**STABILITY OF DYNAMIC INVERSION CONTROL LAWS
APPLIED TO NONLINEAR AIRCRAFT PITCH-AXIS MODELS**

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Stability of Dynamic Inversion Control Laws Applied to Nonlinear Aircraft Pitch-Axis Models

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Abstract

A global stability result is presented for nonlinear aircraft pitch-axis models with dynamic inversion control laws. The result is applied to the F-14 aircraft model. Using this result, we are able to sketch a stability region of the F-14 model in terms of the engine thrust and the selected pitch attitude.

1 Introduction

Dynamic inversion is a nonlinear control technique that has been applied by Honeywell and others to a variety of realistic aerospace vehicle models with reasonably good results. The list of study applications includes models of the F-14 aircraft, the HARV F-18 aircraft, a McDonnell Douglas model of the NASP vehicle, and a General Dynamics model of a next-generation booster vehicle. The main advantage of dynamic inversion over more conventional linear control techniques is its applicability to the full nonlinear vehicle models.

The theory of dynamic inversion (and nonlinear control in general) is not well developed. Most of what we know about dynamic inversion theory is summarized in the references [1], [2], [3] and [10].

This paper describes a global stability result for dynamic inversion applied to nonlinear aircraft pitch-axis models. The point is to examine the technique from a mathematical point of view and try to understand why it works. The control laws studied here are simpler than those we would implement in practice. Both the control laws and the airplane model have a special form introduced to simplify analysis, but most of the analysis can be augmented to apply to models of more general type.

The structure of the paper is as follows. The first seven sections are expository, concentrating on statements of results and explanations instead of detailed analysis. In Section 2 we establish the notation and the form of nonlinear airplane model. In Section 3 we define the equilibrium set and introduce the concept of residualized aerodynamic functions. In Section 4 we present the structure of a dynamic inversion controller and describe an associated positively invariant set for the closed-loop dynamics. In Section 5 we present a stability result based on some properties of the residualized aerodynamic functions. In Section 6, as a comparison to the nonlinear theory, we apply the dynamic inversion control to linear airplane models. A necessary and sufficient condition is given for the resulted closed loop linear system being globally stable. In section 7, we apply the stability results obtained in section 5 to the F-14 aircraft model. Robustness analysis of the stability results presented in section 5 is carried out in section 8.

The last three sections are more technical, concentrating on analytical details left out of the first seven sections. In section 9, we construct the positively invariant set alluded to in Section 4. A rigorous proof of our main stability result in section 5 is provided in section 10 for the interested reader. In Section 11 we present a brief analysis of the relation mentioned in Section 6 between open-loop zeros and closed-loop poles that is associated with dynamic inversion control.

2 Equations of Motion

We concentrate on aircraft pitch-axis models. The body-axis coordinate system is used. See Figure 1. There are four states:

U = component of velocity in the aircraft longitudinal (x) axis

W = component of velocity in the aircraft vertical (z) axis

Q = vehicle pitch-rate

θ = vehicle pitch attitude relative to local horizontal.

The equations of motion are:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} U \\ W \\ Q \\ \theta \end{pmatrix} = & \begin{pmatrix} -WQ \\ UQ \\ 0 \\ Q \end{pmatrix} + \begin{pmatrix} -g \sin(\theta) \\ g \cos(\theta) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \\ 0 \\ 0 \end{pmatrix} + \\ & + \frac{1}{2} \rho V^2 S \begin{pmatrix} C_x(\alpha)/m \\ C_z(\alpha)/m \\ \bar{c} C_M(\alpha)/I_Y \\ 0 \end{pmatrix} + \frac{1}{2} \rho V^2 S \begin{bmatrix} C_{x,\delta}(\alpha)/m \\ C_{z,\delta}(\alpha)/m \\ \bar{c} C_{M,\delta}(\alpha)/I_Y \\ 0 \end{bmatrix} \delta \end{aligned} \quad (2.1)$$

The variables in equation (2.1) have the following meaning:

g = gravitational acceleration (constant),

T = thrust (control input),

m = vehicle mass (constant),

ρ = air density (assumed constant here),

V = speed = $\sqrt{U^2 + W^2}$,

α = angle of attack = $\arctan(\frac{W}{U})$,

I_Y = vehicle pitch inertia (constant),

δ = elevator angle (control input),

$\bar{c}$ = mean aerodynamic chord (constant)

$C_x(\alpha)$, $C_z(\alpha)$, $C_M(\alpha)$ = aerodynamic functions for $\delta = 0$,

$C_{x,\delta}(\alpha)$, $C_{z,\delta}(\alpha)$, $C_{M,\delta}(\alpha)$ = aerodynamic functions due to nonzero δ .

S = wing area .

FIGURE 1: BODY-AXIS COORDINATE SYSTEM

The two control inputs T and δ are assumed to be limited to values within a fixed interval.

A reasonable set of ranges for a fighter is $0 \leq T \leq mg$ and $-20 \text{ degrees} \leq \delta \leq 20 \text{ degrees}$. Although the aerodynamic functions usually depend on Mach as well as α , we neglect the Mach dependence here.

3 The Equilibrium Set

The model (2.1) has the following form:

$$\dot{x} = f(x, u) \quad (3.1)$$

where x is a vector in R^n and u is a vector in R^m . Let $\bar{U}$ denote the set of allowed control values in R^m . Define the equilibrium set

$$\bar{M} = \{(x, u) \mid f(x, u) = 0, \quad u \in \bar{U}\}.$$

Projecting $\bar{M}$ onto the first factor x we obtain M , the set of equilibrium states. Note that $\bar{M}$ and M depend on the specified control limits.

The equilibrium set of a system with m inputs is typically m -dimensional. The system represented by (2.1) has two inputs and M is two-dimensional. We use T and θ as coordinates on M .

Suppose T and θ are fixed. At equilibrium $Q = 0$, the only unspecified terms in equation (2.1) are the aerodynamic forces and moment. However, the direct aerodynamic force terms in the U , W degrees of freedom must exactly balance the gravity and thrust forces for an equilibrium to exist. Therefore, the aerodynamic force vector in the U , W space is completely determined, even though we do not know the point (U, W) .

We introduce the notion of residualized aerodynamic functions.

Definition 3.1 *For the model represented by equation (2.1), the residualized aerodynamic functions are $\bar{C}_x(\alpha)$, $\bar{C}_z(\alpha)$ defined by:*

$$\begin{pmatrix} \bar{C}_x(\alpha) \\ \bar{C}_z(\alpha) \end{pmatrix} = \begin{pmatrix} C_x - C_{x,\delta} C_M / C_{M,\delta} \\ C_z - C_{z,\delta} C_M / C_{M,\delta} \end{pmatrix} (\alpha). \quad (3.2)$$

The meaning of the residualized aerodynamic functions is simple: at any equilibrium state the aerodynamic force vector in the U, W space is:

$$\begin{pmatrix} \overline{F}_x(U, W) \\ \overline{F}_z(U, W) \end{pmatrix} = \frac{1}{2}\rho V^2 S \begin{pmatrix} \overline{C}_x(\alpha) \\ \overline{C}_z(\alpha) \end{pmatrix} \quad (3.3)$$

as can be verified by solving the equation $\dot{Q} = 0$ in (2.1). A problem may arise at values of α for which $C_{M,\delta}(\alpha)$ is zero or (relatively) very small in magnitude. We cannot expect stability for all aircraft in all possible flight conditions, so for purposes of analysis we consider two subproblems:

1. Assume unlimited control authority and work globally, OR
2. Restrict attention to a subset of the states on which control authority is adequate.

