THE KAM THEORY OF SYSTEMS WITH SHORT RANGE INTERACTIONS I

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IMA Preprint Series #32

June 1983

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The KAM Theory of Systems with Short Range Interactions I

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Abstract

The existence of quasiperiodic trajectories for Hamiltonian systems consisting of long chains of nearly identical subsystems, with interactions which decay rapidly with increasing distance between the interacting components, is studied. Such models are of interest in statistical mechanics. It is shown that nonergodic motions persist for much larger perturbations than previous work indicated. If the number of degrees of freedom of the system is N , the allowed perturbation decreases only as an inverse power of N , as the number of degrees of freedom increases, rather than the inverse power of $N!$ which previous estimates yielded.

1. Introduction

The Kolmogorov, Arnol'd, Moser (KAM) Theory [15, 1, 16] proves that "small" perturbations of integrable Hamiltonian systems possess "large" sets of initial conditions for which the trajectories remain quasiperiodic. In this paper we discuss how the "strength" of the allowed perturbation varies with the number of degrees of freedom, N , in the system. (We give precise meanings to the words in quotation marks below.) Classical estimates for a general analytic perturbation of strength ϵ_0 require

$$\epsilon_0 < C(N!)^{-\alpha} \tag{1.1}$$

to ensure that the theory applies. Here C is a constant depending on all the parameters of the system except N , and [11] gives a value of $\alpha = 31$.

Recent numerical experiments [4, 6, 3, 10] indicate that at least in systems with short range interactions, perturbations much larger than those permitted by (1.1) still give rise to quasiperiodic motion. In the present paper we initiate a study of such systems and show that for a class of Hamiltonians with short range interactions the perturbation need only satisfy

$$\epsilon_0 < C'N^{-\alpha'}, \tag{1.2}$$

to ensure the existence of quasiperiodic trajectories. We obtain a value of $\alpha' = 160$.

Nearly thirty years ago [8] it was pointed out that the existence of such trajectories is at variance with the commonly held belief that large systems behave ergodically. It seems that the rapid decay (with increasing N) of the estimate (1.1) ensuring applicability of the KAM theory has discouraged examination of its importance for statistical mechanics. Even the estimate (1.2) gives only a small region of quasiperiodic behavior as N becomes large, but the exponent of N in (1.2) is certainly not optimal, and we hope

that the improvement in the range of applicability of the theory as compared to (1.1) will encourage further research, both to determine the optimal power of N in (1.2), and also to determine whether or not there may be a finite allowed perturbation as $N \rightarrow \infty$, for which the motions are not ergodic (although possibly not quasiperiodic either).

To state our results we introduce the following notation. Let V be the sphere of radius r in $\mathbb{R}^N$, and $T^N = [0, 2\pi]^N$, with opposite ends identified. A Hamiltonian in "action-angle" form is a function $H(\underline{I}, \underline{\phi}) : V \times T^N \rightarrow \mathbb{R}$. We write the Hamiltonian as

$$H(\underline{I}, \underline{\phi}) = h^0(\underline{I}) + f^0(\underline{I}, \underline{\phi}) . \quad (1.3)$$

with $f^0(\underline{I}, \underline{\phi})$ a perturbation of the integrable Hamiltonian $h^0(\underline{I})$. Since $f^0(\underline{I}, \underline{\phi})$ is periodic it may be expanded in a Fourier series

$$f^0(\underline{I}, \underline{\phi}) = \sum_{\underline{\nu} \in \mathbb{Z}^N} f_{\underline{\nu}}^0(\underline{I}) e^{i\underline{\nu} \cdot \underline{\phi}} \quad (1.4)$$

Defining $\underline{z}^{\underline{\nu}} = \prod_{j=1}^N z_j^{\nu_j} = \prod_{j=1}^N e^{i\nu_j \phi_j}$ we can regard $f^0(\underline{I}, \underline{\phi})$ as a function on $\mathbb{C}^{2N}$, which we write as $f^0(\underline{I}, \underline{z})$. We demand that $H(\underline{I}, \underline{z})$ be analytic on

$$W(\rho_0, \xi_0; V) \equiv \bigcup_{\underline{I} \in V} \left\{ (\underline{I}', \underline{z}) \mid (\underline{I}', \underline{z}) \in \mathbb{C}^{2N}, e^{-\xi_0} < |z_j| < e^{\xi_0}, |I'_j - I_j| < \rho_0; \right. \\ \left. \forall j = 1, \dots, N \right\} . \quad (1.5)$$

Given a Hamiltonian define E_0, ϵ_0 , by

$$\sup \left| \partial h^0 / \partial \underline{I} \right| \leq E_0 \quad (1.6)$$

$$\sup \left\{ \left| \frac{\partial f^0}{\partial \underline{I}}(\underline{I}, \underline{z}) \right| + \rho_0^{-1} \left| \frac{\partial f^0}{\partial \underline{\phi}}(\underline{I}, \underline{z}) \right| \right\} \leq \epsilon_0 \quad (1.7)$$

In (1.7), $\frac{\partial}{\partial \underline{z}} \equiv -i \left(z_1 \frac{\partial}{\partial z_1}, \dots, z_N \frac{\partial}{\partial z_N} \right)$. Also, by $\partial^2 f / \partial \phi_i \partial \phi_j$ we mean $-z_i \frac{\partial}{\partial z_i} \left(z_j \frac{\partial f}{\partial z_j} \right)$, the only possible confusion coming when $i = j$. For any $\underline{v} \in \mathbb{C}^{2N}$ define $|\underline{v}| = \sum_{j=1}^N |v_j|$. For a matrix M , $|M| \equiv \sum_{i,j=1}^N |M_{ij}|$. The factor of ρ_0^{-1} in the second term of (1.7) is to keep the dimensions of the two terms the same, since as was pointed out in [11] this greatly simplifies the resulting estimates. Also, we will assume for convenience that $\rho_0 < E_0$.

The Hamiltonians we consider consist of almost independent, almost identical subsystems, lying along a line, with interactions which decrease rapidly in strength as the distance between the points of interaction increases. As a prototype consider

$$H(\underline{I}, \underline{\phi}) = \frac{1}{2} \underline{I} \cdot \underline{I} + \epsilon \sum_{i=1}^{N-1} \cos(\phi_{i+1} - \phi_i) . \quad (1.8)$$

More generally we require:

(a) nearly independent, nearly identical subsystems:

Define $\underline{\omega}^0(\underline{I}) = \frac{\partial h^0}{\partial \underline{I}}(\underline{I})$. We require

$$\sup \left| \frac{\partial \omega_i^0}{\partial I_j} \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{|i-j|}; \quad i \neq j \quad (1.9)$$

and

$$\frac{\partial \omega_i^0}{\partial I_i}(\underline{I}) = 1 + \chi_i^0(\underline{I}) \quad (1.10)$$

with $\sup_{i, \underline{I}} \chi_i^0(\underline{I}) \leq B_1 N^{-1}$, and B_1 some universal constant, say 2^{-3} . These conditions are satisfied for (1.8).

(b) weak, short range interactions:

The Fourier coefficients of $f_{\underline{I}, \underline{z}}^0$ must satisfy

$$\sup_{\underline{z}} \left| f_{\underline{z}}^0(\underline{I}) \right| \leq \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{d(\text{supp } \underline{z})} e^{-\xi_0 |\underline{z}|}, \quad (1.11)$$

where $d(\text{supp } \underline{z})$ = distance between the two most widely separated points in $\text{supp } \underline{z}$ (regarding $\underline{z}$ as a function on $[1, N] \cap \mathbb{Z}$). We also require that if ℓ is the point in $\text{supp } \underline{z}$ farthest from j

$$\sup \left| \frac{\partial}{\partial I_j} f_{\underline{z}}^0(\underline{I}) \right| \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{|\ell-j|} e^{-\xi_0 |\underline{z}|} \quad (1.12)$$

and finally

$$\sup \left| \frac{\partial^2 f_{\underline{z}}^0}{\partial I_i \partial I_j}(\underline{I}) \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{|i-j|} e^{-\xi_0 |\underline{z}|} \quad (1.13)$$

In each of the inequalities (1.9) - (1.13), the supremum is taken over $W(\rho_0, \xi_0, V)$.

Also note that (1.8) obeys (1.7) and (1.11) - (1.13) if we take $\rho_0 = 1$, and

$$\epsilon < \min \left(\frac{2\xi_0}{N} e^{-2\xi_0}, 2\xi_0^2 e^{-2\xi_0} \right).$$

We require a great deal of analyticity in the angular variables $\underline{z}$, enough to set

$$\xi_0 = B_2 \ell \ln N + 33, \quad (1.14)$$

with $B_2 = 2^9$. This is again satisfied for (1.8), for any N , but this is one restriction on our theorem which it would be very nice to weaken—at least to remove the N dependence.

Although we are most interested in the case where N is very large, we will only assume that $N \geq 14$, and even this restriction could be removed at the expense of changing some of the constants appearing in the theorem below. Under assumptions (1.6) - (1.14) we obtain

Theorem 1.1. There exist constants $B, \beta, \gamma, \sigma > 0$ such that when

$$\epsilon_0 < \rho_0 B \lambda^\beta (E_0 \rho_0^{-1})^{-\gamma} N^{-\sigma}, \quad (1.15)$$

there exists a set $\Gamma \subset V$, and a change of variables $(\underline{I}', \underline{z}') \longleftrightarrow (\underline{I}, \underline{z})$ which is one to one and canonical on $\Gamma \times T^N$ and such that if $C : (\underline{I}', \underline{z}') \longrightarrow (\underline{I}, \underline{z})$ we may write

$$\hat{H}(\underline{I}', \underline{z}') \equiv H \circ C(\underline{I}', \underline{z}) = \hat{h}^0(\underline{I}') + \hat{f}^0(\underline{I}', \underline{z}') .$$

There are in addition $\hat{\rho}_0, \hat{\xi}_0 > 0$ such that $\hat{H}(\underline{I}', \underline{z}')$ is analytic on $W(\hat{\rho}_0, \hat{\xi}_0, \Gamma)$, with

$$\sup \left| \frac{\partial \hat{h}^0}{\partial \underline{I}'} \right| \leq \tilde{E}_0 \leq 2E_0, \quad (1.16)$$

$$\sup \left\{ \left| \frac{\partial \hat{f}^0}{\partial \underline{I}'} \right| + \hat{\rho}_0^{-1} \left| \frac{\partial \hat{f}^0}{\partial \underline{z}'} \right| \right\} \leq \hat{\epsilon}_0 \leq \rho_0 (\epsilon_0 \rho_0^{-1})^N, \quad (1.17)$$

$$\frac{\partial^2 \hat{h}^0}{\partial I_i'^2}(\underline{I}') = 1 + \hat{\chi}_i(\underline{I}') \quad (1.18)$$

with

$$\sup_{i, \underline{I}} |\hat{\chi}_i(\underline{I})| \leq 2^{-2} N^{-1} \quad (1.19)$$

and

$$\sup \left| \frac{\partial^2 \hat{h}^0}{\partial I_i' \partial I_j'}(\underline{I}') \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{\frac{1}{8} |i-j|}; \quad \text{if } i \neq j \quad (1.20)$$

Finally, we have $\text{vol } \Gamma \geq (1 - \lambda) \text{vol } V$. In (1.16) - (1.20) the suprema run over the set $W(\hat{\rho}_0, \hat{\xi}_0, \Gamma)$.

As will be seen in the course of the proof below it suffices to pick B some small universal constant, $\beta = 12$, $\gamma = 18$, and $\sigma = 160$. Note also that Theorem 1.1 is not a KAM type theorem since $\hat{H}(\underline{I}, \underline{z})$ is not an integrable system. To complete the proof we need:

Theorem 1.2: Given the Hamiltonian $\hat{H}(\underline{I}, \underline{z})$ constructed in Theorem 1.1, analytic on $W(\hat{\rho}_0, \hat{\xi}_0; \Gamma)$ and satisfying (1.16) - (1.20), there are constants $\hat{B}, \hat{\beta}, \hat{\gamma}, \hat{\sigma}, \hat{k}, \hat{\xi}$, and $\hat{\mu}$ such that if

$$\hat{\epsilon}_0 < \hat{\rho}_0^{\hat{B}\hat{\beta}} (1-\lambda)^{\hat{\xi}} (\hat{\rho}_0^{-1} \hat{E}_0)^{-\hat{\gamma}} (\rho_0^{-1} E_0)^{\hat{k}} N^{-\hat{\sigma}N} e^{-\hat{\mu}N}, \quad (1.21)$$

then there exists $\hat{\Gamma} \subset \Gamma$ and N, C^∞ functions, $I'_1, \dots, I'_N$ such that $I'_1, \dots, I'_N$ are first integrals for the perturbed motions starting in $\hat{\Gamma}$. In addition there are N, C^∞ functions $\phi'_1, \dots, \phi'_N$ such that the change of variables $\hat{C}: (\underline{I}', \underline{\phi}') \rightarrow (\underline{I}, \underline{\phi})$ is one to one and canonical on $\Gamma \times T^N$, and

$$\hat{H} \circ \hat{C}(\underline{I}', \underline{\phi}') = h(\underline{I}'). \quad (1.22)$$

Finally, $\text{vol } \hat{\Gamma} \geq (1-\lambda) \text{vol } \Gamma$.

It suffices to take $\hat{B}$ some universal constant, $\hat{\beta} = 17$, $\hat{\xi} = 17$, $\hat{\gamma} = 2$, $\hat{k} = 16$, $\hat{\sigma} = 14$ and $\hat{\mu} = 185$. The desired KAM theorem is

Corollary 1.3. Let $H(\underline{I}, \underline{z})$ be analytic on $W(\rho_0, \xi_0, V)$ and satisfy (1.6) - (1.14). Then there exists $B' > 0$ such that if

$$\epsilon_0 < \rho_0^{B'} (E_0 \rho_0^{-1})^{18} \lambda^8 (1-\lambda)^2 N^{-160} \quad (1.23)$$

there exists a set $\hat{\Gamma} \subset V$ and $2N, C^\infty$ functions, $I'_1, \dots, I'_N, \phi'_1, \dots, \phi'_N$ such that $I'_1, \dots, I'_N$ are first integrals for motions of the perturbed system starting in $\hat{\Gamma}$, and the change of variables $\tilde{C}: (\underline{I}', \underline{\phi}') \rightarrow (\underline{I}, \underline{\phi})$ is one to one and canonical on $\hat{\Gamma} \times T^N$. Furthermore

$$H \circ \tilde{C}(\underline{I}', \underline{\phi}') = h(\underline{I}'),$$

and $\text{vol } \hat{\Gamma} \geq (1-2\lambda) \text{vol } V$.

Proof: The corollary follows immediately if we take $\lambda = \hat{\lambda}$ and $\tilde{C} = C \circ \hat{C}$, where C and $\hat{C}$ are the two canonical transformations constructed in Theorems 1.1 and 1.2, since (1.23) implies that both (1.15) and (1.22) are satisfied, using the definition of $\hat{\rho}_0$ that emerges from section 2, and the fact that $\hat{\xi}_0$ can be chosen to be one (which also comes from section 2).

We close this section with some comments of a general nature. First we note that Theorem 1.2 is almost precisely the theorem of Chierchia and Gallavotti [5] except that instead of assuming a bound on the anisochrony parameter $\eta_0 = \sup \left| (\partial h^0 / \partial \underline{I} \partial \underline{I})^{-1} \right|$ we will derive a bound on this quantity from (1.16) - (1.20). This gives better control over the constant $\hat{\gamma}$.

Our second comment concerns (1.10). This condition looks very restrictive but we point out two respects in which it is somewhat less so than it may at first appear. First is the trivial observation that we may replace the one on the r. h. s. of (1.10) by any positive constant, just by changing the scale of $\underline{I}$, and possibly altering the constant B_1 . Secondly suppose $h^0(\underline{I}) = \sum_{i=1}^N \tilde{h}^0(I_i)$, with $\tilde{h}^0(I_i)$ analytic in a ball of radius $\tilde{r}$ in $\mathbb{C}$. Then if $\tilde{h}^{0''}(0) > 0$, we may write

$$\frac{\partial \omega_i^0}{\partial I_i}(I_i) = \tilde{h}^{0''}(0) + \tilde{\chi}_i(I_i) , \quad (1.25)$$

and $\tilde{\chi}_i(I_i)$ will obey the bound (1.11) provided $\tilde{r}$ is sufficiently small. One might also wonder how the restrictions (1.11) - (1.13) arise. It would seem more general perhaps to drop the factors of $\epsilon_0 \rho_0$, ϵ_0 , and $\epsilon_0 \rho_0^{-1}$ in front of (1.11) - (1.13). In fact the proof goes through with very few modifications if one makes this change, and the choices used in (1.11) - (1.13) are just one convenient choice of the many possible. Note further, that if one chose to make the change suggested above one could recover the present form by a slight redefinition of ϵ_0 and ρ_0 .

