THE LYAPUNOV SPECTRUM AND STABLE MANIFOLDS
FOR STOCHASTIC LINEAR DELAY EQUATIONS*

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CORRECTIONS TO THE ARTICLE

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Pages 32 & 36:

Assertion (vii) of Theorem 1 should read

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \log ||X(t, \omega, \cdot)||_{L(M_2)} \quad \text{for all } \omega \in \Omega^* .$$

Proof:

We need only prove that $\lambda^* \leq \lambda_1$.

Since the convergence

$$\lim_{k \rightarrow \infty} [X(kr, \omega, \cdot)^* \circ X(kr, \omega, \cdot)]^{1/2k} = \Lambda(\omega) \quad \text{a.a. } \omega \in \Omega$$

is in the uniform operator norm, than taking operator norms and logarithms one gets

$$\begin{aligned} \log ||\Lambda(\omega)||_{L(M_2)} &= \lim_{k \rightarrow \infty} \log ||[X(kr, \omega, \cdot)^* \circ X(kr, \omega, \cdot)]^{1/2k}||_{L(M_2)} \\ &\geq \lim_{k \rightarrow \infty} \log [||X(kr, \omega, \cdot)||_{L(M_2)}^2]^{1/2k} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \log ||X(kr, \omega, \cdot)||_{L(M_2)} \\ &= \lambda^* r \end{aligned}$$

for a.a. $\omega \in \Omega$. But $\Lambda(\omega)$ is compact positive (self-adjoint), so $||\Lambda(\omega)||$ is the top eigenvalue e^{μ_1} of $\Lambda(\omega)$. Hence

$$\lambda^* r \leq \log ||\Lambda(\omega)|| = \mu_1$$

$$\text{i.e. } \lambda^* \leq \frac{\mu_1}{r} = \lambda_1 .$$

§1. Introduction

In this article we wish to consider questions of almost sure structural stability for stochastic linear delay equations where diffusion coefficients do not depend on the past history of the state. Our main objective is to prove a "stable manifold theorem" for stochastic (linear) delay equations with ordinary (non-delayed) diffusion coefficients. The "stable manifold theorem" is well-known for stochastic (non-linear) o.d.e.'s (Arnold & Wihstutz [2] , Arnold, Kliemann & Oeljeklaus [3], Carverhill [7],[10],[11], Baxendale [4],[5]). Although (moment) stability issues for stochastic hereditary equations were discussed by some authors (Kushner [25], Mizel & Trutzer [28], Mohammed [30], [31], Mohammed, Scheutzow & Weizsäcker [32]), so far little is known regarding almost sure structural stability for stochastic hereditary equations e.g. the one-dimensional pathological equation [29] (§3, pp. 144-149).

§2. Existence of the Stochastic Semi-flow

Our approach draws heavily on an infinite-dimensional version of the Oseledec multiplicative ergodic theorem developed by Ruelle [35] within a Hilbert space context. Certain extensions of Ruelle's results to Banach spaces are given by Mañé [26] and Thieullen [36]. We will however use Ruelle's results only.

Let us now formulate the class of stochastic linear delay equations to be considered. Take $r \geq 0$ and $J := [-r, 0]$. Denote by $\mathbb{R}^n$ n -dimensional Euclidean space with the Euclidean norm $|\cdot|$. Following Delfour & Mitter [13], [14], we shall let $M_2 := \mathbb{R}^n \times L^2(J, \mathbb{R}^n)$ stand for the separable real Hilbert space of all

pairs (v, η) , $v \in \mathbb{R}^n$, $\eta \in L^2(J, \mathbb{R}^n)$ given the Hilbert norm

$$\|(v, \eta)\|_{M_2}^2 = |v|^2 + \int_{-r}^0 |\eta(s)|^2 ds.$$

Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ the canonical filtered probability space where Ω is the space of all continuous paths $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^m$ such that $\omega(0) = 0$ given the compact-open topology, $\mathcal{F}$ is the Borel σ -field of Ω and $\mathcal{F}_t$ ($t \geq 0$) is the σ -field generated by the evaluations $\{\rho_u: u \leq t\}$, $\rho_u: \Omega \rightarrow \mathbb{R}^m$, $\rho_u(\omega) = \omega(u)$, $u \in \mathbb{R}^+$, $\omega \in \Omega$. Wiener measure on Ω is denoted by P and $W: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^m$,

$$W(t, \omega) = \omega(t) \quad t \in \mathbb{R}, \omega \in \Omega$$

is standard m -dimensional Brownian motion. For a measurable process $x: [-r, \infty) \times \Omega \rightarrow \mathbb{R}^n$ the segment $x_t: J \times \Omega \rightarrow \mathbb{R}^n$, $t \in \mathbb{R}^+$, is defined by

$$x_t(s) := x(t+s) \quad s \in J.$$

Consider the Itô stochastic f.d.e. (s.f.d.e.)

$$\left. \begin{aligned} dx(t) &= H(x(t), x_t)dt + g(x(t))dW(t) & t > 0 \\ x(0) &= v \in \mathbb{R}^n \\ x_0 &= \eta \in L^2(J, \mathbb{R}^n) \end{aligned} \right\} \quad (I)$$

where the drift $H: M_2 \rightarrow \mathbb{R}^n$ is continuous linear and the diffusion $g: \mathbb{R}^n \rightarrow L(\mathbb{R}^m, \mathbb{R}^n)$ is linear. Note that $L(\mathbb{R}^m, \mathbb{R}^n)$ is the space of all linear maps $\mathbb{R}^m \rightarrow \mathbb{R}^n$ furnished with the uniform operator norm

$$\|A\| := \sup \{ |A(v)| : v \in \mathbb{R}^m, |v| = 1 \}, \quad A \in L(\mathbb{R}^m, \mathbb{R}^n).$$

The stochastic differential dW in (I) is to be understood in the classical Itô sense. (Itô [21], [22], Ikeda & Watanabe [20]).

Under the above conditions it is known that the s.f.d.e. (I) admits a unique non-explosive solution $(v, \eta) x: [-r, \infty) \times \Omega \rightarrow \mathbb{R}^n$ with the following properties:

- (i) $(v, \eta)_X(0) = v$, $(v, \eta)_X(s) = \eta(s)$ a.e. $s \in J$, a.s. on Ω ;
- (ii) $(v, \eta)_X | \mathbb{R}^+ \times \Omega$ has continuous sample paths;
- (iii) for each $t \in \mathbb{R}^+$, $(v, \eta)_X(t, \cdot): \Omega \rightarrow \mathbb{R}^n$ and $(v, \eta)_{X_t}: \Omega \rightarrow L^2(J, \mathbb{R}^n)$ are $\mathcal{F}_t$ -measurable;
- (iv) the M_2 -valued process

$$(v, \eta)_{X(t, \cdot)} := ((v, \eta)_X(t, \cdot), (v, \eta)_{X_t}(\cdot)) \quad t \in \mathbb{R}^+$$

is a Feller process with a.s. sample paths continuous (Mohammed [29], pp. 226-228, cf. Mizel & Trutzer [28]);

- (v) for each $t \in \mathbb{R}^+$ and every $(v, \eta) \in M_2$, $(v, \eta)_{X(t, \cdot)} \in \mathcal{L}^2(\Omega, M_2)$ and the map

$$\begin{aligned} \mathbb{R}^+ \times M_2 &\rightarrow \mathcal{L}^2(\Omega, M_2) \\ (t, (v, \eta)) &\mapsto (v, \eta)_{X(t, \cdot)} \end{aligned}$$

is jointly continuous and Lipschitz in the M_2 -variable uniformly with respect to t in compact subsets of $\mathbb{R}^+$.

Further regularity properties analogous to those in Mohammed ([29], pp. 149-164) can be proven, but we shall not go into these here.

Our first crucial step in the construction of the stochastic 'semi-flow' of (I) is to establish an almost surely continuous version of the trajectory field

$$\left\{ (v, \eta)_{X(t, \cdot)} := ((v, \eta)_X(t, \cdot), (v, \eta)_{X_t}(\cdot)) : t \in \mathbb{R}^+, (v, \eta) \in M_2 \right\}. \quad (\text{II})$$

The existence of 'nice' versions in the non-delay case ($r = 0$, s.o.d.e.) has been well studied by several authors e.g. the "diffeomorphism property" in Kunita [23], [24], Baxendale [4], [6], Bismut [8], Ikeda & Watanabe [20] and Elworthy [16]. It is, however, unfortunate that such nice versions of the trajectory field may never exist if the diffusion term in an s.f.d.e. is allowed to look into the past (Mohammed [29], §3, pp. 144-147).

However, the trajectory field of our regular s.f.d.e. (I) does admit continuous linear versions which are in fact compact for $t \geq r$. Indeed we have

Theorem 1:

The trajectory field $\{^{(v, \eta)}X(t, \cdot) : t \in \mathbb{R}^+, (v, \eta) \in M_2\}$ of the linear s.f.d.e. (I) has a Borel-measurable version $X: \mathbb{R}^+ \times \Omega \times M_2 \rightarrow M_2$ with the following properties:

- (i) for each $(v, \eta) \in M_2$, $X(t, \cdot, (v, \eta)) = ^{(v, \eta)}X(t, \cdot)$ a.s.;
- (ii) for each $t \in \mathbb{R}^+$ and $(v, \eta) \in M_2$, $X(t, \cdot, (v, \eta)) : \Omega \rightarrow M_2$ is $\mathcal{F}_t$ -measurable and belongs to $\mathcal{L}^2(\Omega, M_2; \mathcal{F}_t, P)$;
- (iii) there is a Borel set $\Omega_0 \subset \Omega$ of full P -measure such that for all $\omega \in \Omega_0$ the map

$$X(\cdot, \omega, \cdot) : \mathbb{R}^+ \times M_2 \rightarrow M_2$$

is continuous;

- (iv) for each $t \in \mathbb{R}^+$ and every $\omega \in \Omega_0$, the map

$$X(t, \omega, \cdot) : M_2 \rightarrow M_2$$

is continuous linear; for each $\omega \in \Omega_0$, the map $\mathbb{R}^+ \ni t \mapsto X(t, \omega, \cdot) \in L(M_2)$ is continuous.

- (v) for each $t \geq r$ and all $\omega \in \Omega_0$, the map $X(t, \omega, \cdot) : M_2 \rightarrow M_2$ is compact.

Remark:

If W is one-dimensional or if g satisfies the Frobenius condition

$$g[g(v)(z_1)](z_2) = g[g(v)(z_2)](z_1) \quad v \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}^m, \quad (1)$$

a version of the above theorem is proved in Mohammed ([29], pp. 121-133).

There, the space M_2 is replaced by the Banach space $C(J, \mathbb{R}^n)$ of all continuous initial data $J \rightarrow \mathbb{R}^n$ given the supremum norm.

Proof of Theorem 1.

Let $\varphi: \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the stochastic flow of the linear s.o.d.e.

$$\left. \begin{aligned} d\varphi(t) &= g(\varphi(t))dW(t) & t > 0 \\ \varphi(0) &= v \in \mathbb{R}^n \end{aligned} \right\} \quad (\text{III})$$

Then φ is Borel-measurable and for a.a. $\omega \in \Omega$ and all $t \in \mathbb{R}^+$ the map

$$\begin{aligned} \varphi(t, \omega, \cdot): \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ v &\mapsto \varphi(t, \omega, v) \end{aligned}$$

is invertible linear. Indeed if $GL(\mathbb{R}^n) \subset L(\mathbb{R}^n)$ is the general linear group, then the process

$$\begin{aligned} \mathbb{R}^+ \times \Omega &\rightarrow GL(\mathbb{R}^n) \\ (t, \omega) &\mapsto \varphi(t, \omega, \cdot) \end{aligned}$$

is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and has continuous sample paths (Arnold [1]).

Define the random family of time-dependent retarded f.d.e.'s

$\tilde{H}: \mathbb{R}^+ \times \Omega \times M_2 \rightarrow \mathbb{R}^n$ by

$$\tilde{H}(t, \omega, v, \eta) = \varphi(t, \omega, \cdot)^{-1} [H(\varphi(t, \omega, v), \varphi_t(\cdot, \omega, \cdot) \circ (\text{id}_J, \eta))] \quad (2)$$

for $v \in \mathbb{R}^n$, $\eta \in L^2(J, \mathbb{R}^n)$ where $\varphi_t(\cdot, \omega, \cdot): J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $(\text{id}_J, \eta): J \rightarrow J \times \mathbb{R}^n$ are defined by

$$\varphi_t(\cdot, \omega, \cdot)(s, v) = \begin{cases} \varphi(t+s, \omega, v): & t+s \geq 0, & t \geq 0 \\ v & -r \leq t+s < 0, & s \in J, \end{cases} \quad (3)$$

$$(\text{id}_J, \eta)(s) = (s, \eta(s)) \quad s \in J. \quad (4)$$

For each $t \in \mathbb{R}^+$ and a.a. $\omega \in \Omega$, the maps

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n & L^2(J, \mathbb{R}^n) &\rightarrow L^2(J, \mathbb{R}^n) \\ v &\mapsto \varphi(t, \omega, v), & \eta &\mapsto \varphi_t(\cdot, \omega, \cdot) \circ (\text{id}_J, \eta) \end{aligned}$$

are continuous linear and therefore by composition so is the map

$$\begin{aligned} M_2 &\rightarrow \mathbb{R}^n \\ (v, \eta) &\mapsto \tilde{H}(t, \omega, v, \eta). \end{aligned}$$

Note also that $\tilde{H}$ is Borel-measurable and for a.a. $\omega \in \Omega$,

$$\tilde{H}(\cdot, \omega, \cdot, \cdot): \mathbb{R}^+ \times M_2 \rightarrow \mathbb{R}^n$$

is continuous. Indeed for each $t \in \mathbb{R}^+$ and a.a. $\omega \in \Omega$, let

$$K(t, \omega) := \max \left[\sup_{\substack{s \in J \\ t+s > 0}} \|\varphi(t+s, \omega, \cdot)\|, 1 \right] \quad (5)$$

and

$$L(t, \omega) := \|\varphi(t, \omega, \cdot)^{-1}\| \quad (6)$$

where $\|\cdot\|$ denotes the operator norm on $L(\mathbb{R}^n)$. Then it is easy to see from (2) that

$$|\tilde{H}(t, \omega, v, \eta)| \leq L(t, \omega) K(t, \omega) \|H\| \|v, \eta\|_{M_2} \quad (7)$$

where $\|H\|$ is the operator norm of $H: M_2 \rightarrow \mathbb{R}^n$.

As the maps

$$\begin{aligned} \mathbb{R}^+ &\rightarrow \mathbb{R}, & \mathbb{R}^+ &\rightarrow \mathbb{R} \\ t &\mapsto K(t, \omega), & t &\mapsto L(t, \omega) \end{aligned}$$

are continuous for a.a. $\omega \in \Omega$, the retarded f.d.e.

$$\left. \begin{aligned} y'(t) &= \tilde{H}(t, \omega, y(t), y_t) \\ y(0) &= v \in \mathbb{R}^n, \quad y_0 = \eta \in L^2(J, \mathbb{R}^n) \end{aligned} \right\} \quad t > 0 \quad (IV)$$

admits a unique non-explosive solution $(v, \eta) y(\cdot, \omega): [-r, \infty) \rightarrow \mathbb{R}^n$. Indeed (7) implies the existence of a unique local solution with explosion time $\tau = \tau(v, \eta, \omega)$. To see that $\tau = \infty$, suppose not. Using (7) and (IV), it is easy to show that for a.a. $\omega \in \Omega$, $0 < t < \tau$,

$$\begin{aligned} \| (y(t, \omega), y_t(\cdot, \omega)) \|_{M_2}^2 &= |y(t, \omega)|^2 + \int_{t-r}^t |y(u, \omega)|^2 du \\ &\leq 2|v|^2 + 2t \|H\|^2 \int_0^t L(u, \omega)^2 K(u, \omega)^2 \| (y(u, \omega), y_u(\cdot, \omega)) \|_{M_2}^2 du \\ &\quad + \|\eta\|_{\mathcal{H}}^2 + 2r|v|^2 + 2rt \|H\|^2 \int_0^t L(u, \omega)^2 K(u, \omega)^2 \| (y(u, \omega), y_u(\cdot, \omega)) \|_{M_2}^2 du \\ &\leq (2r+1) \| (v, \eta) \|_{M_2}^2 + (2r+1) t \|H\|^2 \int_0^t L(u, \omega)^2 K(u, \omega)^2 \| (y(u, \omega), y_u(\cdot, \omega)) \|_{M_2}^2 du \quad (7)' \end{aligned}$$

Gronwall's lemma then gives

$$\| (y(t, \omega), y_t(\cdot, \omega)) \|_{M_2}^2 \leq (2r+1) \| (v, \eta) \|_{M_2}^2 \exp \left[(2r+1) t \|H\|^2 \int_0^t L(u, \omega)^2 K(u, \omega)^2 du \right] \quad (8)$$

a.a. $\omega \in \Omega$ and for all $0 < t < \tau$. Therefore $\{|y'(t, \omega)|: 0 < t < \tau\}$ is bounded and so

$$[0, \tau) \ni t \mapsto (v, \eta) y(t, \omega) \in \mathbb{R}^n$$

is uniformly continuous for a.a. $\omega \in \Omega$. This implies that

$$[0, \tau) \ni t \mapsto (y(t, \omega), y_t(\cdot, \omega)) \in M_2$$

is also uniformly continuous for a.a. $\omega \in \Omega$. Taking the limit

$\lim_{t \rightarrow \tau^-} (y(t, \omega), y_t(\cdot, \omega)) \in M_2$ as initial data for (IV), then by local existence we

may extend $(v, \eta) y$ to a larger interval than $[0, \tau]$. This contradicts the

maximality of τ . Hence $\tau = \infty$. (Cf. Hale [18], pp. 141-143), Delfour & Mitter

[13]). Furthermore if we fix $t_0 \geq r$ then the above argument also shows that for

a.a. $\omega \in \Omega$ the set $\{ \|(v, \eta) y'(t, \omega)\| : 0 \leq t \leq t_0, \|(v, \eta)\|_{M_2} \leq 1 \}$ is bounded.

