u)|_{L^2}$$

and

$$\begin{aligned}
|f(u_n) - f(u)|_{L^2} &= \left| \int_0^1 f'((1-s)u + su_n) ds (u_n - u) \right|_{L^2} \\
&\leq |f'((1-s)u + su_n)|_{L^\infty} |u_n - u|_{L^2} \\
&\leq \alpha_0 (1 + |(1-s)u + su_n|_{L^\infty}^m) |u_n - u|_{L^2}.
\end{aligned}$$

Since $|u_n|_{L^\infty}$ is uniformly bounded by Sobolev imbedding (3.22), we easily see that $w = f(u)$. This completes the proof of Lemma. $\square$

Now we can prove our main theorem of this section.

THEOREM 3.5. *Let $k_1 > \frac{K_1}{2}, k_2 > \frac{3}{2}$, then the global attractor of (2.9) coincide with that of (2.35), i.e., $\mathcal{A}_{RD} = J(\mathcal{A}_\gamma)$. In particular, the injection J provides a flow homeomorphism between $\mathcal{A}$ and $\mathcal{A}_{RD}$.*

Proof. Obviously $J(\mathcal{A}_\gamma) \subset \mathcal{A}_{RD}$. Moreover $\mathcal{A}_{RD}$ is a bounded invariance set in $J(V)$. But since $\mathcal{A}_\gamma$ is the global attractor of (2.9), $J(\mathcal{A}_\gamma)$ is the maximal bounded invariant set in $J(V)$. Therefore $\mathcal{A}_{RD} \subset J(\mathcal{A})_\gamma$. The second statement follows directly from Proposition 2.5. $\square$

4. FURTHER RESULTS.

In this section we prove the existence of inertial manifold for the transformed equation (2.35). We assume $k_1 > \frac{K_1}{2}, k_2 > \frac{3}{2}$, and $\gamma \geq 2\rho$ ($\gamma < \infty$) so that the global attractor $\mathcal{A}_{RD}$ of (2.35) is simply the image of the global attractor $\mathcal{A}_\infty$ of (2.1). In other words (2.35) preserves the long-time dynamics of (2.1). In particular, from Corollary 2.3 there exists a positive constant $\rho_6 > 0$ such that

$$|\mathbf{A}^{\frac{1}{2}}\mathbf{U}| < \rho_6, \quad \text{for all } \mathbf{U} \in \mathcal{A}_{RD}.$$

We recall the equation (2.36), i.e.,

$$(4.1) \quad \frac{d}{dt}\mathbf{U} = -\mathbf{A}\mathbf{U} + \tilde{\mathbf{F}}(\mathbf{U}).$$

In order to prove the existence of inertial manifold, it is usually required of truncating nonlinear term, thereby considering a modified equation which provides the same long-time dynamics near the global attractor but is different for large norm in $D(\mathbf{A}^{\frac{1}{2}})$. The modified equation of (4.1) is

$$(4.2) \quad \frac{d\mathbf{U}}{dt} + \mathbf{A}\mathbf{U} = \mathbf{F}(\mathbf{U})$$

where

$$\mathbf{F}(\mathbf{U}) = \phi\left(\frac{|\mathbf{A}^{\frac{1}{2}}\mathbf{U}|^2}{\rho_6^2}\right)\tilde{\mathbf{F}}(\mathbf{U}).$$

Because $\phi(\cdot) = 1$ in the ball containing $\mathcal{A}_{RD}$, we also recover all the dynamics near the attractor.

Now we observe that the operator $\mathbf{A}$ has a block diagonal form, i.e.,

$$\mathbf{A} = \begin{bmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix},$$

where $*$ denote some differential operators. Hence it is easy to see that A has compact resolvent and it has discrete spectrum $\{\lambda_n\}_{n \geq 0}$ with multiple index, where

$$\lambda_n = \left(\frac{2\pi n}{L}\right)^2.$$

Let P be the projection associated with the first $(N + 1)$ eigenvalues of A and $Q = I - P$. Then we easily see that

$$(4.3) \quad P(H) \subset D(A), \quad H = P(H) \oplus Q(H)$$

and $P(H)$, $Q(H)$ is invariant under A (see Dunford and Schwartz (1957)). Moreover $\dim P(H) = 3N + 2$. In fact PH has natural basis

$$(\phi_n, 0, 0), \quad (0, \psi_n, 0), \quad (0, 0, \psi_n),$$

where

$$\phi_n = \sin \frac{2\pi n}{L} x, \quad \psi_n = \cos \frac{2\pi n}{L} x \quad \text{for } n \geq 1, \quad \psi_0 = 1.$$

The inertial manifold for the equation (4.2) is constructed as a graph from $P(H)$ to $Q(H) \cap D(A^{\frac{1}{2}})$.