Our third comment concerns the numerical experiments mentioned earlier [4, 6, 3, 10]. These seem to show that nonergodic behavior persists in the system for a finite perturbation even when N becomes very large. Even

more surprising, the maximum perturbation per mode (ϵ_0/N in our notation) for which nonergodic trajectories survive seems to be almost independent of N for N larger than about ten. While the exponent of N which we quote in (1.23) is certainly not optimal, and might be reduced by more careful estimates, it is by no means clear how the present methods could yield an N -independent estimate for the allowed perturbation. Thus a very interesting unresolved question becomes can one extend the KAM theory to explain the experimental results? If not, can one perhaps show that there is some nonquasiperiodic motion, but also nonergodic motion in these systems? If the answer to either of these questions is yes, one should also ask what implications such motions have for statistical mechanics.

As we mentioned, the proof of Theorem 1.2 is a straightforward application of the KAM machinery. The proof of Theorem 1.1, on the other hand, proceeds by making a finite number of canonical transformations which successively reduce the strength of the interaction while including the effect of interactions on longer and longer distance scales. This multiple length scale analysis appears trivial in the case of (1.8) which has only nearest neighbor interactions. However, the first change of variables produces interactions of arbitrarily long range, making the multiple length scales necessary. To study the decay of the interactions we generate, we borrow an idea from statistical mechanics and field theory [17, 2] which is introduced in section four.

The KAM theory is constructed so that the sequence of canonical transformations may be repeated indefinitely, but the transformations used in Theorem 1.1 permit only a finite number of iterations. We iterate until the characteristic length scale is of order N (the size of the system), and then appeal to Theorem 1.3. This procedure is reminiscent of the renormalization group where one starts with a system in the critical region and iterates the renormalization group transformation a finite number of times, until one is far enough from the critical region to treat the system by other means (e.g., perturbation theory). Because of this

similarity we shall often refer to the transformed Hamiltonian as the renormalized Hamiltonian, and the interactions generated by this procedure as the renormalized interactions. The similarity between KAM theory and the renormalization group has been commented on previously by Doveil and Escande [7], Gallavotti [11] and Kadanoff [13]. Also, that part of our analysis which studies the effects of small denominators on longer and longer length scales is reminiscent of the work of Fröhlich and Spencer [9] on electron localization.

At a number of points we bound derivatives by what we refer to as dimensional estimates. When the derivative so estimated is a derivative with respect to $\underline{I}$, this is just the observation that if $f(\underline{I})$ is an analytic function on the domain D , then $\left| \frac{\partial f}{\partial \underline{I}}(\underline{I}_0) \right| \leq \sup_D |f(\underline{I})| / \text{dist}(\underline{I}_0, \partial D)$. The case of derivatives with respect to ϕ are somewhat more delicate and we refer the reader to Appendix 2 for an explanation.

We close this section with an outline of the remainder of the paper. In section two we outline the proof of Theorem 1.1. Section three defines the sequence of canonical transformations used to prove Theorem 1.1 and shows that the resulting change of variables is well defined. Section four examines in greater detail the change of variables defined in section three. In section five we prove that the renormalized interactions decay nearly as fast as those in our original Hamiltonian, and finally in section six we estimate the amount of phase space we lose in this procedure.

2. An Outline of the Proof of Theorem 1.1

Theorem 1.1 is proven by constructing a sequence of Hamiltonians $H^k(\underline{I}, \underline{z})$, each of which will obey a short range condition similar to (1.11), but with the additional requirement that the size (in the sense of (1.7)) of the nonintegrable part has been reduced with respect to $H^{k-1}(\underline{I}, \underline{z})$. To specify the $H^k(\underline{I}, \underline{z})$ more precisely, define

$$\begin{aligned}
 C &= 2^{11}(\rho_0 \lambda)^{-1} N^2 \\
 \delta &= 36 \\
 L_k &= 8(3/2)^k \\
 M_k &= \delta^{-1} |\ln \epsilon_0 \rho_0^{-1}| (3/2)^k \\
 \epsilon_k &= \rho_0 (\epsilon_0 \rho_0^{-1})^{(3/2)^k} \\
 E_{k+1} &= E_k + \epsilon_k \\
 \eta_k &= \left(1 - \sum_{j=0}^{k-1} \beta_j - (3/32)k \ln(3/2) / \ln N \right) \\
 \beta_k &= (1/3) 2^{-4} (3/2)^{-k} + \frac{(1/2) \ln 3/2}{\ln N} \\
 \xi_{k+1} &= \xi_k - 4\delta \\
 \rho_{k+1} &= \rho_k 2^{-6} (E_k C N)^{-1} \exp(-2M_k - 2L_k) .
 \end{aligned} \tag{2.1}$$

In each case, $k \in \{0, 1, \dots, k_0\}$, where $k_0 = 1 + \text{integer part of } [(\ln N)(\ln 3/2)^{-1}]$.

Define $\mathbf{X}_k = \{ \underline{z} \in \mathbb{Z}^N \mid |\underline{z}| \leq M_k \text{ and } d(\text{supp } \underline{z}) \leq L_k \}$. Given a function $h(\underline{I})$ defined on a region $V \subset \mathbb{R}^N$ the "resonant vectors of h in V on the scale k " are

$$\begin{aligned}
 R(k, h, V) &= \left\{ \underline{I} \in V \mid \left| \left\langle \frac{\partial h}{\partial \underline{I}}(\underline{I}), \underline{z} \right\rangle^{-1} \right| \geq C \exp[(3/2)|\underline{z}| + L_k] \right. \\
 &\quad \left. \text{for some } \underline{z} \in \mathbf{X}_k \right\} .
 \end{aligned} \tag{2.2}$$

Here, $\langle, \rangle$ is the usual inner product for N -vectors.

In Section 3 we define a decreasing sequence of regions $V \supset V_0 \supset \dots \supset V_{k_0}$, which in Section 6 are shown to satisfy $\text{vol}(V_k \setminus V_{k+1}) \leq (1/4)\lambda e^{-(1/2)(3/2)^{k+1}} \text{vol } V$, and transformations $C^k, \tilde{C}^k$ which are 1-1 and canonical on V_k . These transformations are analytic on $W(4\rho_{k+1}, \xi_k - 2\delta, V_k)$ and

$$\begin{aligned} C^k : W(2\rho_{k+1}, \xi_k - 3\delta, V_k) &\longrightarrow W(4\rho_{k+1}, \xi_k - 2\delta, V_k) \\ \tilde{C}^k : W(2\rho_{k+1}, \xi_k - 3\delta, V_k) &\longrightarrow W(4\rho_{k+1}, \xi_k - 2\delta, V_k) , \end{aligned} \quad (2.4)$$

with $W(4\rho_{k+1}, \xi_k - 2\delta, V_k) \subset W(\rho_k, \xi_k, V_{k-1})$. On their common domain of definition $C^k \circ \tilde{C}^k = \tilde{C}^k \circ C^k = \text{identity}$.

The sequence of Hamiltonians is defined by

$$H^{k+1}(\underline{I}, \underline{z}) = H^k \circ C^k(\underline{I}, \underline{z}) \equiv h^{k+1}(\underline{I}) + f^{k+1}(\underline{I}, \underline{z}) \quad (2.5)$$

for $(\underline{I}, \underline{z}) \in W(\rho_{k+1}, \xi_{k+1}, V_k)$. The procedure for splitting H^{k+1} into its integrable (h^{k+1}) and nonintegrable (f^{k+1}) parts is given in Section 3. One then shows that the sequence of Hamiltonians verifies

$$\begin{aligned} \sup \left| \frac{\partial f^k}{\partial \underline{I}}(\underline{I}, \underline{z}) \right| + \rho_k^{-1} \left| \frac{\partial f^k}{\partial \underline{z}}(\underline{I}, \underline{z}) \right| &\leq \epsilon_k , \\ \sup \left| \frac{\partial h^k}{\partial \underline{I}} \right| &\leq E_k , \end{aligned} \quad (2.6)$$

where the supremum runs over $W(\rho_k, \xi_k, V_{k-1})$. Defining $\omega^k(\underline{I}) = (\partial h^k / \partial \underline{I})(\underline{I})$ one has

$$\frac{\partial \omega_i^k}{\partial \underline{I}_i}(\underline{I}) = 1 + \chi_i^k(\underline{I}) , \quad (2.7)$$

with

$$\sup |\chi_i^k(\underline{I})| \leq B_1 N^{-1} + \sum_{j=0}^{k-1} 2\epsilon_j \rho_j^{-1} , \quad \text{for all } i = 1, \dots, N ,$$

and

$$\sup \left| \frac{\partial \omega_i^k}{\partial I_j}(\underline{I}) \right| \leq \theta(k; i, j) \quad \text{for all } i \neq j \quad (2.8)$$

with

$$\theta(k; i, j) = (\epsilon_0 \rho_0^{-1}) \left\{ (\epsilon_0 \rho_0^{-1})^{|i-j|} + \sum_{m=0}^{k-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_m)|i-j|} \right\}$$

Finally each of the Hamiltonians obeys a short range condition, namely, if $f_{\underline{I}}^k(\underline{I})$ are the coefficients in the Laurent expansion for $f^k(\underline{I}, \underline{z})$,

$$\sup \left| f_{\underline{I}}^k(\underline{I}) \right| \leq \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\text{supp } \underline{z})} e^{-\xi_k |\underline{z}|} \quad (2.9a)$$

$$\sup \left| \frac{\partial f_{\underline{I}}^k}{\partial I_j}(\underline{I}) \right| \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|\ell-j|} e^{-\xi_k |\underline{z}|} \quad (2.9b)$$

$$\sup \left| \frac{\partial^2 f_{\underline{I}}^k}{\partial I_i \partial I_j}(\underline{I}) \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} e^{-\xi_k |\underline{z}|}, \quad (2.9c)$$

with the site ℓ on the l.h.s. of (2.9b) the site in $\text{supp } \underline{z}$ furthest from j . In each case the supremum is over $W(\rho_k, \xi_k, V_{k-1})$.

The existence of such an iterative procedure is sufficient to establish

Theorem 1.1. Define

$$\tilde{C} = \tilde{C}^{k_0} \circ \tilde{C}^{k_0-1} \circ \dots \circ \tilde{C}^0 \quad \text{and} \quad C = C^0 \circ C^1 \circ \dots \circ C^{k_0} \quad (2.10)$$

Take $\Gamma = \text{pr}_1(C(V_{k_0} \times T^N))$, with $\text{pr}_1(\underline{I}, \emptyset) = \underline{I}$, and

$$\hat{H}(\underline{I}', \underline{z}') = H^{k_0}(\underline{I}', \underline{z}') = H \circ C(\underline{I}', \underline{z}') . \quad (2.11)$$

Estimates (1.16) - (1.19) of Theorem 1.1 follow from (2.6) - (2.8) combined with the fact that $2 \sum_j \epsilon_j \rho_j^{-1} < 2^{-3}$ (which follows easily from the definition of ϵ_j and ρ_j), and the fact that $(1 - \eta_m) \geq (1/4)$ for $m = 0, 1, \dots, k_0$. Also,

$$\text{vol } \Gamma = \text{vol } V_{k_0} \geq \left\{ 1 - \sum_{m=0}^{k_0} 2^{-2} \lambda \exp \left[-(1/2)(3/2)^k \right] \right\} \text{vol } V \geq (1 - \lambda) \text{vol } V , \quad (2.12)$$

where we used the fact that the canonical transformations preserve phase space volume, and our estimate on $\text{vol}(V_k \setminus V_{k-1})$.

3. The Inductive Step

The iterative procedure described in the previous section starts by taking $H^0(\underline{I}, \underline{z}) = H(\underline{I}, \underline{z})$, with $H(\underline{I}, \underline{z})$ the initial Hamiltonian of Section 1. Estimates (2.6) - (2.9) follow from (1.9) - (1.13). We assume that the Hamiltonians, H^{j+1} , have been constructed for $j \leq k-1 < k_0 - 1$, satisfying the bounds of Section 2. In the present section we construct C^k , $\tilde{C}^k$, and H^{k+1} .

Define $\tilde{\rho}_k = 8\rho_{k+1} = 2^{-3} \rho_k (E_k CN)^{-1} \exp[(-2)(M_k + L_k)]$. Let $B \equiv \{\underline{I} \mid \text{dist}(\underline{I}, \partial V_{k-1}) \leq \tilde{\rho}_k\}$, $\bar{V}_k \equiv (V_{k-1} \setminus B) \setminus R(k, h, V_{k-1})$ and $V_k = \bigcup_{\underline{I} \in \bar{V}_k} S(\tilde{\rho}_k/2, \underline{I})$. Here $S(r, \underline{I}) = \{\underline{I}' \in \mathbb{R}^N \mid |\underline{I} - \underline{I}'| < r\}$, and $\text{dist}(\underline{I}, \underline{I}') = |\underline{I} - \underline{I}'|$. Note that for $\underline{v} \in \mathbb{X}_k$,

$$|\langle \omega^k(\underline{I}), \underline{v} \rangle^{-1}| \leq 2C e^{(3/2)|\underline{v}|} e^{L_k} \quad (3.1)$$

for all $\underline{I} \in W(\tilde{\rho}_k, \xi_k, V_k)$, since for any such $\underline{I}$ there is an $\underline{I}' \in \bar{V}_k$ with a path $\gamma \subset W(\tilde{\rho}_k, \xi_k, V_k)$, connecting $\underline{I}$ and $\underline{I}'$, and such that γ is made up of at most $2N$ pieces, of length at most $\tilde{\rho}_k$, along which only one coordinate varies. Thus,

$$\begin{aligned} |\langle \omega^k(\underline{I}), \underline{v} \rangle^{-1}| &= |\langle \omega^k(\underline{I}'), \underline{v} \rangle^{-1}| \\ &\times \left\{ \left| 1 + \langle \omega^k(\underline{I}'), \underline{v} \rangle^{-1} \int_{\gamma} d\underline{I}'' \left\langle \frac{\partial \omega^k}{\partial \underline{I}}(\underline{I}''), \underline{v} \right\rangle \right| \right\}. \end{aligned} \quad (3.2)$$

By (2.2), $|\langle \omega^k(\underline{I}'), \underline{v} \rangle^{-1}| < C \exp[(3/2)|\underline{v}| + L_k]$ for $\underline{I}' \in \bar{V}_k$. By a dimensional estimate $\left| \int_{\gamma} d\underline{I}'' \langle \partial \omega^k / \partial \underline{I}(\underline{I}''), \underline{v} \rangle \right| \leq 2N |\underline{v}| \tilde{\rho}_k (\rho_k - \tilde{\rho}_k)^{-1} E_k \leq 2N |\underline{v}| \tilde{\rho}_k \rho_k^{-1} E_k$, on $W(\tilde{\rho}_k, \xi_k, V_k)$. These two observations, combined with the definition of $\tilde{\rho}_k$ imply that the second term in braces in (3.2) is bounded in magnitude by $(1/2)$, and (3.1) follows immediately.

Define (on $W(\tilde{\rho}_k, \xi_k, V_k)$) the generating function for our change of variables

$$\Phi^k(\underline{I}', \underline{z}) = \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \left(\frac{f_{\underline{\nu}}^k(\underline{I}')}{i < \underline{\omega}_k(\underline{I}') \cdot \underline{\nu} >} \right) \underline{z}^{\underline{\nu}} \quad (3.3)$$

Specify the change of variables $(\underline{I}', \underline{z}') \longleftrightarrow (\underline{I}, \underline{z})$ by

$$\begin{aligned} \underline{I} &= \underline{I}' + \frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}) \\ \underline{z}' &= \underline{z} \exp \left[i \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right] \end{aligned} \quad (3.4)$$

We show below that the implicit function theorem allows us to invert (3.4) to obtain $(\underline{I}', \underline{z}')$ in terms of $(\underline{I}, \underline{z})$ or vice versa.

The choice of $\Phi^k(\underline{I}', \underline{z})$ is motivated by classical perturbation theory [11]. Unlike previous studies we introduce two cutoffs, one to take advantage of the decay arising from analyticity ($|\underline{\nu}| \leq M$, an "ultraviolet cutoff") and one to take advantage of the decay with distance ($d(\text{supp } \underline{\nu}) \leq L_k$, an "infrared cutoff").