Hence by Ascoli's theorem the family

$$\{ (v, \eta) y_{t_0}'(\cdot, \omega) : \|(v, \eta)\|_{M_2} \leq 1 \}$$

is relatively compact in $C(J, \mathbb{R}^n)$. Using the natural (compact) linear embedding

$$\begin{aligned} C(J, \mathbb{R}^n) &\rightarrow M_2 \\ \eta &\mapsto (\eta(0), \eta) \end{aligned}$$

it follows that the set $\{ (v, \eta) y_{t_0}'(\cdot, \omega), (v, \eta) y_{t_0}'(\cdot, \omega) : \|(v, \eta)\|_{M_2} \leq 1 \}$ is

relatively compact in M_2 .

We now seek a solution $x: [-r, \infty) \times \Omega \times M_2 \rightarrow \mathbb{R}^n$ of our s.f.d.e. (I) in the form

$$x(t, \omega, (v, \eta)) = \begin{cases} \varphi(t, \omega, (v, \eta) y(t, \omega)) & t \geq 0 \\ \eta(t) & \text{a.e. } t \in [-r, 0) \end{cases} \quad (9)$$

First let us check that $x(t, \cdot, (v, \eta))$ is $(\mathcal{F}_t)$ -adapted for $t \in \mathbb{R}^+$. For each $t \in \mathbb{R}^+$ and $(v, \eta) \in M_2$ the right-hand side $\tilde{H}(t, \cdot, (v, \eta))$ of (IV) is $(\mathcal{F}_t)$ -measurable; so a standard successive approximation argument shows that $(v, \eta) y(t, \omega)$ is jointly measurable in (t, ω) and for each $t \in \mathbb{R}^+$, $(v, \eta) y(t, \cdot)$ is $\mathcal{F}_t$ -adapted. Now the fundamental solution

$$\begin{aligned} \mathbb{R}^+ \times \Omega &\rightarrow GL(\mathbb{R}^n) \\ (t, \omega) &\mapsto \varphi(t, \omega, \cdot) \end{aligned}$$

is $(\mathcal{F}_t)_{t \geq 0}$ -adapted. Indeed it is the unique solution of the s.o.d.e.

$$\left. \begin{aligned} d\varphi(t, \cdot, \cdot) &= \sum_{i=1}^m [g^i \varphi(t, \cdot, \cdot)](\cdot) (e_i) dW_i(t), \quad t > 0 \\ \varphi(0, \cdot, \cdot) &= \text{id}_{\mathbb{R}^n} \end{aligned} \right\} \quad (V)$$

in $L(\mathbb{R}^n)$ where

$$W(t) = \sum_{i=1}^m W_i(t) e_i, \quad t > 0,$$

with $\{e_i\}_{i=1}^m$ a canonical basis of $\mathbb{R}^m$, $\{W_i\}_{i=1}^m$ mutually independent one-dimensional Brownian motions.

From the composition (9), it follows that the process $x(\cdot, \cdot, (v, \eta))$ is $(\mathcal{F}_t)$ -adapted, $t \geq 0$. To see that it is actually a solution of (I) we shall invoke Bismut's generalized Itô formula for stochastic flows (Bismut [7], Carverhill & Elworthy [12]). Using the fact that $\{(v, \eta) y(t, \cdot); t \geq 0\}$ is $(\mathcal{F}_t)$ -adapted and has C^1 sample paths, we can take Itô differentials of (9) to get

$$\begin{aligned} dx(t, \omega, (v, \eta)) &= \varphi(t, \omega, (v, \eta) y'(t)) dt + (\omega) g(\varphi(t, \cdot, (v, \eta) y(t))) dW(t), \quad t > 0 \\ &= H(\varphi(t, \omega, (v, \eta) y(t, \omega)), \varphi_t(\cdot, \omega, \cdot) \circ (id_J, (v, \eta) y_t(\cdot, \omega))) dt \\ &\quad + (\omega) g[\varphi(t, \cdot, (v, \eta) y(t, \cdot))] dW(t), \quad t > 0 \quad \text{a.s.} \end{aligned} \quad (10)$$

Now if $t > 0$, then for a.a. $s \in J$ we have

$$\begin{aligned} \{\varphi_t(\cdot, \omega, \cdot) \circ (id_J, (v, \eta) y_t(\cdot, \omega))\}(s) &= \begin{cases} \varphi(t+s, \omega, (v, \eta) y_t(s, \omega)), & t+s \geq 0 \\ (v, \eta) y_t(s, \omega), & -r \leq t+s < 0 \end{cases} \\ &= \begin{cases} \varphi(t+s, \omega, (v, \eta) y(t+s, \omega)), & t+s \geq 0 \\ \eta(t+s), & -r \leq t+s < 0 \end{cases} \\ &= x_t(\cdot, \omega, (v, \eta))(s), \quad \text{a.a. } \omega \in \Omega. \end{aligned}$$

Therefore (10) becomes

$$dx(t, \omega, (v, \eta)) = H(x(t, \omega, (v, \eta)), x_t(\cdot, \omega, (v, \eta))) dt + (\omega) g[x(t, \cdot, (v, \eta))] dW(t)$$

i.e., $x(\cdot, \cdot, (v, \eta))$ satisfies the s.f.d.e. (I) and

$$\begin{aligned}
X(t, \omega, (v, \eta)) &:= (x(t, \omega, (v, \eta)), x_t(\cdot, \omega, (v, \eta))) \\
&= (\varphi(t, \omega, {}^{(v, \eta)}y(t, \omega)), \varphi_t(\cdot, \omega, \cdot) \circ (\text{id}_J, {}^{(v, \eta)}y_t(\cdot, \omega))) \quad (11)
\end{aligned}$$

is a version of its trajectory field.

It remains to show that X satisfies all the requirements of the theorem.

We begin by proving that X is jointly (Borel)-measurable in $(t, \omega, (v, \eta))$.

Consider the space $C_{C, L}(J \times \mathbb{R}^n, \mathbb{R}^n)$ of all maps $A: J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $A(s, \cdot) \in L(\mathbb{R}^n)$ for every $s \in J$ and $A(\cdot, v) \in C(J, \mathbb{R}^n)$ for all $v \in \mathbb{R}^n$. Then $C_{C, L}(J \times \mathbb{R}^n, \mathbb{R}^n)$ is a Banach space (isomorphic to $C(J, L(\mathbb{R}^n))$) under the norm

$$\|A\| = \sup_{s \in J} \sup_{\substack{v \in \mathbb{R}^n \\ |v| \leq 1}} |A(s, v)|.$$

It is also easy to see that the composition

$$\begin{aligned}
\psi: C_{C, L}(J \times \mathbb{R}^n, \mathbb{R}^n) \times \mathcal{Z}^2(J, \mathbb{R}^n) &\rightarrow \mathcal{Z}^2(J, \mathbb{R}^n) \\
(A, \eta) &\mapsto A \circ (\text{id}_J, \eta)
\end{aligned}$$

is continuous bilinear. Now for each $s \in J$, the map

$$\begin{aligned}
\mathbb{R}^+ \times \Omega &\rightarrow L(\mathbb{R}^n) \\
(t, \omega) &\mapsto \varphi_t(s, \omega, \cdot)
\end{aligned}$$

is Borel-measurable. But the Borel field of $C_{C, L}(J \times \mathbb{R}^n, \mathbb{R}^n)$ is identified with that of $C(J, L(\mathbb{R}^n))$ and the latter is generated by evaluations, so it follows immediately that the map

$$\begin{aligned}
\phi: \mathbb{R}^+ \times \Omega &\rightarrow C_{C, L}(J \times \mathbb{R}^n, \mathbb{R}^n) \\
(t, \omega) &\mapsto \varphi_t(\cdot, \omega, \cdot)
\end{aligned}$$

is Borel-measurable (Parthasarathy [33], pp. 212-213) and has continuous sample paths. The successive approximation argument used to solve (IV) gives also that the trajectory field

$$Y(t, \omega, (v, \eta)) := ({}^{(v, \eta)}y(t, \omega), {}^{(v, \eta)}y_t(\cdot, \omega)) \quad (12)$$

is Borel-measurable on $\mathbb{R}^+ \times \Omega \times M_2$. Furthermore (8) implies that for $t \in \mathbb{R}^+$ and a.a. $\omega \in \Omega$, $Y(t, \omega, \cdot): M_2 \rightarrow M_2$ is continuous linear. Therefore the composition

$$X(t, \omega, (v, \eta)) = (\varphi(t, \omega, {}^{(v, \eta)}y(t, \omega)), \psi(\theta(t, \omega), {}^{(v, \eta)}y_t(\cdot, \omega))) \quad (13)$$

is measurable on $\mathbb{R}^+ \times \Omega \times M_2$ and $X(t, \omega, \cdot): M_2 \rightarrow M_2$ is continuous linear for $t \in \mathbb{R}^+$, a.a. $\omega \in \Omega$. Also for a.a. $\omega \in \Omega$, $X(\cdot, \omega, \cdot)$ is continuous on $\mathbb{R}^+ \times M_2$. This proves assertions (iii) and (iv) of the theorem.

Assertions (i) and (ii) follow from the basic existence and uniqueness theorem for s.f.d.e.'s ((iii) and (v) of this section).

Finally, it remains to prove the last statement (v) of the theorem. Fix $t \geq r$ and pick any universal Borel set $\Omega_0 \subset \Omega$ of full P -measure such that $\varphi(t, \omega, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear for all $\omega \in \Omega_0$. Take $\omega \in \Omega_0$. Then the following maps are continuous linear:

$$\begin{array}{ll} M_2 \rightarrow \mathbb{R}^n, & \mathcal{L}^2(J, \mathbb{R}^n) \rightarrow \mathcal{L}^2(J, \mathbb{R}^n) \\ (v, \eta) \mapsto \varphi(t, \omega, {}^{(v, \eta)}y(t, \omega)), & \xi \mapsto \psi(\theta(t, \omega), \xi). \end{array}$$

Therefore the composition (13) yields compactness of $X(t, \omega, \cdot): M_2 \rightarrow M_2$ as we have already established that the map

$$\begin{array}{l} M_2 \rightarrow \mathcal{L}^2(J, \mathbb{R}^n) \\ (v, \eta) \mapsto {}^{(v, \eta)}y_t(\cdot, \omega) \end{array}$$

is compact linear. This completes the proof of the theorem. $\square$

§3. Smooth Approximations. The Cocycle

We wish to obtain suitable approximations of the sample flow $\{X(t, \omega, \cdot): M_2 \rightarrow M_2, t \geq 0, \omega \in \Omega\}$ constructed in §2 by using piece-wise linear

approximations $\{W^k\}_{k=1}^\infty$ of the driving Brownian motion W in (I). (Cf. Ikeda & Watanabe [20], pp. 418-429, McShane [27], Elworthy [16]). To satisfy our purposes the corresponding approximating flows $X^k(t, \omega, \cdot): M_2 \rightarrow M_2$, $k \geq 1$ will converge for a.a. ω to $X(t, \omega, \cdot)$ in the uniform operator topology on $L(M_2)$.

For each $k \geq 1$, invoke a Stratonovich correction term in (V) and let $\varphi^k: \mathbb{R}^+ \times \Omega \rightarrow L(\mathbb{R}^n)$ be the fundamental matrix solution of the o.d.e.

$$\left. \begin{aligned} d\varphi^k(t, \cdot, \cdot) &= \sum_{i=1}^m [g \circ \varphi^k(t, \cdot, \cdot)](\cdot)(e_i)(W_1^k)'(t)dt \\ &\quad - \frac{1}{2} \sum_{i=1}^m (g \circ \{[g \circ \varphi^k(t, \cdot, \cdot)](\cdot)(e_i)\})(\cdot)(e_i)dt \\ \varphi^k(0, \cdot, \cdot) &= \text{id}_{\mathbb{R}^n}. \end{aligned} \right\} \quad (\text{VI}_k)$$

Note that the Itô s.d.e. (V) in $L(\mathbb{R}^n)$ is equivalent to the Stratonovich s.d.e.

$$\left. \begin{aligned} d\varphi(t, \cdot, \cdot) &= \sum_{i=1}^m [g \circ \varphi(t, \cdot, \cdot)](\cdot)(e_i) \circ dW_1(t) \\ &\quad - \frac{1}{2} \sum_{i=1}^m (g \circ \{[g \circ \varphi(t, \cdot, \cdot)](\cdot)(e_i)\})(\cdot)(e_i)dt \\ \varphi(0, \cdot, \cdot) &= \text{id}_{\mathbb{R}^n}. \end{aligned} \right\} \quad (\text{VII})$$

In $(\text{VI})_k$ and (VII), the notation $[g \circ A](\cdot)(e_i)$, $A \in L(\mathbb{R}^n)$, stands for the map

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ v &\mapsto [g \circ A](v)(e_i) \end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a canonical basis of $\mathbb{R}^m$ (Elworthy [16], pp. 74-75, Ikeda & Watanabe [20], pp. 100-102). It is known that for a.a. $\omega \in \Omega$,

$$\lim_{k \rightarrow \infty} \varphi^k(t, \omega, \cdot) = \varphi(t, \omega, \cdot) \quad (14)$$

in $L(\mathbb{R}^n)$ uniformly in $t \in [0, T]$, for $0 < T < \infty$ ([16], [20]).

The following approximation theorem is sufficient to yield the cocycle property for the trajectory field version $X: \mathbb{R}^+ \times \Omega \times M_2 \rightarrow M_2$ constructed in Theorem 1 of §2.

Theorem 2:

For each integer $k \geq 1$, let $\{(v, \eta) y^k(t, \omega): t \geq -r, \omega \in \Omega, (v, \eta) \in M_2\}$ be the unique solutions of the random family of r.f.d.e.'s

$$\left. \begin{aligned} (y^k)'(t) &= \tilde{H}_k(t, \omega, y^k(t), y_t^k) & t > 0 \\ (v, \eta) y^k(0) &= v, & (v, \eta) y_0^k &= \eta \end{aligned} \right\} \quad (\text{VIII})$$

where

$$\tilde{H}_k(t, \omega, v, \eta) = \varphi^k(t, \omega)^{-1} [H(\varphi^k(t, \omega)(v), \varphi_t^k(\cdot, \omega) \circ (\text{id}_J, \eta))] \quad (15)$$

$\{\varphi_t^k(\cdot, \omega) \circ (\text{id}_J, \eta)\}(s) = \varphi^k(t+s, \omega)(\eta(s))$ a.a. $s \in J$, $t \geq 0$, $v \in \mathbb{R}^n$, $\eta \in \mathcal{L}^2(J, \mathbb{R}^n)$. Set

$$X^k(t, \omega, (v, \eta)) := (\varphi^k(t, \omega)((v, \eta) y^k(t, \omega)), \varphi_t^k(\cdot, \omega) \circ (\text{id}_J, (v, \eta) y_t^k(\cdot, \omega))), \quad t \geq 0 \quad (16)$$

If $0 < T < \infty$ and B is a bounded set in M_2 , then for a.a. $\omega \in \Omega$ we have

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{(v, \eta) \in B} \|X^k(t, \omega, (v, \eta)) - X(t, \omega, (v, \eta))\|_{M_2} = 0. \quad (17)$$

Proof:

Fix $0 < T < \infty$ and $B \subset M_2$ a bounded set throughout the proof.

