A RANDOM BOUNDARY VALUE PROBLEM MODELING
SPATIAL VARIABILITY IN POROUS MEDIA FLOW

BY

EDMUND DIKOW

AND

URLICH HORNUNG

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A RANDOM BOUNDARY VALUE PROBLEM MODELING SPATIAL VARIABILITY IN POROUS MEDIA FLOW*

EDMUND DIKOW[†] AND ULRICH HORNUNG[‡]

Abstract. A steady-state flow problem through a porous 3D-domain is considered. Under the assumption that the permeability (or hydraulic conductivity for water-flow) is a random field a random elliptic boundary value problem has to be solved. A mathematical methodology to prove existence and uniqueness is presented. In addition, the statistical moments of the total flux through the domain - considered as a random variable - are calculated in terms of deterministic boundary value problems.

1. Introduction. In the literature dealing with flow through porous media the assumption adopted by most authors is that Darcy's law holds

$$(1.1) \quad \vec{q} = -K \nabla u.$$

Here, the symbols have the following meaning:

$$\begin{aligned} \vec{q} &= \text{Darcy's velocity} , \\ K &= \text{permeability or conductivity,} \\ u &= \text{pressure potential} . \end{aligned}$$

In the absence of sources or sinks one obtains for an incompressible fluid in a saturated rigid medium the continuity equation

$$(1.2) \quad \nabla \cdot \vec{q} = 0.$$

Hence, together with (1.1) the elliptic differential equation

$$(1.3) \quad \nabla \cdot (K \nabla u) = 0$$

comes up which has to be augmented by appropriate boundary conditions to form a boundary value problem in a domain in space. If the permeability K is a known function of the space variable x , the usual theory of linear elliptic equations applies.

In the engineering literature it has been pointed out for many years that the permeability K is a rapidly oscillating function in space, see, e.g., Warren/Price (1961) or Bakr et al. (1978). These spatial variations made the approach obvious of assuming that K is a random field. Another aspect is that in practical applications there is always only a limited number of measurements of K available; and these measurements contain errors. Therefore, it is natural to say that not a function

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[†]Fak. f. Inf. UniBwM, Werner-Heisenberg-Wey 39, D-8014 Neubiberg, West Germany

[‡]SCHI, P.O. Box 1222, D-8014 Neubiberg, West Germany

K is known but an estimate of the moments of a spatial stochastic process can be given. This way of thinking about spatially distributed data has been well known in the geosciences; and the statistical methods developed there have been introduced to hydrology in recent years, see, e.g., Delhomme (1978). There are many research papers dealing with the question how to estimate K and its statistical properties.

Once the assumption has been made that K is not a function of the space variable x alone but is simultaneously a random variable, the question comes up how to set up a framework in which equation (1.3) can be treated as a random partial differential equation.

In hydrology, the Fourier-transform has been applied to equation (1.3), see, e.g., Bakr (1976). This means, the differential equation is considered in all space $\mathbb{R}^n$. It turns out that for stationary random fields K the spectral density can be used to obtain some insight into the structure of possible solutions. The drawbacks of this approach are that precise mathematical statements about existence, uniqueness, stability, etc. are not available, and that it is not applicable to problems in general finite domains with prescribed boundary conditions.

This paper answers the question of which impact the information about the statistical properties of the conductivity K has onto the solution of the differential equation (1.3). The main results are that, firstly, random boundary value problems can be solved in a precise mathematical sense, and that, secondly, relevant functionals of the solution, such as the total flux Q through a given domain, and their moments, i.e. expectation and variance, can be calculated.

The formulas presented here to approximate the moments of Q have well established error estimates. They are given in terms of solutions of deterministic boundary value problems only. Thus, in principle, it is possible to calculate these quantities without using Monte-Carlo-techniques. One can therefore hope that in the near future it will become possible to apply the results of this paper to random boundary problems and thus to make the Monte-Carlo-approach obsolete in this context.