Derivatives of $\Phi^k(\underline{I}', \underline{z})$ are estimated on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$.

$$\left| \frac{\partial \Phi^k}{\partial \underline{\phi}} \right| \leq \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \left| \frac{f_{\underline{\nu}}^k(\underline{I}')}{i < \underline{\omega}_k(\underline{I}'), \underline{\nu} >} \right| |\underline{\nu}| \cdot |\underline{z}^{\underline{\nu}}| \quad (3.5)$$

Bound $|\underline{\nu}| \left| \frac{f_{\underline{\nu}}^k(\underline{I}')}{i < \underline{\omega}_k(\underline{I}'), \underline{\nu} >} \right|$ by $2C \exp[(3/2)|\underline{\nu}| + L_k] \epsilon_k \rho_k e^{-\xi_k |\underline{\nu}|}$, using (2.6) and and (3.1). Also, $|\underline{z}^{\underline{\nu}}| \leq \exp(\xi_k - \delta) |\underline{\nu}|$ on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. Thus,

$$\sup \left| \frac{\partial \Phi^k}{\partial \underline{\phi}}(\underline{I}', \underline{z}) \right| \leq \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} 2C \epsilon_k \rho_k \exp[-(\delta - (3/2))|\underline{\nu}|] e^{L_k}. \quad (3.6)$$

The number of vectors $\underline{\nu} \in \mathbb{Z}^N$ with $|\underline{\nu}| = M$ and $d(\text{supp } \underline{\nu}) = L$ is bounded by $N \cdot 2^{2M} 2^L$. This estimate is derived by fixing the leftmost point in $\text{supp } \underline{\nu}$ to be i . If there are j sites in $\text{supp } \underline{\nu}$, then there are at most $\binom{L}{j}$ ways of choosing them. Furthermore, if $\text{supp } \underline{\nu}$ is fixed there are at most 2^{2M^j} vectors $\underline{\nu}$, with $|\underline{\nu}| = M$, and specified support. Summing j from 1 to L , and estimating the number of choices for i by N yields the stated result. Thus the r.h.s. of (3.6) is bounded by

$$\begin{aligned}
& \sum_{L=1}^{L_k} \sum_{M=1}^{M_k} \sum_{\substack{\underline{\nu} \in \mathbb{X}_k: \\ |\underline{\nu}| = M \\ d(\text{supp } \underline{\nu}) = L}} 2C\epsilon_k \rho_k e^{-(\delta - 3/2)M} e^{L_k} \\
& \leq 2C\epsilon_k \rho_k e^{L_k} \sum_{L=1}^{L_k} \sum_{M=1}^{M_k} N 2^{L_k} 2^{2M} e^{-(\delta - 3/2)M} \\
& \leq 2^2 C\epsilon_k \rho_k e^{(1 + \ln 2)L_k} N.
\end{aligned} \tag{3.7}$$

In the last step we have bounded the sum over M by a geometric series and used the fact that $\exp[(3/2) + 2 \ln 2 - \delta] < (1/2)$.

In a similar fashion,

$$\begin{aligned}
\sup \left| \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right| & \leq \left| \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq 0}} \left\{ \left(\frac{\partial f^k(\underline{I}')}{\partial \underline{I}'} \right) (i < \underline{\omega}^k(\underline{I}'), \underline{\nu} >)^{-1} \right. \right. \\
& \quad \left. \left. - \frac{f^k(\underline{I}')}{(i < \underline{\omega}^k(\underline{I}'), \underline{\nu} >)^2} \left(i \frac{\partial}{\partial \underline{I}'} < \underline{\omega}^k(\underline{I}'), \underline{\nu} > \right) \right\} \underline{z}^{\underline{\nu}} \right|
\end{aligned} \tag{3.8}$$

The first term in braces is bounded by $2C\epsilon_k \exp[(3/2)|\underline{\nu}| + L_k] e^{-\xi_k |\underline{\nu}|}$ on $W(\tilde{\rho}_k, \xi_k, V_k)$ by (2.6) and (3.1). The second term is bounded in magnitude by

$\epsilon_k \rho_k \left(2C \exp[(3/2)|\underline{z}| + L_k] \right)^2 |\underline{z}| E_k (\rho_k - \tilde{\rho}_k)^{-1} e^{-\xi_k |\underline{z}|} \leq$
 $\leq 2^3 \epsilon_k E_k C^2 |\underline{z}| \exp(3|\underline{z}| + 2L_k - \xi_k |\underline{z}|)$, where the first estimate used (2.6), (3.1),
 and a dimensional estimate on $(\partial \omega^k / \partial \underline{I})(\underline{I}')$, and the second inequality used the fact
 that $(\rho_k - \tilde{\rho}_k)^{-1} \leq 2\rho_k^{-1}$. Thus,

$$\begin{aligned}
 \sup \left| \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z}) \right| &\leq \sum_{\substack{\underline{z} \in \mathbb{X}_k \\ \underline{z} \neq 0}} \left[2\epsilon_k C \exp[(3/2)|\underline{z}| + L_k] \right. \\
 &\quad \left. + 2^3 \epsilon_k E_k C^2 |\underline{z}| \exp[(3/2)|\underline{z}| + 2L_k] \right] e^{-\delta |\underline{z}|} \quad (3.9) \\
 &\leq 2^6 \epsilon_k E_k C^2 N \exp[(2 + \ln 2)L_k] ,
 \end{aligned}$$

where we bounded the sum over $\underline{z}$ as in (3.7), and used the fact that $E_k > E_0 > C^{-1}$,
 (which follows from our assumption that $\rho_0 < E_0$) to bound $(1 + E_k C)$ by $2E_k C$.

The implicit function theorem allows us to invert (3.4) as

$$\begin{aligned}
 \underline{z} &= \underline{z}' \exp(\underline{\Delta}(\underline{I}', \underline{z}')) \\
 \underline{I}' &= \underline{I} + \underline{\Xi}'(\underline{I}, \underline{z})
 \end{aligned} \quad (3.10)$$

with $\underline{\Xi}'(\underline{I}, \underline{z})$ analytic in $W(\tilde{\rho}_k/2, \xi_k - \delta, V_k)$ and $\underline{\Delta}(\underline{I}', \underline{z}')$ analytic in $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$
 provided

$$\sup_{j, k} \left| \frac{\partial^2 \Phi^k}{\partial I_j \partial \phi_k}(\underline{I}', \underline{z}) \right| \leq (2N)^{-1} \quad (3.11)$$

$$\sup \left| \frac{\partial \Phi^k}{\partial \phi}(\underline{I}', \underline{z}) \right| \leq \tilde{\rho}_k / 2^3 N .$$

In both cases the supremum is over $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. Inequalities (3.11) follow
 from the implicit function theorem of Appendix 3 of [11] as modified in Appendix 1

of the present work. Combining (3.7) with (A.12) (and strengthening the first inequality in (3.11) somewhat for our later convenience) we see that (3.4) may be inverted whenever

$$2^6 \epsilon_k E_k C_N^2 e^{3L_k} \leq (2^{10} N)^{-1} \quad (3.12)$$

and

$$2^3 \epsilon_k \rho_k C_N e^{3L_k} \leq 2^{-3} \rho_k N^{-1}$$

A little algebra, and the definitions of ϵ_k , ρ_k , $\tilde{\rho}_k$, C , E_k and L_k show that inequalities (3.12) are implied by

$$\epsilon_0 < \rho_0 2^{-40} e^{-24(E_0 \rho_0^{-1})^{-1} \lambda^2 N^{-6}} \quad (3.13)$$

and

$$\epsilon_0 < \rho_0 2^{-40} e^{-4\delta(E_0 \rho_0^{-1})^{-2} \lambda^3 N^{-8}}$$

respectively. Each of these inequalities follows in turn from (1.15). We implicitly used the fact that the $E_k < 2E_0$, for all k , in deriving the second inequality in (3.13) which follows from (2.1) and the observation that

$$\sum_{m=0}^k \epsilon_k < \rho_0 \sum_{m=0}^k (\epsilon_0 \rho_0^{-1})^{(3/2)^k} < 2\epsilon_0 < E_0 \text{ provided } \epsilon_0 < \rho_0^{-1} 2^{-2} \text{ and } \epsilon_0 < E_0 2^{-1},$$

both of which follow from (1.15).

From the definitions of $\underline{\Delta}(\underline{I}', \underline{z}')$ and $\underline{\Xi}'(\underline{I}, \underline{z})$ we see that

$$\underline{\Delta}(\underline{I}', \underline{z}') = - \frac{\partial \Phi^k}{\partial \underline{I}'} (\underline{I}', \underline{z})$$

and

$$\underline{\Xi}'(\underline{I}, \underline{z}) = - \frac{\partial \Phi^k}{\partial \underline{z}} (\underline{I}', \underline{z}) \quad (3.14)$$

on $W(\tilde{\rho}_k/2, \xi_k^{-2\delta}; V_k)$. We also define

$$\underline{\Delta}'(\underline{I}, \underline{z}) = \frac{\partial \Phi^k}{\partial \underline{I}'} (\underline{I} + \underline{\Xi}'(\underline{I}, \underline{z}), \underline{z})$$

and

$$\underline{\Xi}(\underline{I}', \underline{z}') = \frac{\partial \Phi^k}{\partial \underline{z}} [\underline{I}', \underline{z}' \exp(i \underline{\Delta}(\underline{I}', \underline{z}'))] , \quad (3.15)$$

the first being defined on $W(\tilde{\rho}_k/2, \xi_k - \delta, V_k)$ and the second on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$.

We obtain transformations

$$\begin{aligned} C^k : (\underline{I}', \underline{z}') &\longrightarrow (\underline{I}, \underline{z}) \text{ with } \begin{cases} \underline{I} = \underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}') \\ \underline{z} = \underline{z}' \exp(i\underline{\Delta}(\underline{I}', \underline{z}')) \end{cases} \\ \tilde{C}^k : (\underline{I}, \underline{z}) &\longrightarrow (\underline{I}', \underline{z}') \text{ with } \begin{cases} \underline{I}' = \underline{I} + \underline{\Xi}'(\underline{I}, \underline{z}) \\ \underline{z}' = \underline{z} \exp(i\underline{\Delta}'(\underline{I}, \underline{z})) \end{cases} \end{aligned} \quad (3.16)$$

mapping $W(\tilde{\rho}_k/4, \xi_k - 3\delta, V_k)$ into $W(\tilde{\rho}_k/2, \xi_k - 2\delta, V_k)$ with C^k and $\tilde{C}^k$ real and canonical on $V_k \times T^N$ and satisfying $C^k \circ \tilde{C}^k = \tilde{C}^k \circ C^k = \text{identity}$ on their common domain.

Define

$$\begin{aligned} H^{k+1}(\underline{I}', \underline{z}') &\equiv H^k \circ C^k(\underline{I}', \underline{z}') \\ &= h^k(\underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}')) + f^k(\underline{I}' + \underline{\Xi}(\underline{I}', \underline{z}'), \underline{z}' \exp(i\underline{\Delta}(\underline{I}', \underline{z}')))) \\ &\equiv h^{k+1}(\underline{I}') + f^{k+1}(\underline{I}', \underline{z}') \end{aligned} \quad (3.17)$$

with

$$h^{k+1}(\underline{I}') = h^k(\underline{I}') + f_0^k(\underline{I}') , \quad (3.18)$$

and

$$f^{k+1}(\underline{I}', \underline{z}') = H^{k+1}(\underline{I}', \underline{z}') - h^{k+1}(\underline{I}') \quad (3.19)$$

Let

$$f^k[\leq]_{(\underline{I}, \underline{z})} = \sum_{\substack{\underline{\nu} \in \mathbf{X}_k \\ \underline{\nu} \neq \underline{0}}} f_{\underline{\nu}}^k(\underline{I}) \underline{z}^{\underline{\nu}} \quad (3.20)$$

and

$$f^k[\geq]_{(\underline{I}, \underline{z})} = \sum_{\underline{\nu} \notin \mathbf{X}_k} f_{\underline{\nu}}^k(\underline{I}) \underline{z}^{\underline{\nu}} .$$

Since (3.3) insures that

$$\omega^k(\underline{I}') \cdot \frac{\partial \Phi^k}{\partial \phi}(\underline{I}', \underline{z}) + f^{k[\leq]}(\underline{I}', \underline{z}) - f_0^k(\underline{I}') = 0 \quad (3.21)$$

we may rewrite (3.19), using the fundamental theorem of calculus, as

$$f^{k+1}(\underline{I}', \underline{z}') = f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \quad (3.22)$$

where

$$f^I(\underline{I}', \underline{z}') = \int_0^1 dt \int_0^t ds \sum_{i,j=1}^N \frac{\partial^2 h^k}{\partial I_i \partial I_j}(\underline{I}' + s \underline{\Xi}) \Xi_i \Xi_j, \quad (3.23)$$

$$f^{II}(\underline{I}', \underline{z}') = \int_0^1 dt \sum_{j=1}^N \frac{\partial f^{k[\leq]}}{\partial I_j}(\underline{I}' + t \underline{\Xi}, \underline{z}' \exp(i \underline{\Delta})) \Xi_j, \quad (3.24)$$

$$\text{and } f^{III}(\underline{I}', \underline{z}') = f^{k[\geq]}(\underline{I}' + \underline{\Xi}, \underline{z}' \exp(i \underline{\Delta})). \quad (3.25)$$

We have omitted the arguments on the functions $\underline{\Xi}(\underline{I}', \underline{z}')$ and $\underline{\Delta}(\underline{I}', \underline{z}')$ to save space. Using (2.6) we see that on $W(\rho_k, \xi_k, V_{k-1})$ one has

$$\sup \left| \frac{\partial h^{k+1}}{\partial \underline{I}'}(\underline{I}') \right| \leq \sup \left\{ \left| \frac{\partial h^k}{\partial \underline{I}'}(\underline{I}') \right| + \left| \frac{\partial f_0^k}{\partial \underline{I}'}(\underline{I}') \right| \right\} \leq E_k + \epsilon_k = E_{k+1}. \quad (3.26)$$

Bound $\underline{\Xi}(\underline{I}', \underline{z}')$ on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ by (3.7), estimate $\frac{\partial^2 h^k}{\partial I_i \partial I_j}(\underline{I}' + s \underline{\Xi}(\underline{I}', \underline{z}'))$ by $2\rho_k^{-1} E_k$, by combining (2.6) with a dimensional estimate on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$, and bound the number of terms in the sum in (3.23) by N^2 to obtain

$$\sup |f^I(\underline{I}', \underline{z}')| \leq 2^3 \epsilon_k^2 \rho_k E_k C_k^2 e^{4L_k N^4}, \quad (3.27)$$

on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$. To bound (3.24) we note that $\left| \frac{\partial f_{\underline{I}'}^k}{\partial I_j}(\underline{I}' + t \underline{\Xi}, \underline{z}' e^{i \underline{\Delta}}) \right| < \epsilon_k e^{-\xi_k |\underline{z}'|}$

on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$. Thus, on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$

$$\sup \left| \frac{\partial f^k}{\partial I_j} (\underline{I}' + t \underline{\Xi}, z' \exp(i \underline{\Delta})) \right| \leq \sum_{\substack{\underline{\nu} \in \mathbf{X}_k \\ \underline{\nu} \neq 0}} \epsilon_k e^{-\delta |\underline{\nu}|} \leq \epsilon_k \cdot 2^{L_k+1} \cdot N \quad (3.28)$$

where the last inequality bounded the number of terms in the sum over $\underline{\nu}$ as in (3.7). Thus, (3.24) is bounded on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ by

$$\sup |f^{\text{II}}(\underline{I}', \underline{z}')| \leq 2^3 \epsilon_k^2 \rho_k C e^{3L_k} N^2. \quad (3.29)$$

Finally, since (2.6) and (2.9) imply that $f^k(\underline{I}' + \underline{\Xi}) \leq \min \left(\epsilon_k \rho_k e^{-\xi_k |\underline{\nu}|}, \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\text{supp } \underline{\nu})} e^{-\xi_k |\underline{\nu}|} \right)$ on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ we bound $f^{\text{III}}(\underline{I}', \underline{z}')$ on $W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$ by

$$\begin{aligned} \sup |f^{\text{III}}(\underline{I}', \underline{z}')| &\leq \sum_{L=L_k+1}^N \sum_{M=1}^{\infty} 2^{2M} 2^{L_N} (\epsilon_0 \rho_0)^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)L} e^{-2\delta M} \\ &\quad + \sum_{L=1}^{L_k} \sum_{M=M_k+1}^{\infty} 2^{2M} 2^{L_N} \epsilon_k \rho_k e^{-2\delta M} \\ &\leq \left\{ (\epsilon_0 \rho_0^N) (2\epsilon_0 \rho_0^{-1})^{(1-\eta_k)L_k+1} + 2^{L_k+2} N \epsilon_k \rho_k e^{(2 \ln 2 - 2\delta)(M_k+1)} \right\} \end{aligned} \quad (3.30)$$

where the first inequality used the fact that if $\underline{\nu} \notin \mathbf{X}_k$, then either $d(\text{supp } \underline{\nu}) > L_k$, or $d(\text{supp } \underline{\nu}) \leq L_k$ and $|\underline{\nu}| \geq M_k + 1$, and then bounded the number of terms with $d(\text{supp } \underline{\nu}) = L$, $|\underline{\nu}| = M$ by $N \cdot 2^L 2^{2M}$. In the second inequality we just summed the geometric series using the fact that $2(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)}$ and $e^{-2\delta}$ are both less than 2^{-2} , provided $\epsilon < \rho_0 2^{-4}$. (Note that $1 - \eta_k \geq 2^{-2}$ for $k \leq k_0$).