We first show that $\tilde{H}_k(t, \omega, (v, \eta)) \rightarrow \tilde{H}(t, \omega, (v, \eta))$ as $k \rightarrow \infty$ uniformly in $(t, (v, \eta)) \in [0, T] \times B$ for a.a. ω . Without loss of generality take $B = B_1 \times B_2$ where $B_1 \subset \mathbb{R}^n$, $B_2 \subset \mathcal{L}^2(J, \mathbb{R}^n)$ are bounded sets. Consider for a.a. ω , $t \geq 0$, $\eta \in B_2$ the expression

$$\begin{aligned} \|\varphi_t^k(\cdot, \omega) \circ (\text{id}_J, \eta) - \varphi_t(\cdot, \omega) \circ (\text{id}_J, \eta)\|_{\mathcal{L}^2}^2 &= \int_{-r}^0 |[\varphi^k(t+s, \omega) - \varphi(t+s, \omega)](\eta(s))|^2 ds \\ &\leq \sup_{s \in J} \|\varphi^k(t+s, \omega) - \varphi(t+s, \omega)\|_{L(\mathbb{R}^n)}^2 \cdot \|\eta\|_{\mathcal{L}^2}^2. \end{aligned}$$

Letting $k \rightarrow \infty$ in the above inequality and using (14), one gets

$$\lim_{k \rightarrow \infty} \sup_{\eta \in B_2} \sup_{t \in [0, T]} \|\varphi_t^k(\cdot, \omega) \circ (\text{id}_J, \eta) - \varphi_t(\cdot, \omega) \circ (\text{id}_J, \eta)\| = 0 \quad (18)$$

for a.a. $\omega \in \Omega$.

To simplify notation, let

$$\begin{aligned} \Psi^k(t, \omega, v, \eta) &:= H(\varphi^k(t, \omega)(v), \varphi_t^k(\cdot, \omega) \circ (\text{id}_J, \eta)) \\ \Psi(t, \omega, v, \eta) &:= H(\varphi(t, \omega)(v), \varphi_t(\cdot, \omega) \circ (\text{id}_J, \eta)). \end{aligned}$$

Since H is continuous linear (uniformly continuous), (14) and (18) imply that for a.a. $\omega \in \Omega$

$$\Psi^k(t, \omega, v, \eta) \rightarrow \Psi(t, \omega, v, \eta) \quad (19)$$

as $k \rightarrow \infty$ uniformly for $(t, (v, \eta)) \in [0, T] \times B$. Furthermore by continuity of the inverse map

$$\begin{aligned} GL(\mathbb{R}^n) &\rightarrow GL(\mathbb{R}^n) \\ A &\mapsto A^{-1} \end{aligned}$$

it follows from (14) that for a.a. ω

$$\lim_{k \rightarrow \infty} \varphi^k(t, \omega, \cdot)^{-1} = \varphi(t, \omega, \cdot)^{-1} \quad (20)$$

uniformly for $t \in [0, T]$. Now

$$\begin{aligned} |\tilde{H}_k(t, \omega, v, \eta) - \tilde{H}(t, \omega, v, \eta)| &= |\varphi^k(t, \omega)^{-1}(\varphi^k(t, \omega, v, \eta)) - \varphi(t, \omega)^{-1}(\varphi(t, \omega, v, \eta))| \\ &\leq \|\varphi^k(t, \omega)^{-1} - \varphi(t, \omega)^{-1}\| |\varphi^k(t, \omega, v, \eta)| + \|\varphi(t, \omega)^{-1}\| |\varphi^k(t, \omega, v, \eta) - \varphi(t, \omega, v, \eta)| \end{aligned}$$

for a.a. ω , $t > 0$, $(v, \eta) \in M_2$. Using (19) we see that for a.a. ω

$$\lim_{k \rightarrow \infty} \sup_{(v, \eta) \in B} \sup_{0 \leq t \leq T} |\tilde{H}_k(t, \omega, v, \eta) - \tilde{H}(t, \omega, v, \eta)| = 0. \quad (21)$$

Secondly, we prove that for a.a. $\omega \in \Omega$,

$$((v, \eta) y^k(t, \omega), (v, \eta) y_t^k(\cdot, \omega)) \rightarrow ((v, \eta) y(t, \omega), (v, \eta) y_t(\cdot, \omega)) \text{ as } k \rightarrow \infty \quad (22)$$

uniformly in $(t, v, \eta) \in [0, T] \times B$. We use a Gronwall-type argument to do so. It is easy to see from (15) that each map $\tilde{H}_k(t, \omega, \cdot, \cdot): M_2 \rightarrow \mathbb{R}^n$ is continuous linear. Let $\|\tilde{H}_k(t, \omega, \cdot, \cdot)\|$ denote its operator norm. By (21),

$$K_1(\omega) := \sup_{0 \leq t \leq T} \sup_{k \geq 1} \|\tilde{H}_k(t, \omega, \cdot, \cdot)\| < \infty$$

for a.a. $\omega \in \Omega$. Now

$$\begin{aligned} & |((v, \eta) y^k(t, \omega) - (v, \eta) y(t, \omega))| = \\ & = \left| \int_0^t [\tilde{H}_k(u, \omega, (v, \eta) y^k(u, \omega), (v, \eta) y_u^k(\cdot, \omega)) - \tilde{H}(u, \omega, (v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega))] du \right| \\ & \leq K_1(\omega) \int_0^t \|((v, \eta) y^k(u, \omega), (v, \eta) y_u^k(\cdot, \omega)) - ((v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega))\|_{M_2} du \\ & + \int_0^t |\tilde{H}_k(u, \omega, (v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega)) - \tilde{H}(u, \omega, (v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega))| du \end{aligned} \quad (23)$$

for $t > 0$, a.a. $\omega \in \Omega$. From inequality (8) (§2) it is easy to see that for a.a. $\omega \in \Omega$ the set $\{((v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega)): 0 \leq u \leq T, (v, \eta) \in B\}$ is bounded in M_2 . Therefore by (21)

$$\lim_{k \rightarrow \infty} \tilde{H}_k(u, \omega, (v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega)) = \tilde{H}(u, \omega, (v, \eta) y(u, \omega), (v, \eta) y_u(\cdot, \omega)) \quad (24)$$

uniformly for $(u, (v, \eta)) \in [0, T] \times B$ for a.a. $\omega \in \Omega$. Indeed, let $\epsilon > 0$. Then there is a $k_0 = k_0(\epsilon, \omega) > 0$ independent of $(u, v, \eta) \in [0, T] \times B$ such that

$$|\tilde{H}_k(u, \omega, {}^{(v, \eta)}y(u, \omega), {}^{(v, \eta)}y_u(\cdot, \omega)) - \tilde{H}(u, \omega, {}^{(v, \eta)}y(u, \omega), {}^{(v, \eta)}y_u(\cdot, \omega))| < \epsilon$$

for all $k \geq k_0$ and $(u, v, \eta) \in [0, T] \times B$. Squaring (23) then yields

$$\begin{aligned} & |{}^{(v, \eta)}y^k(t, \omega) - {}^{(v, \eta)}y(t, \omega)|^2 \leq 2\epsilon^2 T^2 \\ & + 2K_1(\omega)^2 T \int_0^t \|({}^{(v, \eta)}y^k(u, \omega), {}^{(v, \eta)}y_u^k(\cdot, \omega)) - ({}^{(v, \eta)}y(u, \omega), {}^{(v, \eta)}y_u(\cdot, \omega))\|_{M_2}^2 du \end{aligned} \quad (25)$$

for $(t, v, \eta) \in [0, T] \times B$, a.a. $\omega \in \Omega$.

Therefore

$$\begin{aligned} & \|({}^{(v, \eta)}y_t^k(\cdot, \omega) - {}^{(v, \eta)}y_t(\cdot, \omega))\|_{\mathcal{Y}^2}^2 = \int_{t-r}^t |{}^{(v, \eta)}y^k(\lambda, \omega) - {}^{(v, \eta)}y(\lambda, \omega)|^2 d\lambda \leq 2\epsilon^2 T^2 r \\ & + 2K_1(\omega)^2 T r \int_0^t \|({}^{(v, \eta)}y^k(u, \omega), {}^{(v, \eta)}y_u^k(\cdot, \omega)) - ({}^{(v, \eta)}y(u, \omega), {}^{(v, \eta)}y_u(\cdot, \omega))\|_{M_2}^2 du \end{aligned} \quad (26)$$

$(t, v, \eta) \in [0, T] \times B$, a.a. $\omega \in \Omega$.

If we add up (25) and (26) and apply Gronwall's lemma to the resulting inequality, we get

$$\begin{aligned} & \|({}^{(v, \eta)}y^k(t, \omega), {}^{(v, \eta)}y_t^k(\cdot, \omega)) - ({}^{(v, \eta)}y(t, \omega), {}^{(v, \eta)}y_t(\cdot, \omega))\|^2 \\ & \leq 2\epsilon^2 T^2 (1+r) e^{2K_1^2(\omega) T(1+r)t} \end{aligned} \quad (27)$$

for a.a. $\omega \in \Omega$, all $0 \leq t \leq T$, $k \geq k_0$, $(v, \eta) \in B$.

Let $(t, v, \eta) \in [0, T] \times B$, $k \geq k_0$. Then for a.a. $\omega \in \Omega$

$$\begin{aligned} & |\varphi^k(t, \omega) \left[{}^{(v, \eta)}y^k(t, \omega) \right] - \varphi(t, \omega) \left[{}^{(v, \eta)}y(t, \omega) \right]| \leq \|\varphi^k(t, \omega)\| |{}^{(v, \eta)}y^k(t, \omega) - {}^{(v, \eta)}y(t, \omega)| \\ & + \|\varphi^k(t, \omega) - \varphi(t, \omega)\| |{}^{(v, \eta)}y(t, \omega)|. \end{aligned}$$

Using (27) and the boundedness of the sets $\{\|\varphi^k(t, \omega)\| : k \geq 1, 0 \leq t \leq T\}$,

$\{|{}^{(v, \eta)}y(t, \omega)| : (t, v, \eta) \in [0, T] \times B\}$ gives

$$\lim_{k \rightarrow \infty} \varphi^k(t, \omega) \left[{}^{(v, \eta)}y^k(t, \omega) \right] = \varphi(t, \omega) \left[{}^{(v, \eta)}y(t, \omega) \right] \quad (28)$$

for a.a. $\omega \in \Omega$, uniformly for $(t, v, \eta) \in [0, T] \times B$.

A similar argument shows that for a.a. $\omega \in \Omega$

$$\lim_{k \rightarrow \infty} \varphi_t^k(\cdot, \omega) \circ (\text{id}_J, {}^{(v, \eta)}y_t^k(\cdot, \omega)) = \varphi_t(\cdot, \omega) \circ (\text{id}_J, {}^{(v, \eta)}y_t(\cdot, \omega)) \quad (29)$$

uniformly for $(t, v, \eta) \in [0, T] \times B$.

The convergence (17) follows immediately from (28) and (29). This completes the proof of the theorem. $\square$

As a consequence of the theorem, the field $X: \mathbb{R}^+ \times \Omega \times M_2 \rightarrow M_2$ of (I) gives a cocycle over the canonical shift $\theta: \mathbb{R}^+ \times \Omega \rightarrow \Omega$

$$\theta(t, \omega)(u) = \omega(t+u) - \omega(t), \quad t, u \geq 0, \quad \omega \in \Omega, \quad (30)$$

(cf. Arnold & Wihstutz [2], Carverhill [9], Elworthy [16], Ikeda & Watanabe [20]).

Theorem 3: (The Cocycle Property)

Let $\theta: \mathbb{R}^+ \times \Omega \rightarrow \Omega$ be the shift (30) in Ω . Then there is a universal set $\hat{\Omega} \subset \Omega$ of full Wiener measure such that

$$X(t_2, \theta(t_1, \omega), X(t_1, \omega, (v, \eta))) = X(t_1 + t_2, \omega, (v, \eta)) \quad (31)$$

for all $\omega \in \hat{\Omega}$, all $(v, \eta) \in M_2$, $t_1, t_2 \geq 0$.

Proof:

The idea is simple: We prove the cocycle property (31) when X is replaced by the approximating fields X^k , then use the uniform convergence (17) to pass to the limit.

Fix $k \geq 1$ and $(v, \eta) \in M_2$ till further notice. Define $y^k(t, \omega) := {}^{(v, \eta)}y_t^k(\cdot, \omega)$, $t \geq -r$, $\omega \in \Omega$ as in Theorem 2. Set

$$x^k(t) := {}^{(v, \eta)}x_t^k(\cdot, \omega) = \begin{cases} \varphi^k(t, \omega)(y^k(t, \omega)) & t \geq 0 \\ \eta(t) & \text{a.a. } t \in J \end{cases} \quad (32)$$

Applying the chain rule, suppressing ω and using $(VI)_k$ give

$$\begin{aligned}
 dx^k(t) &= \sum_{i=1}^m [g \circ \varphi^k(t, \cdot)](y^k(t))(e_1)(W_1^k)'(t)dt + \varphi^k(t, \cdot)[\tilde{H}_k(t, \cdot, y^k(t), y_t^k)]dt \\
 &\quad - \frac{1}{2} \sum_{i=1}^m (g \circ [g \circ \varphi^k(t, \cdot)](\cdot)(e_1))(y^k(t))(e_1)dt \\
 &= \sum_{i=1}^m g[\varphi^k(t, \cdot)(y^k(t))](e_1)(W_1^k)'(t)dt \\
 &\quad + H(\varphi^k(t, \cdot)(y^k(t)), \varphi_t^k(\cdot, \cdot) \circ (id_J, y_t^k))dt \\
 &\quad - \frac{1}{2} \sum_{i=1}^m g([g(\varphi^k(t, \cdot)(y^k(t)))](e_1))(e_1)dt \\
 &= H(x^k(t), x_t^k)dt + g(x^k(t))(W^k)'(t)dt - \frac{1}{2} \sum_{i=1}^m (g([g(x^k(t))](e_1)))(e_1)dt.
 \end{aligned}$$

Therefore for each $\omega \in \Omega$, x^k is the unique solution of the r.f.d.e.

$$\left. \begin{aligned}
 \frac{dx^k}{dt} &= H(x^k(t), x_t^k) + g(x^k(t))(W^k)'(t) - \frac{1}{2} \sum_{i=1}^m (g([g(x^k(t))](e_1)))(e_1), \quad t > 0 \\
 x^k(0) &= v, \quad x_0^k = \eta.
 \end{aligned} \right\} (IX)_k$$

Recall that W^k is a piece-wise linear approximation of W viz.

$$(W^k)'(t, \omega) = (\omega^k)'(t), \quad \omega \in \Omega, \quad t \geq 0$$

where

$$\omega^k(t) = k[\omega(\frac{j+1}{k}) - \omega(\frac{j}{k})](t - \frac{j}{k}) + \omega(\frac{j}{k}), \quad \frac{j}{k} \leq t < \frac{j+1}{k}, \quad j = 0, 1, 2, \dots$$

Fix $\omega \in \Omega$ and let $t_1 > 0$. Define

$$\rho(t, \omega) := (v, \eta)_{x^k(t+t_1, \omega)} \text{ for all } t \text{ such that } t+t_1 \geq 0$$

and a.a. $t \in J$ with $-r < t+t_1 < 0$, where $(v, \eta) x^k$ is the solution of $(IX)_k$ defined by (32). Then clearly

$$\left. \begin{aligned} \rho'(t, \omega) &= H(\rho(t, \omega), \rho_t(\cdot, \omega)) + g(\rho(t, \omega))(\omega^k)'(t+t_1) \\ &\quad - \frac{1}{2} \sum_{i=1}^m (g([g(\rho(t, \omega))]e_i))(e_i) \\ \rho(0, \omega) &= (v, \eta) x^k(t_1, \omega), \quad \rho_0(\cdot, \omega) = (v, \eta) x_{t_1}^k(\cdot, \omega). \end{aligned} \right\} \quad (X)$$