Denote by A the set of α corresponding to equilibrium states: A is all angles from 0 to 2π if $C_{M,\delta}(\alpha)$ never vanishes and δ is unlimited (subproblem 1). In practice, elevator deflection angle is limited and its control effectiveness degrades at angles of attack too far from zero, so in real applications the set A turns out to be a proper subset (subproblem 2). In what follows, when we talk about global results, we consider the first subproblem. We consider the second subproblem explicitly in section 9.

The system represented by (2.1) has two inputs and its 2-dimensional equilibrium set M is characterized by solutions of the following algebraic equations

$$\begin{cases} -g \sin(\theta) + T/m + \frac{1}{2m}\rho V^2 S \overline{C}_x(\alpha) = 0 \\ g \cos(\theta) + \frac{1}{2m}\rho V^2 S \overline{C}_z(\alpha) = 0 \end{cases} \quad (3.4)$$

where we consider T and θ as parameters and m, S and g are fixed constants.

Remark 3.1 *For each given pair of (T, θ) , the equilibrium of the system (2.1) is formed by all points*

$$(U, W, Q, \theta) = (U, W, 0, \theta)$$

where

$$U = V \cos(\alpha), \quad W = V \sin(\alpha),$$

and (V, α) is a solution of (3.4) corresponding to the given (T, θ) . Thus for a given pair of (T, θ) , if (3.4) has a unique solution, then the system (2.1) has a unique equilibrium point.

4 Dynamic Inversion

The control problem we consider here is the following:

Statement of the problem: *Given an equilibrium state $\bar{x}$, determine a controller $u = K(x, \bar{x})$ so that $\bar{x}$ is a global attractor for the system*

$$\dot{x} = f(x, K(x, \bar{x})) \quad (4.1)$$

It is clear that any global attractor for (4.1) must be an equilibrium state. Using dynamic inversion we will address this problem for vehicle models having a unique equilibrium point for appropriately chosen engine thrust, T .

The approach we take is to invert the rotational degrees of freedom to a set of stable, second order dynamics. We leave the value of T fixed – we could use T as a control input but we choose not to. The throttle is typically used as a low-bandwidth control that is not changed during dynamic maneuvers. In the rest of this section we show how K is constructed and discuss stability of the attitude dynamics. The big question concerning the stability of the complementary dynamics is addressed in Section 5.

To construct the controller K , first select a desired stable set of second-order linear dynamics for θ :

$$\dot{Q} = -2\zeta\omega Q - \omega^2(\theta - \theta_{cmd}) \quad (4.2)$$

Desirable sets of dynamics for aircraft pitch axis control are described in the MIL-F-8785C specifications document. These dynamics are realized in the closed-loop system if δ takes values according to the following equation:

$$\delta = \frac{\left(\frac{2I_y}{\rho V^2 S \bar{c}} \dot{Q}_{cmd} - C_M(\alpha)\right)}{C_{M,\delta}(\alpha)} \quad (4.3)$$

where $\dot{Q}_{cmd}$ has the value:

$$\dot{Q}_{cmd} = -2\zeta\omega Q - \omega^2(\theta - \theta_{cmd}) \quad (4.4)$$

for suitable values of ζ and ω , and θ_{cmd} is the θ value of the desired equilibrium state. These equations will do the job so long as they give allowed values of δ .

The closed loop system is obtained by inserting (4.3) into (2.1).

$$\frac{d}{dt} \begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} -WQ \\ UQ \end{pmatrix} + \begin{pmatrix} -g \sin(\theta) \\ g \cos(\theta) \end{pmatrix} + \frac{1}{2m} \rho V^2 S \begin{pmatrix} \bar{C}_x \\ \bar{C}_z \end{pmatrix} +$$

$$+\frac{I_Y \dot{Q}_{cmd}}{\bar{c}m C_{M,\delta}(\alpha)} \begin{pmatrix} C_{x,\delta}(\alpha) \\ C_{z,\delta}(\alpha) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} \quad (4.5)$$

$$\frac{d}{dt} \begin{pmatrix} Q \\ \theta \end{pmatrix} = \begin{pmatrix} -2\zeta\omega Q \\ Q \end{pmatrix} + \begin{pmatrix} -\omega^2(\theta - \theta_{cmd}) \\ 0 \end{pmatrix} \quad (4.6)$$

As we see, the dynamics of the pitch rate Q and pitch angle θ have been reduced to linear by our dynamic inversion. Besides, they are decoupled from the dynamics of velocity (U, W) in this closed loop system and converges to $(0, \theta_{cmd})$ according to our design. The remaining question is asymptotic behavior of the so-called complementary dynamics of (U, W) . This will be addressed in the next section.

Remark 4.1 *The surface saturation problem at high α appears in all real aircraft. That is the motivation for using thrust-vectoring for high- α control. The problem of attitude control by dynamic inversion using thrust vectoring was studied by Mike Elgersma in [1].*

In the case that we assume unlimited control authority, we may simply work on the whole state space. When control saturations are considered we restrict our attention to a smaller invariant set. We have identified a particular invariant set closely associated with aircraft flight. This set is a bounded region such that:

1. α lies between the zero-lift value $\alpha_0 < 0$ degrees and the maximum lift-to-drag value α_{max} ,
2. V lies between a minimum cruising speed V_{min} and a maximum speed V_{max} ,
3. the flight-path angle $\gamma = \theta - \alpha$ lies in a fixed range about zero (say ± 10 degrees),
4. T is greater than or equal to the value needed to fly maximum γ at the V_{min} , α_{max} condition
5. θ is constrained to lie in an interval about zero consistent with conditions 1 and 3
6. Q is constrained to have magnitude smaller than a suitable ϵ .

For details, see Section 9.

5 The Stability Result

For pitch-axis models, computer simulations show that the pitch-attitude control strategy presented in Section 4 has excellent stability properties over extreme ranges of initial and transient state conditions. So far we have applied the technique without a rigorous proof that it should work globally. In this section we give a global stability result that explains the results observed for pitch-axis applications.

We sketch the proofs for the unlimited control authority case.

As we have seen in the previous section, we only need to show stability of the complementary dynamics (4.5).

Lemma 5.1 *Suppose the aerodynamic drag is always positive. Then there is a finite disc D centered at the origin in the U, W plane into which all trajectories eventually enter and remain.*

Proof:

From equation (4.5) we can compute the time rate of change of $V^2 = U^2 + W^2$:

$$\frac{d}{dt}(U^2 + W^2) = 2gV \sin(\alpha - \theta) + \frac{1}{m} \left(2TV \cos(\alpha) - \rho V^3 S \overline{C}_D(\alpha) \right) + \frac{2I_Y \dot{Q}_{cmd}}{\bar{c}mC_{M,\delta}} V C_{D,\delta}(\alpha)$$

where $\dot{Q}_{cmd}$ is given by (4.4),

$$\overline{C}_D(\alpha) := - \left(\cos(\alpha) \overline{C}_x(\alpha) + \sin(\alpha) \overline{C}_z(\alpha) \right) \quad (5.1)$$

and

$$C_{D,\delta}(\alpha) := - \left(\cos(\alpha) C_{x,\delta}(\alpha) + \sin(\alpha) C_{z,\delta}(\alpha) \right). \quad (5.2)$$

Likewise,

$$\overline{C}_L(\alpha) := - \left(-\sin(\alpha) \overline{C}_x(\alpha) + \cos(\alpha) \overline{C}_z(\alpha) \right) \quad (5.3)$$

and

$$C_{L,\delta}(\alpha) := - \left(-\sin(\alpha) C_{x,\delta}(\alpha) + \cos(\alpha) C_{z,\delta}(\alpha) \right). \quad (5.4)$$

The expression $\overline{C}_D$ denotes the total drag coefficient including direct surface effects. The first two terms and the last term on the right side are linear in V , while the third is cubic. For sufficiently large V the third term will dominate. By assumption, the coefficient of V^3 is negative, so for V^2 sufficiently large the time rate of change of V^2 is negative. The conclusion follows.