PROPOSITION 4.1. *Under the assumption described above and in Section 2, there exists an inertial manifold $\mathcal{M} = \text{Graph } \Phi$ for the equation (4.2).*

Proof. For the proof we can use the theory developed in Luskin and Sell with a little modification due to the non-selfadjointness of the operator A . We could also appeal to Miklavcic (1990) and Sell and You (1990) which are the most recent and general and also include our case. Instead of giving the full proof, we provide the essential estimates. Because

$$\lambda_0 \leq \dots \leq \lambda_N < \lambda_N + 1 < \lambda_{N+1} - 1 < \lambda_{N+1} \leq \dots,$$

from (Henry (1981), Theorem 1.5.3) one obtains that

$$(4.4) \quad \begin{aligned} |A^{\frac{1}{2}} e^{PA t} U_0| &\leq C_1 e^{(\lambda_N + 1)t} |A^{\frac{1}{2}} U_0| \quad \text{for } t \geq 0 \quad \text{and} \\ |A^{\frac{1}{2}} e^{QA t} U_0| &\leq C_2 e^{(\lambda_{N+1} - 1)t} |A^{\frac{1}{2}} U_0| \quad \text{for } t \leq 0. \end{aligned}$$

Moreover by using Sobolev imbedding $H^1(I) \subset L^\infty(I)$, it is easy to show that

$$F : \mathcal{D}(A^{\frac{1}{2}}) \rightarrow \mathcal{D}(A^{\frac{1}{2}})$$

is globally Lipschitz and the rest is straightforward. $\square$

As a corollary we can conclude that

COROLLARY 4.2. *The essential long-time dynamics of the flow of (2.1) is completely described by the system of ordinary differential equations*

$$(4.5) \quad \frac{d}{dt} \mathbf{PU} + \mathbf{A}\mathbf{PU} = \mathbf{P}\mathbf{F}(\mathbf{PU} + \Phi(\mathbf{PU}))$$

We can write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ and we can prove from the exponential tracking property of inertial manifold that

COROLLARY 4.3. *If $u(t)$ is any solution of (2.9), then there exists a solution $\mathbf{PU}(t)$ of ordinary differential equations (4.5) which satisfies that for some constant C, α ,*

$$|u(t) - (p_1(t) + \Phi_1(\mathbf{PU}))| \leq C e^{-\alpha t} \quad \text{for } t > 0$$

where $\mathbf{PU}(t) = (p_1(t), p_2(t), p_3(t))$.

Let us define

$$(4.6) \quad S = \mathcal{M} \cap J(V),$$

where J is the injection map described in Section 2. From Proposition 2.5, S is an invariant set. We also proved that the essential long-time dynamics of (2.1), (2.9) and (2.35) are fully contained in this cross section. Let us consider the first component of S (denoted by S^u). At this moment we can not prove that S^u is a finite dimensional manifold but we have

THEOREM 4.4. *The set S^u is invariant under the flow of (2.9) and it attracts all the orbits of (2.9).*

Proof. Let $u_0 \in S^u$, then $Ju_0 = (u_0, (u_0)_x, f(u_0)) \in S$. Since $\mathcal{M}$ is invariant under the flow of (2.35) and (4.2), we find from Proposition 2.5 that

$$u(t) \in S_u.$$

The second statement follows directly from the fact $\mathcal{A}_\gamma \subset S_u$. $\square$

REMARK 4.5. *We do not know whether the set S^u attracts all the orbit exponentially. But if we consider the first component of $\mathcal{M}$ i.e.*

$$(4.7) \quad \mathcal{M}^u = \{p_1 + \Phi_1(p_1, p_2, p_3) : p_1, p_2, p_3 \in PH\},$$

then we can prove that this set is a finite dimensional manifold and that it attracts all the bounded sets in V exponentially by using exponential attracting property of the inertial manifold $\mathcal{M}$. But it does not seem to be invariant under any flows.