Suppose $\hat{y}(\cdot, \omega): [-r, \infty) \rightarrow \mathbb{R}^n$ is the unique solution of

$$\left. \begin{aligned} (\hat{y})'(t, \omega) &= \tilde{H}_k(t, \theta(t_1, \omega), \hat{y}(t, \omega), y_t(\cdot, \omega)), \quad t > 0 \\ \hat{y}(0, \omega) &= (v, \eta) x^k(t_1, \omega), \quad \hat{y}_0(\cdot, \omega) = (v, \eta) x_{t_1}^k(\cdot, \omega) \end{aligned} \right\} \quad (XI)$$

and let

$$z(t, \omega) = \begin{cases} p^k(t, \theta(t_1, \omega))(\hat{y}(t, \omega)) & t \geq 0 \\ \hat{y}(t, \omega) & \text{a.a. } t \in [-r, 0]. \end{cases}$$

Using the chain rule gives as before

$$\begin{aligned} z'(t, \omega) &= \sum_{i=1}^m g[p^k(t, \theta(t_1, \omega))(\hat{y}(t, \omega))]e_i (\omega_1^k)'(t, \theta(t_1, \omega)) \\ &\quad + H(p^k(t, \theta(t_1, \omega))(\hat{y}(t, \omega)), p_t^k(\cdot, \theta(t_1, \omega)) \circ (\text{id}_J, \hat{y}_t(\cdot, \omega))) \\ &\quad - \frac{1}{2} \sum_{i=1}^m g([g(p^k(t, \theta(t_1, \omega))(\hat{y}(t, \omega)))]e_i)(e_i) \\ &= H(z(t, \omega), z_t(\cdot, \omega)) + g(z(t, \omega))(\omega^k)'(t+t_1) - \frac{1}{2} \sum_{i=1}^m g([g(z(t, \omega))]e_i)(e_i) \end{aligned} \quad (XII)$$

But

$$\left. \begin{aligned} z(0, \omega) &= \hat{y}(0, \omega) = (v, \eta) x^k(t_1, \omega), \quad z_0(\cdot, \omega) = \hat{y}_0(\cdot, \omega) = (v, \eta) x_{t_1}^k(\cdot, \omega) \end{aligned} \right\}$$

Therefore $z(t, \omega)$, $t \geq -r$, satisfies the initial value problem (X). So by pathwise-uniqueness we must have

$$(z(t, \omega), z_t(\cdot, \omega)) = (\rho(t, \omega), \rho_t(\cdot, \omega)), \quad t \geq 0, \quad \omega \in \Omega.$$

Now by construction of X^k we have

$$X^k(t, \omega, (v, \eta)) = ((v, \eta) X^k(t, \omega), (v, \eta) X_t^k(\cdot, \omega)), \quad t \geq 0.$$

Hence

$$X^k(t+t_1, \omega, (v, \eta)) = (\rho(t, \omega), \rho_t(\cdot, \omega)) \quad t > 0.$$

Also (XII) says that

$$X^k(t, \theta(t_1, \omega), z(0, \omega), z_0(\cdot, \omega)) = (z(t, \omega), z_t(\cdot, \omega)) \quad t \geq 0,$$

i.e.

$$\begin{aligned} X^k(t, \theta(t_1, \omega), X^k(t_1, \omega, (v, \eta))) &= (z(t, \omega), z_t(\cdot, \omega)) \\ &= (\rho(t, \omega), \rho_t(\cdot, \omega)) \\ &= X^k(t+t_1, \omega, (v, \eta)) \end{aligned}$$

for all $\omega \in \Omega$, $t \geq 0$. To complete the proof of the cocycle property (31), take $k \rightarrow \infty$ in the above identity and use Theorem 2.

□

§4. Existence of the Lyapunov Spectrum

This is the final section. Having constructed the regular cocycle (X, θ) in §2 and §3, we are now ready to characterize the almost sure exponential growth rates

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|, \quad \omega \in \Omega, \quad (v, \eta) \in M_2$$

for the trajectory field X of the linear s.f.d.e. (I). We shall prove that the above limit takes values within a fixed (at most) countable set of Lyapunov exponents $\{\lambda_i\}_{i=1}^{\infty}$ (Theorem 1). The associated Oseledec random filtration of the Hilbert space M_2 is established in the above theorem as a

consequence of Ruelle's infinite-dimensional version of the Oseledec multiplicative ergodic theorem (Ruelle [34],[35]).

Corollary 2 of the Oseledec ergodic theorem 1 gives a version of the stable-manifold theorem for the linear s.f.d.e. (I) in the hyperbolic case $\lambda_i \neq 0$ for all $i \geq 1$.

To prove our main result on Lyapunov exponents we need the following sequence of lemmas which all use the notations of §2 and §3.

Lemma 1:

For each integer $k \geq 1$, and any real number $a > 0$,

$$\int_{\Omega} \sup_{0 \leq t \leq a} \|\varphi(t, \omega, \cdot)^{-1}\|_{L(\mathbb{R}^n)}^{2k} dP(\omega) < \infty.$$

Proof:

Recall that $\varphi(t, \cdot) \in L(\mathbb{R}^n)$ satisfies the Itô s.o.d.e.

$$d\varphi(t, \cdot) = \sum_{i=1}^m [g \circ \varphi(t, \cdot)](\cdot)(e_i) dW_i(t), \quad t > 0, \quad \varphi(0, \cdot) = \text{id}_{\mathbb{R}^n}. \quad (V)$$

Let $\hat{\varphi}$ be the unique solution of the following Itô s.o.d.e. in $L(\mathbb{R}^n)$:

$$\left. \begin{aligned} d\hat{\varphi}(t, \cdot) &= -\sum_{i=1}^m \hat{\varphi}(t, \cdot) \circ (g(\cdot)(e_i)) dW_i(t) + \sum_{i=1}^m \hat{\varphi}(t, \cdot) \circ (g \circ (g(\cdot)(e_i)))(\cdot)(e_i) dt \\ \hat{\varphi}(0, \cdot) &= \text{id}_{\mathbb{R}^n} \end{aligned} \right\} \quad (XIII)$$

In (V) and (XIII), $(e_i)_{i=1}^m$ a canonical basis of $\mathbb{R}^m$ and $W(t) = \sum_{i=1}^m W_i(t)e_i$, $t > 0$.

Applying Itô's formula to the process $(\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot): t \geq 0)$ gives

$$\begin{aligned}
d[\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot)] &= d\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot) + \hat{\varphi}(t, \cdot) d\varphi(t, \cdot) + d\hat{\varphi}(t, \cdot) d\varphi(t, \cdot) \\
&= -\sum_{i=1}^m \hat{\varphi}(t, \cdot) \circ \{g(\cdot)(e_i)\} \circ \varphi(t, \cdot) dW_1(t) \\
&\quad + \sum_{i=1}^m \hat{\varphi}(t, \cdot) \circ \{g \circ \{g(\cdot)(e_i)\}(\cdot)(e_i) \circ \varphi(t, \cdot) dt \\
&\quad + \sum_{i=1}^m \hat{\varphi}(t, \cdot) \circ [g \circ \varphi(t, \cdot)](\cdot)(e_i) dW_1(t) \\
&\quad - \sum_{i,j=1}^m \hat{\varphi}(t, \cdot) \circ \{g(\cdot)(e_i)\} \circ [(g \circ \varphi(t, \cdot))(\cdot)(e_j)] dW_1(t) dW_j(t), \quad t > 0
\end{aligned}
\tag{33}$$

Note that if $v \in \mathbb{R}^n$, then

$$\begin{aligned}
[\{g(\cdot)(e_i)\} \circ \varphi(t, \cdot)](v) &= g(\cdot)(e_i)(\varphi(t, \cdot)(v)) = g(\varphi(t, \cdot)(v))(e_i) \\
&= [\{g \circ \varphi(t, \cdot)\}(\cdot)(e_i)](v).
\end{aligned}$$

Hence

$$\{g(\cdot)(e_i)\} \circ \varphi(t, \cdot) = \{g \circ \varphi(t, \cdot)\}(\cdot)(e_i), \quad 1 \leq i \leq m. \tag{34}$$

A similar computation yields

$$\begin{aligned}
(g \circ \{g(\cdot)(e_i)\})(\cdot)(e_i) \circ \varphi(t, \cdot) &= (g \circ [\{g \circ \varphi(t, \cdot)\}(\cdot)(e_i)])(\cdot)(e_i) \\
&= \{g(\cdot)(e_i)\} \circ [\{g \circ \varphi(t, \cdot)\}(\cdot)(e_i)]
\end{aligned}
\tag{35}$$

Therefore from (33), (34) and (35) we get

$$d[\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot)] = 0, \quad t > 0$$

i.e. $\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot)$ is a constant a.s. for all $t \geq 0$. In particular

$$\hat{\varphi}(t, \cdot) \circ \varphi(t, \cdot) = \hat{\varphi}(0, \cdot) \circ \varphi(0, \cdot) = \text{id}_{\mathbb{R}^n} \quad \text{a.s.}$$

for all $t \geq 0$. Thus

$$\hat{\varphi}(t, \cdot) = \varphi(t, \cdot)^{-1}, \quad t > 0.$$

From (XIII), it follows that $\{\hat{\varphi}(t, \cdot): t > 0\}$ is a semi-martingale in $L(\mathbb{R}^n)$ and the integrability result

$$E \sup_{0 \leq t \leq a} \|\hat{\varphi}(t, \cdot)\|^{2k} < \infty, \quad 0 < a < \infty,$$

is an easy consequence of standard s.o.d.e. estimates (Gihman & Skorohod [17], pp. 48-50). □

Lemma 2:

If $0 < a < \infty$, then

$$\int_{\omega \in \Omega} \sup_{0 \leq t_1, t_2 \leq a} \|\varphi(t_1, \theta(t_2, \omega), \cdot)\|_{L(\mathbb{R}^n)}^{2k} dP(\omega) < \infty, \quad k = 1, 2, 3, \dots$$

Proof:

Fix an integer $k > 0$. Since $\{\varphi(t, \cdot, \cdot): t \geq 0\}$ is a cocycle over the shift $\{\theta(t, \cdot): t \geq 0\}$, then for a.a. $\omega \in \Omega$

$$\varphi(t_1, \theta(t_2, \omega), \cdot) = \varphi(t_1 + t_2, \omega, \cdot) \circ \varphi(t_2, \omega, \cdot)^{-1}, \quad t_1, t_2 > 0$$

(Arnold & Wihstutz [2], Carverhill [9]).

From Hölder's inequality,

$$\begin{aligned} & \int_{\Omega} \sup_{0 \leq t_1, t_2 \leq a} \|\varphi(t_1, \theta(t_2, \omega), \cdot)\|^{2k} dP(\omega) \\ & \leq \int_{\Omega} \sup_{0 \leq t_1, t_2 \leq a} \|\varphi(t_1 + t_2, \omega, \cdot)\|^{2k} \sup_{0 \leq t_2 \leq a} \|\varphi(t_2, \omega, \cdot)^{-1}\|^{2k} dP(\omega) \\ & \leq \left[\int_{\Omega} \sup_{0 \leq t \leq 2a} \|\varphi(t, \omega, \cdot)\|^{4k} dP(\omega) \right]^{1/2} \left[\int_{\Omega} \sup_{0 \leq t_2 \leq a} \|\varphi(t_2, \omega, \cdot)^{-1}\|^{4k} dP(\omega) \right]^{1/2} \\ & < \infty \end{aligned}$$

because of Lemma 1. □

In the next two lemmas, $\log^+$ indicates the function

$$\log^+ u = \max(\log u, 0), \quad u > 0.$$

Lemma 3:

If $(v, \eta)y(\cdot, \omega): [-r, \infty) \rightarrow \mathbb{R}^n$ is the solution of the r.f.d.e. (IV), then

$$\int_{\Omega} \sup_{0 \leq t_1, t_2 \leq a} \log^+ \sup_{\|(v, \eta)\|_{M_2} \leq 1} \|((v, \eta)y(t_1, \theta(t_2, \omega)), (v, \eta)y_{t_1}(\cdot, \theta(t_2, \omega)))\|_{M_2} dP(\omega) < \infty$$

for every $0 < a < \infty$.

Proof:

Refer to the proof of Theorem 1, §2. Let $\tau = a$ in inequality (8), take sup over $\{(v, \eta): (v, \eta) \in M_2, \|(v, \eta)\|_{M_2} \leq 1\}$ and $\log^+$ to obtain

$$\begin{aligned} & \log^+ \sup_{\|(v, \eta)\|_{M_2} \leq 1} \|((v, \eta)y(t_1, \theta(t_2, \omega)), (v, \eta)y_{t_1}(\cdot, \theta(t_2, \omega)))\|^2 \\ & \leq \log(2r+1) + (2r+1)a\|H\|^2 \int_0^{t_1} L(u, \theta(t_2, \omega))^2 K(u, \theta(t_2, \omega))^2 du, \quad 0 \leq t_1, t_2 \leq a, \end{aligned}$$

where

$$L(u, \theta(t_2, \omega)) = \|\varphi(u, \theta(t_2, \omega), \cdot)^{-1}\|,$$

$$K(u, \theta(t_2, \omega)) = \max \left[\sup_{\substack{s \in J \\ u+s \geq 0}} \|\varphi(u+s, \theta(t_2, \omega), \cdot)\|, 1 \right].$$

From the relation

$$\varphi(u, \theta(t_2, \omega), \cdot)^{-1} = \varphi(t_2, \omega, \cdot) \circ \varphi(t_2+u, \omega, \cdot)^{-1}, \quad \text{a.a. } \omega \in \Omega, u, t_2 \geq 0,$$

and Hölder's inequality, we get

$$\begin{aligned} & \left[\int_{\Omega} \sup_{0 \leq u, t_2 \leq a} L(u, \theta(t_2, \omega))^4 dP(\omega) \right]^2 \leq \\ & \leq \int_{\Omega} \sup_{0 \leq t_2 \leq a} \|\varphi(t_2, \omega, \cdot)\|^8 dP(\omega) \int_{\Omega} \sup_{0 \leq t \leq 2a} \|\varphi(t, \omega, \cdot)^{-1}\|^8 dP(\omega) \\ & < \infty \end{aligned} \tag{36}$$

because of Lemma 1 ($k = 4$) and Lemma 2 ($t_2 = 0, k = 4$).

Also using Lemma 2 for $k = 2$, we have

$$\begin{aligned} \int_{\Omega} \sup_{0 \leq u, t_2 \leq a} K(u, \theta(t_2, \omega))^4 dP(\omega) &\leq 8+8 \int_{\Omega} \sup_{0 \leq u, t_2 \leq a} \sup_{\substack{s \in J \\ u+s \geq 0}} \|\varphi(u+s, \theta(t_2, \omega), \cdot)\|^4 dP(\omega) \\ &\leq 8+8 \int_{\Omega} \sup_{0 \leq u, t_2 \leq a} \|\varphi(u, \theta(t_2, \omega), \cdot)\|^4 dP(\omega) < \infty. \end{aligned} \quad (37)$$

The lemma now follows from the following estimates:

$$\begin{aligned} &\int_{\Omega} \sup_{0 \leq t_1, t_2 \leq a} \log^+ \sup_{\|(v, \eta)\| \leq 1} \|({}^{(v, \eta)}y(t_1, \theta(t_2, \omega)), {}^{(v, \eta)}y_{t_1}(\cdot, \theta(t_2, \omega)))\|_{M_2}^2 dP(\omega) \\ &\leq \log(2r+1) + (2r+1)a \|H\|^2 \int_{\Omega} \sup_{0 \leq t_2 \leq a} \int_0^a L(u, \theta(t_2, \omega))^2 K(u, \theta(t_2, \omega))^2 du dP(\omega) \\ &\leq \log(2r+1) + \\ &+ (2r+1)a^2 \|H\|^2 \left[\int_{\Omega} \sup_{0 \leq u, t_2 \leq a} L(u, \theta(t_2, \omega))^4 dP(\omega) \right]^{1/2} \left[\int_{\Omega} \sup_{0 \leq u, t_2 \leq a} K(u, \theta(t_2, \omega))^4 dP(\omega) \right]^{1/2} \\ &< \infty. \end{aligned}$$

□

Lemma 4:

$$\int_{\Omega} \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|X(t_1, \theta(t_2, \omega), \cdot)\|_{L(M_2)} dP(\omega) < \infty \quad (38)$$

Proof:

