NONUNIQUENESS FOR A HYPERBOLIC SYSTEM: CAVITATION
IN NONLINEAR ELASTODYNAMICS

BY

K. A. PERICAK-SPECTOR

AND

SCOTT J. SPECTOR

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

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1. Introduction

We let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and consider the problem: Find $u: \Omega \times [0, T] \rightarrow \mathbb{R}^n$ that satisfies

$$\begin{aligned} \operatorname{div} \tilde{S}(\nabla u) &= u_{tt} \quad \text{in} \quad \Omega \times [0, T], \\ u(x, t) &= \lambda x \quad \text{for} \quad x \in \partial\Omega \text{ and } t \in [0, T], \\ u(x, 0) &= \lambda x \quad \text{for} \quad x \in \Omega, \\ u_t(x, 0) &= 0 \quad \text{for} \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where

$$\tilde{S}(F) := F + h'(\det F) \operatorname{adj} F \tag{1.2}$$

with $h'' > 0$, $h''' < 0$, $h'(u) \rightarrow -\infty$ as $u \rightarrow 0^+$, and $h'(u) \rightarrow +\infty$ as $u \rightarrow +\infty$.

We prove that there are $\lambda_1 \rightarrow +\infty$ such that (1.1), with $\lambda = \lambda_1$, has multiple solutions. Moreover we show that each such solution is admissible according to the standard entropy criterion (cf. Lax [12]).

The solutions that we construct are superpositions of special cavitation similarity solutions; i.e., solutions of the form

$$u(x, t) := \frac{\varphi(s)}{s}(x - x_0), \quad s := |x - x_0|/(t - t_0),$$

for $t \geq t_0$, where φ is continuous, piecewise C^2 and satisfies

$$\varphi(0) > 0, \quad \varphi(s) = \lambda s, \quad s \geq \sigma,$$

for some $\sigma > 0$. Thus each such solution is discontinuous, opening a hole in the region Ω .

We note that the above result does not contradict the uniqueness theorem of Wheeler [19] or the more general uniqueness theorems of Dafermos [4]. In particular Dafermos showed that there is a neighborhood of $\bar{u}(x, t) = \lambda x$ in $W^{1, \infty}(\Omega, \mathbb{R}^n)$ in which (1.1) has no other solution whose derivatives are of bounded variation. However, the solutions we construct are discontinuous and hence, by the Sobolev imbedding theorem, not contained in $W^{1, \infty}$.

An important unresolved problem is the precise smoothness¹ of φ (cf. Proposition 6.4). If φ is not C^1 then² the solution u will contain a shock that dissipates energy. In this case our results admit an interesting physical interpretation. Consider the tip of a crack in an elastic material undergoing loading. If the load is sufficiently large then, according to our results, we expect the material to fail; that is, in order to dissipate as much energy as is possible³ the material must attempt to open a cavity at every point near the crack tip. Thus, in practice, we expect to see a large number of cavities form. Of course our results are not directly applicable to this situation since the loading is not

¹Numerical experiments indicate that φ is not C^1 .

²In this case we can also prove that (1.1) has multiple solutions for all large λ .

³Cf. Dafermos [3] who postulates that an admissible solution will dissipate energy (actually entropy) at the highest possible rate. Our explanation of fracture initiation also assumes that there are no other solutions to (1.1).

precisely uniform triaxial⁴ tension near the tip of a crack. However, our explanation of fracture initiation is in qualitative agreement with experimental observations⁵ in certain ductile metals.

Remark. The above interpretation assumes that cavities appear spontaneously. Another explanation for their appearance is that they grow from submicroscopic lattice defects. In elastostatics this distinction is not relevant mathematically since Sivaloganathan [15] (see, also, Horgan & Abeyaratne [9]) has shown that the load required to produce a new cavity is of the same order as the load range where a small cavity will grow rapidly. In elastodynamics no such result is yet available.

Finally, we remark that our work was motivated by the fundamental paper of Ball [1] on cavitation in elastostatics^{6,7}. Ball showed that for a large class of stored energy functions the equilibrium solution $u(x,t) = \lambda x$ is not the global minimizer of the energy for large λ . He also showed that among radial deformations the global minimizer exists and exhibits cavitation. Thus our

⁴If instead the crack is in biaxial tension one might expect cylindrical cavities to form. However, the constitutive relation (1.2) is incompatible with such solutions. We expect that cylindrical cavities can be obtained for appropriate constitutive relations (cf. footnote 8).

⁵Cf., e.g., Cáceres & Wilkinson [2], Elliot [6], Hahn & Rosenfield [7] or Hancock & Mackenzie [8]. In general the application of loads first causes microscopic voids to appear near the tip of a crack. These microvoids grow rapidly until they coalesce and extend the crack into the material.

⁶See, also, Podio-Guidugli, Vergara Caffarelli, & Virga [13] and Stuart [17].

⁷Ball also treated the dynamic problem for incompressible materials. In particular he obtained two solutions (a trivial solution and a cavitation solution) with equal energy to the dead load traction problem.

results are a dynamic version of Ball's for the special constitutive relation (1.2).

2. Notation

Given any vector $\underline{v} \in \mathbb{R}^n$ with components v_i , $i = 1, 2, \dots, n$, we write

$$|\underline{v}| := \left[\sum_{i=1}^n v_i^2 \right]^{1/2}, \quad |\underline{v}|_p := \left[\sum_{i=1}^n |v_i|^p \right]^{1/p},$$

where $1 \leq p < \infty$. We let

$\text{Lin} :=$ space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ with inner product and norms

$$\underline{G} \cdot \underline{H} := \text{trace} (\underline{G} \underline{H}^T), \quad |\underline{H}|_p := [\text{trace} (\underline{H} \underline{H}^T)^{p/2}]^{1/p},$$

respectively, where $\underline{H}^T$ denotes the transpose of $\underline{H}$. We write

$$\text{Lin}^+ := \{\underline{H} \in \text{Lin} : \det \underline{H} > 0\},$$

where $\det$ denotes the determinant. Given two vectors $\underline{a}, \underline{b} \in \mathbb{R}^n$ we write $\underline{a} \otimes \underline{b}$ for the tensor product of $\underline{a}$ and $\underline{b}$; in components

$$(\underline{a} \otimes \underline{b})_{ij} = a_i b_j.$$

We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^n$: for a vector field $\underline{u}$, $\nabla \underline{u}$ is the tensor field with components $(\nabla \underline{u})_{ij} = \partial u_i / \partial x_j$; for a tensor field $\underline{S}$, $\text{div} \underline{S}$ is the vector field with components $\sum_j S_{ij} / \partial x_j$. Given any function $\phi(\underline{a}, \underline{b}, \dots, \underline{g})$ with vector or tensor arguments, we write, e.g., $\partial \phi / \partial \underline{a}$ for the partial Frechet derivative with respect to $\underline{a}$ holding the remaining arguments fixed.