2. The Random Boundary Value Problem. The first objective is to define precisely the mathematical problem to be solved.

2.1 ASSUMPTIONS. (a) Let $G \subset \mathbb{R}^n$ be a bounded domain the boundary of which is split into parts:

$$\partial G = S \cup T \cup \Gamma, \quad \Gamma = \partial S \cap \partial T,$$

where

$$S \text{ and } T \text{ are } C^{2,\alpha} - \text{smooth}$$

and

$$u_0 \in C^{1,\alpha}(\overline{G})$$

with vanishing normal derivative on T , i.e.,

$$\partial_\nu u_0 = 0 \quad \text{on } T.$$

(b) Let $(\Omega, \mathfrak{A}, P)$ be a probability space with countably generated σ -algebra $\mathfrak{A}$ and probability measure P on Ω . The expectation operator is denoted by

$$E \cdots = \int_{\Omega} \cdots dP$$

(c) For all $x \in G$ let

$$K(x, \cdot) : \Omega \rightarrow \mathbb{R}$$

be a random variable, i.e. a measurable function, such that

$$1/\mu \leq K \leq \mu$$

holds $P - a.e.$ with some constant $\mu > 0$.

2.2 NOTATIONS. We use the function spaces

$$\mathfrak{H} = L^2(\Omega, \mathfrak{A}, P),$$

$$C_{0,S}^1(G, \mathfrak{H}) = \{u \in C^1(G, \mathfrak{H}) : u \text{ has compact support in } \overline{G} \setminus S\},$$

$$W_{0,S}^{1,2}(G, \mathfrak{H}) = \text{closure of } C_{0,S}^1(G, \mathfrak{H}) \text{ in the space } W^{1,2}(G, \mathfrak{H}).$$

2.3. DEFINITION. A weak solution of the random boundary value problem (RBVP)

$$\begin{cases} \nabla(K \nabla u) = 0 & \text{in } G \\ u = u_0 & \text{on } S \\ \partial_\nu u = 0 & \text{on } T \end{cases}$$

is a function $u \in W^{1,2}(G, \mathfrak{H})$ such that

$$(a) \quad u - u_0 \in W_{0,S}^{1,2}(G, \mathfrak{H})$$

and the equation

$$(b) \quad \int_G EK \nabla u \cdot \nabla \varphi dx = 0$$

holds for all $\varphi \in W_{0,S}^{1,2}(G, \mathfrak{H})$. This definition goes back to work by Papanicolaou/Varadhan (1982); using the Lax-Milgram lemma one obtains easily the following result.

THEOREM 1. *There is a unique weak solution of the RBVP (2.3).*

By the usual method one also may prove stability, i.e., continuous dependence on the data given.

3. The Random Total Flux. First we define what the random total flux means.

3.1 ASSUMPTION. Let $S_1 \subset S, S_1 \neq \emptyset, S_1 \neq S$ with $\text{dist}(S_1, S \setminus S_1) > 0$ be given, and $\rho \in C^\infty(\mathbb{R}^n)$ with

$$\rho = \begin{cases} 1 & \text{on } S_1 \\ 0 & \text{on } S \setminus S_1. \end{cases}$$

The following statement is almost obvious; it serves as a motivation for the definition to follow.

3.2 LEMMA. If $\vec{q} \in C^1(\overline{G})$ is a vector field with

$$\begin{cases} \nabla \cdot \vec{q} = 0 & \text{in } G \\ \vec{q} \cdot \nu = 0 & \text{on } T \end{cases}$$

then

$$\int_{S_1} \vec{q} \cdot \nu dS = \int_G \vec{q} \cdot \nabla \rho dx$$

where this quantity is independent of the special choice of ρ .