Let $0 \leq t_1, t_2 \leq r$. By the construction of X in the proof of Theorem 1, §2 (equation (11)) we have for a.a. $\omega \in \Omega$ and all $(v, \eta) \in M_2$:

$$\begin{aligned} \|X(t_1, \theta(t_2, \omega), (v, \eta))\|_{M_2}^2 &= \\ &\|(\varphi(t_1, \theta(t_2, \omega), \cdot), {}^{(v, \eta)}y(t_1, \theta(t_2, \omega))), \varphi_{t_1}(\cdot, \theta(t_2, \omega), \cdot) \circ (\text{id}_J, {}^{(v, \eta)}y_{t_1}(\cdot, \theta(t_2, \omega)))\|_{M_2}^2 \\ &\leq \|\varphi(t_1, \theta(t_2, \omega), \cdot)\|^2 |{}^{(v, \eta)}y(t_1, \theta(t_2, \omega))|^2 + \int_{-r}^{-t_1} |{}^{(v, \eta)}y(t_1+s, \theta(t_2, \omega))|^2 ds \\ &\quad + \int_{-t_1}^0 |\varphi(t_1+s, \theta(t_2, \omega), {}^{(v, \eta)}y(t_1+s, \theta(t_2, \omega)))|^2 ds \end{aligned}$$

$$\begin{aligned}
&\leq \| \varphi(t_1, \theta(t_2, \omega), \cdot) \|^2 |^{(v, \eta)} y(t_1, \theta(t_2, \omega))|^2 + \int_{-r}^{-t_1} |^{(v, \eta)} y_{t_1}(s, \theta(t_2, \omega))|^2 ds \\
&\quad + \sup_{\substack{s \in J \\ t_1 + s \geq 0}} \| \varphi(t_1 + s, \theta(t_2, \omega), \cdot) \|^2 \int_{-t_1}^0 |^{(v, \eta)} y_{t_1}(s, \theta(t_2, \omega))|^2 ds \\
&\leq \| \varphi(t_1, \theta(t_2, \omega), \cdot) \|^2 |^{(v, \eta)} y(t_1, \theta(t_2, \omega))|^2 \\
&\quad + K(t_1, \theta(t_2, \omega))^2 \|^{(v, \eta)} y_{t_1}(\cdot, \theta(t_2, \omega))\|_{L^2(J, \mathbb{R}^n)}^2 \\
&\leq \left[\| \varphi(t_1, \theta(t_2, \omega), \cdot) \|^2 + 2 \sup_{\substack{s \in J \\ t_1 + s \geq 0}} \| \varphi(t_1 + s, \theta(t_2, \omega), \cdot) \|^2 \right] \|^{(v, \eta)} y(t_1, \theta(t_2, \omega)), ^{(v, \eta)} y_{t_1}(\cdot, \theta(t_2, \omega))\|_{M_2}^2
\end{aligned}$$

Using the inequality $\log^+ u < u$, $u > 0$, and Lemmas 2 and 3 ($k = 1$) gives

$$\begin{aligned}
&\int_{\Omega} \sup_{0 \leq t_1, t_2 \leq r} \log^+ \| X(t_1, \theta(t_2, \omega), \cdot) \|_{L(M_2)} dP(\omega) \\
&\leq 3 \int_{\Omega} \sup_{0 \leq t_1, t_2 \leq r} \| \varphi(t_1, \theta(t_2, \omega), \cdot) \|^2 dP(\omega) + 2 \\
&\quad + 2 \int_{\Omega} \sup_{0 \leq t_1, t_2 \leq r} \log^+ \sup_{\| (v, \eta) \| \leq 1} \| ^{(v, \eta)} y(t_1, \theta(t_2, \omega)), ^{(v, \eta)} y_{t_1}(\cdot, \theta(t_2, \omega)) \|_{M_2} dP(\omega) \\
&< \infty.
\end{aligned}$$

□

The next lemma summarizes some essentially known results on the ergodic behavior of the shift $\theta(t, \cdot): \Omega \rightarrow \Omega$, $t \geq 0$ under Wiener measure P on Ω . We give the proof, however, for completeness.

Lemma 5:

- (i) $\theta_1 := \theta(1, \cdot): \Omega \rightarrow \Omega$ is P -preserving and ergodic.
- (ii) If $h \in \mathcal{L}^1(\Omega, \mathbb{R}; P)$, then $\lim_{k \rightarrow \infty} \frac{1}{k} h(\theta(kr, \omega)) = 0$ for a.a. $\omega \in \Omega$.

(iii) Let $h: \Omega \rightarrow \mathbb{R}^+$ be $\mathcal{F}$ -measurable and such that

$$\int_{\Omega} \sup_{0 \leq u \leq r} h(\theta(u, \omega)) dP(\omega) < \infty. \quad (39)$$

Then $\lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$ for a.a. $\omega \in \Omega$.

Proof:

(i) An elegant proof of the fact that θ_1 is measure-preserving is given in Elworthy's book ([16], Chapter I, §3, pp. 18-20). To prove ergodicity of θ_1 under P we proceed as follows. Recall that $\mathcal{F} = \text{Borel } \Omega$, $\mathcal{F}_t = \sigma\{\rho_u : 0 \leq u \leq t\}$, the σ -algebra generated by the evaluations

$$\begin{aligned} \rho_u : \Omega &\rightarrow \mathbb{R}^m, \quad 0 \leq u \leq t, \\ \omega &\mapsto \omega(u). \end{aligned}$$

Suppose $f: \Omega \rightarrow \mathbb{R}$ is $\mathcal{F}$ -measurable and invariant under θ_1 i.e.

$$f(\theta_1(\omega)) = f(\omega) \quad P\text{-a.a. } \omega \in \Omega. \quad (40)$$

For θ_1 to be ergodic it is sufficient to prove that $f(\omega) = \text{constant}$ for a.a. $\omega \in \Omega$. Now, since $P \circ \theta_1^{-1} = P$, then (40) implies

$$f(\theta_1^k(\omega)) = f(\omega) \quad \text{a.a. } \omega \in \Omega, \quad k \geq 0. \quad (41)$$

Let $\mathcal{G}_k = \sigma\{(\theta_1^k)^{-1}(B) : B \in \mathcal{F}\}$, the σ -algebra generated by the $\theta_1^k: \Omega \rightarrow \Omega$, $k \geq 0$.

Since $\mathcal{F}$ is generated by the cylinder sets, then clearly $\mathcal{G}_k$ is generated by the family $\{\rho_t \circ \theta_1^k : t \geq 0\}$ for each $k > 0$. Now

$$(\rho_t \circ \theta_1^k)(\omega) = \omega(k+t) - \omega(k) = W(k+t, \omega) - W(k, \omega)$$

for every $k \geq 0$, $t \geq 0$. Hence $\mathcal{F}_k$ and $\mathcal{G}_k$ are independent under P (Elworthy [16], Proposition 3B, p. 19).

Now for each $k > 0$,

$$(\theta_1^{k+1})^{-1}(B) = (\theta_1^k)^{-1}(\theta_1^{-1}(B)) \in \mathcal{G}_k$$

whenever $B \in \mathcal{F}$, so $\mathcal{G}_{k+1} \subseteq \mathcal{G}_k$ and the family $\{\mathcal{G}_k\}_{k=1}^{\infty}$ is decreasing.

We shall show that the tail σ -algebra $\mathcal{G}_{\infty} = \bigcap_{k=1}^{\infty} \mathcal{G}_k$ is trivial in the sense that $P(A) = 0$ or 1 for each $A \in \mathcal{G}_{\infty}$. To see this let $A \in \mathcal{G}_k$ for all $k \geq 1$.

Then A is independent of $\mathcal{F}_k$ for each $k \geq 1$. But $\mathcal{F} = \bigcup_{k=1}^{\infty} \mathcal{F}_k$ and $A \in \mathcal{F}$, so $A \in \mathcal{F}_{k'}$ for some $k' > 0$. Therefore $A \in \mathcal{F}_{k'} \cap \mathcal{G}_{k'}$, and by independence $P(A) = 0$ or $P(A) = 1$. Hence $\mathcal{G}_{\infty}$ is trivial.

Now by (41), f is $\mathcal{G}_k$ -measurable for each $k \geq 1$ i.e. f is $\mathcal{G}_{\infty}$ -measurable. Hence $f(\omega) = \text{constant}$ for a.a. $\omega \in \Omega$. This completes the proof of ergodicity of θ_1 (and $\theta(k, \cdot)$ for all $k > 0$).

(ii) Let $h \in \mathcal{L}^1(\Omega, \mathbb{R}; P)$. Since $\theta(kr, \cdot): \Omega \rightarrow \Omega$ is P -preserving for each integer $k > 0$, the process

$$Z_k(\omega) = h(\theta(kr, \omega)) \quad \omega \in \Omega, \quad k \geq 0$$

is strictly stationary in the sense of Doob ([15], p. 454). Hence by the strong law of large numbers (Birkhoff's ergodic theorem, Doob [15], p. 465), one has

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} h(\theta(kr, \omega)) &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k h(\theta(jr, \omega)) - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k h(\theta((j-1)r, \omega)) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k+1} \sum_{j=0}^k h(\theta(jr, \omega)) - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j'=0}^{k-1} h(\theta(j'r, \omega)) \\ &= h^*(\omega) - h^*(\omega) = 0 \end{aligned}$$

for a.a. $\omega \in \Omega$, where $h^*: \Omega \rightarrow \mathbb{R}$ is some measurable function.

(iii) Let $h: \Omega \rightarrow \mathbb{R}^+$ satisfy the integrability condition (39). For each $t > 0$, let $k = \left\lfloor \frac{t}{r} \right\rfloor$, the greatest integer less than or equal to $\frac{t}{r}$. Then $kr \leq t \leq (k+1)r$, and so

$$\begin{aligned}
\frac{1}{t} h(\theta(t, \omega)) &\leq \frac{1}{kr} h(\theta(t-kr, \theta(kr, \omega))) \\
&\leq \frac{1}{kr} \sup_{0 \leq u \leq r} h(\theta(u, \theta(kr, \omega))) \\
&= \frac{1}{kr} \hat{h}(\theta(kr, \omega)) \quad \text{for all } \omega \in \Omega
\end{aligned}$$

where

$$\hat{h}(\omega) = \sup_{0 \leq u \leq r} h(\theta(u, \omega)) \quad \omega \in \Omega.$$

But condition (39) says that $\hat{h} \in \mathcal{L}^1(\Omega, \mathbb{R}^+; P)$. So by (ii) above we must have

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) \leq \lim_{k \rightarrow \infty} \frac{1}{kr} \hat{h}(\theta(kr, \omega)) = 0 \quad \text{for a.a. } \omega \in \Omega.$$

This completes the proof of the lemma. $\square$

Lemma 6:

(i) Suppose that for a.a. $\omega \in \Omega$, $\lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, \cdot)\|_{L(M_2)}$ exists. Then

for a.a. $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} = \frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, \cdot)\|_{L(M_2)}. \quad (42)$$

(ii) For each $(v, \eta) \in M_2$ let $\lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, (v, \eta))\|_{M_2}$ exist for a.a.

$\omega \in \Omega$. Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| = \frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, (v, \eta))\|_{M_2} \quad \text{a.a. } \omega \in \Omega \quad (43)$$

Proof:

(i) For $t > 0$, let $k = \left\lfloor \frac{t}{r} \right\rfloor$. Then $kr \leq t \leq (k+1)r$. From the cocycle property (Theorem 3, §3),

$$\|X(t, \omega, \cdot)\|_{L(M_2)} \leq \|X(t-kr, \theta(kr, \omega), \cdot)\|_{L(M_2)} \cdot \|X(kr, \omega, \cdot)\|_{L(M_2)}.$$

Hence

$$\begin{aligned}
\frac{1}{t} \log \|X(t, \omega, \cdot)\| &\leq \frac{1}{kr} \log \|X(t-kr, \theta(kr, \omega), \cdot)\| + \frac{1}{kr} \log \|X(kr, \omega, \cdot)\| \\
&\leq \frac{1}{kr} \sup_{0 \leq u \leq r} \log \|X(u, \theta(kr, \omega), \cdot)\| + \frac{1}{kr} \log \|X(kr, \omega, \cdot)\| \\
&= \frac{1}{kr} h(\theta(kr, \omega)) + \frac{1}{kr} \log \|X(kr, \omega, \cdot)\|
\end{aligned} \tag{44}$$

where

$$h(\omega) = \sup_{0 \leq u \leq r} \log^+ \|X(u, \omega, \cdot)\|, \quad \text{for a.a. } \omega \in \Omega.$$

Also the relation

$$X((k+1)r, \omega, \cdot) = X((k+1)r-t, \theta(t, \omega), \cdot) \circ X(t, \omega, \cdot), \quad kr \leq t \leq (k+1)r,$$

implies that

$$\begin{aligned}
\frac{1}{(k+1)r} \log \|X((k+1)r, \omega, \cdot)\| &\leq \frac{1}{t} \log \|X((k+1)r-t, \theta(t, \omega), \cdot)\| + \frac{1}{t} \log \|X(t, \omega, \cdot)\| \\
&\leq \frac{1}{t} \sup_{0 \leq u \leq r} \log \|X(u, \theta(t, \omega), \cdot)\| + \frac{1}{t} \log \|X(t, \omega, \cdot)\| \\
&= \frac{1}{t} h(\theta(t, \omega)) + \frac{1}{t} \log \|X(t, \omega, \cdot)\|
\end{aligned} \tag{45}$$

for a.a. $\omega \in \Omega$.

Combining (44) and (45) yields

$$\begin{aligned}
\frac{1}{(k+1)r} \log \|X((k+1)r, \omega, \cdot)\| &\leq \frac{1}{t} \log \|X(t, \omega, \cdot)\| \\
&\leq \frac{1}{kr} h(\theta(kr, \omega)) + \frac{1}{kr} \log \|X(kr, \omega, \cdot)\| - \frac{1}{t} h(\theta(t, \omega))
\end{aligned} \tag{46}$$

for $kr \leq t \leq (k+1)r$.

Now by Lemma 4, $h \in \mathcal{L}^1(\Omega, \mathbb{R}^+; P)$ and so

$$\lim_{k \rightarrow \infty} \frac{1}{kr} h(\theta(kr, \omega)) = 0 \quad \text{a.a. } \omega \in \Omega. \tag{47}$$

Also from Lemma 4, h satisfies the integrability condition (39) of Lemma 5(iii)

which gives

$$\lim_{t \rightarrow \infty} \frac{1}{t} h(\theta(t, \omega)) = 0 \quad \text{a.a. } \omega \in \Omega. \tag{48}$$

Taking limits in (46) as $k, t \rightarrow \infty$ and using (47), (48) immediately gives the equality (42).

(ii) For $(v, \eta) \in M_2$, $kr \leq t \leq (k+1)r$, it follows as in (i) that

$$\begin{aligned} \frac{1}{(k+1)r} \log \|X((k+1)r, \omega, (v, \eta))\| - \frac{1}{t} h(\theta(t, \omega)) &\leq \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| \\ &\leq \frac{1}{kr} h(\theta(kr, \omega)) + \frac{1}{kr} \log \|X(kr, \omega, (v, \eta))\|, \quad \text{a.a. } \omega \in \Omega. \end{aligned}$$

The required relation (43) is now obtained by taking limits as before. $\square$

The way is now clear to give a proof of the following Oseledec ergodic theorem for the linear s.f.d.e. (I).

Theorem 1: (The Lyapunov Spectrum)

Let $X: \mathbb{R}^+ \times \Omega \times M_2 \rightarrow M_2$ be the regular version of the trajectory field $\{((v, \eta)_x(t), (v, \eta)_{x_t}): t \geq 0, (v, \eta) \in M_2\}$ to the linear s.f.d.e. (I) as given by Theorem 1, §2.

Then there exist

- (a) a universal Borel set $\Omega^* \subset \Omega$ of full Wiener measure,
- (b) a fixed (non-random) sequence of real numbers $\{\lambda_i\}_{i=1}^\infty$,
- (c) a random family $\{E_i(\omega): i \geq 1, \omega \in \Omega^*\}$ of (closed) finite-codimensional subspaces of M_2 ,

with the following properties:

(i) $\lambda_{i+1} < \lambda_i$ for all i and if the spectrum $\{\lambda_i\}_{i=1}^\infty$ is infinite one has $\lim_{i \rightarrow \infty} \lambda_i = -\infty$; otherwise there is a fixed (non-random) integer $N > 1$ such that

$$\lambda_N = -\infty < \lambda_{N-1} < \dots < \lambda_2 < \lambda_1;$$

(ii) for each $\mu \in \mathbb{R}$, the set $\{\lambda_i: \lambda_i > \mu\}$ is finite;

(iii) for each $i \geq 1$, the map $\omega \mapsto E_i(\omega)$ is Borel-measurable into the

Grassmannian structure of M_2 ;

(iv) for each $\omega \in \Omega^*$,

$$E_{i+1}(\omega) \subset E_i(\omega) \subset \dots \subset E_2(\omega) \subset E_1(\omega) = M_2, \quad i \geq 1;$$

- (v) for each $i \geq 1$, $\text{co-dim } E_1(\omega) = n_1$ is fixed independently of $\omega \in \Omega^*$;
- (vi) $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|_{M_2} = \lambda_1, \quad (v, \eta) \in E_1(\omega) \setminus E_{1+1}(\omega), \quad \omega \in \Omega^*$;
- (vii) $\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} < \infty$ for all $\omega \in \Omega^*$;
- (viii) $X(t, \omega, \cdot)(E_1(\omega)) \subseteq E_1(\theta(t, \omega)) \quad \omega \in \Omega^*, \quad t \geq 0, \quad i \geq 1$.