As we have observed in the previous section, the dynamics of Q and θ are decoupled from the dynamics of velocity (U, W) . Therefore, while studying the complementary

dynamics (4.4), we may assume that Q and θ are known functions and henceforth, (4.5) becomes a two-dimensional time varying nonlinear systems. Furthermore, since Q tends to zero and θ tends to θ_{cmd} as t tends to ∞ according to our control design (4.4), the right side of (4.5) will become

$$g \begin{pmatrix} -\sin(\theta_{cmd}) \\ \cos(\theta_{cmd}) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} + \frac{\rho V^2 S}{2m} \begin{pmatrix} \overline{C}_x(\alpha) \\ \overline{C}_z(\alpha) \end{pmatrix}$$

where $\overline{C}_x(\alpha)$ and $\overline{C}_z(\alpha)$ are the residualized aerodynamic functions given by (3.2). This leads us to define the following residualized system:

$$\begin{pmatrix} \dot{U} \\ \dot{W} \end{pmatrix} = g \begin{pmatrix} -\sin(\theta_{cmd}) \\ \cos(\theta_{cmd}) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} \overline{F}_x \\ \overline{F}_z \end{pmatrix} \quad (5.5)$$

where

$$\overline{F} = \begin{pmatrix} \overline{F}_x(U, W) \\ \overline{F}_z(U, W) \end{pmatrix}$$

denotes the residualized aerodynamic force vector defined by (3.3). Mathematically, the residualized system (5.5) is called the limiting system of the system (4.5) (cf. [7]) whose dynamics behavior determines that of (4.5) as we will see later.

Lemma 5.2 *Assume*

$$\text{div}(\overline{F}) = \frac{\partial \overline{F}_x}{\partial U} + \frac{\partial \overline{F}_z}{\partial W} < 0 \quad (5.6)$$

for all (U, W) not equal to $(0, 0)$. Then the only possible closed orbits of the residualized system are equilibria.

Proof: The proof is by contradiction. Let Φ denote the right hand side of equation (5.5). Suppose there is some closed orbit C – either a limit cycle or a homoclinic orbit. Let A denote the interior of C . By Green's theorem in the plane,

$$\int_C \langle \Phi, \nu \rangle = \int_A \text{div}(\Phi) d\text{Area} \quad (5.7)$$

where ν denotes the outward pointing unit normal vector to C . The integrand on the left vanishes identically by construction, while the integrand on the right is equal to $\text{div}(\overline{F})$ which is negative by assumption. This contradiction proves the lemma.

Remark 5.1 *This Lemma (and proof) are given as exercise 1.3 in chapter 2 of [4]. We approach the problem this way because the examples we have looked at (the F-4, F-14, and F-15 aircraft) appear to satisfy the negative divergence condition.*

There is a relatively simple expression for $\text{div}(\bar{F})$ derived in Section 10.

As a corollary of the above lemma, we have the following stability result for the residualized system (5.5).

Proposition 5.1 *Assume that*

1. *the residualized system (5.5) has a unique equilibrium (U^*, W^*) for given T and θ_{cmd} ;*
2. *the residualized aerodynamic force vector $\bar{F}$ satisfies the divergent condition (5.6).*

Then the unique equilibrium (U^, W^*) is a global attractor for the residualized system (5.5), i.e., any solution $(U(t), W(t))$ of the residualized system (5.5) satisfies*

$$\lim_{t \rightarrow \infty} U(t) = U^*, \quad \lim_{t \rightarrow \infty} W(t) = W^*.$$

Proof: By Lemma 5.1 any solution $(U(t), W(t))$ is bounded. Thus its limit set is either an equilibrium or a periodic path. But Lemma 5.2 excludes the periodic path case. Hence the limit set of any solution consists of one point (U^*, W^*) which is the unique equilibrium of (5.5) by the assumption. This is equivalent to saying that (U^*, W^*) is a global attractor of the system (5.5).

Remark 5.2 *It is not difficult to see that (U^*, W^*) is also a stable equilibrium of the system (5.5) under the assumptions of Proposition 5.1.*

Remark 5.3 *An easy computation shows that for given T and θ_{cmd} , the residualized system (5.5) has a unique equilibrium if and only if the equation*

$$\left(\frac{T}{mg} - \sin(\theta_{cmd}) \right) \bar{C}_z(\alpha) - \cos(\theta_{cmd}) \bar{C}_x(\alpha) = 0 \quad (5.8)$$

has only one solution $\alpha = \alpha^$ for given T and θ_{cmd} .*

The divergence condition (5.6) is also called the *dissipative condition*. The following example explains that this dissipative condition alone is not enough to show that the residualized system (5.5) has a global attractor.

Example 5.1 [5]: Consider the following nonlinear system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y + x - x^3 \end{cases}$$

It is readily to see that (i): $\text{div}(F) = -1 < 0$ everywhere and (ii): the system has three equilibrium: $(0, 0)$ (unstable), $(-1, 0)$ (stable) and $(1, 0)$ (stable).

Now we state our main stability result for the closed loop system (4.5)-(4.6).

Theorem 5.1 *Assume that*

1. *the total drag coefficient*

$$\overline{C}_D(\alpha) = C_D(\alpha) - C_{D,\delta}(\alpha) \frac{C_M(\alpha)}{C_{M,\delta}(\alpha)} > 0;$$

2. *the aerodynamic functions satisfy the dissipative condition*

$$3\overline{C}_D(\alpha) + \frac{d\overline{C}_L}{d\alpha}(\alpha) > 0;$$

3. *for a given thrust T and θ_{cmd} , the following equation of α*

$$\left(\frac{T}{mg} - \sin(\theta_{cmd}) \right) \overline{C}_z(\alpha) - \cos(\theta_{cmd}) \overline{C}_x(\alpha) = 0 \quad (5.9)$$

has only one solution $\alpha = \alpha^$.*

Then, for given T and θ_{cmd} , the closed loop system (4.5)-(4.6) has a unique equilibrium $(\theta^, Q^*, U^*, W^*)$ given by*

$$\theta^* = \theta_{cmd}, \quad Q^* = 0, \quad (5.10)$$

and

$$U^* = \sqrt{\frac{-2mg \cos(\theta_{cmd})}{\rho S \overline{C}_z(\alpha^*)}} \cos(\alpha^*), \quad W^* = \sqrt{\frac{-2mg \cos(\theta_{cmd})}{\rho S \overline{C}_z(\alpha^*)}} \sin(\alpha^*) \quad (5.11)$$

where α^ is the unique solution of the equation (5.8).*

Moreover any solution $(\theta(t), Q(t), U(t), W(t))$ of the closed loop system (4.5)-(4.6) satisfies

$$\theta(t) \rightarrow \theta_{cmd}, \quad Q(t) \rightarrow 0, \quad U(t) \rightarrow U^*, \quad W(t) \rightarrow W^*$$

as $t \rightarrow \infty$. That is to say, $(\theta_{cmd}, 0, U^, W^*)$ is a global attractor of the closed loop system (4.5)-(4.6).*