3.3 DEFINITION. If u is a weak solution of (2.3) then we define the total flux through the part S_1 of the boundary as

$$Q = \int_G K \nabla u \cdot \nabla \rho dx;$$

we write shortly

$$Q = \int_{S_1} K \partial_\nu u dS.$$

3.4 LEMMA. For the weak solution u of the RBVP (2.3) the quantity Q is an element of $\mathfrak{H}$.

3.5 ASSUMPTIONS.

- (a) $n = 2$ or $n = 3$.
- (b) The permeability K can be written as

$$K(x) = K_0(x) \exp(\xi(x)),$$

where K_0 is deterministic and ξ is random.

- (c) The function K_0 satisfies

$$K_0 \in C^\alpha(\overline{G})$$

with uniform α -Hölder norm and

$$K_0(x) \geq \lambda_0 > 0$$

uniformly on $\overline{G}$.

(d) For all $x \in G$ the function

$$\xi(x, \cdot) : \Omega \rightarrow \mathbb{R}$$

is $\mathfrak{A}$ -measurable, i.e., it is a random variable, such that

$$-\Lambda \leq \xi(x) \leq \Lambda \quad \text{P-a.e.}$$

with a constant Λ independent of x .

(e) The mapping ξ , considered as

$$\xi : G \rightarrow L^6(\Omega, \mathfrak{A}, P),$$

is uniformly α -Hölder continuous.

(f) For all $x \in \overline{G}$

$$E\xi(x) = 0$$

holds.

(g) There is an α -Hölder-continuous function $R : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\text{Cov}(\xi(x), \xi(y)) = R(x - y)$$

holds for all $x, y \in \overline{G}$.

Under these assumptions one obtains the following lemma and theorem; here the definitions (2.3) and (3.3) have to be modified appropriately.

3.6 LEMMA. (a) *There is a unique weak solution $v_0 \in C^{1,\alpha}(\overline{G})$ of the BVP*

$$\begin{cases} \nabla(K_0 \nabla v_0) = 0 & \text{in } G \\ v_0 = u_0 & \text{on } S \\ \partial_\nu v_0 = 0 & \text{on } T \end{cases}$$

(b) *There is a unique weak solution $v_1 \in C^{1,\alpha}(\overline{G}, \mathfrak{H})$ of the RBVP*

$$\begin{cases} \nabla(K_0 \nabla v_1) = -\nabla(K_0 \xi \nabla v_0) & \text{in } G \\ v_1 = 0 & \text{on } S \\ \partial_\nu v_1 = 0 & \text{on } T \end{cases}$$

(c) *There is a unique weak solution $v_2 \in C^{1,\alpha}(\overline{G}, L^3(\Omega, \mathfrak{A}, P))$ of the RBVP*

$$\begin{cases} \nabla(K_0 \nabla v_2) = -\nabla(K_0 (\frac{1}{2} \xi^2 \nabla v_0 + \xi \nabla v_1)) & \text{in } G \\ v_2 = 0 & \text{on } S \\ \partial_\nu v_2 = 0 & \text{on } T \end{cases}$$

THEOREM 2. *The random total flux is*

$$Q(\xi) = Q_0 + Q'_0(\xi) + \frac{1}{2}Q''_0(\xi, \xi) + r$$

where

$$Q_0 = \int_{S_1} K_0 \partial_\nu v_0 dS$$

is a constant,

$$Q'_0(\xi) = \int_{S_1} K_0 (\xi \partial_\nu v_0 + \partial_\nu v_1) dS$$

is a linear functional of ξ , and

$$Q''_0(\xi, \xi) = \int_{S_1} K_0 (\xi^2 \partial_\nu v_0 + 2\xi \partial_\nu v_1 + 2\partial_\nu v_2) dS$$

is a quadratic functional of ξ ; the remainder $r \in \mathfrak{H}$ satisfies

$$\|r\|_{\mathfrak{H}} \leq C \exp(3\Lambda) |\xi|_{0,\alpha,G,\mathfrak{H}}^3$$

In the latter estimate C depends only on $n, \alpha, \lambda_0, u_0, S$ and G ; $|\cdot|_{0,\alpha,G,\mathfrak{H}}$ is the α -Hölder norm of (3.5e).