Combining (3.27), (3.28) and (3.30) with a pair of dimensional estimates we have

$$\begin{aligned}
& \sup \left| \frac{\partial f^{k+1}}{\partial \underline{I}'}(\underline{I}', \underline{z}') + \rho_{k+1}^{-1} \frac{\partial f^{k+1}}{\partial \underline{z}'}(\underline{I}', \underline{z}') \right| \\
& \leq \rho_{k+1}^{-1} (1 + \delta^{-1}) \left\{ 2^3 \epsilon_k^2 \rho_k E_k C^2 e^{4L_k} N^4 + 2^3 \epsilon_k^2 \rho_k C e^{3L_k} N^2 \right. \\
& \quad \left. + (\epsilon_0 \rho_0 N) (2 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)L_k+1} + 2^{L_k+2} N \epsilon_k \rho_k \exp(2 \ln 2 - 2\delta)(M_k + 1) \right\} \\
& \leq \epsilon_{k+1} ,
\end{aligned} \tag{3.31}$$

on $W(\tilde{\rho}_k/4, \xi_k - 4\delta, V_k) \supset W(\rho_{k+1}, \xi_{k+1}, V_k)$. The last inequality follows by bounding the product of $2\rho_{k+1}^{-1}$ and each of the terms in braces by $2^{-2}\epsilon_{k+1}$. Some tedious manipulations with the definitions (2.1) show that this follows from the three inequalities

$$\begin{aligned}
\epsilon_0 & < \rho_0 2^{-120} e^{-128} (E_0 \rho_0^{-1})^{-6} \lambda^8 N^{-30} \\
\epsilon_0 & < \rho_0 2^{-80} e^{-60} (E_0 \rho_0^{-1})^{-2} \lambda^{-3} N^{-12} \\
\epsilon_0 & < \rho_0 2^{-25} e^{-6} (E_0 \rho_0^{-1})^{-1} \lambda N^{-4} ,
\end{aligned} \tag{3.32}$$

each of which follows from (1.15). Thus we have verified that (2.6) may be iterated.

This section closes by iterating (2.7) and (2.8).

$$\frac{\partial \omega_i^{k+1}}{\partial \underline{I}_i}(\underline{I}') = 1 + \chi_i^k(\underline{I}') + \frac{\partial^2 f_0^k}{\partial \underline{I}_i^2}(\underline{I}') \tag{3.33}$$

Define $\chi_i^{k+1}(\underline{I}') = \chi_i^k(\underline{I}') + (\partial^2 f_0^k / \partial \underline{I}_i^2)(\underline{I}')$. Applying (2.6) and a dimensional estimate we bound $(\partial^2 f_0^k / \partial \underline{I}_i^2)(\underline{I}')$ by $2\epsilon_k \rho_k^{-1}$ on $W(\tilde{\rho}_k, \xi_k, V_k)$ so that

$$\sup \left| \chi_i^{k+1}(\underline{I}') \right| \leq B_1 N^{-1} + \sum_{j=0}^{k-1} 2\epsilon_j \rho_j^{-1} + 2\epsilon_k \rho_k^{-1} \quad (3.34)$$

on $W(\tilde{\rho}_k, \xi_k, V_k) \supset W(\rho_{k+1}, \xi_{k+1}, V_k)$ verifying (2.7). Similarly, by (2.9),

$$\left| (\partial^2 f_0^k / \partial I_i \partial I_j)(\underline{I}') \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} \text{ on } W(\rho_k, \xi_k, V_{k-1}) \text{ so}$$

$$\sup \left| \frac{\partial \omega_i^{k+1}}{\partial I_j}(\underline{I}') \right| \leq \theta(k; i, j) + \sup \left| \frac{\partial^2 f_0^k}{\partial I_i \partial I_j}(\underline{I}') \right| \leq \theta(k+1; i, j) \quad (3.35)$$

on the smaller domain $W(\rho_{k+1}, \xi_{k+1}, V_k)$, verifying (2.8).

4. Decay Estimates

In this section we present a series of lemmas leading to the proof of the decay estimates (2.9) in Section 5. A central idea is the use of a "random walk" expansion [2, 17] which controls derivatives that arise in inverting (3.4). We first estimate $\frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z}) \equiv (z_j/z'_i) \frac{\partial z'_i}{\partial z_j}(\underline{I}', \underline{z})$.

Lemma 4.1: $\frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z})$ is analytic on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ and satisfies

$$\sup \left| \frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z}) \right| \leq \delta_{ij} (1 + 2^{-6} N^{-1}) + (1 - \delta_{ij}) D_2 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \quad (4.1)$$

$$\text{with } D_2 = 2^3 \epsilon_0 C N e^{(2+\ell n 2)L_k} \left[2^{10} E_k C (\epsilon_0 \rho_0^{-1})^{(3/2)^k - 1} \right]^{\beta_k}.$$

Proof: By (3.4), on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$

$$\frac{\partial z'_i}{\partial z_j}(\underline{I}', \underline{z}) = \left\{ \delta_{ij} + z_i \left(i \frac{\partial^2 \Phi^k}{\partial I'_i \partial z_j}(\underline{I}', \underline{z}) \right) \right\} \exp \left(i \frac{\partial \Phi^k}{\partial I'_i}(\underline{I}', \underline{z}) \right), \quad (4.2)$$

so

$$(z_j/z'_i) \frac{\partial z'_i}{\partial z_j}(\underline{I}', \underline{z}) = \left\{ \delta_{ij} + \frac{\partial^2 \Phi^k}{\partial I'_i \partial \phi'_j}(\underline{I}', \underline{z}) \right\}. \quad (4.3)$$

The analyticity of $\frac{\partial \phi'_i}{\partial \phi'_j}$ follows immediately from the fact that $\underline{\Delta}(\underline{I}', \underline{z})$ and

$\frac{\partial^2 \Phi^k}{\partial I'_i \partial \phi'_j}(\underline{I}', \underline{z})$ are analytic on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. By (3.9) and a dimensional estimate on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$

$$\sup \left| \frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z}) \right| \leq 1 + 2^{10} \epsilon_k E_k C^2 N e^{(2+\ell n 2)L_k} \quad (4.4)$$

Note that (3.12) insures that the r.h.s. of (4.4) is bounded by $1 + 2^{-6} N^{-1}$. If $i \neq j$, then on $W(\beta_k, \xi_k^{-\delta}, V_k)$,

$$\begin{aligned} \sup \left| \frac{\partial \phi_i}{\partial \phi_j}(\underline{I}', \underline{z}) \right| &\leq \sup \sum_{\substack{\underline{v} \in \mathbf{X}_k \\ \underline{v} \neq 0}} \left| \left\{ \frac{\partial f_{\underline{v}}^k}{\partial I_i'}(\underline{I}') \left(i < \omega^k(\underline{I}'), \underline{v} > \right)^{-1} \right. \right. \\ &\quad \left. \left. - f_{\underline{v}}^k(\underline{I}') \left(i < \omega^k(\underline{I}'), \underline{v} > \right)^{-2} \left(i \frac{\partial}{\partial I_i'} < \omega^k(\underline{I}'), \underline{v} > \right) \right\} \right| |\nu_j| e^{(\xi_k^{-\delta})|\underline{v}|} \end{aligned} \quad (4.5)$$

By (2.9), and the fact that any term on the r.h.s. of (4.5) with $j \notin \text{supp } \underline{v}$ vanishes,

$$\sup \left| \frac{\partial f_{\underline{v}}^k}{\partial I_i'}(\underline{I}') \right| \leq \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{|i-j|} e^{-\xi_k |\underline{v}|}. \quad \text{Similarly, } \sup \left| \frac{\partial}{\partial I_i'} < \omega^k(\underline{I}'), \underline{v} > \right|$$

$$\leq |\underline{v}| \sup_{\ell \in \text{supp } \underline{v}} \theta(k; i, \ell), \text{ and}$$

$$\sup \left| f_{\underline{v}}^k(\underline{I}') \right| \leq \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{d(\text{supp } \underline{v})} e^{-\xi_k |\underline{v}|} \leq \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{|j-\ell|} e^{-\xi_k |\underline{v}|},$$

for any $\ell \in \text{supp } \underline{v}$. Combining these estimates we bound (4.5) by

$$\begin{aligned} &\sum_{\substack{\underline{v} \in \mathbf{X}_k \\ \underline{v} \neq 0}} \left\{ 2 \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{|i-j|} \exp(|\underline{v}| + L_k) \right. \\ &\quad \left. + 2^2 |\underline{v}| \epsilon_0^{\rho_0} C^2 \exp(2|\underline{v}| + 2L_k) \right. \\ &\quad \left. \times \sup_{\ell \in \text{supp } \underline{v}} \left[\theta(k; \ell, i) (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{|j-\ell|} \right] \right\} e^{-\delta |\underline{v}|}. \end{aligned} \quad (4.6)$$

Use the fact that $|\theta(k; \ell, i)| < (\epsilon_0^{\rho_0})^{-1} (\epsilon_0^{\rho_0})^{-1} (1-\eta_k)^{|\ell-i|}$, and our now standard estimate of $N \cdot 2^{2M} 2^L$ on the number of terms with $|\underline{v}| = M$ and $d(\text{supp } \underline{v}) = L$ to bound (4.6) by

$$\begin{aligned}
& 2\epsilon_0 C(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} (1+\epsilon_0 C)^N \sum_{L=1}^{L_k} \sum_{M=1}^{M_k} M 2^{2M} 2^L \exp[2L_k + (2-\delta)M] \\
& \leq 2^3 \epsilon_0 C^N e^{(2+\ln 2)L_k} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|}.
\end{aligned} \tag{4.7}$$

In the last step we summed the geometric series and used $\epsilon_0 < \rho_0 2^{-11} \lambda N^{-2} = C^{-1}$ (by (1.15)) so that $(1 + \epsilon_0 C) < 2$. Again (3.9) and a dimensional estimate give

$\sup \left| \frac{\partial \phi'_i}{\partial \phi_j} \right| \leq 2^{10} \epsilon_k E_k C^2 N \exp[(2 + \ln 2)L_k]$ on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ (if $i \neq j$). Combining this observation with (4.7) and the fact that if $\chi < \min(c_1, c_2)$ one has $\chi < c_1^{(1-\beta)} c_2^\beta$ for $\beta \in [0, 1]$, we obtain, for $i \neq j$

$$\begin{aligned}
& \sup \left| \frac{\partial \phi'_i}{\partial \phi_j}(\underline{I}', \underline{z}) \right| \\
& \leq \left[2^3 \epsilon_0 C^N e^{(2+\ln 2)L_k} \right]^{(1-\beta_k)} \left[2^{10} \epsilon_k E_k C^2 N e^{(2+\ln 2)L_k} \right]^{\beta_k} \\
& \quad \chi (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|},
\end{aligned} \tag{4.8}$$

which when combined with (4.4) verifies (4.1).

Let $(D\phi')(\underline{I}', \underline{z})$ be the $N \times N$ matrix whose elements are the functions

$$\left[(D\phi')(\underline{I}', \underline{z}) \right]_{ij} = \frac{\partial \phi'_i}{\partial \phi_j}(\underline{I}', \underline{z}). \quad \text{Define } \frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z}') \equiv (z'_j/z'_i) \frac{\partial z_i}{\partial z'_j}(\underline{I}', \underline{z}').$$

Lemma 4.2: If $(D\phi')^{-1}(\underline{I}', \underline{z})$ is the inverse matrix of $(D\phi)(\underline{I}', \underline{z})$ then

$$\frac{\partial \phi'_i}{\partial \phi'_i}(\underline{I}', \underline{z}') = \left[(D\phi')^{-1}(\underline{I}', \underline{z}' \exp(i\Delta(\underline{I}', \underline{z}')) \right]_{ij} \tag{4.9}$$

for $(\underline{I}', \underline{z}') \in W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$.

Proof: If Z' is the Jacobian matrix for the transformation $\underline{z} \rightarrow \underline{z}'(\underline{I}', \underline{z})$, then by the analytic inverse function theorem (see [13] for example),

$$[Z'^{-1}]_{ij} = \frac{\partial z_j}{\partial z'_i}(\underline{I}', \underline{z}'(\underline{I}', \underline{z})). \text{ Elementary linear algebra yields}$$

$$\det(D\phi') = \sum_{\pi} \text{sgn}(\pi) \prod_{j=1}^N (z_j/z'_{\pi(j)}) \frac{\partial z'_j}{\partial z_j} = \prod_{j=1}^N (z_j/z'_j) \det(\mathbf{Z}') , \quad (4.10)$$

with π a permutation of $\{1, \dots, N\}$. Similarly

$$\text{cofactor}(D\phi')_{ij} = \left(\prod_{k \neq j} z_k \right) \left(\prod_{k \neq i} z'_k \right)^{-1} \text{cofactor}(\mathbf{Z}')_{ij} \quad (4.11)$$

It was proved in Appendix 1 that $|\det(D\phi')| \neq 0$. Thus

$$\begin{aligned} [(D\phi')^{-1}(\underline{I}', \underline{z})]_{ij} &= (\det(D\phi'))^{-1} (-1)^{i+j} \text{cofactor}(D\phi')_{ij} \\ &= (z'_i/z_j) (\det(\mathbf{Z}'))^{-1} (-1)^{i+j} \text{cofactor}(\mathbf{Z}')_{ij} \\ &= (z'_i/z_j) \left[\frac{\partial z_j}{\partial z'_i}(\underline{I}', \underline{z}'(\underline{I}', \underline{z})) \right]. \end{aligned} \quad (4.12)$$

Since $(\underline{I}', \underline{z}) \rightarrow (\underline{I}', \underline{z}'(\underline{I}', \underline{z}))$ is invertible on $W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$, with inverse given by (3.10) we may rewrite (4.12) as

$$(z'_i/z_j) \left[\frac{\partial z_j}{\partial z'_i}(\underline{I}', \underline{z}') \right] = \left[(D\phi')^{-1}(\underline{I}', \underline{z}' \exp[i\Delta(\underline{I}', \underline{z}')] \right]_{ij} , \quad (4.13)$$

proving Lemma 4.2.

We now show that $\frac{\partial \phi_i}{\partial \phi'_j}(\underline{I}', \underline{z})$ decays nearly as fast as $\frac{\partial \phi'_i}{\partial \phi'_j}(\underline{I}', \underline{z})$.

Lemma 4.3: On $W(\tilde{\rho}_k, \xi_k^{-3\delta}, V_k)$ one has

$$\sup \left| \frac{\partial \phi_i}{\partial \phi_j} \right| \leq 2^3 \left[2^4 (\epsilon_0^{\rho_0 - 1}) \right]^{(1-\eta_k)(1-\beta_k)|i-j|}. \quad (4.14)$$

The proof of this lemma depends on the following expansion, due in its present form to [2].

Lemma 4.4: Let Λ be a purely diagonal $N \times N$ matrix, M a purely off diagonal one. Suppose also that $\Lambda_{ii} \geq \xi \sum_j |M_{ij}|$ for all i and some $\xi > 1$. Then

$$[(\Lambda - M)^{-1}]_{ij} = \sum_{\Omega: i \rightarrow j} \Lambda_{jj}^{-1} \left(\prod_{\ell \in L} \Lambda_{\ell\ell}^{-n(\ell, \Omega)} \right) \left(\prod_{s \in \Omega} M_s \right). \quad (4.15)$$

On the r.h.s. of (4.14), Ω is a random walk on the lattice $L = \{1, \dots, N\}$, i.e., a set of pairs $\{(i_1, i_2), \dots, (i_k, i_{k+1})\}$, $i_j \in \{1, \dots, N\}$. Each of the pairs is referred to as a step, s , of the walk, $|\Omega|$ is the number of steps in the walk, and $\Omega: i \rightarrow j$ means $i_1 = i$, $i_{k+1} = j$. Finally $M_s = M_{(i_j, i_{j+1})}$, $|s| = |i_{j+1} - i_j|$, and $n(j, \Omega)$ is the number of times j appears as the first element of some step in Ω . Note that the matrix M is not symmetric here (nor is it necessarily positive) as it was in [2], but this just requires us to keep track of the direction of each step in the walk Ω .