Proof:

We shall appeal to Ruelle's infinite-dimensional discrete version of Oseledec's multiplicative ergodic theorem ([35], §2, pp. 252-255).

Suppose the delay $r > 0$. Set

$$T(\omega) := X(r, \omega, \cdot) \in L(M_2), \quad \text{a.a. } \omega \in \Omega.$$

By Theorem 1, §2, $T(\omega): M_2 \rightarrow M_2$ is compact linear for all $\omega \in \Omega_0$ where Ω_0 has full P -measure.

Let $\tau: \Omega \rightarrow \Omega$ be the shift

$$\tau = \theta(r, \cdot).$$

From the cocycle property (Theorem 3, §3) it follows that

$$X(kr, \omega, \cdot) = T(\tau^{k-1}(\omega)) \circ T(\tau^{k-2}(\omega)) \circ \dots \circ T(\tau(\omega)) \circ T(\omega)$$

for every integer $k \geq 1$ and all ω in a set $\hat{\Omega}$ of full P -measure.

In view of Lemma 4, $\int_{\hat{\Omega}} \log^+ \|T(\omega)\|_{L(M_2)} dP(\omega) < \infty$. Since $T: \Omega \rightarrow L(M_2)$ is weakly (and hence strongly) measurable, then Corollary 2.2 (p. 253) of Ruelle [35] applies: there is a set $\hat{\Omega}_0 \in \mathcal{F}$ such that $P(\hat{\Omega}_0) = 1$, $\theta(r, \cdot)(\hat{\Omega}_0) \subseteq \hat{\Omega}_0$, a random family of compact linear maps $\{\Lambda(\omega): \omega \in \hat{\Omega}_0\} \subset L(M_2)$ with the property

$$\lim_{k \rightarrow \infty} [X(kr, \omega, \cdot)^* \circ X(kr, \omega, \cdot)]^{1/2k} = \Lambda(\omega)$$

for all $\omega \in \hat{\Omega}_0$, in the uniform operator norm topology on $L(M_2)$. Note that

$X(kr, \omega, \cdot)^*$ is the Hilbert space adjoint of $X(kr, \omega, \cdot)$. So $\Lambda(\omega)$ is a compact positive self-adjoint operator on M_2 with a discrete (countable) spectrum consisting of positive eigenvalues (with 0 as a possible accumulation point) given by

$$e^{\mu_1(\omega)} > e^{\mu_2(\omega)} > \dots > e^{\mu_i(\omega)} > e^{\mu_{i+1}(\omega)} > \dots$$

where $\{\mu_i(\omega)\}_{i=1}^{\infty}$ decreases to $-\infty$ as $i \rightarrow \infty$, or otherwise terminates at

$\mu_{N(\omega)}(\omega) = -\infty$ for each $\omega \in \hat{\Omega}_0$. Each eigenvalue $e^{\mu_i(\omega)}$ of $\Lambda(\omega)$ has finite multiplicity $m_i(\omega)$ and a corresponding finite-dimensional eigenspace $F_i(\omega)$.

Define the Oseledec filtration of M_2 by

$$E_1(\omega) = M_2$$

$$E_i(\omega) = \left[\bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp,$$

the orthogonal complement of $\bigoplus_{j=1}^{i-1} F_j(\omega)$ in M_2 for $i > 1$ (Ruelle [35], Corollary 2.2). Then

$$M_2 = E_1(\omega) \supset E_2(\omega) \supset \dots \supset E_i(\omega) \supset E_{i+1}(\omega) \dots$$

Also

$$n_i(\omega) := \text{co-dim } E_i(\omega) = \sum_{j=1}^{i-1} m_j(\omega) < \infty, \quad \omega \in \hat{\Omega}_0.$$

But the functions

$$\begin{array}{lll} \hat{\Omega}_0 \rightarrow \mathbb{R}, & \hat{\Omega}_0 \rightarrow \mathbb{Z}^+ & \hat{\Omega}_0 \rightarrow \mathbb{Z}^+ \\ \omega \mapsto \mu_1(\omega) & \omega \mapsto m_1(\omega) & \omega \mapsto N(\omega) \end{array}$$

are invariant under the ergodic measure-preserving transformation $\theta(r, \cdot)$, so, redefining $\hat{\Omega}_0$ if necessary, we may take fixed $\mu_1 \in \mathbb{R}$, $m_1, n_1, N \in \mathbb{Z}^+$ such that

$$\begin{aligned}\mu_1(\omega) &= \mu_1 \\ m_1(\omega) &= m_1 \\ n_1(\omega) &= \sum_{j=1}^{i-1} m_j = n_i\end{aligned}$$

for all $\omega \in \hat{\Omega}_0$, $i > 1$.

Furthermore the Oseledec theorem implies the existence of the limits

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, (v, \eta))\|_{M_2} = \mu_1, \quad (v, \eta) \in E_1(\omega) \setminus E_{i+1}(\omega), \quad \omega \in \hat{\Omega}_0.$$

Using the above limit and Lemma 6(ii) and redefining $\hat{\Omega}_0$ if necessary, one gets

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|_{M_2} = \frac{1}{r} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|X(kr, \omega, (v, \eta))\| = \frac{\mu_1}{r} = \lambda_1, \text{ say,} \quad (49)$$

$i \geq 1$, $\omega \in \hat{\Omega}_0$, $(v, \eta) \in E_1(\omega) \setminus E_{i+1}(\omega)$.

The exponential growth rates $\lambda_1 = \frac{\mu_1}{r}$, $i \geq 1$, of the trajectory field $\{X(t, \omega, (v, \eta)): t \geq 0, (v, \eta) \in M_2\}$ are the Lyapunov exponents of the linear s.f.d.e. (I). This proves statement (vi) of the theorem.

Since the spectrum $\{e^{\mu_1}\}_{i=1}^{\infty}$ of $\Lambda(\omega)$, $\omega \in \hat{\Omega}_0$, is discrete with no accumulation points except possibly 0, then for each $\mu \in \mathbb{R}$ the set $\{e^{\mu_1}: e^{\mu_1} > e^{\mu}\}$ must be finite (or empty). Hence the set $\{\lambda_1: \lambda_1 > \mu\}$ is also finite. Thus (ii) is proved.

The measurability assertion (iii) is also a consequence of Oseledec's theorem (cf. Thieullen [35]).

Define the random field $\lambda: \hat{\Omega}_0 \times M_2 \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$\lambda(\omega, (v, \eta)) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\|, \quad \omega \in \hat{\Omega}_0, \quad (v, \eta) \in M_2. \quad (50)$$

We contend that for a.a. $\omega \in \Omega$,

$$E_1(\omega) = \{(v, \eta) : (v, \eta) \in M_2, \lambda(\omega, (v, \eta)) \leq \lambda_1\}, \quad i \geq 1. \quad (51)$$

To see this, fix ω and let $(v, \eta) \in M_2$ be such that $\lambda(\omega, (v, \eta)) \leq \lambda_1$. Now there is a $j = j(\omega, v, \eta) \geq 1$ such that $(v, \eta) \in E_j(\omega) \setminus E_{j+1}(\omega)$. But by (49), $\lambda(\omega, (v, \eta)) = \lambda_j \leq \lambda_1$. Hence $j \geq i$ and $E_j(\omega) \subseteq E_i(\omega)$. In particular $(v, \eta) \in E_i(\omega)$. Conversely if $(v, \eta) \in E_i(\omega)$, then $(v, \eta) \in E_j(\omega) \setminus E_{j+1}(\omega)$ for some $j \geq i$ with $\lambda(\omega, (v, \eta)) = \lambda_j \leq \lambda_1$. This proves our contention (51).

Note further that if $t' > 0$, then λ is invariant under $(\theta(t', \cdot), X(t', \cdot, \cdot))$ viz.

$$\lambda(\omega, (v, \eta)) = \lambda(\theta(t', \omega), X(t', \omega, (v, \eta))) \quad \text{a.a. } \omega, \quad (v, \eta) \in M_2. \quad (52)$$

This holds because for fixed $t' > 0$, $(v, \eta) \in M_2$ and a.a. $\omega \in \Omega$, the cocycle property implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \theta(t', \omega), X(t', \omega, (v, \eta)))\| &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t+t', \omega, (v, \eta))\| \\ &= \lim_{u \rightarrow \infty} \frac{1}{u-t'} \log \|X(u, \omega, (v, \eta))\| \\ &= \lim_{u \rightarrow \infty} \frac{1}{u} \log \|X(u, \omega, (v, \eta))\| = \lambda(\omega, (v, \eta)). \end{aligned}$$

In particular, if $(v, \eta) \in E_1(\omega)$, then

$$\lambda(\theta(t', \omega), X(t', \omega, (v, \eta))) = \lambda(\omega, (v, \eta)) \leq \lambda_1.$$

Hence by (51), $X(t', \omega, (v, \eta)) \in E_1(\theta(t', \omega))$. This proves assertion (viii).

It remains to prove (vii). Observe that for a.a. $\omega \in \Omega$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \sup_{\|(v, \eta)\| \leq 1} \|X(t, \omega, (v, \eta))\| \\ &= \lim_{t \rightarrow \infty} \sup_{\|(v, \eta)\| \leq 1} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| \\ &\geq \sup_{\|(v, \eta)\| \leq 1} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| = \lambda_1. \end{aligned}$$

To see the existence of the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} < \infty$$

let $z: \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ be the process

$$z(t, \omega) = \log \|X(t, \omega, \cdot)\|_{L(M_2)} \quad t > 0, \omega \in \Omega.$$

It is easy to see from Lemma 4 that

$$\int_{\Omega} \sup_{0 \leq u \leq 1} z^+(u, \omega) dP(\omega) < \infty, \quad \int_{\Omega} \sup_{0 \leq u \leq 1} z^+(1-u, \theta(u, \omega)) dP(\omega) < \infty.$$

The cocycle property also implies that

$$z(t_1 + t_2, \omega) \leq z(t_1, \omega) + z(t_2, \theta(t_1, \omega)) \quad t_1, t_2 \geq 0, \text{ a.a. } \omega \in \Omega.$$

So by Kingman's sub-additive ergodic theorem, there exists $\lambda^* \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, \cdot)\|_{L(M_2)} = \lambda^* < \infty \quad \text{a.a. } \omega \in \Omega.$$

This completes the proof of the theorem. □

Our next result is a corollary of the Oseledec multiplicative ergodic theorem in the hyperbolic case when the Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ does not contain 0. We thus obtain an almost sure exponential dichotomy for the flow X of the linear s.f.d.e. (I) (cf. the deterministic case $g \equiv 0$ in Hale [18]):

Corollary 1:

Assume that the s.f.d.e. (I) is hyperbolic in the sense that the Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ does not contain 0. Then there are random families $\{\mathcal{U}(\omega): \omega \in \Omega^*\}$, $\{\mathcal{J}(\omega): \omega \in \Omega^*\}$ of linear subspaces of M_2 with the following properties:

- (i) the maps $\omega \mapsto \mathcal{U}(\omega)$, $\omega \mapsto \mathcal{J}(\omega)$ are $\mathcal{F}$ -measurable into the Grassmannian of M_2 ;
- (ii) for each $\omega \in \Omega^*$, $\mathcal{U}(\omega)$ and $\mathcal{J}(\omega)$ are mutually orthogonal;
- (iii) $M_2 = \mathcal{U}(\omega) \oplus \mathcal{J}(\omega)$, $\omega \in \Omega^*$;

(vi) each subspace $\mathcal{U}(\omega)$, $\omega \in \Omega^*$, is finite-dimensional with a fixed (non-random) dimension $d \geq 0$; each $\mathcal{S}(\omega)$, $\omega \in \Omega^*$, is closed with finite codimension d .

(v) for $\omega \in \Omega^*$ and $(v, \eta) \in \mathcal{U}(\omega)$, there exist $t_1 = t_1(\omega, v, \eta) > 0$ and a positive number δ_1 , independent of (ω, v, η) , such that

$$\|X(t, \omega, (v, \eta))\|_{M_2} \geq \|(v, \eta)\|_{M_2} e^{\delta_1 t} \quad (53)$$

for all $t \geq t_1$;

(vi) for $\omega \in \Omega^*$ and $(v, \eta) \in \mathcal{S}(\omega)$, there exist $t_2 = t_2(\omega, v, \eta)$ and a positive δ_2 , independent of (ω, v, η) , such that

$$\|X(t, \omega, (v, \eta))\|_{M_2} \leq \|(v, \eta)\|_{M_2} e^{-\delta_2 t} \quad (54)$$

for all $t \geq t_2$.

Proof:

We use the notation of the previous theorem and its proof. Fix $\omega \in \Omega^*$ throughout this proof.

Assuming hyperbolicity, there exists $i_1 \geq 1$ such that $\lambda_1 > \lambda_2 > \dots > \lambda_{i_1} > 0 > \lambda_{i_1+1} > \lambda_{i_1+2} > \dots$. Take $\mathcal{U}(\omega)$ to be the finite-dimensional subspace

$$\mathcal{U}(\omega) = \bigoplus_{j=1}^{i_1} F_j(\omega) = [E_{i_1+1}(\omega)]^\perp$$

with fixed dimension $d = \sum_{j=1}^{i_1} m_j \geq 0$. Also set

$$\mathcal{S}(\omega) = E_{i_1+1}(\omega) = [\mathcal{U}(\omega)]^\perp.$$

Clearly

$$M_2 = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega).$$

By (51),

$$\mathcal{S}(\omega) = \{(v, \eta) : (v, \eta) \in M_2, \lambda(\omega, (v, \eta)) \leq \lambda_{i_1+1}\}.$$

In particular if $(v, \eta) \in \mathfrak{A}(\omega)$ is such that $\|(v, \eta)\| = 1$, then

$$\lambda(\omega, (v, \eta)) \leq \lambda_{i_1+1} < 0$$

i.e. $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| \leq \lambda_{i_1+1}$. Pick $\epsilon > 0$ such that $\lambda_{i_1+1} + \epsilon < 0$ and set $\delta_2 = -(\lambda_{i_1+1} + \epsilon) > 0$. Then there exists $t_2 = t_2(\omega, (v, \eta), \epsilon) > 0$ such that

$$\|X(t, \omega, (v, \eta))\| < e^{-\delta_2 t}$$

for all $t \geq t_2$. Replacing any $(v, \eta) \neq 0$ by $\frac{(v, \eta)}{\|(v, \eta)\|}$ and using the linearity of $X(t, \omega, \cdot)$ gives assertion (vi) of the corollary.

In order to prove (v) we claim first that

$$\mathcal{U}(\omega) \subseteq \{(v, \eta) : (v, \eta) \in M_2, \lambda_{i_1} \leq \lambda(\omega, (v, \eta)) \leq \lambda_1\} \cup \{0\}.$$

To see this, let $0 \neq (v, \eta) \in \mathcal{U}(\omega)$. Then $(v, \eta) \in [E_{i_1+1}(\omega)]^\perp$ because $\Lambda(\omega)$ is self-adjoint. Suppose, if possible, that $\lambda(\omega, (v, \eta)) < \lambda_{i_1}$. Therefore

$\lambda(\omega, (v, \eta)) \leq \lambda_{i_1+1}$ by discreteness of the spectrum. So $(v, \eta) \in E_{i_1+1}(\omega)$ and in fact $(v, \eta) \in E_{i_1+1}(\omega) \cap [E_{i_1+1}(\omega)]^\perp = \{0\}$. Therefore $(v, \eta) = 0$ which is a contradiction. Thus we must have $\lambda(\omega, (v, \eta)) \geq \lambda_{i_1}$, and our claim is proved.

This says that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, \omega, (v, \eta))\| \geq \lambda_{i_1} > 0. \quad (5.5)$$

Now pick $\epsilon > 0$ such that $\delta_1 := \lambda_{i_1} - \epsilon > 0$ and use (5.5) together with the linearity of $X(t, \omega, \cdot)$ as before in order to prove assertion (5.3) of the corollary. This concludes the proof of Corollary 1. □

Remarks:

(i) As to be expected, almost sure asymptotic exponential stability of all trajectories to s.f.d.e. (I) is guaranteed if the top Lyapunov exponent $\lambda_1 < 0$. In this case $\mathbb{M}(\omega) = M_2$, $\omega \in \Omega^*$ and (54) holds for all $(v, \eta) \in M_2$. This classically important situation will be illustrated by a specific case study in a forthcoming article.