Remark 5.4 *The assumption 2 of Theorem 5.1 is equivalent to the divergency condition (5.6) of the residualized aerodynamic force vector $\overline{F}$. Indeed, we have (see section 10 for the divergence computation)*

$$\text{div}(\overline{F}) = -\frac{1}{2}\rho SV \left(3\overline{C}_D + \frac{d\overline{C}_L}{d\alpha} \right) \quad (5.12)$$

All three assumptions of Theorem 5.1 are given in terms of the aerodynamic functions in a form that is convenient to use in applications. In practice, Assumption 1 and Assumption 2 are valid. Hence whether the system is globally stable or not depends on whether the equation (5.9) has a unique solution, which in turn depends on the engine thrust T and the selected pitch attitude command θ_{cmd} . As an example, in section 7, we will apply Theorem 5.1 to the F-14 model. In that case, we are able to sketch a region of the thrust T and the selected pitch attitude command θ_{cmd} in which the corresponding closed loop system (4.5)-(4.6) has a global attractor.

The proof of Theorem 5.1 is provided in section 10 for the interested reader.

6 Linear Models

In the linear case the issue of stability is related to right half plane zeros in the elevator to pitch angle transfer function. For a derivation of this result, see Section 11.

Under the usual assumptions [8] this transfer function is given by

$$\frac{\theta}{\delta} = \frac{A(s + \frac{1}{T_1})(s + \frac{1}{T_2})}{(s^2 + a_{ph}s + b_{ph})(s^2 + a_{sp}s + b_{sp})} \quad (6.1)$$

where both zeros are typically in the left half plane with a few exceptions: there are some conditions where $1/T_1$ is negative and there are post stall conditions where $1/T_2$ is negative.

The application of dynamic inversion to the pitch axis in the linear case gives a closed loop characteristic equation given by

$$\phi_d(s) = (s^2 + 2\zeta\omega s + \omega^2)(s + \frac{1}{T_1})(s + \frac{1}{T_2}) \quad (6.2)$$

so right half plane zeros of θ/δ give unstable closed loop poles but yet

$$\frac{\theta}{\theta_c} = \frac{\omega^2}{(s^2 + 2\zeta\omega s + \omega^2)} \quad (6.3)$$

where ω and ζ are specified. We conclude:

Proposition 6.1 *The dynamic inversion controller discussed in Section 3 for the linear model is stable if and only if T_1 and T_2 are positive.*

Proposition 6.1 makes it easy to check the stability of the closed-loop system by looking at standard tabular data for the open-loop zeros. The open-loop parameters

for the transfer functions of 10 aircraft at 94 total flight conditions is tabulated in Hefley and Jewell's data book [6]. Of these 94 flight conditions there were four with right half plane zeros: the four exceptional values were in the range $-0.027 \leq \frac{1}{T_1} \leq -0.00049$ rad/sec. None of the flight conditions were post stall so $\frac{1}{T_2}$ was positive for all 94 conditions.

These are low frequency instabilities that pilots or autopilots usually stabilize with a combination of additional outer loop feedbacks: the outer-loop feedback maps a combination of altitude, altitude rate, and velocity to a combination of elevator and throttle.

In [8] there are approximate formulas for $\frac{1}{T_1}$ that show that it is the variation of thrust minus drag with velocity that causes the right half plane zeros for 3 of the 4 exceptional flight conditions. Since drag increases with velocity (stabilizing) we conclude the right half plane zeros result in part from propulsion characteristics (consequently, the stability result of Section 5 should not be expected to apply). These 3 cases were the F-4 at Mach 0.6 and 35K ft altitude, and the XB-70 at Mach 0.6 and 20K ft and Mach 0.9 and 40K ft altitude.

The one other exceptional case is interesting. That case is the X-15 at Mach 1.6 and 80K ft altitude. The right half plane zero is not associated with propulsion because the engine was off. We do not know why the stability result of Section 5 does not work here (other than to say that some hypothesis was violated). This flight condition was exceptional in that the angle of attack was above 14 degrees, the largest value of all those listed for the X-15 (there were 10 flight condition listed for the X-15).

7 Application to F-14

As an example, we apply the results obtained in section 5 to analyze stability of a simplified pitch-axis aero table (zero sideslip, fixed Mach, zero slat and flap deflections, dominant static functions only) F-14 aircraft with a wing sweep of 22 degrees (source of aero data: [9]). For the sake of illustration we will apply the data in that reference at a Mach 0.7 flight condition, even though the Mach numbers mentioned in that reference were lower (they were interested in high- α maneuvers).

The basic aerodynamic functions are presented in a tabular form as functions of angle of attack α (see [10]).

These tabular aero data were obtained from wind tunnel testing and are given for every 5 degree from 0 degree to 55 degree. The aero data at other angles of attack are obtained from linear interpolation for simplicity. See Figure 2 for graphs of these

Figure 2: Aerodynamic functions

(a): C_D , C_L and C_M

(b): $C_{D,\delta}$, $C_{L,\delta}$ and $C_{M,\delta}$

aerodynamic functions. The functions C_L , C_D , and C_M are dimensionless, while $C_{L,\delta}$, $C_{D,\delta}$, and $C_{M,\delta}$ are in inverse degrees.

Figure 3 shows that the total drag coefficient (including direct effect) $\bar{C}_D(\alpha)$ is always positive and that the dissipative condition is also satisfied. As a consequence, according to Theorem 5.1, whether the closed loop system (4.5)-(4.6) is stable depends on the engine thrust T and the selected pitch attitude command θ_{cmd} . More precisely, for a given pair of the engine thrust T and the pitch attitude command θ_{cmd} , if equation (5.8) has a unique solution, then the system (4.5)-(4.6) has a unique equilibrium point which is a global attractor.

For any fixed T , the solution α of the equation (5.8) may be considered as a function of θ_{cmd} . As shown in Figure 4-(a) ($T = 0.2mg$), this function is single valued for θ_{cmd} located inside some interval and is multivalued when θ_{cmd} is located in some other interval. Similarly, for any fixed pitch attitude command θ_{cmd} , the solution α of the equation (5.8) is a function of T/mg . Figure 4-(b) is a picture of this function with $\theta_{cmd} = 5$ degree. As we can see from Figure 4, this function is not defined for some values of θ_{cmd} or T/mg . This is because the aerodynamic functions in [9] are only provided for $0 \text{ degree} \leq \alpha \leq 55 \text{ degree}$. In other intervals, the function is either single valued or multivalued (three valued). In the case that the function is single valued, the system corresponding to the given pair of $(T/mg, \theta_{cmd})$ has a unique equilibrium point which is a global attractor. In the case that the function is three valued, the system has three equilibrium points, of which two are stable and one is unstable.