Set $\Lambda_{ij} = \delta_{ij} \frac{\partial \phi_i'}{\partial \phi_j}(\underline{I}', \underline{z})$ and $M_{ij} = (\delta_{ij} - 1) \frac{\partial \phi_i'}{\partial \phi_j}(\underline{I}', \underline{z})$. By (4.3) and a dimensional estimate on $\frac{\partial^2 \Phi^k}{\partial I_i' \partial \phi_j}(\underline{I}', \underline{z})$ we have

$$|\Lambda_{ii}| \geq 1 - 2^{10} \epsilon_k E_k C^2 N e^{3L_k} \geq 1 - 2^{-6} N^{-1}, \quad (4.16)$$

on $W(\tilde{\rho}_k, \xi_k^{-2\delta}, V_k)$, where the last inequality used (3.12). By the same dimensional estimate on $\frac{\partial^2 \Phi^k}{\partial I_i' \partial \phi_j}$ we find

$$|M_{ij}| \leq 2^{10} \epsilon_k E_k C^2 N e^{3L_k} \leq 2^{-6} N^{-1}, \quad (4.17)$$

so $\sum_j |M_{ij}| \leq (1/2) |\Lambda_{ii}|$ for all i , verifying the hypothesis of Lemma 4.4. Combining Lemma 4.2 and Lemma 4.4 we find

$$\frac{\partial \phi_j}{\partial \phi'_i}(\underline{I}', \underline{z}') = \sum_{\Omega: i \rightarrow j} \Lambda_{jj}^{-1} \left(\prod_{\ell \in L} \Lambda_{\ell\ell}^{-n(\ell, \Omega)} \right) \left(\prod_{s \in \Omega} M_s \right), \quad (4.18)$$

where the Λ_{ii} and the M_{ij} are evaluated at $(\underline{I}', \underline{z}(\underline{I}', \underline{z}')) = (\underline{I}', \underline{z}' \exp(i\underline{\Delta}))$. By Lemma 4.1, $\sup |\Lambda_{ii}| \geq 1 - 2^{-6} N^{-1}$ and $\sup |M_{ij}| \leq D_2(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$. Thus, on $W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$

$$\sup \left| \frac{\partial \phi_j}{\partial \phi'_i}(\underline{I}', \underline{z}') \right| \leq \sum_{\Omega: i \rightarrow j} (1 - 2^{-6} N^{-1})^{-1} (1 - 2^{-6} N^{-1})^{-\sum_{\ell} n(\ell, \Omega)} \prod_{s \in \Omega} \left[D_2(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|s|} \right], \quad (4.19)$$

where $|s|$ = length of the step s . Define $L(\Omega) = \sum_{s \in \Omega} |s|$, and $|\Omega| = \sum_{\ell=1}^N n(\ell, \Omega)$. Since every step has length at least one we must have $L(\Omega) \geq |\Omega|$. Furthermore, every walk from i to j must have $L(\Omega) \geq |i-j|$. We rewrite the r.h.s. of (4.19) as

$$\sum_{L=|i-j|}^{\infty} \sum_{n=0}^L \sum_{\substack{\Omega: i \rightarrow j \\ L(\Omega)=L \\ |\Omega|=n}} (1 - 2^{-6} N^{-1})^{-(n+1)} D_2^n(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)L}. \quad (4.20)$$

To estimate the number of walks with n steps and total length L , note that there are at most 2^L ways of distributing the total length among the n steps. Furthermore, there are no more than 2^n walks with n steps of fixed length (each step either to the right or the left) so (4.20) is bounded by

$$\begin{aligned} & \sum_{L=|i-j|}^{\infty} \sum_{n=0}^L 2^L 2^n (1 - 2^{-6} N^{-1})^{-(n+1)} D_2^n(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)L} \\ & \leq 2^3 2^{|i-j|} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \end{aligned} \quad (4.21)$$

provided $D_2 < 2^{-3}$ and $2(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} < (1/2)$. From the definition of D_2 and the fact that $(1-\eta_k)(1-\beta_k) \geq 2^{-2}$ this is seen to follow from

$$\epsilon_0 < \rho_0 2^{-60} (\epsilon_0 \rho_0^{-1})^{-2} \lambda^3 N^{-120} \quad (4.22)$$

both of these being implied by (1.15). But (4.21) implies (4.14), completing the proof of the lemma.

The final estimate of the present section is

Lemma 4.5: On $W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$ one has

$$\sup \left| \frac{\partial \Xi_m}{\partial \phi_i'}(\underline{I}', \underline{z}') \right| \leq \epsilon_0 D_3 \left[2^4 (\epsilon_0 \rho_0^{-1}) \right]^{(1-\eta_k)(1-\beta_k)|i-j|} \quad (4.23)$$

where
$$D_3 = 2^5 \rho_0 C N^2 e^{(2+\ln 2)L_k} \left[(\rho_k \rho_0^{-1}) (\epsilon_0 \rho_0^{-1})^{(3/2)^k - 1} \right]^{\beta_k} .$$

Proof: Recall that on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$,

$$\Xi_j(\underline{I}', \underline{z}') = \frac{\partial \Phi^k}{\partial \phi_j'}(\underline{I}', \underline{z}' \exp(i \underline{\Delta}(\underline{I}', \underline{z}'))). \quad (4.24)$$

Applying the chain rule gives

$$\begin{aligned} \frac{\partial \Xi_j}{\partial \phi_i'}(\underline{I}', \underline{z}') &= \left(i z_i' \frac{\partial z_j}{\partial z_i'}(\underline{I}', \underline{z}') \right) \left(\frac{\partial \Phi^k}{\partial z_j}(\underline{I}', \underline{z}' e^{i \underline{\Delta}}) \right) \\ &\quad + \sum_m \left(i z_i' z_j \frac{\partial z_m}{\partial z_i'}(\underline{I}', \underline{z}') \right) \frac{\partial^2 \Phi^k}{\partial z_m \partial z_j}(\underline{I}', \underline{z}' e^{i \underline{\Delta}}) \\ &= \sum_m \frac{\partial \phi_m}{\partial \phi_i'}(\underline{I}', \underline{z}') \frac{\partial^2 \Phi^k}{\partial \phi_m \partial \phi_j}(\underline{I}', \underline{z}' e^{i \underline{\Delta}}) . \end{aligned} \quad (4.25)$$

Note that when rewritten in terms of the functions $\frac{\partial \phi_m}{\partial \phi_i^1}$, this last expression has the same form as if we had treated Φ^k as a function of $\underline{\phi}$ rather than $\underline{z}$ and applied the chain rule in the $\underline{\phi}$ variables, which is one reason for our definition of the functions $\frac{\partial \phi_i}{\partial \phi_j^1}$. On $W(\tilde{\rho}_k, \xi_k^{-\delta}, V_k)$

$$\left| \frac{\partial^2 \Phi^k}{\partial \phi_m \partial \phi_j^1}(\underline{I}', \underline{z}) \right| \leq \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \left| \frac{f_{\underline{\nu}}^k(\underline{I}')}{i < \omega_k(\underline{I}', \underline{\nu})} \right| |\nu_m| |\nu_j| e^{(\xi_k^{-\delta})|\underline{\nu}|} \quad (4.26)$$

All terms in this sum vanish unless both $m, j \in \text{supp } \underline{\nu}$ so applying (2.9), the r.h.s. of (4.26) is bounded by

$$\sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k)|m-j|} \cdot 2C e^{L_k |\underline{\nu}|} e^{(1-\delta)|\underline{\nu}|} \quad (4.27)$$

The sum over $\underline{\nu}$ is bounded in the standard fashion and we find that (4.26) is less than or equal to

$$2^2 \epsilon_0^{\rho_0} C N e^{(1+\ln 2)L_k} (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k)|m-j|} \quad (4.28)$$

If instead of bounding $\left| f_{\underline{\nu}}^k(\underline{I}') \right|$ by (2.9) we had used (2.6) we would have found that (4.26) is bounded by

$$2^2 \epsilon_k^{\rho_k} C e^{(1+\ln 2)L_k} N. \quad (4.29)$$

Combining (4.28) and (4.29) we find

$$\left| \frac{\partial^2 \Phi^k}{\partial \phi_m \partial \phi_j^1}(\underline{I}', \underline{z}) \right| \leq 2^2 \epsilon_0^{\rho_0} C N e^{(1+\ln 2)L_k} \left[(\rho_k^{\rho_0^{-1}}) (\epsilon_0^{\rho_0^{-1}})^{(3/2)^k - 1} \right]^{\beta_k} \times (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k)(1-\beta_k)|m-j|} \quad (4.30)$$

for all $(\underline{I}', \underline{z}) \in W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. Combining the result of Lemma 4.4 with (4.30) we see that each term on the l.h.s. of (4.25) may be bounded by

$$2^3 \cdot 2^2 \epsilon_{0\rho_0}^{\text{CN}e} (1 + \ln 2)^{L_k} \left[(\rho_k \rho_0^{-1}) (\epsilon_{0\rho_0}^{-1})^{(3/2)^k - 1} \right]_k^{\beta_k} \times (2^4 \epsilon_{0\rho_0}^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \quad (4.31)$$

for $(\underline{I}', \underline{z}') \in W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$. Bounding the number of terms in the sum by N , the l.h.s. of (4.25) is bounded by

$$\epsilon_{0D_3} (2^4 \epsilon_{0\rho_0}^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|},$$

which verifies (4.24).

5. The Renormalized Interactions

In this section we show that the renormalized interactions obey (2.9). The principal tool is

Proposition 5.1: On $W(\rho_{k+1}, \xi_{k+1}, V_k)$ one has

$$\sup \left| \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} f^m(\underline{I}', \underline{z}') \right| \leq (1/3) \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1 - \eta_{k+1})^{|i-j|} \quad (5.1a)$$

$$\sup \left| \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial I'_j} f^m(\underline{I}', \underline{z}') \right| \leq (1/3) \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1 - \eta_{k+1})^{|i-j|} \quad (5.1b)$$

$$\sup \left| \frac{\partial}{\partial I'_i} \frac{\partial}{\partial I'_j} f^m(\underline{I}', \underline{z}') \right| \leq (1/3) \epsilon_0^{\rho_0} (\epsilon_0^{\rho_0})^{-1} (1 - \eta_{k+1})^{|i-j|}, \quad (5.1c)$$

where $m = \text{I, II, III}$.

We prove (5.1a) for the case $m = \text{II}$ below as an illustration. The proofs of the remaining cases are presented in [18]. We first show that these estimates lead to (2.9).

From (3.22) we have

$$f_{\underline{\nu}}^{k+1}(\underline{I}') = \left(\frac{1}{2\pi i} \right)^N \oint \frac{d\underline{z}'}{(\prod_k z'_k)} \underline{z}'^{-\underline{\nu}} \left\{ f^{\text{I}}(\underline{I}', \underline{z}') + f^{\text{II}}(\underline{I}', \underline{z}') + f^{\text{III}}(\underline{I}', \underline{z}') \right\}, \quad (5.2)$$

where each integral on the r. h. s. of (5.2) runs over the contour $|z_i| = 1$. Take i and j respectively the leftmost and rightmost points in $\text{supp } \underline{\nu}$. Rewrite (5.2) as

$$\begin{aligned} f_{\underline{\nu}}^{k+1}(\underline{I}') &= \frac{(-1)^{\nu_i \nu_j}}{\nu_i \nu_j} \left(\frac{1}{2\pi i} \right)^N \oint \frac{d\underline{z}'}{(\prod_k z'_k)} \underline{z}'^{-\underline{\nu}} \\ &\quad \times \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} \left\{ f^{\text{I}}(\underline{I}', \underline{z}') + f^{\text{II}}(\underline{I}', \underline{z}') + f^{\text{III}}(\underline{I}', \underline{z}') \right\}. \end{aligned} \quad (5.3)$$

(This is just the multivariable version of the observation that if $\tilde{g}_n$ are the coefficients in the Laurent expansion of the analytic function $g(z)$, then

$(\widetilde{zg'(z)})_n = \frac{1}{n} \tilde{g}_n$.) Estimate (5.1a) guarantees that on $W(\rho_{k+1}, \xi_{k+1}, V_k)$

$$\sup \left| \frac{1}{|\nu_i| |\nu_j|} \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} \left\{ f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \right\} \right| \leq \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} . \quad (5.4)$$

Thus we have written $f_{\underline{\nu}}^{k+1}(\underline{I}')$ as the $\underline{\nu}^{\text{th}}$ Laurent coefficient of a function bounded in magnitude by $\epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|}$. Cauchy's theorem guarantees that

$$\sup \left| f_{\underline{\nu}}^{k+1}(\underline{I}') \right| \leq \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} e^{-\xi_{k+1}|\underline{\nu}|} , \quad (5.5)$$

with this supremum again running over $W(\rho_{k+1}, \xi_{k+1}, V_{k+1})$. This verifies (2.9a).

The estimates (2.9b) and (2.9c) follow in like fashion. To obtain (2.9b) assume that ℓ is the point in $\text{supp } \underline{\nu}$ farthest from j . Then just as above we find

$$\begin{aligned} \frac{\partial f_{\underline{\nu}}^{k+1}}{\partial I'_j}(\underline{I}') &= \left(\frac{1}{2\pi i} \right)^N \oint \left(\prod_j \frac{d\underline{z}'}{z'_j} \right) \underline{z}'^{-\underline{\nu}} (1/i\nu_\ell) \\ &\quad \times \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial I'_j} \left\{ f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \right\} . \end{aligned} \quad (5.6)$$

By (5.1b) we have

$$\sup \left| \frac{1}{|\nu_\ell|} \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial I'_j} \left\{ f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \right\} \right| \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|\ell-j|} , \quad (5.7)$$

so Cauchy's theorem implies

$$\sup \left| \frac{\partial f_{\underline{\nu}}^{k+1}}{\partial I_j}(\underline{I}') \right| \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|\ell-j|} e^{-\xi_{k+1}|\underline{\nu}|} \quad (5.8)$$

on $W(\rho_{k+1}, \xi_{k+1}, V_k)$.

Finally, by (5.2c) one has

$$\sup \left| \frac{\partial}{\partial I'_i} \frac{\partial}{\partial I'_j} \left\{ f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \right\} \right| \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|}, \quad (5.9)$$

so that

$$\begin{aligned} & \sup \left| \frac{\partial}{\partial I'_i} \frac{\partial}{\partial I'_j} f_{\underline{\nu}}^{k+1}(\underline{I}') \right| \\ & \leq \sup \left| \left(\frac{1}{2\pi i} \right)^N \oint \frac{d\underline{z}'}{\left(\prod_k z'_i \right)} z'^{-\underline{\nu}} \frac{\partial}{\partial I'_i} \frac{\partial}{\partial I'_j} \left\{ f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}') \right\} \right| \\ & \leq \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} e^{-\xi_{k+1}|\underline{\nu}|} \end{aligned} \quad (5.10)$$

on $W(\rho_{k+1}, \xi_{k+1}, V_k)$. This completes the verification of (2.9).

We now show how the estimates of the previous section are used to prove Proposition 5.1. Recall that

$$f^{II}(\underline{I}', \underline{z}') = \sum_{m=1}^N \int_0^1 dt \frac{\partial f^{k[\leq]}}{\partial I_j}(\underline{I}' + t\underline{z}, \underline{z}' e^{i\Delta}) \Xi_j. \quad (5.11)$$

Applying the chain rule (and assuming that $i \neq j$, since if $i = j$, (5.2a) follows immediately from (3.27) - (3.30) and a pair of dimensional estimates), we have

$$\begin{aligned}
\frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} f^{\Pi}(\underline{I}', \underline{z}') &= \int_0^1 dt \left\{ \sum_{m=1}^N \frac{\partial f^k[\leq]}{\partial I_m} (\underline{I}' + t \underline{\Xi}, \underline{z}' e^{i\Delta}) \frac{\partial^2 \Xi_m}{\partial \phi'_i \partial \phi'_j} (\underline{I}', \underline{z}') \right. \\
&+ \int_0^1 dt \sum_{m,n=1}^N \frac{\partial^2 f^k[\leq]}{\partial I_m \partial \phi_n} (\underline{I}' + t \underline{\Xi}, \underline{z}' e^{i\Delta}) \cdot \frac{\partial \phi_n}{\partial \phi'_i} (\underline{I}', \underline{z}') \cdot \frac{\partial \Xi_m}{\partial \phi'_j} (\underline{I}', \underline{z}') \\
&+ \int_0^1 dt \sum_{m,n=1}^N t \frac{\partial^2 f^k[\leq]}{\partial I_m \partial I_n} (\underline{I}' + t \underline{\Xi}, \underline{z}' e^{i\Delta}) \cdot \frac{\partial \Xi_n}{\partial \phi'_i} (\underline{I}', \underline{z}') \cdot \frac{\partial \Xi_m}{\partial \phi'_j} (\underline{I}', \underline{z}') \\
&+ \int_0^1 dt \sum_{m,n=1}^N \frac{\partial^2 f^k[\leq]}{\partial I_m \partial \phi_n} \cdot \frac{\partial \phi_n}{\partial \phi'_j} \cdot \frac{\partial \Xi_m}{\partial \phi'_i} + \int_0^1 dt \sum_{m,n=1}^N \frac{\partial^2 f^k[\leq]}{\partial \phi_n \partial I_m} \cdot \frac{\partial^2 \phi_n}{\partial \phi'_i \partial \phi'_j} \cdot \Xi_m \\
&+ \int_0^1 dt \sum_{m,n,p=1}^N \frac{\partial^3 f^k[\leq]}{\partial \phi_p \partial \phi_n \partial I_m} \cdot \frac{\partial \phi_n}{\partial \phi'_j} \cdot \frac{\partial \phi_p}{\partial \phi'_i} \cdot \Xi_m \\
&+ \int_0^1 dt \sum_{m,n,p=1}^N t \frac{\partial^3 f^k[\leq]}{\partial I_p \partial \phi_n \partial I_m} \cdot \frac{\partial \phi_n}{\partial \phi'_j} \cdot \frac{\partial \Xi_p}{\partial \phi'_i} \cdot \Xi_m \\
&+ \int_0^1 dt \sum_{m,n=1}^N t \frac{\partial^2 f^k[\leq]}{\partial I_m \partial I_n} \cdot \frac{\partial^2 \Xi_n}{\partial \phi'_i \partial \phi'_j} \cdot \Xi_m + \int_0^1 dt \sum_{m,n=1}^N t \frac{\partial^2 f^k[\leq]}{\partial I_m \partial I_n} \cdot \frac{\partial \Xi_n}{\partial \phi'_j} \cdot \frac{\partial \Xi_m}{\partial \phi'_i} \\
&+ \int_0^1 dt \sum_{m,n,p=1}^N t \frac{\partial^3 f^k[\leq]}{\partial I_m \partial I_n \partial \phi_p} \cdot \frac{\partial \phi_p}{\partial \phi'_i} \cdot \frac{\partial \Xi_n}{\partial \phi'_j} \cdot \Xi_m \\
&+ \int_0^1 dt \sum_{m,n,p=1}^N t^2 \frac{\partial^3 f^k[\leq]}{\partial I_m \partial I_n \partial I_p} \cdot \frac{\partial \Xi_n}{\partial \phi'_j} \cdot \frac{\partial \Xi_p}{\partial \phi'_i} \cdot \Xi_m \left. \right\}, \tag{5.12}
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial^2 \phi_n}{\partial \phi'_i \partial \phi'_j} (\underline{I}', \underline{z}') &\equiv i \left[z_n^{-1} z'_i z'_j \frac{\partial^2 z_n}{\partial z'_i \partial z'_j} (\underline{I}', \underline{z}') - z_n^{-2} z'_i z'_j \frac{\partial z_n}{\partial z'_i} (\underline{I}', \underline{z}') \frac{\partial z_n}{\partial z'_j} (\underline{I}', \underline{z}') \right. \\
&\quad \left. + \delta_{ij} z_n^{-1} \frac{\partial z_n}{\partial z'_i} (\underline{I}', \underline{z}') \right]. \tag{5.13}
\end{aligned}$$

We have omitted the arguments in the last few terms of (5.12) to save space. They are the same in each case as those of the corresponding functions in the first few terms. We also recall that the quantities $\frac{\partial \phi}{\partial \phi'_i}(\underline{I}', \underline{z}')$ are functions, defined in Section 4, and not derivatives.