Throughout this paper $\Omega \subset \mathbb{R}^n$ will denote a regular region with boundary, $\partial\Omega$. For $1 \leq p < \infty$ we let $\|\cdot\|_{m,p}$ denote the $W^{m,p}$ -norm on Ω so that, in particular,

$$\|\nabla \tilde{f}\|_{0,p}^p = \int_{\Omega} |\nabla \tilde{f}|_p^p, \quad \|\tilde{f}\|_{1,p}^p = \int_{\Omega} |\tilde{f}|_p^p + \int_{\Omega} |\nabla \tilde{f}|_p^p.$$

We denote by $L^p(\Omega, \mathbb{R}^n)$ the space (of equivalence classes) of $\mathbb{R}^n$ -valued functions whose p -norm to the p -th power is integrable on Ω and define

$$W^{m,p}(\Omega, \mathbb{R}^n) := \text{completion of } \{\tilde{f} \in C^\infty(\Omega, \mathbb{R}^n) : \|\tilde{f}\|_{m,p} < +\infty\}$$

with respect to the $W^{m,p}$ -norm. For $T > 0$ we let

$C^j([0, T], W^{m,p}(\Omega, \mathbb{R}^n))$ be the set of functions $u: [0, T] \rightarrow W^{m,p}(\Omega, \mathbb{R}^n)$ whose first j derivatives exist and are continuous on $[0, T]$.

We let $PC^1 = PC^1([0, \infty), [0, \infty))$ be the set of continuous functions $\varphi: [0, \infty) \rightarrow [0, \infty)$ whose right-hand derivative exists at every $s \geq 0$, whose left-hand derivative exists at every $s > 0$, and whose right-hand and left-hand derivatives are continuous and equal except at a finite number of points, which we denote $0 = s_0^\varphi < s_1^\varphi < \dots < s_{L_\varphi}^\varphi < +\infty$. We write

$$PC^2 := \{\varphi \in PC^1 : \varphi \in C^2([s_i^\varphi, s_{i+1}^\varphi], [0, \infty)), i = 0, 1, 2, \dots, L_\varphi - 1\}.$$

Thus $\varphi \in PC^2$ is continuous and piecewise C^2 .

Finally, we will make use of the following form of the divergence theorem.

Lemma 1.1. Let $\mathfrak{B} \subset (\mathbb{R}^n \times [0, T])$ be a regular region, $\varphi \in C^1(\overline{\mathfrak{B}}, \mathbb{R})$, and $\underline{v} \in C^1(\overline{\mathfrak{B}}, \mathbb{R}^n)$. Then

$$\int_{\mathfrak{B}} [\varphi_t(\underline{x}, t) + \operatorname{div} \underline{v}(\underline{x}, t)] dV(\underline{x}, t) = \int_{\partial \mathfrak{B}} (\underline{v}, \varphi) \cdot \underline{N}(\underline{x}, t) dS(\underline{x}, t),$$

where $\underline{N}$ is the outward unit normal to $\partial \mathfrak{B}$, $dV(\underline{x}, t) = d\underline{x} dt$, and $dS(\underline{x}, t)$ is the standard surface measure.

3. Kinematics

We consider a body that, for convenience, we identify with the region Ω that it occupies in a fixed reference configuration. A deformation $\underline{f}$ of the body is a member of the space

$$\text{Def} := \{ \underline{f} \in W^{1, P}(\Omega, \mathbb{R}^n) : \det \nabla \underline{f} > 0 \text{ a.e.} \}.$$

Given $T > 0$ we call $\underline{u} : [0, T] \rightarrow \text{Def}$ a motion (of duration T) provided that $\underline{u}$ satisfies

$$(i) \quad \underline{u} \in C^0([0, T], W^{1, P}(\Omega, \mathbb{R}^n)),$$

$$(ii) \quad \underline{u} \in C^1([0, T], L^P(\Omega, \mathbb{R}^n)).$$

Remark. The space of deformations has been chosen to allow point singularities (cavities) to develop in the material. A motion is a one-parameter family of deformations that yields a well-defined velocity and whose deformation gradient varies continuously with the parameter. Although these definitions are appropriate for the special fracture problem we herein consider, they are too restrictive to obtain more general fracture results since they prohibit discontinuities in $\underline{f}$ on $n-1$ dimensional surfaces.

Let us now suppose that

$$\Omega = \mathfrak{B}_\rho := \{x \in \mathbb{R}^n : |x| < \rho\}$$

and consider a function $\tilde{f} : \mathfrak{B}_\rho \setminus \{0\} \rightarrow \mathbb{R}^n$ that is radial; i.e.,

$$\tilde{f}(x) := \frac{r(R)}{R} x, \quad R := |x|, \quad (3.1)$$

where $r : [0, \rho) \rightarrow [0, \infty)$.

Lemma 3.1 (Ball [1, p. 566]). Let $\tilde{f}$ be defined by (3.1). Then $\tilde{f} \in \text{Def}$ if and only if r is absolutely continuous on $(0, \rho)$, $r' > 0$ almost everywhere, and

$$\int_0^\rho [|r'(R)|^p + \left| \frac{r(R)}{R} \right|^p] R^{n-1} dR < \infty.$$

In this case the weak derivatives of $\tilde{f}$ are given by

$$\nabla \tilde{f}(x) = \frac{r(R)}{R} \tilde{I} + [r'(R) - \frac{r(R)}{R}] \frac{x}{R} \otimes \frac{x}{R}, \quad \text{a.e. } x \in \mathfrak{B}_\rho. \quad (3.2)$$

We will consider motions that, in addition to being radial, only depend on R/t . With this in mind let $\lambda > 0$ and suppose that $\varphi : [0, \infty) \rightarrow [0, \infty)$ and satisfies $\varphi(s)/s \equiv \lambda$ for all $s \geq \sigma := \rho/T$. Define $\underline{u} : \mathfrak{B}_\rho \setminus \{0\} \times [0, \infty) \rightarrow \mathbb{R}^n$ by

$$\underline{u}(x, t) := \begin{cases} \frac{\varphi(s)}{s} x & t > 0 \\ \lambda x, & t = 0 \end{cases}, \quad s := |x|/t. \quad (3.3)$$

Lemma 3.2. Let $\underline{u}$ be defined by (3.3) and suppose that $p < n$. If $\varphi \in PC^1([0, \infty), [0, \infty))$ and $\dot{\varphi} > 0$ almost everywhere then $\underline{u}$ is a motion.

Proof. Since φ and $\dot{\varphi}$ are bounded and $\dot{\varphi} > 0$ a.e. we apply Lemma 3.1 to conclude that $\underline{u}(\cdot, t) \in \text{Def}$ for every $t \in [0, T]$. Also, by

(3.2) and (3.3),

$$\nabla \underline{u}(\underline{x}, t) = \frac{\varphi(s)}{s} \underline{I} + [\dot{\varphi}(s) - \frac{\varphi(s)}{s}] \frac{\underline{x}}{R} \otimes \frac{\underline{x}}{R} \quad (3.4)$$

for almost every $\underline{x} \in \mathfrak{B}_\rho$ and $t \in (0, T]$.