Figure 3: Assumption 1 and 2

(a): $C_D - C_{D,\delta} C_M / C_{M,\delta} > 0$

(b): $3\bar{C}_D + \frac{d\bar{C}_L}{d\alpha} > 0$

Figure 4:

(a)

(b)

Figure 5: Stability region of F-14

The above analysis enables us to sketch the stability region of the system (4.5)-(4.6) in terms of the thrust T/mg and the selected pitch attitude command θ_{cmd} (see Figure 5). In general, if $(T/mg, \theta_{cmd})$ falls into the white region, then the system has a unique equilibrium which is a global attractor; if $(T/mg, \theta_{cmd})$ falls into the black region, then the system has three equilibrium points of which two are stable and one is unstable. There is no information available in the dot gray region since the aerodynamic functions are only given between 0 degree to 55 degree.

The above analysis works also for other aircrafts provided their aerodynamic function data bases are provided.

8 Robustness Analysis

In practice, the aerodynamic functions are usually only known through tabular data which are obtained from wind tunnel testing and flight test results. Typically these empirical data may have errors up to $\pm 20\%$. It is therefore important to analyze robustness of the stability results obtained in the previous section.

Let us denote the uncertain aerodynamic functions by

$$C_D(\alpha, \eta_1) = (1 + \eta_1)C_D(\alpha), \quad C_L(\alpha, \eta_2) = (1 + \eta_2)C_L(\alpha),$$

$$C_{D,\delta}(\alpha, \eta_3) = (1 + \eta_3)C_{D,\delta}(\alpha), \quad C_{L,\delta}(\alpha) = (1 + \eta_4)C_{L,\delta}(\alpha),$$

$$C_M(\alpha, \eta_5) = (1 + \eta_5)C_M(\alpha), \quad C_{M,\delta}(\alpha, \eta_2) = (1 + \eta_6)C_{M,\delta}(\alpha)$$

where C_D , C_L , C_M , $C_{D,\delta}$, $C_{L,\delta}$, C_M , and $C_{M,\delta}$ are nominal aerodynamic functions and η_k ($k = 1, 2, \dots, 6$) are constants which are used to measure uncertainty of the aerodynamic functions.

Using the dynamic inversion control law (4.2)-(4.3) based on the nominal aerodynamic functions, we obtain the following closed loop system:

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} U \\ W \end{pmatrix} &= \begin{pmatrix} -WQ \\ UQ \end{pmatrix} + \begin{pmatrix} -g \sin(\theta) \\ g \cos(\theta) \end{pmatrix} + \frac{1}{2m} \rho V^2 S \begin{pmatrix} \bar{C}_x(\alpha, \eta) \\ \bar{C}_z(\alpha, \eta) \end{pmatrix} + \\ &+ \frac{I_Y \dot{Q}_{cmd}}{\bar{c} m C_{M,\delta}(\alpha)} \begin{pmatrix} C_{x,\delta}(\alpha, \eta) \\ C_{z,\delta}(\alpha, \eta) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} \end{aligned} \quad (8.1)$$

$$\frac{d}{dt} \begin{pmatrix} Q \\ \theta \end{pmatrix} = \begin{pmatrix} (1 + \eta_6) \dot{Q}_{cmd} \\ Q \end{pmatrix} + \begin{pmatrix} (\eta_5 - \eta_6) \frac{\rho V^2 S \bar{c}}{2 I_Y} C_M(\alpha) \\ 0 \end{pmatrix} \quad (8.2)$$

where $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$.

If there is no uncertainty on the pitch moment (i.e., $\eta_5 = \eta_6 = 0$), then the dynamics of the pitch attitude and the pitch rate are still decoupled from those of the velocities. We shall consider this case here. The stability result just demonstrated in section 5 for the nominal aerodynamic functions works immediately for the perturbed C_L , C_D , $C_{L,\delta}$ and $C_{D,\delta}$. The following is an example of an application to the F-14 where those aerodynamic functions are varied by $\pm 20\%$ but with the certain pitch moment.

Figure 6 shows that the total drag coefficient is still positive even with error up to $\pm 20\%$ and the dissipative condition is also satisfied with the perturbed C_L , C_D , $C_{L,\delta}$ and $C_{D,\delta}$.

Based on Theorem 5.1 we are still able to sketch a stability region with respect to the engine thrust T and the selected pitch attitude command θ_{cmd} (see Figure 8). In the nominal case, the stability region consists of three parts. In the region I, the α component of the equilibrium is between 0 degree and 7.5 degree. It is between 26 degree and 35 degree in region II, and is between 43 degree and 55 degree in region III. When the aerodynamic lift and drag functions have errors up to 20%, comparing to the nominal case (see Figure 7 and Figure 8), the stability region I and II shrink

Figure 6: Robustness of Assumption 1 and 2

(a): $C_D - C_{D,\delta} C_M / C_{M,\delta} > 0$

(b): $3\bar{C}_D + \frac{d\bar{C}_L}{d\alpha} > 0$

and the whole region III melt into the uncertain region. It leads us to conclude that the stability of the equilibrium with lower angle of attack is more robust than those of the equilibrium with higher angle of attack.

In practice, we need to consider the uncertainty of the pitch moment. In this case, the dynamics of the pitch attitude and the pitch rate are no longer decoupled from those of the velocities. It would force us to consider an essentially three dimensional fully nonlinear dynamic system, which is of course a much more challenging problem. We do have some results in this case, but does not present them here. In any case, our numerical simulations have shown that the uncertain system (8.1)-(8.2) ($\eta_5 \neq 0$, $\eta_6 \neq 0$) has stability properties similar to those of the nominal system. We plan to present a rigorous analysis in a separate paper.

9 The Invariant Set

In this section we produce a special bounded subset Σ of the state space that is positively invariant for the closed-loop dynamics. The assumption is made that the time-dependent attitude commands issued to the controller lie within a small

Figure 7: Stability region of F-14

Figure 8: Robust Stability Region of F-14

range of values (e.g. ± 4 degrees) near zero degrees. In addition to this restriction, the aerodynamic models and the controller must satisfy a variety of conditions if the invariance condition is to hold. Intuitively, the set Σ can be thought of as a benign set of flight conditions inside which the aircraft will remain if the attitude commands are kept moderately small.

Recall that a positively invariant subset is one that contains the future trajectories associated with each of its points. For the rest of this section we simply use the expression *invariant set* to mean *positively invariant set*.

To define the set Σ , we switch coordinates on the velocity vector from rectangular (U, W) to polar (V, α) . The dynamic equations for V and α are:

$$\frac{dV}{dt} = -g \sin(\gamma) + \frac{1}{m} \left[\cos(\alpha)T - \frac{1}{2}\rho V^2 S(C_D(\alpha) + C_{D,\delta}(\alpha)\delta) \right] \quad (9.1)$$

$$\frac{d\alpha}{dt} = Q + \frac{1}{mV} \left[mg \cos(\gamma) - T \sin(\alpha) - \frac{1}{2}\rho V^2 S(C_L(\alpha) + C_{L,\delta}(\alpha)\delta) \right] \quad (9.2)$$

The θ and Q values of points in Σ lie in the intervals:

$$\theta_{min} < \theta < \theta_{max}, \quad Q_{min} < Q < Q_{max} \quad (9.3)$$

The conditions on the ranges will be established in the analysis below.

The V and α values of points in Σ will lie in intervals:

$$\alpha_0 < \alpha < \alpha_{max}, \quad V_{min} < V < V_{max} \quad (9.4)$$

where the parameters α_0 , α_{max} , V_{min} , V_{max} are defined as follows:

- α_0 = zero-lift angle of attack;
- α_{max} = maximum lift/drag angle of attack;
- V_{min} = minimum cruising speed;
- V_{max} = maximum speed for γ between γ_{min} and γ_{max} .