Finally we explain how the dynamics in S^u is described by lower dimensional ordinary differential equations than (4.5). Suppose $u_0 \in S^u$, then $u(t) \in S^u$ and $J(u(t)) \in S$. Let $J(u(t)) = (u(t), v(t), w(t))$, then since $Ju(t) \in \mathcal{M}$,

$$(4.8) \quad \begin{aligned} u(t) &= p_1(t) + \Phi_1(p_1(t), p_2(t), p_3(t)) \\ v(t) &= p_2(t) + \Phi_2(p_1(t), p_2(t), p_3(t)) \\ w(t) &= p_3(t) + \Phi_3(p_1(t), p_2(t), p_3(t)) \end{aligned}$$

where $\mathbf{PU}(t) = (p_1(t), p_2(t), p_3(t))$ is a solution of the inertial form (4.5) with initial condition

$$\mathbf{PU}(0) = \mathbf{PJ}(u_0).$$

Now $v(t) = (u(t))_x$, $w(t) = f(u(t))$ and $\mathbf{PU} = (p_1(t), p_2(t), p_3(t))$ is a solution of the ordinary differential equations

$$(4.9) \quad \frac{d}{dt}\mathbf{PU} = -\mathbf{APU} + \mathbf{PF}(u(t), u(t)_x, f(u(t)))$$

where $u(t) = p_1(t) + \Phi_1(p_1(t), p_2(t), p_3(t))$. Furthermore, by an elementary calculation we obtain that if

$$p_1(t) = \sum_{1 \leq n \leq N} b_n \phi_n \quad \text{and} \quad p_2(t) = a_1 + \sum_{1 \leq n \leq N} c_n \psi_n,$$

then $p_2(t) = (p_1(t))_x$ and

$$a_1 = 0, \quad c_n = \frac{2\pi n}{L} b_n \quad \text{for} \quad 1 \leq n \leq N.$$

This implies that $p_2(t)$ does not enter in the system of ODEs (4.9). Only $p_3(t)$ is involved (in a complicated manner). Hence we have come to the conclusion;

THEOREM 4.6. *The essential long-time dynamics of the flow of (2.1) is completely described by the system of ordinary differential equations*

$$(4.10) \quad \frac{d}{dt}\mathbf{PU} + \mathbf{APU} = \mathbf{PF}(J(u))$$

where $\mathbf{PU} = (p_1, (p_1)_x, p_3)$, and $u(t) = p_1(t) + \Phi_1(p_1(t), (p_1(t))_x, p_3(t))$.

REFERENCES

- S-N. CHOW AND K. LU (1988), *Invariant manifolds for flows in Banach spaces*, J. Differential Equations, 74, pp. 285-317.
- P. CONSTANTIN, C. FOIAS, B. NICOLAENKO AND R. TEMAM (1989), *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Appl. Math. Sciences, Springer Verlag, New York.
- N. DUNFORD AND J.T. SCHWARTZ (1957), *Linear Operator 1*, Interscience, NY.
- C. FOIAS, G.R. SELL AND R. TEMAM (1988), *Inertial Manifolds for Nonlinear Evolutionary Equations*, J. Differential Equations, 73, pp. 309-353.
- A. FRIEDMAN (1964), *Partial differential equations of parabolic type*, Prentice-Hall, Eaglewood Cliffs, New Jersey.
- J. K. HALE (1988), *Asymptotic Behavior of dissipative Systems*, Math. Surveys and Monograph, Amer. Math. Soc, Providence, R.I..
- D. HENRY (1981), *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer Verlag, New York.
- J.L. LIONS (1969), *Quelques Méthodes de Résolution des Problèmes aux Limites non linéaires*, Gauthier Villars, Paris.
- M. LUSKIN AND G.R. SELL (1989), *Parabolic Regularization and Inertial Manifolds*, IMA Preprint.
- J. MALLET-PARET AND G.R. SELL (1988), *Inertial manifolds for reaction diffusion equations in higher space dimensions*, IMA Preprint No.331; J. Amer. Math. Soc., 1.
- M. MIKLAVCIC (1990), *A sharp condition for existence of an inertial manifold*, IMA Preprint, 604.
- A. PAZY (1983), *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol 44, Springer Verlag, New York.
- G.R. SELL AND Y. YOU (1990), *Inertial manifolds: The non-self adjoint case*, Preprint.
- R. TEMAM (1988), *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer Verlag, New York.
- R. TEMAM (1979), *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS Regional Conference Series, No. 41, SIAM, Philadelphia.