(ii) Since the almost sure growth rates

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log || (v, \eta)_{x(t, \cdot), (v, \eta)_{x_t(\cdot, \cdot)}} ||_{M_2}$$

for the s.f.d.e. (I) (§2) are independent of the choice of the regular version X in Theorem 1 (§2), it is clear that the Lyapunov spectrum $\{\lambda_i\}_{i=1}^\infty$, the Oseledec filtration $\{E_i(\omega)\}_{i=1}^\infty$, and the random families $U(\omega)$, $S(\omega)$ are also independent of X .

(iii) With more work, it is possible to show that the approximation property (17) and the cocycle property (31) (§3) both hold for all ω in a universal set $\hat{\Omega}$ of full Wiener measure independent of T, B , $t_1, t_2 \geq 0$ which is shift-invariant viz.

$\theta(t, \cdot)(\hat{\Omega}) \subseteq \hat{\Omega}$, $t \geq 0$. E.g. take

$\hat{\Omega} = \{\omega: \omega \in \Omega, \lim_{k \rightarrow \infty} \phi^k(t, \omega, \cdot) = \phi(t, \omega, \cdot) \in L(\mathbb{R}^n) \text{ uniformly for } t \text{ in compact subsets of } \mathbb{R}^+, \phi(t, \omega, \cdot) \in GL(\mathbb{R}^n) \forall t \geq 0,$

$\phi(t_2, \theta(t_1, \omega), \cdot) \circ \phi(t_1, \omega, \cdot) = \phi(t_1 + t_2, \omega, \cdot) \forall t_1, t_2 \geq 0\}$.

(iv) By essentially the same methods, all the theorems in this article also hold for linear s.f.d.e.'s with several discrete delays in the drift viz.

$$dx(t) = H(x(t-d_1), x(t-d_2), \dots, x(t-d_N), x(t), x_t) dt + g(x(t)) dw(t), \quad t > 0$$

where $0 < d_1 < d_2 < d_3 < \dots < d_N \leq r$, $H: (\mathbb{R}^n)^N \times M_2 \longrightarrow \mathbb{R}^n$, continuous linear.

The exponential dichotomy in Corollary 1 has one snag:

The splitting

$$M_2 = U(\omega) \oplus S(\omega) \quad \text{a.a. } \omega \in \Omega$$

is not completely invariant under the cocycle (X, θ) . Indeed while the stable manifold $S(\omega)$ is always invariant

$$X(t, \omega, \cdot)(S(\omega)) \subseteq S(\theta(t, \omega)) \quad t \geq 0 \quad \text{a.a. } \omega \in \Omega$$

the unstable subspaces $U(\omega)$ are not generally invariant when X is not self-adjoint. Note, however, that

$$X^*(t, \omega, \cdot)(U(\theta(t, \omega))) \subseteq U(\omega) \quad t \geq 0 \quad \text{a.a. } \omega \in \Omega.$$

Therefore, heuristically speaking, if the shift $\theta(t, \cdot)$ were invertible with inverse $\theta(t, \cdot)^{-1}$ one would have

$$X^*(t, \theta(t, \cdot)^{-1}(\omega), \cdot)(U(\omega)) \subseteq U(\theta(t, \cdot)^{-1}(\omega)).$$

This suggests that we should look at the adjoint stochastic flow

$X^*(t, \theta(t, \cdot)^{-1}(\omega), \cdot)$ for a suitably defined "inverse" of the shift θ . In order to do this it seems necessary that we should enlarge the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ to a new space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{-\infty < t < \infty}, \tilde{P})$ so that the above construction makes sense. Luckily this can be done without loss of generality and in a very natural manner as follows:

Let $\tilde{\Omega}$ be the Wiener space of all continuous paths $\tilde{\omega}: \mathbb{R} \rightarrow \mathbb{R}^m$, $\tilde{\omega}(0) = 0$, $\tilde{\mathcal{F}}$ its Borel field, $\tilde{P}$ Wiener measure on $\tilde{\Omega}$. For each $-\infty < t < \infty$ denote by $\tilde{\mathcal{F}}_t$ the completion of the σ -algebra generated by all evaluations

$$\begin{aligned} \tilde{\rho}_u: \tilde{\Omega} &\longrightarrow \mathbb{R} \quad , \quad u \leq t \\ \tilde{\omega} &\longmapsto \tilde{\omega}(u) \quad . \end{aligned}$$

Brownian motion $\tilde{W}: \mathbb{R} \times \tilde{\Omega} \rightarrow \mathbb{R}^m$ is defined by

$$\tilde{W}(t, \omega) := \tilde{\omega}(t) \quad t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega}$$

over the whole line. For $t \in \mathbb{R}$ define the shift $\tilde{\theta}(t, \cdot): \tilde{\Omega} \rightarrow \tilde{\Omega}$ by

$$\tilde{\theta}(t, \tilde{\omega})(u) = \tilde{\omega}(t+u) - \tilde{\omega}(t) \quad u, t \in \mathbb{R}, \tilde{\omega} \in \tilde{\Omega} .$$

It is easy to check that $\{\tilde{\theta}(t, \cdot) : t \in \mathbb{R}\}$ is a one-parameter group of $\tilde{P}$ -preserving transformations on $\tilde{\Omega}$ (cf. Elworthy [16] pp. 18-19). The stochastic flows $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $X : \mathbb{R}^+ \times \Omega \times M_2 \longrightarrow M_2$ may be extended to stochastic flows $\tilde{\varphi} : \mathbb{R}^+ \times \tilde{\Omega} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\tilde{X} : \mathbb{R}^+ \times \tilde{\Omega} \times M_2 \longrightarrow M_2$ in two equivalent ways:

either by taking these to satisfy (i) of Lemma 7 below as a defining measure-theoretic relation or otherwise take them to be the stochastic flows associated with the s.d.e.'s

$$\left. \begin{aligned} d\tilde{\varphi}(t) &= g(\tilde{\varphi}(t))d\tilde{W}(t) & t > 0 \\ \tilde{\varphi}(0) &= v \in \mathbb{R}^n \end{aligned} \right\} \quad (\text{III})$$

$$\left. \begin{aligned} d\tilde{x}(t) &= H(\tilde{x}(t), \tilde{x}_t)dt + g(\tilde{x}(t))d\tilde{W}(t) & t > 0 \\ \tilde{x}(0) &= v \in \mathbb{R}^n, \quad \tilde{x}_0 = \eta \in L^2(J, \mathbb{R}^n) \end{aligned} \right\} \quad (\text{I})$$

following the constructions in §2, §3. Define $\pi : \tilde{\Omega} \longrightarrow \Omega$ to be the restriction map

$$\pi(\tilde{\omega}) := \tilde{\omega}|_{\mathbb{R}^+}.$$

It is clear that $\tilde{P} \cdot \pi^{-1} = P$ (Elworthy [16] pp. 18-19).

Thus we have

Lemma 7:

(i) If $\tilde{\varphi}, \tilde{X}$ are defined through (III), (I), then

$$\tilde{\varphi}(t, \tilde{\omega}, \cdot) = \varphi(t, \pi(\tilde{\omega}), \cdot)$$

$$\tilde{X}(t, \tilde{\omega}, \cdot) = X(t, \pi(\tilde{\omega}), \cdot)$$

for $\tilde{P}$ -a.a. $\tilde{\omega} \in \tilde{\Omega}$ and all $t \in \mathbb{R}^+$.

(ii) $(\tilde{\varphi}(t, \cdot, \cdot), \tilde{\theta}(t, \cdot), (\tilde{X}(t, \cdot, \cdot), \tilde{\theta}(t, \cdot)))$, $t \in \mathbb{R}^+$ are cocycles viz. there is a universal set $\hat{\tilde{\Omega}}$ of full P -measure in $\tilde{\Omega}$ such that

$$\tilde{\varphi}(t_2, \tilde{\theta}(t_1, \tilde{\omega}), \cdot) \circ \varphi(t_1, \tilde{\omega}, \cdot) = \tilde{\varphi}(t_1 + t_2, \tilde{\omega}, \cdot)$$

$$\tilde{X}(t_2, \tilde{\theta}(t_1, \tilde{\omega}), \cdot) \circ \tilde{X}(t_1, \tilde{\omega}, \cdot) = \tilde{X}(t_1 + t_2, \tilde{\omega}, \cdot)$$

for all $\tilde{\omega} \in \hat{\tilde{\Omega}}$, $t_1, t_2 \geq 0$. Also $\tilde{\theta}(t, \cdot)(\hat{\tilde{\Omega}}) \subseteq \hat{\tilde{\Omega}}$ for all $t \geq 0$.

(iii) For each $t \geq 0$ and $\tilde{\omega} \in \tilde{\Omega}$,

$$\pi\{\tilde{\theta}(t, \tilde{\omega})\} = \theta(t, \pi(\tilde{\omega}))$$

(iv) For fixed $r_1, r_2 > 0$, we have

$$\int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r_1 \\ 0 \leq t_2 \leq r_2}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(t_2, \tilde{\omega}), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}) < \infty \quad (56)$$

and

$$\int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r_1 \\ 0 \leq t_2 \leq r_2}} \|\tilde{\varphi}(t_1, \tilde{\theta}(t_2, \tilde{\omega}), \cdot)\|_{L(\mathbb{R}^n)}^{2k} d\tilde{P}(\tilde{\omega}) < \infty, \quad k = 1, 2, 3, \dots \quad (57)$$

(v) For $-\infty < t_1 < t_2 < \infty$, the random field

$$\xi_{t_1, t_2}(\tilde{\omega}, v) := \tilde{\varphi}(t_2 - t_1, \tilde{\theta}(t_1, \tilde{\omega}), v) \quad \tilde{\omega} \in \tilde{\Omega}, \quad v \in \mathbb{R}^n$$

is the unique stochastic flow of the stochastic integral equation

$$\xi_{t_1, t_2} = v + \int_{t_1}^t g\{\xi_{t_1, u}\} d\tilde{W}(u)$$

(vi) Let $P_1 : M_2 \longrightarrow \mathbb{R}^n$ be the projection of $M_2 = \mathbb{R}^n \times \mathbb{L}^2(J, \mathbb{R}^n)$ onto its first factor. For $-\infty < t_1 < t < \infty$, the process

$$t_1 \tilde{x}(t, \omega, v, \eta) := \begin{cases} P_1[\tilde{X}(t - t_1, \tilde{\theta}(t_1, \tilde{\omega}), v, \eta)] & t \geq t_1 \\ \eta(t - t_1) & t_1 - r < t < t_1 \end{cases}$$

is the unique solution of the equation

$$\tilde{x}(t) = \begin{cases} v + \int_{t_1}^t H(\tilde{x}(u), \tilde{x}_u) du + \int_{t_1}^t g(\tilde{x}(u)) d\tilde{W}(u) & t \geq t_1 \\ \eta(t - t_1) & t_1 - r < t < t_1, \quad (v, \eta) \in M_2 \end{cases}$$

Proof:

We first prove assertion (iii). Let $t, u \geq 0$. Then

$$[\pi\{\tilde{\theta}(t, \tilde{\omega})\}](u) = \tilde{\omega}(t+u) - \tilde{\omega}(u) = \pi(\tilde{\omega})(t+u) - \pi(\tilde{\omega})(u) = \theta(t, \pi(\tilde{\omega}))(u).$$

To prove (i) note that $\tilde{W}(t, \tilde{\omega}) = W(t, \pi(\tilde{\omega}))$ for $t > 0$, $\tilde{\omega} \in \tilde{\Omega}$. Therefore it follows easily from (III) that the process $t \mapsto \varphi(t, \pi(\tilde{\omega}), \cdot)$ satisfies the s.o.d.e. (III). Hence by uniqueness we must have $\tilde{\varphi}(t, \tilde{\omega}, \cdot) = \varphi(t, \pi(\tilde{\omega}), \cdot)$ $t \geq 0$, a.a. $\tilde{\omega} \in \tilde{\Omega}$. The corresponding relation for $\tilde{X}$ follows similarly.

To check (ii) let $\hat{\Omega}$ be a $\theta(t, \cdot)$ -invariant universal set of full P -measure in Ω with the property

$$\varphi(t_2, \theta(t_1, \omega), \cdot) \circ \varphi(t_1, \omega, \cdot) = \varphi(t_1 + t_2, \omega, \cdot)$$

$$X(t_2, \theta(t_1, \omega), \cdot) \circ X(t_1, \omega, \cdot) = X(t_1 + t_2, \omega, \cdot)$$

for all $\omega \in \hat{\Omega}$, $t_1, t_2 \geq 0$. Now $\tilde{P}[\pi^{-1}(\hat{\Omega})] = P(\hat{\Omega}) = 1$. So pick $\tilde{\omega} \in \tilde{\hat{\Omega}} := \pi^{-1}(\hat{\Omega})$ and observe that

$$\begin{aligned} \tilde{X}(t_2, \tilde{\theta}(t_1, \tilde{\omega}), \cdot) \circ \tilde{X}(t_1, \tilde{\omega}, \cdot) &= X(t_2, \pi[\tilde{\theta}(t_1, \tilde{\omega})], \cdot) \circ X(t_1, \pi(\tilde{\omega}), \cdot) \\ &= X(t_2, \theta(t_1, \pi(\tilde{\omega})), \cdot) \circ X(t_1, \pi(\tilde{\omega}), \cdot) \\ &= X(t_1 + t_2, \pi(\tilde{\omega}), \cdot) = \tilde{X}(t_1 + t_2, \tilde{\omega}, \cdot), \quad t_1, t_2 \geq 0. \end{aligned}$$

Similarly for $\tilde{\varphi}$. Note also that if $t \geq 0$ and $\tilde{\omega} \in \tilde{\hat{\Omega}}$, then $\pi(\tilde{\omega}) \in \hat{\Omega}$ and $\pi\{\tilde{\theta}(t, \tilde{\omega})\} = \theta(t, \pi(\tilde{\omega})) \in \hat{\Omega}$. Hence $\tilde{\theta}(t, \tilde{\omega}) \in \pi^{-1}(\hat{\Omega}) = \tilde{\hat{\Omega}}$ i.e. $\tilde{\theta}(t, \cdot)(\tilde{\hat{\Omega}}) \subseteq \tilde{\hat{\Omega}}$, $t \geq 0$.

To establish the first estimate in (iv) observe that (i) and lemma (4) give

$$\begin{aligned} & \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r_1 \\ 0 \leq t_2 \leq r_2}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(t_2, \tilde{\omega}), \cdot)\|_{L(M_2)} d\tilde{P}(\omega) \\ &= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r_1 \\ 0 \leq t_2 \leq r_2}} \log^+ \|X(t_1, \theta(t_2, \pi(\tilde{\omega})), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}) \\ &= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r_1 \\ 0 \leq t_2 \leq r_2}} \log^+ \|X(t_1, \theta(t_2, \omega), \cdot)\|_{L(M_2)} dP(\omega) < \infty \end{aligned}$$

because $\tilde{P} \circ \pi^{-1} = P$. The corresponding integrability property for $\tilde{\varphi}$ is similarly proved, using Lemma 2.

The proofs of assertions (v), (vi) follow routinely from (i) so we shall omit them.

This concludes the proof of the lemma. □

The exponential dichotomy in Corollary 1 can now be refined to give a fully-fledged stable-manifold theorem with a cocycle-invariant splitting. The method of proof we use is an application of discrete-time techniques developed by Ruelle in [34], [35].

Corollary 2: (The Stable-Manifold Theorem).

Suppose the s.f.d.e. (I) is hyperbolic. Then there exist random families $\{\hat{U}(\tilde{\omega}), \tilde{S}(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}^*\}$, of linear subspaces of M_2 with the following properties:

(i) The maps $\tilde{\omega} \mapsto \hat{U}(\tilde{\omega}), \tilde{\omega} \mapsto \tilde{S}(\tilde{\omega})$ are $\tilde{\mathcal{F}}$ -measurable into the Grassmannian of M_2 .

(ii) $\tilde{\Omega}^*$ has full Wiener measure in $\tilde{\Omega}$.

(iii) $M_2 = \hat{U}(\tilde{\omega}) \oplus \tilde{S}(\tilde{\omega}), \tilde{\omega} \in \tilde{\Omega}^*$.