The point we emphasize is that a factor of

$$(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} \left[1 - \left((3/32) \ell_n(3/2) \right) / \ell_n N \right]^{|i-j|}$$

can be extracted from each term in (5.12), which leads to the decay claimed in Proposition 5.1. Many of the factors in question were bounded in Section 4. The remainder are controlled by the following lemmas.

Lemma 5.2. On $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$ one has

$$\sup \left| \frac{\partial^{2k} f^{[\leq]} }{\partial \phi_i \partial \phi_j}(\underline{I}, \underline{z}) \right| \leq D_4 \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} |i-j| \quad (5.14a)$$

$$\sup \left| \frac{\partial^{2k} f^{[\leq]} }{\partial \phi_i \partial I_j}(\underline{I}, \underline{z}) \right| \leq D_5 \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} |i-j| \quad (5.14b)$$

$$\sup \left| \frac{\partial^{2k} f^{[\leq]} }{\partial I_j \partial I_i}(\underline{I}, \underline{z}) \right| \leq D_6 \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)} |i-j| \quad (5.14c)$$

with $D_4 = 2^{L_k+1} N \left[(\rho_k \rho_0^{-1}) (\epsilon_0 \rho_0^{-1}) (3/2)^{k-1} \right]^{\beta_k},$

$$D_5 = 2^{L_k+1} N \left[(\epsilon_0 \rho_0^{-1}) (3/2)^{k-1} \right]^{\beta_k},$$

and $D_6 = 2^{L_k+1} N \left[(\rho_k^{-1} \rho_0) (\epsilon_0 \rho_0^{-1}) (3/2)^{k-1} \right]^{\beta_k}.$

Proof:

$$\begin{aligned}
\sup \left| \frac{\partial^2 f^k}{\partial \phi_i \partial \phi_j} (\underline{I}, \underline{z}) \right| &= \sup_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \left| \sum_{\underline{\nu}} f_{\underline{\nu}}^k(\underline{I}(\nu_i, \nu_j)) \underline{z}^{\underline{\nu}} \right| \\
&\leq \sum_{L=|i-j|}^{L_k} \sum_{M=1}^{M_k} \epsilon_{0\rho_0} (\epsilon_{0\rho_0}^{-1})^{(1-\eta_k)|i-j|} \cdot N \cdot 2^L \cdot 2^{2M} M^2 e^{-\delta M},
\end{aligned} \tag{5.15}$$

where we used (2.9) and the fact that all terms in (5.15) for which either i or j are not in $\text{supp } \underline{\nu}$ vanish. The last two sums are easily performed and yield

$$\sup \left| \frac{\partial^2 f^k}{\partial \phi_i \partial \phi_j} (\underline{I}, \underline{z}) \right| \leq 2^{L_k+1} \epsilon_{0\rho_0}^N (\epsilon_{0\rho_0}^{-1})^{(1-\eta_k)|i-j|} \tag{5.16}$$

If in (5.15) we bound $f_{\underline{\nu}}^k$ by (2.6) instead of (2.9) we find

$$\sup \left| \frac{\partial^2 f^k}{\partial \phi_i \partial \phi_j} (\underline{I}, \underline{z}) \right| \leq 2^{L_k+1} \epsilon_{k\rho_k}^N, \tag{5.17}$$

so that by combining (5.16) and (5.17) we find

$$\begin{aligned}
&\sup \left| \frac{\partial^2 f^k}{\partial \phi_i \partial \phi_j} (\underline{I}, \underline{z}) \right| \\
&\leq 2^{L_k+1} \epsilon_{0\rho_0}^N \left[(\rho_{k\rho_0}^{-1}) (\epsilon_{0\rho_0}^{-1})^{(3/2)^k - 1} \right]^{\beta_k} (\epsilon_{0\rho_0}^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|},
\end{aligned} \tag{5.18}$$

Verifying (5.14a).

Similarly,

$$\begin{aligned}
\sup \left| \frac{\partial^2 f^k}{\partial \phi_i \partial I_j} (\underline{I}, \underline{z}) \right| &= \sup \left| \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{\partial f_{\underline{\nu}}^k}{\partial I_j} (\underline{I}) (\nu_i) \underline{z}^{\underline{\nu}} \right| \\
&\leq \sum_{L=1}^{L_k} \sum_{M=1}^{M_k} N 2^{L+2M} \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} M e^{-\delta M} \\
&\leq 2^{L_k+1} \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|},
\end{aligned} \tag{5.19}$$

since in the first equality $i \in \text{supp } \underline{\nu}$ for every nonvanishing term and

$$\left| \frac{\partial f_{\underline{\nu}}^k}{\partial I_j} (\underline{I}) \right| \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|j-\ell|} e^{-\xi_k |\underline{\nu}|} \leq \epsilon_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} e^{-\xi_k |\underline{\nu}|}$$

by (2.9), where ℓ is the point in $\text{supp } \underline{\nu}$ most distant from j . If we replace the bound on $\frac{\partial f_{\underline{\nu}}^k}{\partial I_j}$ coming from (2.9) by that of (2.6) we can bound the r.h.s. of (5.19) by $2^{L_k+1} \epsilon_k N$ and (5.14b) follows by combining this estimate with (5.19).

Finally one has

$$\begin{aligned}
\sup \left| \frac{\partial^2 f^k}{\partial I_i \partial I_j} (\underline{I}, \underline{z}) \right| &\leq \sup \left| \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{\partial^2 f_{\underline{\nu}}^k}{\partial I_i \partial I_j} (\underline{I}) \underline{z}^{\underline{\nu}} \right| \\
&\leq 2^{L_k+1} N \epsilon_0 \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|},
\end{aligned} \tag{5.20}$$

using (2.9) to bound $\frac{\partial^2 f_{\underline{\nu}}^k}{\partial I_i \partial I_j}$, and our standard estimate on the sum over $\underline{\nu}$. If

we use (2.6) plus a dimensional estimate to bound $\left| \frac{\partial^2 f}{\partial I_i \partial I_j} \right|$ by $2 \rho_k^{-1} \epsilon_k$ on

$W(\tilde{\rho}_k, \xi_k^{-\delta}, V_k)$ we can bound the l.h.s. of (5.20) by $2^{L_k+2} \epsilon_k \rho_k^{-1} N$, and (5.14c) follows by combining this bound with (5.20).

Lemma 5.3: On $W(\rho_{k+1}, \xi_{k+1}, V_k)$

$$\sup \left| \frac{\partial^2 \phi_j}{\partial \phi'_k \partial \phi'_i} (\underline{I}', \underline{z}') \right| \leq D_7 (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k) |k-i|} \quad (5.21)$$

for any $j = 1, \dots, N$ and $D_7 = 2^{17} E_k \text{CN} \tilde{\rho}_k^{-1} \rho_0 \left[\rho_k \rho_0^{-1} (\epsilon_0 \rho_0^{-1})^{(3/2)^k - 1} \right]^{-\beta_k}$

Proof: Note that $\frac{\partial^2 \phi_j}{\partial \phi'_k \partial \phi'_i} (\underline{I}', \underline{z}') = i z'_k \frac{\partial}{\partial z'_k} \left(z'_i z'_j^{-1} \frac{\partial z_j}{\partial z'_i} (\underline{I}', \underline{z}') \right)$. We will assume $i > j > k$. If $i \geq k \geq j$, then (5.21) follows by combining Lemma 4.3 with a dimensional estimate on $W(\tilde{\rho}_k, \xi_k^{-4\delta}, V_k)$. Similarly, if $k \geq i \geq j$ note that

$$z'_k \frac{\partial}{\partial z'_k} \left(z'_i z'_j^{-1} \frac{\partial z_j}{\partial z'_i} (\underline{I}', \underline{z}') \right) = z'_i \frac{\partial}{\partial z'_i} \left(z'_k z'_j^{-1} \frac{\partial z_j}{\partial z'_i} (\underline{I}', \underline{z}') \right) \text{ and apply Lemma 4.3 and a}$$

dimensional estimate to this last quantity. Using (4.18) to reexpress

$\left(z'_i z'_j^{-1} \frac{\partial z_j}{\partial z'_i} (\underline{I}', \underline{z}') \right)$ we obtain

$$\begin{aligned} \frac{\partial^2 \phi_j}{\partial \phi'_k \partial \phi'_i} (\underline{I}', \underline{z}') &= \sum_{\Omega: i \rightarrow j} \left\{ \left[\sum_{\ell \in L} (-n(\ell, \Omega) - \delta_{\ell, j}) \left(z'_k \frac{\partial}{\partial z'_k} (\Lambda_{\ell\ell}) \right) / (\Lambda_{\ell\ell}) \right. \right. \\ &\quad \left. \left. \times \Lambda_{jj}^{-1} \prod_{\ell \in L} \Lambda_{\ell\ell}^{-n(\ell, \Omega)} \prod_{s \in \Omega} M_s \right] \right. \\ &\quad \left. + \Lambda_{jj}^{-1} \left[\prod_{\ell \in L} \Lambda_{\ell\ell}^{-n(\ell, \Omega)} \right] \left[\sum_{s' \in \Omega} z'_k \frac{\partial}{\partial z'_k} (M_{s'}) \prod_{\substack{s \in \Omega \\ s \neq s'}} M_s \right] \right\}, \end{aligned} \quad (5.22)$$

with the factors of $\Lambda_{\ell\ell}$ and M_s on the r.h.s. of (5.22) evaluated at

$(\underline{I}', \underline{z}') e^{i\Delta(\underline{I}', \underline{z}')}$.

By the chain rule,

$$iz'_k \frac{\partial}{\partial z'_k} \left(\frac{\partial^2 \Phi^k}{\partial I_\ell \partial \phi_m} (I', \underline{z}' e^{i\Delta}) \right) = \sum_{n=1}^N \left(\frac{\partial^3 \Phi^k}{\partial \phi_n \partial \phi_m \partial I_\ell} (I', \underline{z}' e^{i\Delta}) \right) \left(\frac{\partial \phi_n}{\partial \phi'_k} (I', \underline{z}') \right) \quad (5.23)$$

Note again the efficacy of our functions $\frac{\partial \phi_n}{\partial \phi'_k}$. On $W(\tilde{\rho}_k/2, \xi_k - 2\delta, V_k)$

$$\sup \left| \frac{\partial^3 \Phi^k}{\partial \phi_n \partial \phi_m \partial I_\ell} (I', \underline{z}') \right| = \sup_{\substack{\underline{\nu} \in \mathbf{X}_k \\ \underline{\nu} \neq \underline{0}}} \left| \sum_{\underline{\nu} \in \mathbf{X}_k} \left\{ \frac{\partial f_{\underline{\nu}}^k(I')}{\partial I_\ell} \langle i \omega^k(I), \underline{\nu} \rangle^{-1} \right. \right. \quad (5.24)$$

$$\left. + i f_{\underline{\nu}}^k(I') \left\langle i \frac{\partial \omega^k(I')}{\partial I_\ell}, \underline{\nu} \right\rangle \left(i \langle \omega^k(I'), \underline{\nu} \rangle \right)^{-2} \right\} (-\nu_n \nu_m) \underline{z}^\nu \right|$$

$$\leq 2^{6\tilde{\rho}_k-1} E_k C^2 N e^{3L_k(\epsilon_0 \rho_0)(\epsilon_0 \rho_0^{-1})^{(1-\eta_k)|\ell-m|}},$$

where we used (2.9), (3.1), and a dimensional estimate to control the quantities in braces on the r.h.s. of the first equality, and then used our standard estimate to control the sum over $\underline{\nu}$. On the other hand, a dimensional estimate on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ combined with (4.3), (4.5), and (4.7) shows that

$$\sup \left| \frac{\partial^3 \Phi^k}{\partial \phi_n \partial \phi_m \partial I_\ell} (I', \underline{z}) \right| \leq 2^8 \epsilon_0 C N e^{3L_k(\epsilon_0 \rho_0^{-1})^{(1-\eta_k) \max(|\ell-m|, |\ell-n|)}} \quad (5.25)$$

If we combine (5.24), (5.25), and Lemma 4.3, we see that the r.h.s. of (5.23) is bounded by

$$2^{11} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k \left[2^4 (\epsilon_0 \rho_0^{-1}) \right]^{(1-\eta_k)(1-\beta_k) \max(|k-\ell|, |k-m|)}} \quad (5.26)$$

where we bounded the number of terms in the sum on the r.h.s. of (5.23) by N .

Since $\Lambda_{\ell\ell}(I', \underline{z}' e^{i\Delta}) = \frac{\partial^2 \Phi^k}{\partial I_\ell \partial \phi_\ell} (I', \underline{z}' e^{i\Delta})$ and

$M_{\ell m}(\underline{I}', \underline{z}' e^{i\Delta}) = (\delta_{\ell m} - 1) \frac{\partial^2 \Phi^k}{\partial I_\ell \partial \rho_m}(\underline{I}', \underline{z}' e^{i\Delta}),$ (5.26) immediately gives a bound on $z'_k \frac{\partial}{\partial z'_k}(\Lambda_{\ell\ell})$ and $z'_k \frac{\partial}{\partial z'_k}(M_{\ell m}).$

For a given walk Ω , let $\bar{\ell}$ be the leftmost point, ℓ , with $n(\ell, \Omega) \neq 0$. Similarly $\bar{r}$ is the rightmost point, ℓ , with $n(\ell, \Omega) \neq 0$. Let $\bar{\Omega} = \{\bar{\ell}, \bar{\ell}+1, \dots, \bar{r}-1, \bar{r}\}$. Substituting our estimates for derivatives of $\Lambda_{\ell\ell}$ and M_s into (5.22), using the estimates on $\Lambda_{\ell\ell}$ and M_s from Section 4, and noting that $\sum_{\ell} n(\ell, \Omega) = \sum_{s' \in \Omega} 1 = |\Omega|$ we bound the r.h.s. of (5.22) by

$$2^{12} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k} \sum_{\Omega: i \rightarrow j} \left\{ (2|\Omega|+1) \left[2^4 (\epsilon_0 \rho_0^{-1}) \right]^{(1-\eta_k)(1-\beta_k) \text{dist}(\bar{\Omega}, k)} \right\} \quad (5.27)$$

$$\times D_2^{|\Omega|-1} (1-2^{-6} N^{-1})^{-(|\Omega|+1)} \prod_{s \in \Omega} \left[2^4 (\epsilon_0 \rho_0^{-1}) \right]^{(1-\eta_k)(1-\beta_k) |s|},$$

where we used the fact that we may extract a factor of

$$2^{11} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k} \left[2^4 \epsilon_0 \rho_0^{-1} \right]^{(1-\eta_k)(1-\beta_k) \text{dist}(\bar{\Omega}, k)} \quad \text{from } z'_k \frac{\partial}{\partial z'_k}(\Lambda_{\ell\ell}) \text{ if}$$

$$n(\ell, \Omega) \neq 0, \text{ and a factor of } 2^{11} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k} \left[2^4 \epsilon_0 \rho_0^{-1} \right]^{(1-\eta_k)(1-\beta_k) [\text{dist}(\bar{\Omega}, k) + |s|]}$$

from $z'_k \frac{\partial}{\partial z'_k}(M_s)$. We break the sum in (5.27) into two parts, those terms for which the length, $L(\Omega)$, of the walk is greater than or equal to $|k-i|$, and those for which it is less than $|k-i|$.

In the first case, discard the factor of $\left[2^4 \epsilon_0 \rho_0^{-1} \right]^{(1-\eta_k)(1-\beta_k) \text{dist}(\bar{\Omega}, k)}$, and bound the number of walks with n steps and length L by $2^n 2^L$ as in Section 4. Then that part of (5.27) is bounded by

$$2^{12} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k} \sum_{L=|k-i|}^{\infty} \sum_{n=0}^L 2^n (2n+1) 2^{n+1} D_2^{n-1}$$

$$\times 2^L (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)L} \quad (5.28)$$

$$\leq 2^{17} D_2^{-1} \epsilon_0 E_k C^2 N^2 \rho_k^{-1} \rho_0 e^{3L_k} (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k) |k-i|}.$$

To estimate the part of the sum with $L(\Omega) < |k-i|$ extract from each term a factor of $(2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)[\text{dist}(\bar{\Omega}, j) + L(\Omega)]} \leq (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|k-i|}$. Then the sum is bounded by

$$\begin{aligned}
& 2^{12} \epsilon_0 E_k C^2 N \tilde{\rho}_k^{-1} \rho_0 e^{3L_k} (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|k-i|} \\
& \times \sum_{L=|i-j|}^{|k-i|-1} \sum_{n=0}^L 2^L 2^n (2n+1) D_2^{n-1} \cdot 2^n \quad (5.29) \\
& \leq 2^{16} \epsilon_0 E_k C^2 N D_2^{-1} \tilde{\rho}_k^{-1} \rho_0 e^{3L_k} (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|k-i|}.
\end{aligned}$$

Combining (5.28) and (5.29) yields (5.21).