To show $\underline{u} \in C^0((0, T], W^{1,p}(\mathfrak{B}_\rho, \mathbb{R}^n))$ we note that, by (3.3) and (3.4),

$$\begin{aligned} \|\underline{u}(\cdot, t) - \underline{u}(\cdot, t_0)\|_{1,p}^p &= \mathfrak{s}_n \int_0^\rho |t \varphi(\frac{R}{t}) - t_0 \varphi(\frac{R}{t_0})|^p R^{n-1} dR \\ &+ \mathfrak{s}_n \int_0^\rho |\dot{\varphi}(\frac{R}{t}) - \dot{\varphi}(\frac{R}{t_0})|^p R^{n-1} dR \\ &+ (n-1) \mathfrak{s}_n \int_0^\rho |t \varphi(\frac{R}{t}) - t_0 \varphi(\frac{R}{t_0})|^p R^{n-p-1} dR, \end{aligned} \quad (3.5)$$

where $\mathfrak{s}_n$ denotes the surface area of the unit ball in $\mathbb{R}^n$. Since φ is piecewise smooth each of the integrands in (3.5) tends to zero pointwise almost everywhere as $t \rightarrow t_0$. Thus, since φ and $\dot{\varphi}$ are bounded and $n > p$, we apply the Lebesgue dominated convergence theorem to obtain the desired result.

Similarly, at $t_0 = 0$,

$$\begin{aligned} \|\underline{u}(\cdot, t) - \underline{u}(\cdot, 0)\|_{1,p}^p &= \mathfrak{s}_n \int_0^\rho |t \varphi(\frac{R}{t}) - \lambda R|^p R^{n-1} dR \\ &+ \mathfrak{s}_n \int_0^\rho |\dot{\varphi}(\frac{R}{t}) - \lambda|^p R^{n-1} dR \\ &+ (n-1) \mathfrak{s}_n \int_0^\rho |t \varphi(\frac{R}{t}) - \lambda R|^p R^{n-p-1} dR. \end{aligned}$$

Once again each of the integrands tends to zero pointwise a.e. as $t \rightarrow 0^+$ and hence

$$\lim_{t \rightarrow 0^+} \| \underline{u}(\cdot, t) - \underline{u}(\cdot, 0) \|_{1, p} = 0.$$

Therefore $\underline{u} \in C^0([0, T], W^{1, p}(\mathfrak{B}_\rho, \mathbb{R}^n))$.

To show $\underline{u} \in C^1([0, T], L^p(\mathfrak{B}_\rho, \mathbb{R}^n))$ let

$$\underline{v}(\underline{x}, t) := \begin{cases} \frac{1}{R} [\varphi(s) - s \dot{\varphi}(s)] \underline{x}, & t > 0 \\ 0, & t = 0 \end{cases} \quad (3.6)$$

so that

$$\underline{v}(\underline{x}, t) = \frac{\partial}{\partial t} \underline{u}(\underline{x}, t) \text{ for a.e. } \underline{x} \in \mathfrak{B}_\rho \text{ and } t \in [0, T]. \quad (3.7)$$

We now show that $\underline{v}(\cdot, t) = \frac{d}{dt} \underline{u}(\cdot, t)$ in $L^p(\mathfrak{B}_\rho, \mathbb{R}^n)$ for every $t \in [0, T]$ and that $\underline{v} \in C^0([0, T], L^p(\mathfrak{B}_\rho, \mathbb{R}^n))$ to finish this proof.

By (3.6) we find that, for $t_0 > 0$,

$$\begin{aligned} \| \underline{v}(\cdot, t) - \underline{v}(\cdot, t_0) \|_{0, p}^p &= \\ &= \int_0^\rho | \varphi(\frac{R}{t}) - \varphi(\frac{R}{t_0}) + \frac{R}{t_0} \dot{\varphi}(\frac{R}{t_0}) - \frac{R}{t} \dot{\varphi}(\frac{R}{t}) |^p R^{n-1} dR, \end{aligned}$$

and

$$\| \underline{v}(\cdot, t) - \underline{v}(\cdot, 0) \|_{0, p}^p = \int_0^\rho | \varphi(\frac{R}{t}) - \frac{R}{t} \dot{\varphi}(\frac{R}{t}) |^p R^{n-1} dR.$$

Since $\varphi(s) = s \dot{\varphi}(s)$ for $s \geq \sigma$ it is not difficult to see that each of the above integrands tends to zero pointwise a.e. and thus that $\underline{v} \in C^0([0, T], L^p(\mathfrak{B}_\rho, \mathbb{R}^n))$.

Finally, we consider, for $t > 0$,

$$\begin{aligned} \| [\underline{u}(\cdot, t+h) - \underline{u}(\cdot, t)]/h - \underline{v}(\cdot, t) \|_{0, p}^p &= \\ &= \int_0^\rho \left| \left[\frac{(t+h)\varphi(\frac{R}{t+h}) - t\varphi(\frac{R}{t})}{h} - \varphi(\frac{R}{t}) + \frac{R}{t} \dot{\varphi}(\frac{R}{t}) \right] \right|^p R^{n-1} dR \end{aligned} \quad (3.8)$$

and

$$\| [u(\cdot, h) - u(\cdot, 0)]/h - v(\cdot, 0) \|_{0, p}^p = \int_0^\rho | [h \varphi(\frac{R}{h}) - \lambda R]/h |^p R^{n-1} dR. \quad (3.9)$$

It follows from (3.6) and (3.7) that each of the above integrands tends to zero pointwise a.e. as $h \rightarrow 0$. If, in addition, each of the integrands are bounded, uniformly in h , by an integrable function the Lebesgue dominated convergence theorem will once again give the desired result, i.e., that $v(\cdot, t) = \frac{d}{dt} u(\cdot, t)$ in $L^p(\mathfrak{A}_\rho, \mathbb{R}^n)$.

Recall that φ is piecewise smooth and hence Lipschitz. Consequently there is an $L > 0$ such that

$$\begin{aligned} |t\varphi(\frac{R}{t+h}) - t\varphi(\frac{R}{t})| &\leq L t \left| \frac{R}{t+h} - \frac{R}{t} \right| \\ &= L |h| \frac{R}{t+h}, \end{aligned}$$

and hence the integrand in (3.8) is uniformly bounded. The uniform bound on the integrand in (3.9) will follow from the estimate

$$|h \varphi(\frac{R}{h}) - \lambda R| \leq |h| (\lambda \sigma + \sup_{s \in [0, \sigma]} |\varphi(s)|), \quad (3.10)$$

which we will establish for all $R \in [0, R_0]$ and $h > 0$.