The values of α_0 and α_{max} are determined by the aerodynamic functions of the vehicle; they do not depend on thrust setting (recall, the thrust is constant) or the control laws. Typically, α_0 is negative one or two degrees, while α_{max} is less than ten degrees. As part of our construction, inside Σ the values of θ are limited, so $\gamma = \theta - \alpha$ takes values between γ_{min} and γ_{max} . The minimum cruising speed is the steady-state speed associated with flying at α_{max} .

To verify the invariance of a set Σ defined by the above inequalities it is necessary to check the four conditions:

- (i): $\alpha = \alpha_0 \rightarrow \dot{\alpha} > 0$
- (ii): $\alpha = \alpha_{max} \rightarrow \dot{\alpha} < 0$
- (iii): $V = V_{min} \rightarrow \dot{V} > 0$
- (iv): $V = V_{max} \rightarrow \dot{V} < 0$

By examination of equations 9.1 and 9.2 we will show that condition (i) through (iv) are satisfied, provided that θ and Q are restricted to lie in sufficiently small ranges. It is necessary to make some assumptions about the aerodynamic models: some motivation for these assumptions is provided by an example. The example is discussed in Section 9.1, the verification of the invariance condition is postponed until Section 9.2.

9.1 An Example

The example is the simplified pitch-axis model of the F-14 which was considered in section 7. The flight condition is at Mach 0.7 at 10,000 feet altitude above sea level. For the sake of definiteness, let us assume the temperature at altitude that day is 450 degrees Rankine, and the air density is 0.00156 slugs/ ft^3 . Then the speed of sound is roughly 1040 feet per second, the airspeed 728 feet/second, and dynamic pressure ($= \frac{1}{2}\rho V^2$) is roughly 413 pounds/ ft^2 . The wing-area S is 565 ft^2 , and we assume the weight of the vehicle is 60,000 lbs. At equilibrium the product of dynamic pressure and S balances the weight, corresponding to a C_L value of about 0.26. Interpolating from the aero data table [7] we see that α at equilibrium should be near 2 degrees. At $\alpha = 2$ the drag is approximately 16% of the lift, so the equilibrium thrust should be near 10,000 lbs.

For the purpose of our example we do not assume the thrust is set at the equilibrium value – instead, we assume a total thrust of 15,000 lbs. At the flight condition just described this level is a thrust excess of 5000 lbs (net acceleration of 5000/60000 = one-twelfth of a g). In such a condition the vehicle speed V is increasing provided that γ is sufficiently small.

We now make several observations about the aerodynamic data that will be assumed in the next section.

Observation 9.1: *The ratio $C_M/C_{M,s}$ has magnitude smaller than four degrees for all α between 0 and 10 degrees. This means that very little surface deflection is required (no more than four degrees) to move the equilibrium angle of attack over the table range of 10 degrees.*

Observation 9.2: *The drag coefficient C_D is nearly two orders of magnitude greater than the control-surface deflection drag coefficient $C_{D,s}$. Except for α in the*

range from -2 to 2 degrees, the lift coefficient C_L is an order of magnitude greater than the control-surface deflection lift coefficient.

From Observations 9.1 and 9.2 it follows that the residualized aerodynamic functions are little different from the ordinary functions, when $|\alpha| < 10$ degree, except possibly the lift coefficient in the range between -2 and 2 degrees.

Observation 9.3: As already mentioned, the zero-lift angle of attack α_0 is some negative value not far from 0, and α_{max} is between 0 and 10 degrees. We will assume α_{max} is 5 degrees.

9.2 Verification of Conditions for the Invariant Set

We now turn to the details of verifying conditions (i) through (iv).

Condition (i): When α is near α_0 the thrust and lift terms in equation (9.2) are small. In our example the thrust is only 25% of the vehicle weight, while dynamic pressure multiplied by $C_{L,\delta}$ only amounts to 10% of the weight. If δ is kept smaller than 5 degrees and γ smaller than (say) 30 degrees, then the term $g \cos(\gamma)/V$ is positive and has magnitude more than twice that of the other $\frac{1}{mV}$ terms. If we add the assumption:

$$Q_{min} > -\frac{g \cos(\gamma_{max})}{2V_{max}}, \quad (9.5)$$

then condition (i) is satisfied. The value of V_{max} is generated in the analysis of Condition (iv) below – that value is 1680 feet/sec. The resulting limit on Q_{min} is -.0096 radians per second, or about $-\frac{1}{2}$ degrees per second.

Condition (ii): When α is near α_{max} , both the thrust and the aerodynamics produce a negative effect on $\dot{\alpha}$. It turns out that the aero term is the significant factor here – we can pick V_{min} sufficiently large so that the lift at α_{max} exceeds mg by a comfortable margin, say the ratio of lift to weight at α_{max} is at least $1 + x$ at V_{min} . Then by imposing the constraint

$$Q_{max} < \frac{xg}{V_{min}} \quad (9.6)$$

condition (ii) is satisfied. In the process of arriving at (9.6) we had to assume:

$$V_{min} > \sqrt{\frac{2(1+x)mg}{\rho S(C_L(\alpha_{max}) - \delta_{max}C_{L,\delta}(\alpha_{max}))}}. \quad (9.7)$$

In the example, suppose α_{max} is 5 degrees. Let $x = 0.1$. Assume δ_{max} is 5 degrees. Then

$$C_L(\alpha_{max}) - \delta_{max}C_{L,\delta} = .47 \quad (9.8)$$

and, by (9.7), V_{min} is around 565 feet/sec. Then by (9.6) we need to restrict Q_{max} to be no greater than 0.0057 radians per second, or roughly $\frac{1}{3}$ degrees per second.

Condition (iii): This condition is satisfied whenever:

$$T > \frac{mg \sin(\gamma_{max}) + \frac{1}{2} \rho V_{min}^2 S \bar{C}_D(\alpha_{max})}{\cos(\alpha_{max})} \quad (9.9)$$

For our example, T is 15000 lbs, mg is 60000 lbs, and the second term in the numerator is less than 8200 lbs. The cosine of 5 degrees very close to 1, so if $mg \sin(\gamma_{max})$ is less than 6000 lbs the inequality (9.9) is satisfied. We conclude that if γ_{max} is less than six degrees, the condition is satisfied.

Because α is assumed bounded between -2 and 5 degrees, the γ_{max} constraint is met if θ is never bigger than four degrees.

Condition (iv): We have only to pick a suitable value for V_{max} . We assume θ is close to 0 (say $|\theta| < 4$ degrees and α is also bounded, so the gravity term plus thrust is bounded by 0.25 g. A rough estimate of V_{max} is obtained by setting the maximum gravity plus thrust value, 25000 lbs, equal to the drag force at the minimum drag α , or:

$$\frac{\rho V_{max}^2 S C_{D_0}}{2} = 5000. \quad (9.10)$$

From the table we do not know exactly what C_{D_0} is, so let us assume a reasonable value: $C_{D_0} = 0.02$. We solve (9.10) to find that $V_{max} = 1680$ ft/sec is the top speed compatible with the assumed model and flight condition.

The conditions for the four inequalities have been checked.

In practice, the value of X_{max} must be smaller for the model to apply, but 1680 ft/sec is only an upper bound so there is no logical error in the argument.