#	Author/s	Title
746	Chris Grant,	Solutions to evolution equations with near-equilibrium initial values
747	Mario Taboada and Yuncheng You,	Invariant manifolds for retarded semilinear wave equations
748	Peter Rejto and Mario Taboada,	Unique solvability of nonlinear Volterra equations in weighted spaces
749	Hi Jun Choe,	Holder regularity for the gradient of solutions of certain singular parabolic equations
750	Jack D. Dockery,	Existence of standing pulse solutions for an excitable activator-inhibitory system
751	Jack D. Dockery and Roger Lui,	Existence of travelling wave solutions for a bistable evolutionary ecology model
752	Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo,	Singular perturbation problems with a compact support semilinear term
753	Emad A. Fatemi,	Numerical schemes for constrained minimization problems
754	Y. Kuang and H.L. Smith,	Slowly oscillating periodic solutions of autonomous state-dependent delay equations
755	Emad A. Fatemi,	A new splitting method for scalar conservation laws with stiff source terms
756	Hi Jun Choe,	A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities
757	Haitao Fan,	A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids
758	T.A. Osborn and F.H. Molzahn,	The Wigner–Weyl transform on tori and connected graph propagator representations
759	Avner Friedman and Bei Hu,	A free boundary problem arising in superconductor modeling
760	Avner Friedman and Wenxiong Liu,	An augmented drift-diffusion model in semiconductor device
761	Avner Friedman and Miguel A. Herrero,	Extinction and positivity for a system of semilinear parabolic variational inequalities
762	David Dobson and Avner Friedman,	The time-harmonic Maxwell equations in a doubly periodic structure
763	Hi Jun Choe,	Interior behaviour of minimizers for certain functionals with nonstandard growth
764	Vincenzo M. Tortorelli and Epifanio G. Virga,	Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation
765	Nikan B. Firoozye and Robert V. Kohn,	Geometric parameters and the relaxation of multiwell energies
766	Haitao Fan and Marshall Slemrod,	The Riemann problem for systems of conservation laws of mixed type
767	Joseph D. Fehribach,	Analysis and application of a continuation method for a self-similar coupled Stefan system
768	C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi,	Dissipativity of numerical schemes
769	D.D. Joseph, T.Y.J. Liao and J.-C. Saut,	Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity
770	Chris Grant,	Solutions to evolution equations with near-equilibrium initial values
771	B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu,	Convergence of finite volume methods
772	N.G. Lloyd and J.M. Pearson,	Computing centre conditions for certain cubic systems
773	João Palhoto Matos,	Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
774	L.A. Peletier & W.C. Troy,	Self-similar solutions for infiltration of dopant into semiconductors
775	H. Scott Dumas and James A. Ellison,	Nekhoroshev’s theorem, ergodicity, and the motion of energetic charged particles in crystals
776	Stathis Filippas and Robert V. Kohn,	Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
777	Patricia Bauman, Nicholas C. Owen and Daniel Phillips,	Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
778	Patricia Bauman, Nicholas C. Owen and Daniel Phillips,	Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity
779	Jack Carr and Robert Pego,	Self-similarity in a coarsening model in one dimension
780	J.M. Greenberg,	The shock generation problem for a discrete gas with short range repulsive forces
781	George R. Sell and Mario Taboada,	Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
782	T. Subba Rao,	Analysis of nonlinear time series (and chaos) by bispectral methods
783	Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy,	Vortex rings of one fluid in another free fall
784	Oscar Bruno, Avner Friedman and Fernando Reitich,	Asymptotic behavior for a coalescence problem
785	Johannes C.C. Nitsche,	Periodic surfaces which are extremal for energy functionals containing curvature functions
786	F. Abergel and J.L. Bona,	A mathematical theory for viscous, free-surface flows over a perturbed plane
787	Gunduz Caginalp and Xinfu Chen,	Phase field equations in the singular limit of sharp interface problems
788	Robert P. Gilbert and Yongzhi Xu,	An inverse problem for harmonic acoustics in stratified oceans
789	Roger Fosdick and Eric Volkmann,	Normality and convexity of the yield surface in nonlinear plasticity
790	H.S. Brown, I.G. Kevrekidis and M.S. Jolly,	A minimal model for spatio-temporal patterns in thin film flow