If we let $s = R/h$ then

$$\begin{aligned} |h \varphi(\frac{R}{h}) - \lambda R| &= |h| |\varphi(s) - \lambda s| \\ &\leq |h| \sup_{s \in [0, \sigma]} |\varphi(s) - \lambda s|, \end{aligned}$$

since $\varphi(s) \equiv \lambda s$ for $s \geq \sigma$, and (3.10) is now clear. ■

4. The Constitutive Relation

We assume that the body is hyperelastic with C^2 response function $W: \Omega \times \text{Lin}^+ \rightarrow [0, \infty)$. W gives the stored energy

$$W(\underline{x}, \nabla \underline{f}(\underline{x}))$$

at almost every $\underline{x} \in \Omega$ when the body is deformed by $\underline{f} \in \text{Def}$. The Piola-Kirchhoff stress $\underline{S}: \Omega \times \text{Lin}^+ \rightarrow \text{Lin}$ is given by

$$\underline{S}(\underline{x}, \underline{F}) := \frac{\partial}{\partial \underline{F}} W(\underline{x}, \underline{F}),$$

while the Cauchy stress $\underline{T}: \Omega \times \text{Lin}^+ \rightarrow \text{Lin}$ is defined by

$$\underline{T}(\underline{x}, \underline{F}) := \underline{S}(\underline{x}, \underline{F}) \underline{F}^T / \det \underline{F}. \quad (4.1)$$

We suppose that the body in its reference configuration is homogeneous and isotropic so that there is a symmetric function $\phi: (0, \infty)^n \rightarrow [0, \infty)$ that satisfies

$$W(\underline{x}, \underline{F}) = \phi(\lambda_1, \lambda_2, \dots, \lambda_n) \quad (4.2)$$

for all $\underline{F} \in \text{Lin}^+$, where λ_i are the eigenvalues of $(\underline{F}\underline{F}^T)^{1/2}$. In addition the density of the body in the reference configuration will be a constant that, for convenience, we assume is 1.

We note that, by (3.2), the eigenvalues of $(\underline{F}\underline{F}^T)^{1/2}$ for a radial deformation are given by

$$\lambda_1 = r'(R), \quad \lambda_2 = \lambda_3 = \dots = \lambda_n = r(R)/R. \quad (4.3)$$

To simplify notation we therefore define $\hat{\phi}, \hat{\phi}_1, \hat{\phi}_{1j}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by

$$\hat{\phi}(\alpha, \lambda) := \phi(\alpha, \lambda, \dots, \lambda), \quad (4.4)$$

$$\hat{\phi}_1(\alpha, \lambda) := \phi_{,1}(\alpha, \lambda, \dots, \lambda), \quad \hat{\phi}_{1j}(\alpha, \lambda) := \phi_{,1j}(\alpha, \lambda, \dots, \lambda).$$

Most of our results will require the stored energy function to have the form

$$\phi(\lambda_1, \dots, \lambda_n) = \frac{1}{2} \sum_{i=1}^n \lambda_i^2 + h\left(\frac{n}{\pi} \lambda_1\right), \quad (4.5)$$

where $h \in C^3(\mathbb{R}^+, \mathbb{R})$ and satisfies

$$\begin{aligned} h''(u) &> 0, & h'''(u) &< 0, \\ \lim_{u \rightarrow 0^+} h'(u) &= -\infty, & \lim_{u \rightarrow +\infty} h'(u) &= +\infty. \end{aligned} \quad (4.6)$$

Further, we let $H > 0$ be the (unique) real number that satisfies

$$h'(H) = 0. \quad (4.7)$$

5. The Dynamic Problem

Let $\lambda > 0$ and consider the problem: Find a motion u that satisfies

$$\begin{aligned} \operatorname{div} \tilde{S}(\nabla u) &= u_{tt} \quad \text{in} \quad \Omega \times [0, T], \\ u(x, t) &= \lambda x \quad \text{for} \quad x \in \partial\Omega \text{ and } t \in [0, T], \\ u(x, 0) &= \lambda x \quad \text{for} \quad x \in \Omega, \\ u_t(x, 0) &= 0 \quad \text{for} \quad x \in \Omega. \end{aligned} \quad (5.1)$$

We say that a motion u is a ~~weak solution~~ of (5.1) provided u satisfies (5.1)_{3,4} for almost every $x \in \dot{\Omega}$, u satisfies (5.1)₂ (in the sense of trace) for every $t \in [0, T]$,

$$\tilde{S}(\nabla u(\cdot, t)) \in L^1(\Omega, L^1)$$

for every $t \in [0, T]$, and

$$\int_0^T \int_{\Omega} [\nabla v \cdot \tilde{S}(\nabla u) - v_t \cdot u_t] dx dt = 0 \quad (5.2)$$

for every $\underline{u} \in C_0^\infty(\Omega \times (0, T), \mathbb{R}^n)$. Clearly,

$$\bar{u}(\underline{x}, t) := \lambda \underline{x} \text{ for all } \underline{x} \in \Omega \text{ and } t \geq 0$$

is one such weak solution. We will show that $\bar{u}$ is not unique.

We now restrict our attention to radial motions of a homogeneous isotropic ball of radius ρ . If we let $\underline{u}(\underline{x}, t) = (r(R, t)/R)\underline{x}$ we find that

$$\nabla \underline{u}(\underline{x}, t) = \frac{r(R, t)}{R} \underline{I} + [r_R(R, t) - \frac{r(R, t)}{R}] \frac{\underline{x}}{R} \otimes \frac{\underline{x}}{R}$$

and hence that (cf., e.g., Ball [1, p. 568])

$$\underline{S}(\nabla \underline{u}) = \hat{\phi}_2 \underline{I} + (\hat{\phi}_1 - \hat{\phi}_2) \frac{\underline{x}}{R} \otimes \frac{\underline{x}}{R}, \quad (5.3)$$

where $\hat{\phi}_1 = \hat{\phi}_1(r_R(R, t), r(R, t)/R)$. Thus (5.1)₁ reduces to

$$\frac{\partial}{\partial R}(R^{n-1}\hat{\phi}_1) - (n-1)R^{n-2}\hat{\phi}_2 = R^{n-1}r_{tt}. \quad (5.4)$$

If we let $r(R, t) = t\varphi(R/t)$, $s = R/t$, and note that $\partial/\partial R = t^{-1}d/ds$, $r_R = \dot{\varphi}$, $r_t = \varphi - s\dot{\varphi}$, and $r_{tt} = R^2\ddot{\varphi}/t^3$, we find that (5.4) reduces to

$$\frac{d}{ds}[s^{n-1}\hat{\phi}_1] - (n-1)s^{n-2}\hat{\phi}_2 = s^{n+1}\ddot{\varphi}$$

or, equivalently,

$$(s^{2-\phi_{11}}(s))\ddot{\varphi}(s) = \frac{n-1}{s}[\phi_{12}(s)(\dot{\varphi}(s) - \frac{\varphi(s)}{s}) + \phi_1(s) - \phi_2(s)], \quad (5.5)$$

where

$$\phi_{ij}(s) := \hat{\phi}_{ij}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = \phi_{ij}(\dot{\varphi}(s), \frac{\varphi(s)}{s}, \frac{\varphi(s)}{s}, \dots, \frac{\varphi(s)}{s}).$$

Any sufficiently smooth motion of the form (3.3) will consequently satisfy (5.1)₁ (in a neighborhood of $(\underline{x}, t)$) provided φ satisfies (5.5) (in a neighborhood of $s = |\underline{x}|/t$).