The bottom line for our example is this: let Σ be the set such that:

$$-2 \text{ degrees} < \alpha < 5 \text{ degrees} ,$$

$$565 \text{ ft/sec} < V < V_{max}, \quad V_{max} < 1680 \text{ ft/sec},$$

$$-4 \text{ degrees} < \theta < 4 \text{ degrees} ,$$

$$-\frac{1}{2} \text{ degrees/sec} < Q < \frac{1}{3} \text{ degrees/sec},$$

T is fixed, and suppose the controller keeps θ and Q inside those ranges for all points in Σ . Suppose all the other parameters in the model are as specified in section 9.1. Then the set Σ is an positive invariant set.

The analysis of Section 5 can be applied to the stability of the controller inside Σ .

Concerning the assumption about the controller – it is possible to make a dynamic inversion controller that satisfies the requirements for keeping Q and θ bounded inside of Σ (see Section 3). Such a controller would not be acceptable for piloted flight requiring quick maneuvers, but it would be quite acceptable for an autopilot/cruise mode of flight.

One final observation about invariant sets for this problem: the set Σ is not so large as one might want for a given application, however, it is only one of many invariant sets that can be found. If one works harder to define a special control strategy, one should be able to find a larger set. The set constructed here is simply the first one that came to mind: it was selected because its boundary is coordinate hyperplanes that are easy to describe analytically (e.g. $\alpha = \alpha_0$).

10 Proof of Theorem 5.1

In this section we provide a rigorous proof of the Theorem 5.1 for the interested readers.

As we had observed earlier, (4.6) is decoupled from (4.5). Besides, $Q(t) \rightarrow 0$ and $\theta(t) \rightarrow \theta_{cmd}$ as $t \rightarrow \infty$ according to the designed feedback control (4.3). It becomes obvious that the complementary system (4.5) is *asymptotic autonomous* and its *limiting equation* is given by the residualized system (5.5). The following proposition relates stability of the system (5.5) to that of the system (4.5).

Proposition 10.1 *Under the assumptions of proposition 5.1, there is a neighborhood G of (U^*, W^*) in the $U - W$ plane such that for any $(U_0, W_0) \in G$, the solution (U, W) of (4.5) with $U(0) = U_0$ and $W(0) = W_0$ satisfies*

$$\lim_{t \rightarrow \infty} U(t) = U^*, \quad \lim_{t \rightarrow \infty} W(t) = W^*.$$

Proof: This is a direct corollary of Markus' Theorem [11] and the fact that (U^*, W^*) is not only a global attractor for the residualized system (5.5) but also is a stable equilibrium (see Remark 5.2).

The above proposition states that (U^*, W^*) is a local attractor to the time varying system (4.5). Our goal is to show that (U^*, W^*) is also a global attractor to (the system (4.5), which is achieved by invoking Yoshizawa's following theorem.

Consider the system

$$\dot{x} = f(x, t) + \hat{g}(x, t) \tag{10.1}$$

Assume Σ is an open set in R^N and $f(x, t)$, $\hat{g}(x, t)$ are continuous on $\Sigma \times I$. In addition, assume that if $x(t)$ is continuous and bounded on $[t_0, \infty)$, then

$$\int_{t_0}^{\infty} |\hat{g}(s, x(s))| ds < +\infty \quad (10.2)$$

Theorem 10.1 (Yoshizawa, [14])

Suppose Ω is a nonempty closed subset of Σ and (i). $f(x, t)$ converges to $h(x)$ as $t \rightarrow \infty$ uniformly on any compact subset of Ω ; (ii). for any $\epsilon > 0$ and $y \in \Omega$, there exists a $\delta(\epsilon, y) > 0$ and a $T(\epsilon, y) > 0$ such that if $|x - y| < \delta(\epsilon, y)$ and $t > T(\epsilon, y)$, then

$$|f(x, t) - f(y, t)| < \epsilon.$$

If $x(t, t_0, x_0)$ is a bounded solution of (10.1) and attends to Ω as $t \rightarrow \infty$. Then the positive limit set Γ^+ of $x(t, t_0, x_0)$ is a positive invariant set of equation

$$\dot{x} = h(x), \quad x \in \Omega \quad (10.3)$$

contained in Ω .

Proof of Theorem 5.1: To apply Yoshizawa's Theorem to the system (4.5), we set $N = 2$, $\Omega = R^2$,

$$f = \begin{pmatrix} -WQ \\ UQ \end{pmatrix} + g \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} + \frac{\rho V^2 S}{2m} \begin{pmatrix} \overline{C}_x(\alpha) \\ \overline{C}_z(\alpha) \end{pmatrix}$$

and

$$\hat{g} = -\frac{I_Y \dot{Q}_{cmd}}{m C_{M, \delta}} \begin{pmatrix} C_{x, \delta} \\ C_{z, \delta} \end{pmatrix}.$$

Thus

$$h = g \begin{pmatrix} -\sin(\theta_{cmd}) \\ \cos(\theta_{cmd}) \end{pmatrix} + \begin{pmatrix} T/m \\ 0 \end{pmatrix} + \begin{pmatrix} F_x \\ F_z \end{pmatrix}.$$

The assumptions of Yoshizawa's Theorem are obviously satisfied. Let $(U, W)(t)$ be a solution of (4.5). It is bounded according to Lemma 5.1. Then it follows from Yoshizawa's Theorem that the positive set Γ^+ of $(U, W)(t)$ is a positive invariant set of equation (10.3). However, it is from Proposition 5.1 that the unique equilibrium (U^*, W^*) of (5.5) is contained inside the closure of any positive invariant set of a solution of (10.3). In particular, $(U^*, W^*) \in \Gamma^+$. On the other had, by Proposition 10.1, there is a neighborhood G of (U^*, W^*) such that if $(U, W)(t)$ enters into G , then $(U, W)(t) \rightarrow (U^*, W^*)$ as $t \rightarrow \infty$. But we know now that (U^*, W^*) is a limit point

of $(U, W)(t)$, that is to say, $(U, W)(t)$ is assured to enter into G at some time $t_0 > 0$. This completes the proof. $\square$

Finally, we compute an expression for $\text{div}(\bar{F})$ in terms of the total drag and lift functions $\bar{C}_D$ and $\bar{C}_L$. We ignore the Mach dependence of these coefficients, so that they depend only on α .