- 791 **Chao-Nien Chen**, On the uniqueness of solutions of some second order differential equations
- 792 **Xinfu Chen and Avner Friedman**, The thermistor problem for conductivity which vanishes at large temperature
- 793 **Xinfu Chen and Avner Friedman**, The thermistor problem with one-zero conductivity
- 794 **E.G. Kalnins and W. Miller, Jr.**, Separation of variables for the Dirac equation in Kerr Newman space time
- 795 **E. Knobloch, M.R.E. Proctor and N.O. Weiss**, Finite-dimensional description of doubly diffusive convection
- 796 **V.V. Pukhnachov**, Mathematical model of natural convection under low gravity
- 797 **M.C. Knaap**, Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents
- 798 **Stathis Filippas and Wenxiong Liu**, On the blowup of multidimensional semilinear heat equations
- 799 **A.M. Meirmanov**, The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution
- 800 **Bo Guan and Joel Spruck**, Interior gradient estimates for solutions of prescribed curvature equations of parabolic type
- 801 **Hi Jun Choe**, Regularity for solutions of nonlinear variational inequalities with gradient constraints
- 802 **Peter Shi and Yongzhi Xu**, Quasistatic linear thermoelasticity on the unit disk
- 803 **Satyanad Kichenassamy and Peter J. Olver**, Existence and non-existence of solitary wave solutions to higher order model evolution equations
- 804 **Dening Li**, Regularity of solutions for a two-phase degenerate Stefan Problem
- 805 **Marek Fila, Bernhard Kawohl and Howard A. Levine**, Quenching for quasilinear equations
- 806 **Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii**, Global existence of weak solutions for interface equations coupled with diffusion equations
- 807 **Mark J. Friedman and Eusebius J. Doedel**, Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study
- 808 **Mark J. Friedman**, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
- 809 **Peter W. Bates and Songmu Zheng**, Inertial manifolds and inertial sets for the phase-field equations
- 810 **J. López Gómez, V. Márquez and N. Wolanski**, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
- 811 **Xinfu Chen and Fahuai Yi**, Regularity of the free boundary of a continuous casting problem
- 812 **Eden, A., Foias, C., Nicolaenko, B. and Temam, R.**, Inertial sets for dissipative evolution equations Part I: Construction and applications
- 813 **Jose-Francisco Rodrigues and Boris Zaltzman**, On classical solutions of the two-phase steady-state Stefan problem in strips
- 814 **Viorel Barbu and Srdjan Stojanovic**, Controlling the free boundary of elliptic variational inequalities on a variable domain
- 815 **Viorel Barbu and Srdjan Stojanovic**, A variational approach to a free boundary problem arising in electrophotography
- 816 **B.H. Gilding and R. Kersner**, Diffusion-convection-reaction, free boundaries, and an integral equation
- 817 **Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez**, On the Barenblatt equation of elastoplastic filtration
- 818 **Avner Friedman and Bei Hu**, The Stefan problem with kinetic condition at the free boundary
- 819 **M.A. Grinfeld**, The stress driven instabilities in crystals: mathematical models and physical manifestations
- 820 **Bei Hu and Lihe Wang**, A free boundary problem arising in electrophotography: solutions with connected toner region
- 821 **Yongzhi Xu, T. Craig Poling, and Trent Brundage**, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean
- 822 **Steven J. Altschuler**, Singularities of the curve shrinking flow for space curves
- 823 **Steven J. Altschuler and Matthew A. Grayson**, Shortening space curves and flow through singularities
- 824 **Tong Li**, On the Riemann problem of a combustion model
- 825 **L.A. Peletier & W.C. Troy**, Self-similar solutions for diffusion in semiconductors
- 826 **C.J. van Duijn, L.A. Peletier & R.J. Schotting**, On the analysis of brine transport in porous media
- 827 **Minkyu Kwak**, Finite dimensional description of convective reaction-diffusion equations
- 828 **Minkyu Kwak**, Finite dimensional inertial forms for the 2D Navier-Stokes equations
- 829 **Victor A. Galaktionov and Sergey A. Posashkov**, On some monotonicity in time properties for a quasilinear parabolic equation with source
- 830 **Victor A. Galaktionov**, Remark on the fast diffusion equation in a ball
- 831 **Hi Jun Choe and Lihe Wang**, A regularity theory for degenerate vector valued variational inequalities
- 832 **Vladimir I. Oliker and Nina N. Uraltseva**, Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.
- 833 **S. Kamin and W. Liu**, Large time behavior of a nonlinear diffusion equation with a source
- 834 **Shoshana Kamin and Juan Luis Vazquez**, Singular solutions of some nonlinear parabolic equations
- 835 **Bernhard Kawohl and Robert Kersner**, On degenerate diffusion with very strong absorption