We next consider a solution of (5.1) that contains a shock; i.e., a discontinuity in $\underline{u}_t$, on the surface $\mathfrak{s}(t)$ that is propagating with speed $\mathfrak{s}(\underline{x}, t)$ in the direction $\underline{n}(\underline{x}, t)$. It is known (cf., e.g., Truesdell & Toupin [18, p. 545]) that such a solution must satisfy the Rankine-Hugoniot jump condition

$$[[\mathfrak{S}\underline{n}]] + \mathfrak{s}[[\underline{u}_t]] = \underline{Q} \text{ on } \mathfrak{s}(t), \quad (5.6)$$

where, e.g., $[[\underline{u}_t]]$ denotes the jump in $\underline{u}_t$ across the surface $\mathfrak{s}(t)$. For the radial motion of a radial shock we find, with the aid of (5.3), that the jump condition reduces to

$$[[\hat{\phi}_1]] + \mathfrak{s}[[r_t]] = 0 \text{ on } \mathfrak{s}(t).$$

In particular, if $r(R, t) = t \varphi(R/t)$ and $\mathfrak{s}(t) = \mathfrak{s}t\mathfrak{s}$ ($\mathfrak{s}$ denotes the unit sphere in $\mathbb{R}^n$), where $\mathfrak{s}$ denotes both the (constant) speed of the shock and the location of the discontinuity in $\dot{\varphi}$, the jump condition reduces to

$$[[\hat{\phi}_1(\mathfrak{s})]] = \mathfrak{s}^2 [[\dot{\varphi}(\mathfrak{s})]]$$

or, equivalently,

$$\hat{\phi}_1(\dot{\varphi}_+(\mathfrak{s}), \frac{\varphi(\mathfrak{s})}{\mathfrak{s}}) - \hat{\phi}_1(\dot{\varphi}_-(\mathfrak{s}), \frac{\varphi(\mathfrak{s})}{\mathfrak{s}}) = \mathfrak{s}^2 [\dot{\varphi}_+(\mathfrak{s}) - \dot{\varphi}_-(\mathfrak{s})]. \quad (5.7)$$

Here

$$\dot{\varphi}_{\pm}(\mathfrak{s}) := \lim_{s \rightarrow \mathfrak{s}_{\pm}} \dot{\varphi}(s).$$

Any sufficiently smooth motion of the form (3.3) will consequently satisfy (5.6) (on the surface $\mathfrak{s}(t) = \mathfrak{s}t\mathfrak{s}$) provided φ satisfies (5.7).

We next consider a radial motion $y(x, t) = (r(R, t)/R)x$ that satisfies $r(0, t) > 0$ for $t > 0$. In such a motion a hole of radius $r(0, t)$ develops in the material and the boundary of this hole should constitute a new traction free surface in the material; i.e., we expect that the radial component of the Cauchy stress to be zero on this surface. By (4.1) and (5.3) we find that we should require

$$\lim_{R \rightarrow 0} \left[\frac{R}{r(R, t)} \right]^{n-1} \hat{\phi}_1(r_R(R, t), \frac{r(R, t)}{R}) = 0,$$

or, assuming that $r(R, t) = t \varphi(R/t)$,

$$\lim_{s \rightarrow 0} \left[\frac{s}{\varphi(s)} \right]^{n-1} \hat{\phi}_1(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = 0.$$

Theorem 5.1. Let $\varphi \in PC^2([0, \infty), [0, \infty))$ with discontinuities in $\dot{\varphi}$ at $0 = s_0 < s_1 < \dots < s_L =: \sigma$ and suppose that φ satisfies the following:

- (i) $\varphi(s) = \lambda s$ for all $s \geq \sigma$;
- (ii) $\dot{\varphi}(s) > 0$ for every $s \neq s_i$, $i = 0, 1, \dots, L$;
- (iii) $(s^2 - \phi_{11}(s))\ddot{\varphi}(s) = \frac{n-1}{s}[\phi_{12}(s)(\dot{\varphi}(s) - \frac{\varphi(s)}{s}) + \phi_1(s) - \phi_2(s)]$ for every $s \neq s_i$, $i = 0, 1, \dots, L$;
- (iv) $\hat{\phi}_1(\dot{\varphi}_+(s_i), \frac{\varphi(s_i)}{s_i}) - \hat{\phi}_1(\dot{\varphi}_-(s_i), \frac{\varphi(s_i)}{s_i}) = s_i^2(\dot{\varphi}_+(s_i) - \dot{\varphi}_-(s_i))$ for $i = 1, 2, \dots, L$;
- (v) $\lim_{s \rightarrow 0} \left[\frac{s}{\varphi(s)} \right]^{n-1} \hat{\phi}_1(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = 0$;
- (vi) $s^{n-1} \hat{\phi}_2(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \in L^1((0, \sigma), \mathbb{R})$.

Then

$$y(x, t) := \begin{cases} \frac{\varphi(s)}{s} x, & t > 0 \\ \lambda x, & t = 0 \end{cases}, \quad s := |x|/t$$

is a weak solution of (5.1) in $\mathfrak{B}_\rho \times [0, T]$, $T := \rho/\sigma$.

Proof. Let $\varphi \in PC^2$ and satisfy (i)-(vi). Then, by Lemma 3.2, $\underline{u}$ is a motion that satisfies (5.1)₃ and (5.1)₄. Also, by (i), $\underline{u}$ will satisfy (5.1)₂.

Next, suppose that $\underline{u} \in C_0^\infty(\mathfrak{B}_\rho \times (0, T), \mathbb{R}^n)$. Then there is an $\epsilon > 0$ such that $\underline{u}$ has compact support in $\mathfrak{B}_\rho \times (\epsilon, T)$. Let $\rho_0 := 0$, $\rho_{L+1} := \rho$, $\rho_i(t) := s_i t$,

$$\mathcal{A}_i := \{(\underline{x}, t) \in \mathfrak{B}_\rho \times (\epsilon, T) : |\underline{x}| = \rho_i(t)\},$$

$i = 1, 2, \dots, L$, and

$$\mathfrak{R}_i := \{(\underline{x}, t) \in \mathfrak{B}_\rho \times (\epsilon, T) : \rho_i(t) \leq |\underline{x}| \leq \rho_{i+1}(t)\},$$

$i = 0, 1, \dots, L$. Note that

$$\tilde{N}^i := (1 + s_i^2)^{-1/2}(\underline{n}, -s_i) \quad (\underline{n} := \underline{x}/|\underline{x}|) \quad (5.8)$$

is one of the unit normals to $\mathcal{A}_i$ and that for any $\varphi \in C_0^\infty(\mathfrak{B}_\rho \times (\epsilon, T), \mathbb{R})$ one can parametrize the space-time surface integral

$$\int_{\mathcal{A}_i} \varphi(\underline{x}, t) dS(\underline{x}, t) = \int_{\mathfrak{B}_{s_i T} \setminus \mathfrak{B}_{s_i \epsilon}} \varphi(\underline{x}, |\underline{x}|/s_i) (1 + s_i^2)^{1/2} d\underline{x}. \quad (5.9)$$