Lemma 5.4: On $W(\rho_{k+1}, \xi_{k+1}, V_k)$

$$\sup \left| \frac{\partial^2 \Xi_m}{\partial \phi'_i \partial \phi'_j}(\underline{I}', \underline{z}') \right| \leq \epsilon_0 D_8 \left[2^8 \epsilon_0 \rho_0^{-1} \right]^{(1-\eta_k)(1-\beta_k)|i-j|} \quad (5.30)$$

for $m = 1, \dots, N$, $D_8 = 2^9 \rho_0 C N e^{2L_k} D_7$.

Proof: Recall that on $W(\tilde{\rho}_k, \xi_k^{-2\delta}, V_k)$, $\Xi_m(\underline{I}', \underline{z}') = \frac{\partial \Phi^k}{\partial \phi_m}(\underline{I}', \underline{z}' e^{i\Delta})$. By the chain rule,

$$\begin{aligned}
\frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} \Xi_m(\underline{I}', \underline{z}') &= \sum_{n=1}^N \left(\frac{\partial^2 \Phi^k}{\partial \phi_n \partial \phi_m}(\underline{I}', \underline{z}' e^{i\Delta}) \right) \left(\frac{\partial^2 \phi_n}{\partial \phi'_i \partial \phi'_j}(\underline{I}', \underline{z}') \right) \\
&+ \sum_{n,p=1}^N \left(\frac{\partial^3 \Phi^k}{\partial \phi_p \partial \phi_n \partial \phi_m}(\underline{I}', \underline{z}' e^{i\Delta}) \right) \left(\frac{\partial \phi_n}{\partial \phi'_j}(\underline{I}', \underline{z}') \right) \left(\frac{\partial \phi_p}{\partial \phi'_i}(\underline{I}', \underline{z}') \right)
\end{aligned} \quad (5.31)$$

On $W(\tilde{\rho}_k, \xi_k^{-\delta}, V_k)$

$$\frac{\partial^2 \Phi^k}{\partial \phi_n \partial \phi_m}(\underline{I}', \underline{z}) = \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{f_{\underline{z}}^k(\underline{I}')}{i < \omega_k(\underline{I}'), \underline{\nu} >} (-\nu_n \nu_m) \underline{z}^{\underline{\nu}}, \quad (5.32)$$

so

$$\begin{aligned} \sup \left| \frac{\partial^2 \Phi^k}{\partial \phi_n \partial \phi_m} \right| &\leq \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} 2C e^{|\underline{\nu}|} e^{L_k |\underline{\nu}|^2} e^{-\delta |\underline{\nu}|} (\epsilon_0^{\rho_0}) (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k) |n-m|} \\ &\leq 2^2 (\epsilon_0^{\rho_0}) C N e^{2L_k (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k) |n-m|}}, \end{aligned} \quad (5.33)$$

the first inequality resulting from applying (2.9) to bound the factor of $f_{\underline{\nu}}^k(\underline{I}')$ and the fact that all terms in the sum with either m or $n \notin \text{supp } \underline{\nu}$ vanish. The second inequality uses our standard bound to control the sum over $\underline{\nu}$.

Combining a pair of dimensional estimates on $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$ with (3.7) and (5.33) we also have

$$\sup \left| \frac{\partial^2 \Phi^k}{\partial \phi_n \partial \phi_m}(\underline{I}', \underline{z}) \right| \leq 2^6 \epsilon_k^{\rho_k} C N e^{2L_k}, \quad (5.34)$$

$$\sup \left| \frac{\partial^3 \Phi^k}{\partial \phi_n \partial \phi_m \partial \phi_p}(\underline{I}', \underline{z}) \right| \leq 2^6 (\epsilon_0^{\rho_0}) C N e^{2L_k (\epsilon_0^{\rho_0^{-1}})^{(1-\eta_k) |m-n|}}. \quad (5.35)$$

Inequality (5.35) combined with Lemma 4.3 allows us to bound each of the N^2 terms in the second sum in (5.31) by $2^9 \epsilon_0^{\rho_0} C N e^{2L_k [2^4 \epsilon_0^{\rho_0^{-1}}]^{(1-\eta_k)(1-\beta_k) |i-j|}}$ on $W(\tilde{\rho}_k, \xi_k - 3\delta, V_k)$. Similarly, on $W(\rho_{k+1}, \xi_{k+1}, V_k)$, (5.34) and Lemma 5.3 combine to bound each of the terms in the first sum by $2^6 \epsilon_k^{\rho_k} C N D_7 e^{2L_k (2^8 \epsilon_0^{\rho_0^{-1}})^{(1-\eta_k)(1-\beta_k) |i-j|}}$. Combining these two estimates leads immediately to (5.30).

We are now in a position to bound each of the terms in (5.12). For instance the first term is bounded by

$$N (2^{L_k+1} \epsilon_k N) (\epsilon_0 D_8) (2^8 \epsilon_0^{\rho_0^{-1}})^{(1-\eta_k)(1-\beta_k) |i-j|}, \quad (5.36)$$

on $W(\rho_{k+1}, \xi_{k+1}, V_k)$, where the factor of N bounds the number of terms in the

sum, $\frac{\partial f^k[\leq]}{\partial I_m}$ is bounded by (3.28) and $\frac{\partial^2 \Xi_m}{\partial \phi_i' \partial \phi_j'}$ is bounded by (5.30). Some algebra shows that (5.36) is bounded by

$$2^{-6}(\epsilon_0 \rho_0)(\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} \quad (5.37)$$

provided $\epsilon_0 < \rho_0 2^{-165} e^{-130} \lambda^7 (E_0 \rho_0^{-1})^{-6} N^{-150}$ which is satisfied by (1.15).

In like fashion bound the second and fourth terms in (5.12), using (5.14b) and Lemmas 4.3 and 4.5 by

$$\begin{aligned} N^2 D_5 \epsilon_0 2^3 \epsilon_0 D_3 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\ \leq 2^{-6} \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} \end{aligned} \quad (5.38)$$

provided $\epsilon_0 < \rho_0 e^{-40} (E_0 \rho_0^{-1})^{-3} \lambda^3 N^{-100}$, which again is insured by (1.15).

We bound the third and the ninth terms by

$$\begin{aligned} D_6 \epsilon_0 \rho_0^{-1} (\epsilon_0 D_3)^2 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\ \leq 2^{-6} \epsilon_0 \rho_0 (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|} \end{aligned} \quad (5.39)$$

provided $\epsilon_0 < \rho_0 2^{-144} e^{-144} (E_0 \rho_0^{-1})^{-8} \lambda^8 N^{-100}$ which again follows from (1.15).

Continuing in this vein we bound the fifth term by

$$N^2 (2^4 2^{L_k+1} \epsilon_k N) (D_7) (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} (2^2 \epsilon_k \rho_k C N e^{2L_k}) , \quad (5.40)$$

the sixth term by

$$2N^3 \rho_k^{-1} D_4 \epsilon_0 \rho_0^2 C \epsilon_k \rho_k e^{2L_k} N \cdot 2^3 (2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} , \quad (5.41)$$

the seventh and the tenth terms are bounded by

$$(N^3 2\tilde{\rho}_k^{-1})(2^3 \epsilon_0 D_3)(\epsilon_0 D_5)(2^2 \epsilon_k \rho_k \text{CN} e^{2L_k})(2^4 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \quad (5.42)$$

the eighth term by

$$N^2 2\tilde{\rho}_k^{-1} (2^{L_k+1} \epsilon_k N)(2^2 \epsilon_k \rho_k \text{CN} e^{2L_k}) \epsilon_0 D_8 (2^8 \epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \quad (5.43)$$

and the eleventh term by

$$2\tilde{\rho}_k^{-1} N^3 (\epsilon_0 D_3)^2 (2^2 \epsilon_k \rho_k \text{CN} e^{2L_k}) (\epsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}. \quad (5.44)$$

A fair amount of tedious algebra shows that (1.15) implies that each of the quantities

(5.40) - (5.44) is bounded by $2^{-6} (\epsilon_0 \rho_0) (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|}$. Putting the estimates on all these terms together we find

$$\sup \left| \frac{\partial}{\partial \phi'_i} \frac{\partial}{\partial \phi'_j} f^{\Pi}(\underline{I}', \underline{z}') \right| \leq (11) (2^{-6} (\epsilon_0 \rho_0) (\epsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})|i-j|}),$$

on $W(\rho_{k+1}, \xi_{k+1}, V_k)$, verifying the bound of (5.1a) in the case $m = \Pi$.

6. An Estimate on the Volume of Phase Space Lost at the k^{th} Iterative Step

We complete the induction argument of Section 2 by estimating $\text{vol}(V_{k-1} \setminus V_k)$. (Set $V_{-1} = V$.) Note that $\bar{V}_k \subset V_k$ and $\bar{V}_k$ is obtained from V_{k-1} by first omitting the set of points, B , whose distance from the boundary of V_{k-1} is less than $\tilde{\rho}_k$, and then omitting all resonant points, $R(k, h^k, V_{k-1})$; thus

$$\text{vol}(V_{k-1} \setminus V_k) \leq \text{vol } B + \text{vol } R(k, h^k, V_{k-1}) . \quad (6.1)$$

It was shown in [5, 11] that

$$\text{vol } B \leq \left[1 - (1 - \tilde{\rho}_k / \rho_k)^N \right] \text{vol } V \quad (6.2)$$

$$\leq (1/2)(E_0 \rho_0^{-1})^{-1} \lambda (2^{11} N^2)^{-1} \exp \left[-2(M_k + L_k) \right] \text{vol } V ,$$

where the last inequality just used the definition of $\tilde{\rho}_k$.

If $\underline{\omega}^k$ is single-valued,

$$\text{vol}(R(k, h^k, V_{k-1})) = \int_{\underline{\omega}^k(R(k, h^k, V_{k-1}))} \left| \det \left[\frac{\partial \underline{\omega}^k}{\partial \underline{I}} \right]^{-1} \right| d\underline{\omega} \quad (6.3)$$

The single-valuedness of $\underline{\omega}^k$ follows from (2.7) and (2.8), and the fundamental theorem of calculus. Pick a path, γ , contained in $W(\rho_0, \xi_0, V)$ from $\underline{I}$ to $\underline{I}'$, of length at most $3|\underline{I} - \underline{I}'|$, and made up of line segments along which only one co-ordinate of $\underline{I}$ is allowed to vary. Then

$$|\underline{\omega}^0(\underline{I}) - \underline{\omega}^0(\underline{I}')| = \left| \int_{\gamma} d\underline{I}'' \cdot \frac{\partial \underline{\omega}^0}{\partial \underline{I}}(\underline{I}'') \right|. \quad (6.4)$$

By (2.6) and (2.7) we see that the integral on the r.h.s. of (6.4) satisfies

$$(1 - 3B_1 N^{-1} - N \epsilon_0 \rho_0^{-1}) |\underline{I} - \underline{I}'| \leq \left| \int_{\gamma} d\underline{I}'' \cdot \frac{\partial \underline{\omega}^0}{\partial \underline{I}}(\underline{I}'') \right| \leq (1 + 3B_1 N^{-1} + N \epsilon_0 \rho_0^{-1}) |\underline{I} - \underline{I}'| \quad (6.5)$$

It was shown in Appendix H of [11] that

$$\left| \sum_{j=0}^k \left(\frac{\partial f_0^j}{\partial \underline{I}} (\underline{I}') - \frac{\partial f_0^j}{\partial \underline{I}} (\underline{I}) \right) \right| \leq 3 |\underline{I} - \underline{I}'| \sum_{j=0}^k (\epsilon_j \rho_j^{-1}). \quad (6.6)$$

This last quantity is less than $6 |\underline{I} - \underline{I}'| \epsilon_0 \rho_0^{-1}$ since $(\epsilon_j \rho_j^{-1}) \leq (\epsilon_0 \rho_0^{-1})^j$, and $\epsilon_0 < \rho_0/2$. Thus

$$\begin{aligned} |\underline{\omega}^k(\underline{I}') - \underline{\omega}^k(\underline{I})| &= \left| \underline{\omega}^0(\underline{I}) - \underline{\omega}^0(\underline{I}') + \sum_{j=0}^{k-1} \left(\frac{\partial f_0^j}{\partial \underline{I}} (\underline{I}) - \frac{\partial f_0^j}{\partial \underline{I}} (\underline{I}') \right) \right| \\ &\geq (1 - 3B_1 N^{-1} - 6N \epsilon_0 \rho_0^{-1}) |\underline{I} - \underline{I}'| \\ &\geq (1/2) |\underline{I} - \underline{I}'|, \end{aligned} \quad (6.7)$$

since $\epsilon_0 < 2^{-4} \rho_0 N$, which insures $\underline{\omega}^k$ is single-valued.

We now estimate $\sup \left| \det \left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}} \right)^{-1} \right|$ on $W(\rho_k, \xi_k, V_{k-1})$.

$$\det \left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}} \right)^{-1} = \exp \left\{ -\text{tr} \ln \left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}} \right) \right\} = \exp \left\{ -\text{tr} \ln(1 - \mathbb{D}) \right\} \quad (6.8)$$

where from (2.7) and (2.8) we see that $\mathbb{D}$ is a matrix whose diagonal entries are bounded in magnitude by $B_1 N^{-1} + 2 \sum_{j=0}^{k-1} \epsilon_j \rho_j^{-1}$ and whose off diagonal entries are bounded by $\theta(k; i, j) \leq \epsilon_0 \rho_0^{-1}$ by (1.15). Hence, $\mathbb{D}^2$ will be a matrix with diagonal entries bounded by $2B_1 N^{-2}$, and off diagonal entries bounded by $(\epsilon_0 \rho_0^{-1}) N^{-1}$, provided $\epsilon_0 < \rho_0 2^{-4} N^{-2}$, which again follows from (1.15). An easy induction argument shows

$$[\mathbb{D}^k]_{ij} \leq \begin{cases} 2B_1 N^{-k} & \text{if } i = j \\ (\epsilon_0 \rho_0^{-1}) N^{-(k-1)} & \text{if } i \neq j \end{cases}. \quad (6.9)$$

Thus, $\text{tr} \mathbb{D}^k \leq 2B_1 N^{-(k-1)}$, and hence

$$\sup \left| \det \left(\frac{\partial \underline{\omega}^k}{\partial \underline{I}} \right)^{-1} \right| \leq \exp \sum_{j=1}^{\infty} \frac{(1)}{j} \sup \text{tr } \mathbb{D}^j \leq \exp(2^2 B_1) . \quad (6.10)$$

Inserting the estimate in (6.3) gives

$$\text{vol}(\mathbb{R}(k, h^k, V_{k-1})) \leq \exp(2^2 B_1) \int_{\underline{\omega}^k(\mathbb{R}(k, h^k, V_{k-1}))} d\underline{\omega} \quad (6.11)$$

$$\leq 2^2 \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \int_{\underline{\omega}:} d\underline{\omega} : \\ |\langle \underline{\omega}, \underline{\nu} \rangle| \leq (2C)^{-1} e^{-(3/2)|\underline{\nu}|} e^{-L_k} \\ \underline{\omega} \in \underline{\omega}^k(\mathbb{R}(k, h^k, V_{k-1}))$$

But

$$\int_{\substack{|\langle \underline{\omega}, \underline{\nu} \rangle| \leq (2C)^{-1} e^{-(3/2)|\underline{\nu}|} e^{-L_k} \\ \underline{\omega} \in \underline{\omega}^k(\mathbb{R}(k, h^k, V_{k-1}))}} d\underline{\omega} \leq \int_{\substack{|\langle \underline{\omega}, \underline{\nu} \rangle| \leq (2C)^{-1} e^{-(3/2)|\underline{\nu}|} e^{-L_k} \\ \|\underline{\omega} - \underline{\omega}^k(\underline{0})\| \leq r(1+\eta)}} d\underline{\omega} \quad (6.12) \\ \leq (2C)^{-1} e^{-(3/2)|\underline{\nu}|} e^{-L_k} \pi^{(N-1)/2} [r(1+\eta)]^{N-1} / \Gamma[1+(N-1)/2] .$$

The first of these inequalities follows by noting that (6.5), (6.6), and the fact that $V_{k-2} \subset V = \text{sphere of radius } r$ imply

$$\|\underline{\omega}^k(\underline{I}) - \underline{\omega}^k(\underline{0})\| \leq (1 + 3B_1 N^{-1} + 6\epsilon_0 \rho_0^{-1} N) \|\underline{I}\| \leq (1 + \eta) r , \quad (6.13)$$

with $\eta = 3B_1 N^{-1} + 6\epsilon_0 \rho_0^{-1} N$, and we used the fact that $|\underline{x}| \leq N \|\underline{x}\|$.