4. The Moments of the Total Flux.

4.1 NOTATION. Let $\sigma^2 = R(0)$.

4.2 LEMMA. (a) For each $y \in \overline{G}$ there is a unique weak solution $v_y \in C^{1,\alpha}(\overline{G})$ of the BVP

$$\begin{cases} \nabla(K_0 \nabla v_y) = -\nabla(R(\cdot - y) \nabla v_0) & \text{in } G \\ v_y = 0 & \text{on } S \\ \partial_\nu v_y = 0 & \text{on } T. \end{cases}$$

The vector field $f = \nabla v_x(x)$ is in $C^\alpha(\overline{G})$.

(b). There is a unique weak solution $w \in C^{1,\alpha}(\overline{G})$ of the BVP

$$\begin{cases} \nabla(K_0 \nabla w) = -\nabla(f + \frac{1}{2}\sigma^2 \nabla v_0) & \text{in } G \\ w = 0 & \text{on } S \\ \partial_w w = 0 & \text{on } T. \end{cases}$$

THEOREM 3. *The expectations of the terms in Theorem 2 are*

$$EQ'_0(\xi) = 0$$

and

$$EQ''_0(\xi, \xi) = \int_{S_1} K_0 (\sigma^2 \partial_\nu v_0 + 2f \cdot \nu + 2\partial_\nu w) dS.$$

The latter formula can, at least in principle, be evaluated numerically. The expectation of the remainder satisfies a similar estimate as in theorem 2 given for r itself.

4.3 LEMMA. For each $y \in \overline{G}$ and $1 \leq j \leq n$ there is a unique weak solution $V_{y,j} \in C^{1,\alpha}(\overline{G})$ of the BVP

$$\begin{cases} \nabla(K_0 \nabla V) = -\nabla(K_0 \overrightarrow{h}) & \text{in } G \\ V = 0 & \text{on } S \\ \partial_\nu V = 0 & \text{on } T, \end{cases}$$

where $\overrightarrow{h}_{y,j}$ is defined as

$$\overrightarrow{h}_{y,j}(x) = \partial_j v_x(y) \nabla v_0(x).$$

4.4 NOTATION. We use the function

$$g^{i,j}(x, y) = \partial_i V_{y,j}(x).$$

THEOREM 4. The variance of $Q'_0(\xi)$ in Theorem 2 is given by

$$\begin{aligned} \text{Var } Q'_0(\xi) = & \int_{S_1} \int_{S_1} K_0(x) K_0(y) (R(x-y) \partial_\nu v_0(x) \partial_\nu v_0(y) \\ & + \partial_\nu v_0(x) \nu(y) \cdot \nabla v_x(y) + \partial_\nu v_0(y) \nu(x) \cdot \nabla v_y(x) \\ & + \sum_{i,j} \nu^i(x) \nu^j(y) g^{i,j}(x, y)) dS_x dS_y. \end{aligned}$$

5. Perspectives. The proofs of the lemmas and theorems of §3 and 4 are given in Dikow (1986). An application of these ideas to problems in rectangular domains is discussed in Dikow (1987); there Fourier-series are used to simplify the numerical calculations.

It should be pointed out here that the approach presented in this paper works on a macro-scale. A very important open question related to porous media flow is how to derive Darcy's law from more basic physical assumptions in a medium with stochastic geometry on a micro-scale. Relevant works in this direction are the use of homogenization for periodic structures, see Tartar (1980), and flow outside of small balls which are randomly positioned, see Rubinstein (1986).

Another aspect is to apply the technique presented here to miscible displacement problems and to deal with hydrodynamic dispersion. There are papers on this subject, see, e.g., Dagan (1984), Gelhar (1984), and Tompson/Gray (1986). Their results have still to be put onto a sound mathematical basis.

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