Now, in $\mathfrak{R}_i$, $i \neq 0$, $\nabla \underline{u} \cdot \underline{S} = \text{div}(\underline{S}^T \underline{u}) - \underline{u} \cdot \text{div } \underline{S}$, $\underline{u}_t \cdot \underline{u}_t = (\underline{u} \cdot \underline{u}_t)_t - \underline{u} \cdot \underline{u}_{tt}$. Therefore, we find, with the aid of the divergence theorem (Lemma 1.1), (5.8), (5.9), and the fact that $\underline{u}(\underline{x}, t) = \underline{Q}$ when $t = \epsilon$ or $t = T$, that, for $i = 1, 2, \dots, L-1$,

$$\begin{aligned}
\int_{\mathcal{R}_1} [\nabla \tilde{y} \cdot \tilde{S}(\nabla u^1) - u_t^1 \cdot \tilde{y}_t] dV(x, t) &= - \int_{\mathcal{R}_1} \tilde{y} \cdot [d \operatorname{div} \tilde{S}(\nabla u^1) - u_{tt}^1] dV(x, t) \\
&+ \int_{\mathcal{B}_{s_{i+1}T} \setminus \mathcal{B}_{s_{i+1}\epsilon}} \tilde{y} \cdot [\tilde{S}(\nabla u^1)_n + s_{i+1} u_t^1] dx \\
&- \int_{\mathcal{B}_{s_1T} \setminus \mathcal{B}_{s_1\epsilon}} \tilde{y} \cdot [\tilde{S}(\nabla u^1)_n + s_1 u_t^1] dx,
\end{aligned}$$

where $u^1 := u|_{\mathcal{R}_1}$. Also, since $y(x, t) = 0$ when $|x| = \rho$, the above equation will be satisfied when $i = L$ provided that the second integral on the right-hand side is deleted. Therefore, since y has compact support in $\mathcal{B}_\rho \times (\epsilon, T)$, since u satisfies (iii) and (iv), and since

$$\bigcup_{i=0}^L \mathcal{R}_i = \mathcal{B}_\rho \times (\epsilon, T),$$

we discover that

$$\begin{aligned}
\int_0^T \int_{\mathcal{B}_\rho} [\nabla \tilde{y} \cdot \tilde{S}(\nabla u) - u_t \cdot \tilde{y}_t] dx dt &= \int_{\mathcal{R}_0} [\nabla \tilde{y} \cdot \tilde{S}(\nabla u) - u_t \cdot \tilde{y}_t] dV(x, t) \\
&- \int_{\mathcal{B}_{s_1T} \setminus \mathcal{B}_{s_1\epsilon}} \tilde{y} \cdot [\tilde{S}(\nabla u^1)_n + s_1 u_t^1] dx.
\end{aligned} \tag{5.10}$$

We note that, by (v) and (vi) (cf. (5.3)),

$$\tilde{S}(\nabla u(\cdot, t)) \in L^1(\mathcal{B}_\rho, \operatorname{Lin})$$

for every $t \in [0, T]$. Thus, in order to show that the left-hand side of (5.10) is zero we will prove that

$$\lim_{\delta \rightarrow 0^+} I_\delta = 0,$$

$$I_\delta := \int_0^T \int_{\mathbb{R}_\rho \setminus \mathbb{R}_\delta} [\nabla \underline{v} \cdot \underline{S}(\nabla \underline{u}) - \underline{u}_t \cdot \underline{v}_t] dx dt,$$

and the desired result will follow from the Lebesgue dominated convergence theorem.

We write $dx = R^{n-1} dR d\Theta$, where $d\Theta$ denotes the standard $n-1$ dimensional measure on the surface of the unit sphere $\mathbb{S}$. Then by Lemma 1.1, the fact that $\underline{u}$ satisfies (iii) in $\mathbb{R}_0 \setminus [\mathbb{R}_\delta \times (\epsilon, T)]$, and since $\underline{v}(\underline{x}, t) = 0$ when $t = \epsilon$ or $t = T$, we find that

$$\begin{aligned} \int_{\mathbb{R}_0 \setminus [\mathbb{R}_\delta \times (\epsilon, T)]} [\nabla \underline{v} \cdot \underline{S}(\nabla \underline{u}^0) - \underline{u}_t^0 \cdot \underline{v}_t] dV(\underline{x}, t) &= \int_{\mathbb{R}_{S_1 T} \setminus \mathbb{R}_{S_1 \epsilon}} \underline{v} \cdot [\underline{S}(\nabla \underline{u}^0) \underline{n} + S_1 \underline{u}_t^0] d\underline{x} \\ &- \int_{\epsilon}^T dt \int_{\mathbb{S}} \underline{v} \cdot \underline{S}(\nabla \underline{u}^0) \underline{n} \delta^{n-1} d\Theta. \end{aligned} \quad (5.11)$$

If we combine (5.10) and (5.11) and use the jump condition (iv) we conclude that

$$I_\delta = - \int_{\epsilon}^T \int_{\mathbb{S}} \underline{v} \cdot \underline{S}(\nabla \underline{u}) \underline{n} \delta^{n-1} d\Theta.$$

Now, by (5.3), $\underline{S} \underline{n} = \underline{x} \cdot \hat{\phi}_1 / R$ and hence the integrand in the last equation is dominated by

$$\delta^{n-1} |\hat{\phi}_1(\varphi(\frac{\delta}{t}), \frac{t}{\delta} \varphi(\frac{\delta}{t}))| \sup_{\mathbb{R}_\rho \times (0, T)} |\underline{v}|.$$

Since φ is piecewise C^2 we find, with the aid of (v), that the last expression goes to zero uniformly for $t \in [\epsilon, T]$ and hence $I_\delta \rightarrow 0$ as $\delta \rightarrow 0^+$. ■

6. Existence

We will now show that, for appropriate choices of $\lambda > 0$, we can construct a piecewise C^2 function φ that satisfies (i)-(vi) of Theorem 5.1 for the special stored energy function (4.5). We begin by noting that, in this case, (iii) and (v) of Theorem 5.1 are equivalent to

$$\ddot{\varphi}(s) = \frac{\frac{n-1}{s} \left[\dot{\varphi}(s) - \frac{\varphi(s)}{s} \right] \left[1 + \dot{\varphi}(s) \left(\frac{\varphi(s)}{s} \right)^{2n-3} h''(u(s)) \right]}{s^2 - 1 - \left[\frac{\varphi(s)}{s} \right]^{2n-2} h''(u(s))}, \quad (6.1)$$

$$\lim_{s \rightarrow 0^+} \tau(s) = 0, \quad \tau(s) := u(s) \left[\frac{\varphi(s)}{s} \right]^{2-2n} + h'(u(s)), \quad (6.2)$$

where $u(s) := \dot{\varphi}(s)(\varphi(s)/s)^{n-1}$, the specific volume, and τ is the radial component of the Cauchy stress.

Theorem 6.1. Let $n \geq 3$.⁸ Suppose that the stored energy satisfies (4.5)-(4.7). Then there exist $\lambda_1 \rightarrow +\infty$ and $T_1 > 0$ such that (5.1), with $\lambda = \lambda_1$, has a weak solution of the form

$$\tilde{u}^1(\tilde{x}, t) := \begin{cases} \frac{\varphi(s)}{s} \tilde{x}, & t \in (0, T_1] \\ \lambda_1 \tilde{x}, & t = 0 \end{cases}, \quad s := |\tilde{x}|/t,$$

where $\varphi \in PC^2$ satisfies (6.1), (6.2), and $\varphi(0) > 0$.