The second estimate in (6.12) is just a geometrical estimate on the volume of a slice of thickness $(2C)^{-1} e^{-(3/2)|\underline{\nu}|} e^{-L_k}$ out of an N -dimensional sphere. Since $\text{vol } V = (\pi^{N/2} r^N) / \Gamma(1 + N/2)$ we obtain

$$\text{vol}(R(k, h^k, V_{k-1})) \leq 2^2 \pi^{-(1/2)} (2Cr)^{-1} \left\{ \Gamma(1+N/2) / \Gamma[1 - (N-1)/2] \right\} (1+\eta)^{N-1} \quad (6.14)$$

$$\times \left\{ \sum_{\substack{\underline{v} \in \mathbf{X}_k \\ \underline{v} \neq \underline{0}}} e^{-(3/2)|\underline{v}|_e} e^{-L_k} \right\} \text{vol } V .$$

Use again the fact that

$$\sum_{\substack{\underline{v} \in \mathbf{X}_{k-1} \\ \underline{v} \neq \underline{0}}} e^{-(3/2)|\underline{v}|_e} e^{-L_k} \leq 2^{5N(2/e)} e^{L_k} . \quad (6.15)$$

Standard recursion relations (see, e. g., [12]) for the Γ function and a little algebra gives

$$\Gamma(1+N/2) / \Gamma[1 + (N-1)/2] \leq 2^2 [(N/4) + 1]^{(1/2)} \quad (6.16)$$

If $\epsilon_0 < \rho_0 2^{-6} N^{-2}$ one has $(1+\eta)^{N-1} \leq \exp(3B_1 + 6\epsilon_0 \rho_0^{-1} N^2) \leq 2$. Thus, (6.11) is bounded by

$$\begin{aligned} \text{vol}(V_{k-1} \setminus V_k) &\leq 2^5 (Cr)^{-1} 2^2 [(N/4) + 1]^{1/2} \cdot 2^{3N(2/e)} e^{L_k} \text{vol } V \\ &\leq 2^{10} (C\rho_0)^{-1} N^{3/2} (2/e)^{L_k} , \end{aligned} \quad (6.17)$$

since $\rho_0 < r$ by assumption. Combining (6.2) and (6.17),

$$\begin{aligned} \text{vol}(V_{k-1} \setminus V_k) &\leq \left[2^{-12} (E_0 \rho_0^{-1})^{-1} \lambda N^{-2} e^{-2M_k} e^{-2L_k} + 2^{10} (C\rho_0)^{-1} N^{3/2} (2/e)^{L_k} \right] \text{vol } V \\ &\leq (1/4) \lambda \exp \left[-(1/2)(3/2)^k \right] \text{vol } V , \end{aligned}$$

as claimed in Section 2. In the last inequality we used the definition of C and L_k .

Appendix A. The Implicit Function Theorem

We wish to invert the equation

$$\underline{z}' = \underline{z} \exp \left(i \frac{\partial \Phi^k}{\partial \underline{I}} (\underline{I}', \underline{z}) \right) . \quad (\text{A.1})$$

Let $\mathbf{Z}'$ denote the Jacobian matrix $[\mathbf{Z}']_{ij} = \frac{\partial z'_i}{\partial z_j} (\underline{I}', \underline{z})$. We demonstrated in

Section 4 that the matrix $D\phi'$ with $[D\phi']_{ij} = (z_j/z'_i) \left[\frac{\partial z'_i}{\partial z_j} \right] (\underline{I}', \underline{z})$ satisfies

$$\det(D\phi') = \left(\prod_{j=1}^N z_j/z'_j \right) \det(\mathbf{Z}') . \quad (\text{A.2})$$

If one writes $(D\phi')_{ij} = \delta_{ij} + \sigma_{ij}$, then (4.3) implies

$$\sigma_{ij} = \frac{\partial^2 \Phi^k}{\partial I'_i \partial \phi_j} = \sum_{\substack{\underline{\nu} \in \mathbf{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{\partial}{\partial I'_i} \left\{ \frac{f_{\underline{\nu}}^k(\underline{I}')}{i_{<\underline{\omega}^k(\underline{I}'), \underline{\nu}>}} \right\} (i\nu_j) \underline{z}^{\underline{\nu}} . \quad (\text{A.3})$$

This is essentially the same sum we bounded in (3.8) and using the same techniques we did there we find

$$\sup |\sigma_{ij}| \leq 2^6 \epsilon_k E_k C_k^2 N e^{3L_k} \quad \text{on } W(\tilde{\rho}_k, \xi_k^{-\delta}, V_k) \quad (\text{A.4})$$

If

$$(2^6 \epsilon_k E_k C_k^2 N e^{3L_k}) N < (1/2) , \quad (\text{A.5})$$

which is guaranteed by

$$\epsilon_0 < \rho_0 2^{-15} e^{-12} (E_0 \rho_0^{-1})_{\lambda N}^{-3} \quad (\text{A.6})$$

(which in turn follows from (1.15)), one has $\sum_j |\sigma_{ij}| < (1/2)$ which by a standard linear algebra result implies $|\det(D\phi')| \neq 0$. Also, $|(z'_j/z_j)| \neq 0$ on $W(\tilde{\rho}_k, \xi_k, V_k)$ so

$$|\det(\mathbb{Z}')| = \left| \prod_{j=1}^N (z'_j/z_j) \det(D\phi') \right| \neq 0, \quad (\text{A.7})$$

on $W(\mathcal{P}_k, \xi_k - \delta, V_k)$.

We now show that the map (A.1) is one to one. Define

$$\begin{aligned} \text{Dist}(z'_{1j}, z'_{2j}) &= \left| \ln |z'_{1j}| - \ln |z'_{2j}| + \text{Im} \left(\frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_2) - \frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_1) \right) \right| \\ &+ \left| \arg z'_{1j} - \arg z'_{2j} + \text{Re} \left(\frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_1) - \frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_2) \right) \right|. \end{aligned} \quad (\text{A.8})$$

From (A.1) it is clear that $\underline{z}'_1 = \underline{z}'_2$ iff $\sum_{j=1}^N \text{Dist}(z'_{1j}, z'_{2j}) = 0$. Let γ be the path connecting $\underline{z}_1$ to $\underline{z}_2$, consisting of N pieces γ_i , with each γ_i in turn consisting of two parts $\gamma_i^r + \gamma_i^a$. The path γ_i joins the point $(z_{21}, \dots, z_{2i-1}, z_{1i}, \dots, z_{1N})$ to the point $(z_{21}, \dots, z_{2i-1}, z_{2i}, z_{1i+1}, \dots, z_{1N})$, and along γ_i only the i^{th} co-ordinate of $\underline{z}$ varies. Assume for illustration that $|z_{2i}| < |z_{1i}|$ and $\arg z_{2i} > \arg z_{1i}$. All other possible situations are handled analogously. Then

$$\gamma_i^a \equiv \left\{ (z_{21}, \dots, z_{2i-1}, |z_{1i}| e^{i\alpha}, z_{1i+1}, \dots, z_{1N}) : \arg z_{1i} \leq \alpha \leq \arg z_{2i} \right\} \quad (\text{A.9})$$

$$\gamma_i^r \equiv \left\{ (z_{21}, \dots, z_{2i-1}, r e^{i \arg z_{2i}}, z_{1i+1}, \dots, z_{1N}) : |z_{1i}| \geq r \geq |z_{2i}| \right\}.$$

(see Figure A.I).

By the fundamental theorem of calculus one has

$$\left| \frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_1) - \frac{\partial \Phi^k}{\partial I'_j}(\mathbb{I}', \underline{z}_2) \right| \leq \sum_{k=1}^N \left| \left\{ \int_{\gamma_k} dz_k \cdot \frac{\partial \Phi^k}{\partial I'_j \partial z_k}(\mathbb{I}', \underline{z}) \right\} \right| \quad (\text{A.10})$$

(cont'd)

Figure A.1. The integration path.

(A.10 cont'd)

$$\leq \sum_{k=1}^N \left| \left\{ \int_{\gamma_k^a} \frac{dz_k}{iz_k} \cdot \frac{\partial \Phi^k}{\partial \mathbb{I}'_j \partial \phi_k}(\mathbb{I}', \underline{z}) + \int_{\gamma_k^r} \frac{dz_k}{iz_k} \cdot \frac{\partial \Phi^k}{\partial \mathbb{I}_j \partial \phi_k}(\mathbb{I}', \underline{z}) \right\} \right|$$

Now note that if $(\mathbb{I}', \underline{z}_1)$ and $(\mathbb{I}', \underline{z}_2) \in W(\tilde{\rho}_k, \xi_k - \delta, V_k)$, then $(\mathbb{I}', \underline{z}) \in W(\tilde{\rho}_k, \xi_k - \delta, V_k)$ for all $(\mathbb{I}', \underline{z}) \in \gamma$.

By (A.4),

$$\sup \left| \frac{\partial^2 \Phi^k}{\partial \mathbb{I}'_j \partial \phi_k}(\mathbb{I}', \underline{z}) \right| \leq 2^6 \epsilon_k E_k C^2 N e^{3L_k}, \quad (\text{A.11})$$

on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. One also has

$$\left| \int_{\gamma_j^a} \frac{dz_j}{z_j} \right| = |\arg z_{1j} - \arg z_{2j}|, \quad \left| \int_{\gamma_j^r} \frac{dz_k}{z_k} \right| = |\ell n |z_{1j}| - \ell n |z_{2j}||. \quad (\text{A.12})$$

Inequality (A.11) allows us to bound the r.h.s. of (A.10) by

$$2^6 \epsilon_k E_k C^2 N e^{3L_k} \sum_{k=1}^N \left\{ \left| \arg |z_{1k}| - \arg |z_{2k}| \right| + \left| \ell n |z_{1k}| - \ell n |z_{2k}| \right| \right\} . \quad (\text{A.13})$$

Summing both sides of (A.8) from $j = 1$ to N and using (A.13) we find

$$\begin{aligned} \sum_{j=1}^N \text{Dist}(z'_{1j}, z'_{2j}) &\geq \sum_{j=1}^N \left\{ \left| \ell n |z_{1j}| - \ell n |z_{2j}| \right| + \left| \arg z_{1j} - \arg z_{2j} \right| \right\} \\ &\quad \times (1 - 2^6 \epsilon_k E_k N^2 C^2 e^{3L_k}) \\ &\neq 0 , \end{aligned} \quad (\text{A.14})$$

if $\underline{z}_1 \neq \underline{z}_2$ and

$$2^6 \epsilon_k E_k C^2 N^2 e^{3L_k} \leq (1/2) . \quad (\text{A.15})$$

Thus, (A.1) is one to one, and since it also has nonzero determinant on $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$, standard inverse functions theorems (e.g., [13, Theorem I.7.6]) guarantee that (A.1) has an analytic inverse on the image, B , of $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$. Denote the inverse map by

$$\underline{z}(\underline{I}', \underline{z}') = \underline{z}' e^{i\Delta(\underline{I}', \underline{z}')} . \quad (\text{A.16})$$

From (A.1) we see that $\Delta(\underline{I}', \underline{z}') = -\frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z})$, so

$$\sup_B |\Delta(\underline{I}', \underline{z}')| \leq \sup \left| \frac{\partial \Phi^k}{\partial \underline{I}'} \right| \leq 2^6 \epsilon_k E_k C^2 N e^{3L_k} \leq \delta/2 , \quad (\text{A.17})$$

the last inequality following from (A.16). Hence, as $(\underline{I}', \underline{z})$ varies over $W(\tilde{\rho}_k, \xi_k - \delta, V_k)$, $(\underline{I}', \underline{z}')$ must cover at least $W(\tilde{\rho}_k, \xi_k - 2\delta, V_k)$, so that $\Delta(\underline{I}', \underline{z}')$ is analytic on that domain.

The argument for inverting $\underline{I} = \underline{I}' + \frac{\partial \Phi^k}{\partial \underline{I}'}(\underline{I}', \underline{z})$ is identical to that in [11] so we do not reproduce it.

Appendix B. A Lemma on Dimensional Bounds

Lemma B.1: If $e^{-(\xi-\delta)} < |z_0| < e^{\xi-\delta}$ there is a circle $C(z_0)$ centered at z_0 , with the property that for all $z \in C(z_0)$, $e^{-\xi} < |z| < e^{\xi}$ and $|(z/z_0) - 1| \geq 2^{-4}$.

Proof: Let $C(z_0)$ be the circle of radius $R = |z_0|(1 - e^{-\delta})^2$. It is easy to check that this circle is contained in the annulus $e^{-\xi} < |z| < e^{\xi}$.

Write $z_0 = |z_0|e^{i\theta_0}$, and $z = |z|e^{i\theta_1}$. By the law of cosines,
 $(z/z_0) = \left[1 + (1 - e^{-\delta})^4 + 2(1 - e^{-\delta})^2 \cos \theta\right]^{1/2} e^{i\theta}$ where we have set $\theta_1 - \theta_0 = \theta$.
 An easy computation yields

$$\begin{aligned} |(z/z_0) - 1| = & \left\{ \left[1 - \left(1 + (1 - e^{-\delta})^4 + 2(1 - e^{-\delta})^2 \cos \theta \right)^{1/2} \right]^2 \right. \\ & \left. + 2 \left(1 + (1 - e^{-\delta})^4 + 2(1 - e^{-\delta})^2 \cos \theta \right) (1 - \cos^2 \theta) \right\}^{1/2} \quad (\text{B.1}) \end{aligned}$$

Note that each of the two terms in braces is nonnegative.

Consider the three cases: I) $0 \leq \theta \leq \pi/4$; II) $\pi/4 < \theta < 7\pi/8$;
 III) $7\pi/8 \leq \theta \leq \pi$. The symmetry of the cosine function makes it unnecessary to consider $\pi \leq \theta \leq 2\pi$.

Case I) The first term in (B.1) is bounded below by

$$2 + (1 - e^{-\delta})^4 + \sqrt{2}(1 - e^{-\delta})^2 - 2 \left(1 + (1 - e^{-\delta})^4 + 2(1 - e^{-\delta})^2 \right)^{1/2} \geq 2^{-3}, \quad (\text{B.2})$$

verifying the bound in the lemma. (Recall that $\delta = 8$.)

Case II) Consider just the second term in (B.1). This bounded below by

$$2 \left(1 - \cos^2(7\pi/8) \right) \left(1 + (1 - e^{-\delta})^4 - 2(1 - e^{-\delta})^2 \cos(7\pi/8) \right)^{1/2} \geq 2^{-4}. \quad (\text{B.3})$$

Case III) Again we consider the first term, which this time is bounded below by

$$1 + 1 + (1 - e^{-\delta})^4 - 2(1 - e^{-\delta})^2 - 2 \left(1 + (1 - e^{-\delta})^4 + 2(1 - e^{-\delta})^2 \cos(7\pi/8) \right)^{1/2} \geq 2^{-4}, \quad (\text{B.4})$$

completing the proof of the lemma.

Lemma B.2: If $f(z)$ is analytic on the annulus $e^{-\xi} < |z| < e^{\xi}$, then for $e^{-(\xi-\delta)} < |z_0| < e^{\xi-\delta}$ one has

$$\left| \frac{\partial f}{\partial \phi}(z_0) \right| = \left| z_0 \frac{\partial f}{\partial z}(z_0) \right| \leq 2^4 \sup |f(z)| \quad (\text{B.5})$$

and

$$\left| \frac{\partial^2 f}{\partial \phi^2}(z_0) \right| \leq 2^{10} \sup |f(z)|, \quad (\text{B.6})$$

where the supremum is taken over $e^{-(\xi-\delta)} < |z| < e^{(\xi-\delta)}$.

Proof: By Cauchy's Theorem

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2\pi i} \oint_{C(z_0)} \frac{dz}{(z-z_0)} \cdot \frac{f(z)}{(z-z_0)}, \quad (\text{B.7})$$

and the contour $C(z_0)$ is the circle constructed in Lemma B.1. Thus,

$$\left| z_0 \frac{\partial f}{\partial z_0} \right| \leq \frac{1}{2\pi} \oint_{C(z_0)} \left| \frac{dz}{z-z_0} \right| \cdot \frac{|f(z)|}{|z/z_0 - 1|} \leq 2^4 \sup |f(z)|, \quad (\text{B.8})$$

by Lemma B.1. The bound on $\frac{\partial^2 f}{\partial \phi^2}(z_0)$ follows by noting that

$$\frac{\partial^2 f}{\partial \phi^2}(z_0) = z_0 \frac{\partial f}{\partial z}(z_0) + z_0^2 \frac{\partial^2 f}{\partial z^2}(z_0), \quad (\text{B.9})$$

and applying Cauchy's Theorem and Lemma B.1 separately to each term.

The extension to the case where $\underline{z} \in \mathbb{C}^N$ is immediate and left to the reader.

Acknowledgments

This problem was suggested to me by Giovanni Gallavotti who I thank for many helpful discussions. I also wish to acknowledge useful conversations with P. Collet, R. de la Llave and T. Spencer.

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