(iv) For each $t \geq 0$,

$$\tilde{X}(t, \tilde{\omega}, \cdot)(\tilde{S}(\tilde{\omega})) \subseteq \tilde{S}(\tilde{\theta}(t, \tilde{\omega})) \tag{58}$$

$$\tilde{X}(t, \tilde{\omega}, \cdot)(\hat{U}(\tilde{\omega})) = \hat{U}(\tilde{\theta}(t, \tilde{\omega})) \tag{59}$$

for a.a. $\tilde{\omega} \in \tilde{\Omega}^*$.

(v) Every subspace $\hat{U}(\tilde{\omega}), \tilde{\omega} \in \tilde{\Omega}^*$, is finite-dimensional with a fixed (non-random) dimension $d \geq 0$. Each $\tilde{S}(\tilde{\omega}), \tilde{\omega} \in \tilde{\Omega}^*$ is closed with finite codimension d .

(vi) For a.a. $\tilde{\omega} \in \tilde{\Omega}$ and $t \geq 0$, the restriction

$$\tilde{X}(t, \tilde{\omega}, \cdot)|_{\hat{U}(\tilde{\omega})} : \hat{U}(\tilde{\omega}) \longrightarrow \hat{U}(\tilde{\theta}(t, \tilde{\omega}))$$

is a linear homeomorphism onto.

(vii) For $\tilde{\omega} \in \tilde{\Omega}^*$ and $(v, \eta) \in \hat{U}(\tilde{\omega})$, there exist $t_1 = t_1(\tilde{\omega}, v, \eta) > 0$ and a positive number δ_1 , independent of $(\tilde{\omega}, v, \eta)$ such that

$$||\tilde{X}(t, \tilde{\omega}, v, \eta)||_{M_2} \geq ||(v, \eta)||_{M_2} e^{\delta_1 t} \quad (60)$$

for all $t \geq t_1$.

(viii) For $\tilde{\omega} \in \tilde{\Omega}^*$ and $(v, \eta) \in \tilde{S}(\tilde{\omega})$ there exist $t_2 = t_2(\omega, v, \eta) > 0$ and a positive δ_2 , independent of $(\tilde{\omega}, v, \eta)$ such that

$$||\tilde{X}(t, \tilde{\omega}, v, \eta)||_{M_2} \leq ||(v, \eta)||_{M_3} e^{-\delta_2 t} \quad (61)$$

for all $t \geq t_2$.

Remark:

In the absence of diffusion ($g \equiv 0$) the above results specialize to well-known classical results of J. Hale ([18]) obtained via semi-group techniques.

Proof:

From Lemma 7 (iv) we know that the extended cocycle $\{(\tilde{X}(t, \cdot, \cdot), \tilde{\theta}(t, \cdot)) : t \geq 0\}$ satisfies the integrability condition of Lemma 4. Furthermore for $t \geq r$ and a.a. $\tilde{\omega} \in \tilde{\Omega}$ the maps $\tilde{X}(t, \tilde{\omega}, \cdot) : M_2 \longrightarrow M_2$ are compact. Hence, as in the proof of Theorem 1, the Oseledec ergodic theorem yields compact positive self-adjoint operators $\tilde{\Lambda}(\tilde{\omega}) : M_2 \longrightarrow M_2$, a.a. $\tilde{\omega} \in \tilde{\Omega}$ such that

$$\lim_{k \rightarrow \infty} [\tilde{X}(kr, \tilde{\omega}, \cdot)^* \circ \tilde{X}(kr, \tilde{\omega}, \cdot)]^{\frac{1}{2k}} = \tilde{\Lambda}(\tilde{\omega}) \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}$$

in the uniform operator norm. Now

$$\begin{aligned} [\tilde{X}(kr, \tilde{\omega}, \cdot)^* \circ \tilde{X}(kr, \tilde{\omega}, \cdot)]^{\frac{1}{2k}} &= [X(kr, \pi(\tilde{\omega}), \cdot)^* \circ X(kr, \pi(\tilde{\omega}), \cdot)]^{\frac{1}{2k}} \\ &\longrightarrow \Lambda(\pi(\tilde{\omega})) \end{aligned}$$

in $L(M_2)$, where $\Lambda(\omega)$ is defined in the proof of Theorem 1. So by uniqueness of limits in $L(M_2)$ we must have

$$\tilde{\Lambda}(\tilde{\omega}) = \Lambda(\pi(\tilde{\omega})) \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

Also

$$\sigma(\tilde{\Lambda}(\tilde{\omega})) = \sigma(\Lambda(\pi(\tilde{\omega}))) = \{e^{\mu_i} : i \geq 1\} \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}$$

where $\mu_i = r\lambda_i$ and $\{\lambda_i\}_{i=1}^{\infty}$ is the Lyapunov spectrum of the s.f.d.e. (I).

By hyperbolicity, let $i_1 > 0$ be such that

$$\lambda_1 > \lambda_2 > \dots > \lambda_{i_1} > 0 > \lambda_{i_1+1} > \dots$$

For $j \geq 1$ let $\tilde{F}_j(\tilde{\omega})$ be the eigenspace of $\tilde{\Lambda}(\tilde{\omega})$ corresponding to e^{μ_j} . Then $\tilde{F}_j(\tilde{\omega}) = F_j(\pi(\tilde{\omega}))$ a.a. $\tilde{\omega}$ and $\dim \tilde{F}_j(\tilde{\omega}) = \dim F_j(\pi(\tilde{\omega})) =$ multiplicity of λ_j , a fixed integer for a.a. $\tilde{\omega}$.

Take

$$\tilde{S}(\tilde{\omega}) := \left[\begin{array}{c} i_1 \\ \oplus \\ \tilde{F}_j(\tilde{\omega}) \end{array} \right]_{j=1}^{\perp} = E_{i_1+1}(\pi(\tilde{\omega})) \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

These are our candidates for the stable manifolds. They satisfy the invariance property

$$\tilde{X}(t, \tilde{\omega}, \cdot)(\tilde{S}(\tilde{\omega})) \subseteq \tilde{S}(\tilde{\theta}(t, \tilde{\omega})) \quad t \geq 0, \text{ a.a. } \tilde{\omega} \in \tilde{\Omega}$$

because of Theorem 1 (viii). The exponential estimate (viii) is proved in exactly the same way as in (vi) of Corollary 1. The map $\tilde{\omega} \mapsto \tilde{S}(\tilde{\omega})$ is measurable into the Grassmannian of M_2 because of Theorem 1 (iii).

In order to construct the unstable manifolds $\hat{U}(\omega)$ for a.a. $\tilde{\omega} \in \tilde{\Omega}$ we first consider the family of operators

$$\hat{X}(t, \tilde{\omega}, \cdot) := \tilde{X}^*(t, \tilde{\theta}(-t, \tilde{\omega}), \cdot) \quad t \geq 0, \tilde{\omega} \in \tilde{\Omega}.$$

Since $\tilde{\theta}(-t, \cdot)$ is $\tilde{P}$ -preserving, it is clear from the compactness of $\tilde{X}(t, \tilde{\omega}, \cdot)$, $t \geq r$ a.a. $\tilde{\omega} \in \tilde{\Omega}$ that for a.a. $\tilde{\omega} \in \tilde{\Omega}$ and all $t \geq r$ each map $\hat{X}(t, \tilde{\omega}, \cdot)$ is compact linear. Furthermore note that $\{(\hat{X}(t, \cdot, \cdot), \tilde{\theta}(-t, \cdot)) : t \geq 0\}$ is a cocycle; indeed from the corresponding property for $\tilde{X}$ one gets for $t_1, t_2 \geq 0$ and a.a. $\tilde{\omega}$ the following

$$\begin{aligned}
& \hat{X}(t_2, \tilde{\theta}(-t_1, \tilde{\omega}), \cdot) \circ \hat{X}(t_1, \tilde{\omega}, \cdot) \\
&= \tilde{X}^*(t_2, \tilde{\theta}(-t_2, \tilde{\theta}(-t_1, \tilde{\omega})), \cdot) \circ \tilde{X}^*(t_1, \tilde{\theta}(-t_1, \tilde{\omega}), \cdot) \\
&= [\tilde{X}(t_1, \tilde{\theta}(-t_1, \tilde{\omega}), \cdot) \circ \tilde{X}(t_2, \tilde{\theta}(-t_1-t_2, \tilde{\omega}), \cdot)]^* \\
&= [\tilde{X}(t_1, \tilde{\theta}(t_2, \tilde{\omega}'), \cdot) \circ \tilde{X}(t_2, \tilde{\omega}', \cdot)]^*, \quad \tilde{\omega}' = \tilde{\theta}(-t_1-t_2, \tilde{\omega}) \\
&= \tilde{X}^*(t_1+t_2, \tilde{\theta}(-t_1-t_2, \tilde{\omega}), \cdot) \\
&= \hat{X}(t_1+t_2, \tilde{\omega}, \cdot).
\end{aligned}$$

We next verify that the adjoint stochastic flow $\hat{X}$ satisfies the integrability condition

$$\int_{\tilde{\Omega}} \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|\hat{X}(t_1, \tilde{\theta}(-t_2, \tilde{\omega}), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}) < \infty \quad (62)$$

needed to apply Theorem 1 (cf. Lemma 4). To see this one notes that the above integral is equal to

$$\begin{aligned}
& \int_{\tilde{\Omega}} \sup_{0 \leq t_1, t_2 \leq r} \log^+ \|\tilde{X}^*(t_1, \tilde{\theta}(-(t_1+t_2), \tilde{\omega}), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r \\ 0 \leq t_2 \leq 2r}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(-t_2, \tilde{\omega}), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}) \\
&= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r \\ 0 \leq t_2 \leq 2r}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(-t_2, \tilde{\theta}(2r, \tilde{\omega}')), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}') \quad , \quad \tilde{\omega} = \tilde{\theta}(2r, \tilde{\omega}') \\
&= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r \\ 0 \leq t_2 \leq 2r}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(2r-t_2, \tilde{\omega}'), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}') \\
&= \int_{\tilde{\Omega}} \sup_{\substack{0 \leq t_1 \leq r \\ 0 \leq t_2' \leq 2r}} \log^+ \|\tilde{X}(t_1, \tilde{\theta}(t_2', \tilde{\omega}'), \cdot)\|_{L(M_2)} d\tilde{P}(\tilde{\omega}'), \quad t_2' = 2r-t_2,
\end{aligned}$$

which is finite by (56) of Lemma 7. Therefore by Theorem 1 there is a random family of compact positive self-adjoint operators $\hat{\Lambda}(\tilde{\omega})$, a.a. $\tilde{\omega} \in \tilde{\Omega}$ such that

$$\lim_{k \rightarrow \infty} [\hat{X}(kr, \tilde{\omega}, \cdot)^* \circ \hat{X}(kr, \tilde{\omega}, \cdot)]^{\frac{1}{2k}} = \hat{\Lambda}(\tilde{\omega}) \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

From the discussion in Ruelle [35] (§3.5, pp. 261-262) we know that

$$\sigma(\hat{\Lambda}(\tilde{\omega})) = \sigma(\tilde{\Lambda}(\tilde{\omega})) = \{e^{r\lambda_i} : i \geq 1\}$$

for a.a. $\tilde{\omega} \in \tilde{\Omega}$.

For each $j \geq 1$ let $\hat{F}_j(\tilde{\omega})$ be the eigenspace of $\hat{\Lambda}(\tilde{\omega})$ corresponding to the eigenvalue $e^{r\lambda_j}$. With $i_1 > 0$ defined before, set

$$\hat{U}(\tilde{\omega}) := \bigoplus_{j=1}^{i_1} \hat{F}_j(\tilde{\omega}) \quad \text{a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

Then as in [35] (§3.5 pp. 261-262) it follows that

$$M_2 = \mathfrak{F}(\tilde{\omega}) \oplus \hat{U}(\tilde{\omega}) \quad ,$$

$$\mathfrak{F}(\tilde{\omega}) \cap \hat{U}(\tilde{\omega}) = \{0\} \quad ,$$

and $\dim \hat{F}_j(\tilde{\omega}) = \dim \tilde{F}_j(\tilde{\omega})$, $j \geq 1$, a.a. $\tilde{\omega} \in \tilde{\Omega}$. In particular

$$\dim \hat{U}(\tilde{\omega}) = \dim U(\pi(\tilde{\omega})) = d, \text{ fixed for a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

To prove the invariance property (59) for $\hat{U}(\tilde{\omega})$ observe that the conclusion (viii) of Theorem 1 gives the following invariance relation for the adjoint stochastic flow

$$\hat{X}(t, \tilde{\omega}, \cdot) (\hat{U}(\tilde{\omega})^\perp) \subseteq [\hat{U}(\tilde{\theta}(-t, \tilde{\omega}))]^\perp \quad t \geq 0,$$

a.a. $\tilde{\omega} \in \tilde{\Omega}$. Taking adjoints yields

$$\hat{X}^*(t, \tilde{\omega}, \cdot) (\hat{U}(\tilde{\theta}(-t, \tilde{\omega}))) \subseteq \hat{U}(\tilde{\omega})$$

i.e.

$$\tilde{X}(t, \tilde{\theta}(-t, \tilde{\omega}), \cdot) (\hat{U}(\tilde{\theta}(-t, \tilde{\omega}))) \subseteq \hat{U}(\tilde{\omega}), \quad t \geq 0, \text{ a.a. } \tilde{\omega} \in \tilde{\Omega}.$$

In the above relation put $\tilde{\omega}' = \tilde{\theta}(-t, \tilde{\omega})$ to obtain

$$\tilde{X}(t, \tilde{\omega}', \cdot) (\hat{U}(\tilde{\omega}')) \subseteq \hat{U}(\tilde{\theta}(t, \tilde{\omega}')) \quad t \geq 0, \text{ a.a. } \tilde{\omega}' \in \tilde{\Omega}$$

which is the desired property (59).

We now prove the estimate (60). Fix $\tilde{\omega}$ in a set $\tilde{\Omega}^*$ of full $\tilde{P}$ -measure.

Take $0 \neq (v, \eta) \in \hat{U}(\tilde{\omega})$ and suppose if possible that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log ||\tilde{X}(t, \tilde{\omega}, v, \eta)||_{M_2} \leq \lambda_{i_1+1}.$$

This says that $(v, \eta) \in \tilde{E}_{i_1+1}(\tilde{\omega}) = \tilde{S}(\tilde{\omega})$. Therefore $(v, \eta) \in \tilde{S}(\tilde{\omega}) \cap \hat{U}(\tilde{\omega}) = \{0\}$.

Hence we must have $(v, \eta) = 0$, a contradiction. So by discreteness of the Lyapunov spectrum we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log ||\tilde{X}(t, \tilde{\omega}, v, \eta)||_{M_2} \geq \lambda_{i_1}$$

for all $(v, \eta) \in \hat{U}(\tilde{\omega})$. This easily implies the exponential estimate in (60).

To prove assertion (vi) of the corollary, fix $t \geq 0$ and $\tilde{\omega} \in \tilde{\Omega}^*$. Suppose there is a $(v, \eta) \in \hat{U}(\tilde{\omega})$ such that $\tilde{X}(t, \tilde{\omega}, v, \eta) = 0$. Then by the cocycle property $\tilde{X}(t', \tilde{\omega}, v, \eta) = 0$ for all $t' \geq t$. Pick $t_1(\tilde{\omega}, v, \eta) > 0$ so that (60) is satisfied. If $t_0 > \max(t, t_1(\tilde{\omega}, v, \eta))$, then

$$0 = ||\tilde{X}(t_0, \tilde{\omega}, v, \eta)|| \geq ||(v, \eta)||_{M_2} e^{\delta_1 t_0}$$

Therefore $(v, \eta) = 0$ and $\tilde{X}(t, \tilde{\omega}, \cdot) | \hat{U}(\tilde{\omega})$ is injective for every $t \geq 0$. Since $\dim \hat{U}(\tilde{\omega}) = \dim \hat{U}(\tilde{\theta}(t, \tilde{\omega})) = d$, the map $\tilde{X}(t, \tilde{\omega}, \cdot) | \hat{U}(\tilde{\omega}) : \hat{U}(\tilde{\omega}) \longrightarrow \hat{U}(\tilde{\theta}(t, \tilde{\omega}))$ is surjective. Indeed its inverse satisfies

$$||[\tilde{X}(t, \tilde{\omega}, \cdot) | \hat{U}(\tilde{\omega})]^{-1}(v, \eta)|| \leq ||(v, \eta)|| e^{-\delta_1 t}$$

for $(v, \eta) \in \hat{U}(\tilde{\theta}(t, \tilde{\omega}))$, $t \geq t_1$.

This completes the proof of the corollary. □

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