⁸The requirement $n \geq 3$ is due to the quadratic growth of ϕ in the individual eigenvalues (cf. (4.5)). We expect that one can obtain dynamic cavitation in two dimensions for constitutive relations with slower growth.

Remark. If we combine (3.6) and (3.7) we find that

$$\varphi(0) = \lim_{x \rightarrow 0} |u_t(x, t)|.$$

Thus $\varphi(0)$ gives us the velocity of the surface of the cavity.

In order to prove the above theorem we need to establish a number of intermediate results. For the remainder of this section we suppose that the stored energy satisfies (4.5)-(4.7) and that
 $n \geq 3$.

Proposition 6.2. Let $v > 0$. Then there exists an $s_0 > 0$ and a unique $\varphi \in C^2([0, s_0], \mathbb{R})$ that satisfies (6.1), (6.2), and $\varphi(0) = v$. In addition, for all $s \in (0, s_0)$,

$$\begin{aligned} \varphi(s) &> 0, \quad \dot{\varphi}(s) > 0, \\ s^2 - 1 - \left[\frac{\varphi(s)}{s} \right]^{2n-2} h''(u(s)) &< 0, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \dot{\varphi}(s) &< \frac{\varphi(s)}{s}, \\ \dot{\varphi}(0) &= 0, \quad \lim_{s \rightarrow 0^+} u(s) = H, \\ s^{n-1} \hat{\phi}_2(\dot{\varphi}(s), \frac{\varphi(s)}{s}) &\in L^1((0, s_0), \mathbb{R}). \end{aligned} \tag{6.4}^9$$

Proof. Define $\hat{\mathcal{T}}: \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\hat{\mathcal{T}}(u, \omega) = u\omega^{2n-2} + h'(u).$$

Then, by (4.6)₂, $\hat{\mathcal{T}}(\cdot, \omega): \mathbb{R}^+ \rightarrow \mathbb{R}$ is surjective for every $\omega \in \mathbb{R}$ and hence there is a function $\hat{u}: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ that satisfies

⁹Cf. equation (4.7).

$$\hat{\mathcal{T}}(\hat{u}(\mathcal{T}, \omega), \omega) = \mathcal{T}$$

for every $\mathcal{T}, \omega \in \mathbb{R}$. Moreover, since $h \in C^3(\mathbb{R}^+, \mathbb{R})$ and satisfies $h'' > 0$, $\hat{\mathcal{T}} \in C^2(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ and satisfies $\hat{\mathcal{T}}_u > 0$. Therefore, by the implicit function theorem, $\hat{u} \in C^2(\mathbb{R}^2, \mathbb{R}^+)$.

Next, if $\varphi \in C^2((0, s_0), \mathbb{R})$ and satisfies (6.1) and (6.3) then $\mathcal{T} \in C^1((0, s_0), \mathbb{R})$ and satisfies

$$\begin{aligned}\dot{\mathcal{T}} &= \dot{u} \left[\frac{\varphi}{s} \right]^{2-2n} + (2-2n) \frac{u}{s} \left[\frac{\varphi}{s} \right]^{1-2n} \left[\dot{\varphi} - \frac{\varphi}{s} \right] + \dot{u} h'', \\ \dot{u} &= \ddot{\varphi} \left[\frac{\varphi}{s} \right]^{n-1} + (n-1) \frac{\dot{\varphi}}{s} \left[\frac{\varphi}{s} \right]^{n-2} \left[\dot{\varphi} - \frac{\varphi}{s} \right],\end{aligned}$$

and hence

$$\dot{\mathcal{T}} = \ddot{\varphi} \left[\frac{\varphi}{s} \right]^{1-n} \left[1 + \left(\frac{\varphi}{s} \right)^{2n-2} h'' \right] + \frac{n-1}{s} \dot{\varphi} \left[\dot{\varphi} - \frac{\varphi}{s} \right] \left[\left(\frac{\varphi}{s} \right)^{n-2} h'' - \left(\frac{s}{\varphi} \right)^n \right].$$

However, (6.1) is equivalent to

$$\left[1 + \left(\frac{\varphi}{s} \right)^{2n-2} h'' \right] \ddot{\varphi} = s^2 \ddot{\varphi} - \frac{n-1}{s} \left[\dot{\varphi}^2 \left(\frac{s}{\varphi} \right)^n - \left(\frac{s}{\varphi} \right)^{n-2} \right]$$

and therefore $\mathcal{T}$ satisfies

$$\dot{\mathcal{T}} = \frac{s^{n+1}}{\varphi^{n-1}} \ddot{\varphi} - \frac{n-1}{s} \left[\dot{\varphi}^2 \left(\frac{s}{\varphi} \right)^n - \left(\frac{s}{\varphi} \right)^{n-2} \right]. \quad (6.5)$$

We note, that by the construction of $\hat{u}$, the pair $(\varphi, \mathcal{T})$ satisfies

$$\dot{\varphi}(s) = \left[\frac{s}{\varphi(s)} \right]^{n-1} \hat{u}(\mathcal{T}(s), s/\varphi(s)). \quad (6.6)$$

Also (6.1) is equivalent to

$$s \ddot{\varphi} = (n-1)(s \dot{\varphi} - \varphi) \frac{s^{2n-3} + \dot{\varphi} \varphi^{2n-3} h''}{s^{2n-s} s^{2n-2} - \varphi^{2n-2} h''}. \quad (6.7)$$

If we combine (6.5)-(6.7) we conclude that, given $\varphi \in C^2((0, s_0), \mathbb{R})$ that satisfies (6.1) and (6.3), the pair

$(\varphi, \tau) \in C^1((0, s_0), \mathbb{R}^2)$ and satisfies the system of ordinary differential equations

$$\begin{aligned}\dot{\tau}(s) &= P(\tau(s), \varphi(s), s), \\ \dot{\varphi}(s) &= Q(\tau(s), \varphi(s), s),\end{aligned}\tag{6.8}$$

where

$$\begin{aligned}P(\tau, \varphi, s) &:= \frac{s^n}{\varphi^{n-1}} R - (n-1) \left[Q^2 \frac{s^{n-1}}{\varphi^n} - \frac{s^{n-3}}{\varphi^{n-2}} \right], \\ R(\tau, \varphi, s) &:= (n-1)(sQ - \varphi) \frac{s^{2n-3} + Q \varphi^{2n-3} h''(\hat{u}(\tau, s/\varphi))}{s^{2n-s} s^{2n-2} - \varphi^{2n-2} h''(\hat{u}(\tau, s/\varphi))}, \\ Q(\tau, \varphi, s) &:= \left[\frac{s}{\varphi} \right]^{n-1} \hat{u}(\tau, s/\varphi).\end{aligned}\tag{6.9}$$

Conversely, given $(\varphi, \tau) \in C^1((0, s_0), \mathbb{R}^2)$ that satisfies (6.3) and (6.8), one can conclude from (6.8) and (6.9)₃ that $\varphi \in C^2((0, s_0), \mathbb{R})$ and hence that φ is a solution of (6.1). Thus we consider the initial-value problem (6.8),

$$\tau(0) = 0, \quad \varphi(0) = v.\tag{6.10}$$

We note that $\hat{u}$ and h'' are C^1 and hence that P and Q are C^1 in some neighborhood of $(0, v, 0)$ for any $v > 0$. A standard theorem from ordinary differential equations then yields an $\alpha > 0$ and a unique pair $(\varphi, \tau) \in C^1((-\alpha, \alpha), \mathbb{R}^2)$ that satisfies (6.8) and (6.10).