From the definitions:

$$\alpha = \arctan\left(\frac{W}{U}\right), \quad V = \sqrt{U^2 + W^2}$$

it is easy to compute:

$$\frac{\partial \alpha}{\partial U} = \frac{-W}{U^2 + W^2} = \frac{-\sin(\alpha)}{V}, \quad \frac{\partial \alpha}{\partial W} = \frac{U}{U^2 + W^2} = \frac{\cos(\alpha)}{V} \quad (10.4)$$

Also, by definition,

$$\begin{bmatrix} \bar{C}_x \\ \bar{C}_z \end{bmatrix} = \begin{bmatrix} -\cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \begin{bmatrix} \bar{C}_D(\alpha) \\ \bar{C}_L(\alpha) \end{bmatrix} \quad (10.5)$$

The residualized aero force vector $\bar{F}(U, W)$ can be written:

$$\bar{F}(U, W) = \frac{1}{2} \rho S V^2 \begin{bmatrix} \bar{C}_x \\ \bar{C}_z \end{bmatrix}$$

Next we compute a formula for $\text{div}(\bar{F})$.

$$\begin{aligned} \text{div}(\bar{F}) &= \frac{\partial \bar{F}_x}{\partial U} + \frac{\partial \bar{F}_z}{\partial W} \\ &= \rho S U \bar{C}_x + \frac{1}{2} \rho S V^2 \frac{\partial \bar{C}_x}{\partial U} + \rho S W \bar{C}_z + \frac{1}{2} \rho S V^2 \frac{\partial \bar{C}_z}{\partial W}. \end{aligned} \quad (10.6)$$

Now

$$U \bar{C}_x + W \bar{C}_z = V(\cos(\alpha) \bar{C}_x + \sin(\alpha) \bar{C}_z) = -V \bar{C}_D \quad (10.7)$$

So (substituting (10.7) into (10.6)):

$$\begin{aligned} \text{div}(\bar{F}) &= -\rho S V \bar{C}_D + \frac{1}{2} \rho S V^2 \left(\frac{\partial \bar{C}_x}{\partial U} + \frac{\partial \bar{C}_z}{\partial W} \right) \\ &= -\rho S V \bar{C}_D + \frac{1}{2} \rho S V^2 \left(\frac{d \bar{C}_x}{d \alpha} \frac{\partial \alpha}{\partial U} + \frac{d \bar{C}_z}{d \alpha} \frac{\partial \alpha}{\partial W} \right) \end{aligned} \quad (10.8)$$

By equation (10.5) we find:

$$\frac{d\bar{C}_x}{d\alpha} = \sin(\alpha)\bar{C}_D - \cos(\alpha)\frac{d\bar{C}_D}{d\alpha} + \cos(\alpha)\bar{C}_L + \sin(\alpha)\frac{d\bar{C}_L}{d\alpha}, \quad (10.9)$$

$$\frac{d\bar{C}_z}{d\alpha} = -\cos(\alpha)\bar{C}_D - \sin(\alpha)\frac{d\bar{C}_D}{d\alpha} + \sin(\alpha)\bar{C}_L - \cos(\alpha)\frac{d\bar{C}_L}{d\alpha} \quad (10.10)$$

Substituting equations (10.9), (10.10) and (10.4) into (10.8) the result is:

$$\text{div}(\bar{F}) = -\frac{1}{2}\rho SV \left(3\bar{C}_D + \frac{d\bar{C}_L}{d\alpha} \right).$$

11 Closed-Loop Poles Associated with Dynamic Inversion

In Section 6 we assumed that the zeros of the open-loop plant become poles of the closed-loop system when dynamic inversion is employed. In this section we present some analysis that justifies that assumption. The argument here is similar to the one presented in [12].

Consider the linear system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where the dimension of y is m , is the same as the dimension of u and the dimension of x is n .

Assume that an integer k can be found such that

$$CA^{k-1}B \quad \text{is full rank}$$

and

$$CA^{i-1}B = 0 \quad \text{for } i = 1, 2, \dots, k-1$$

and define relative degree to be the integer k . In the case of a single input, u and output y , such an integer can always be found. In the multiple input, multiple output case, it is possible for $CA^{k-1}B$ to be less than full rank, but not zero for some k . In this case the relative degree would be different for the different input output directions.

The dynamic inversion approach to control y is to use u to obtain a desired value of the k th derivative of y with respect to time. Since

$$\frac{d^k y}{dt^k} = CA^k x + CA^{k-1} Bu$$

this is accomplished by letting

$$u_c = (CA^{k-1}B)^{-1} \left(\frac{d^k y_d}{dt^k} - CA^k x_m \right)$$

where the subscript c denotes command, d denotes desired, and m denotes measurement. Note that if $u = u_c$ (perfect actuation) and $x_m = x$ (perfect sensing)

$$\frac{d^k y}{dt^k} = \frac{d^k y_d}{dt^k}. \quad (11.1)$$

The closed loop equations for this control approach are

$$\begin{aligned} \dot{x} &= Ax + B(CA^{k-1}B)^{-1} \left(\frac{d^k y_d}{dt^k} - CA^k x_m \right) \\ \dot{x} &= \left[A - B(CA^{k-1}B)^{-1}CA^k \right] x + B(CA^{k-1}B)^{-1} \frac{d^k y_d}{dt^k}. \end{aligned} \quad (11.2)$$

By (11.1) there are km closed loop poles at the origin. The following is a proof that the other $n - km$ poles are the open loop finite zeros.

The $n - km$ finite zeroes of the transfer function from u to y satisfy

$$\begin{bmatrix} \xi_i^T & w_i^T \end{bmatrix} \begin{bmatrix} A - z_i I & B \\ C & 0 \end{bmatrix} = [0 \ 0] \quad (11.3)$$

where z_i is a finite zero and $[\xi_i^T \ w_i^T]$ is the corresponding zero direction for $i = 1, 2, \dots, n - km$.

The block generalized eigenvalue problem (11.3) for the open loop zeroes can be written as two equations

$$\xi^T A + w^T C = z \xi^T \quad (11.4)$$

$$\xi^T B = 0. \quad (11.5)$$

Multiply (11.4) on the right by B to obtain

$$\xi^T AB + w^T CB = z \xi^T B$$

then using (11.5)

$$\xi^T AB + w^T CB = 0$$

If $k = 1$, then

$$w^T = -\xi^T A^k B \left(C A^{k-1} B \right)^{-1}. \quad (11.6)$$

If $k > 1$ then $CB = 0$ and from (11.6)

$$\xi^T AB = 0, \quad (11.7)$$

next multiply (11.4) by AB to obtain

$$\xi^T A^2 B + w^C AB = z \xi^T AB$$

then using (11.7)

$$\xi^T A^2 B + w^T CAB = 0. \quad (11.8)$$

If $k = 2$, then

$$w^T = -\xi^T A^k B \left(C A^{k-1} B \right)^{-1}. \quad (11.9)$$

If $k > 2$ then $CAB = 0$ and from (11.8)

$$\xi^T A^2 B = 0, \quad (11.10)$$

next multiply (11.4) by $A^2 B$ to obtain

$$\xi^T A^3 B + w^C A^2 B = z \xi^T A^2 B,$$

then using (11.10)

$$\xi^T A^3 B + w^T CA^2 B = 0.$$

If $k = 3$ then

$$w^T = -\xi^T A^k B \left(C A^{k-1} B \right)^{-1}. \quad (11.11)$$

This procedure can be continued if $k > 3$, and in general

$$w^T = -\xi^T A^k B \left(C A^{k-1} B \right)^{-1}. \quad (11.12)$$

Note that (11.6) for $k = 1$ is the same as (11.9) for $k = 2$ is the same as (11.11) for $k = 3$.

Next plug (11.12) into (11.4) to obtain

$$\xi^T A - \xi^T A^k B \left(C A^{k-1} B \right)^{-1} C = z \xi^T$$

and multiply by A^k to obtain

$$\xi^T A^{k+1} - \xi^T A^k B (C A^{k-1} B)^{-1} C A^k = z \xi^T A^k. \quad (11.13)$$

Now define

$$\eta^T = \xi^T A^k$$

and factor (11.13) to obtain

$$\eta^T \left[A - B (C A^{k-1} B)^{-1} C A^k \right] = z \eta^T.$$

Thus η^T is an eigenvector of the closed loop state feedback matrix in (11.2) and the n closed loop poles include the $n - km$ open loop zeros locations and km poles at the origin.

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