To show that φ satisfies (6.3) and (6.4)_{1,2} (and hence that φ is C^2 and satisfies (6.1)), we note that (6.10) implies that φ is strictly positive on some interval $[0, s_1)$. It then follows from (6.8)₂, (6.9)₃, and the construction of $\hat{u}$ that $\dot{\varphi}(0) = 0$, $\varphi(0) = H$, and $\dot{\varphi}(s) > 0$ for all $s \in (0, s_1)$. It is clear that there is a subinterval $(0, s_0)$ upon which (6.3)₂ and (6.4)₁ are satisfied.

Next, we need to show that $\ddot{\varphi}$ has a finite limit as $s \rightarrow 0^+$ in order to conclude that $\varphi \in C^2([0, s_0], \mathbb{R})$. If we multiply (6.7) by $1/s$ we find that the desired result will follow provided that

$$\lim_{s \rightarrow 0^+} \frac{\dot{\varphi}(s)}{s} \text{ exists.}$$

However, by (6.8)₂, (6.9)₃, and the construction of $\hat{u}$ this limit does exist and is equal to zero.

We now consider uniqueness. Let $\varphi \in C^2([0, s_0], \mathbb{R})$ satisfy (6.1), (6.2), and $\varphi(0) = v$. Then $u \in C^1((0, s_0], \mathbb{R})$ and by (4.7) and (6.2)

$$\lim_{s \rightarrow 0^+} u(s) = H.$$

Thus there is a subinterval $[0, s_1]$ upon which $u > 0$, $\varphi > 0$ and hence we conclude that (6.3)₁ is satisfied on $(0, s_1)$. It is then clear that there is a subinterval $(0, s_2)$ upon which (6.3)₂ is also satisfied. It now follows from (6.2) and (6.5) that $(\varphi, \tau) \in C^1([0, s_2], \mathbb{R})$ and satisfies (6.10) and, by a previous argument, that (φ, τ) satisfies (6.8). Therefore (φ, τ) must be the unique solution to (6.8) and (6.10) that we previously constructed.

Finally, by (4.5),

$$s^{n-2} \hat{\phi}_2(\dot{\varphi}, \frac{\varphi}{s}) = \varphi s^{n-3} + h'(u) \dot{\varphi} \varphi^{n-2}$$

and hence (6.4)₃ is satisfied. ■

Proposition 6.3. Let $\varphi \in C^2([0, s_0], \mathbb{R})$ be the unique solution of (6.1), (6.2), and $\varphi(0) = v$. Then φ can be continued uniquely as a solution of (6.1) to a maximal interval $[0, s_M(v))$ where

$$s_M^2 - 1 - \left(\frac{\varphi(s_M)}{s_M} \right)^{2n-2} h''(\varphi(s_M)) = 0. \quad (6.11)$$

Moreover, $\varphi \in C^1([0, s_M], \mathbb{R}) \cap C^2([0, s_M), \mathbb{R})$ and satisfies (6.3) and (6.4)₁ for all $s \in (0, s_M)$.

Proof. The standard continuation theorem for ordinary differential equations implies that a solution can be uniquely continued as long as the right-hand side of (6.1) remains Lipschitz. Thus φ can be continued as a solution of (6.1) to an interval $[0, s_M(v))$; $\varphi \in C^2([0, s_M), \mathbb{R})$; and either $s_M = +\infty$,

$$\lim_{s \rightarrow s_M} \varphi(s) \text{ does not exist,} \quad (6.12)$$

$$\lim_{s \rightarrow s_M} \dot{\varphi}(s) \text{ does not exist,}$$

$$\lim_{s \rightarrow s_M} \varphi(s) = 0, \quad \lim_{s \rightarrow s_M} \varphi(s) = +\infty, \quad (6.13)$$

$$\lim_{s \rightarrow s_M} \dot{\varphi}(s) = 0, \quad \lim_{s \rightarrow s_M} \dot{\varphi}(s) = +\infty,$$

or $\varphi \in C^1([0, s_M], \mathbb{R})$ and (6.11) is satisfied. Moreover, (6.3) must be satisfied on $(0, s_M)$ since the right-hand side of (6.1) is assumed to be Lipschitz at $(s, \varphi(s), \dot{\varphi}(s))$ for every $s \in (0, s_M)$.

To show that (6.4)₁ is satisfied on $(0, s_M)$ suppose that $\alpha := \dot{\varphi}(t_0) = \varphi(t_0)/t_0$ at $t_0 \in (0, s_M)$. Then $\varphi_*(s) := \alpha s$ is a solution of (6.1) with $\varphi_*(t_0) = \alpha t_0$ and $\dot{\varphi}_*(t_0) = \alpha$ and the right-hand side of (6.1) is C^1 in a neighborhood of $(s, \varphi, \dot{\varphi}) = (s, \alpha s, \alpha)$. A standard uniqueness theorem for ordinary differential equations then yields $\varphi(s) = \varphi_*(s) = \alpha s$ on $(0, t_0)$. However, $\varphi_*(0) = 0 < v = \varphi(0)$ and hence no such t_0 can exist.

Next, (6.1), (6.3), and (6.4)₁ imply that $\dot{\varphi} > 0$ and $\ddot{\varphi} > 0$ on $(0, s_M)$. Thus φ and $\dot{\varphi}$ are monotone functions and consequently have left-hand limits as $s \rightarrow s_M^-$. Therefore φ cannot satisfy (6.12). We also note that $(\varphi(s)/s)' = (\dot{\varphi}(s) - \varphi(s)/s)/s < 0$ and hence that

$$0 < \dot{\varphi}(s) < \dot{\varphi}(t) < \frac{\varphi(t)}{t} < \frac{\varphi(s)}{s} < +\infty \quad (6.14)$$

whenever $0 < s < t < s_M$. In particular φ cannot satisfy (6.13).

To show that s_M is finite and hence that φ satisfies (6.11) we suppose instead that $s_M = +\infty$. Then, by (6.14), u is bounded on $(s_0, +\infty)$ and consequently

$$\left[\frac{\varphi(s)}{s} \right]^{2n-2} h''(u(s))$$

is uniformly bounded for $s \in (s_0, +\infty)$. It follows that (6.3)₂ cannot be satisfied on $(s_0, +\infty)$, a contradiction. Therefore $\varphi \in C^1([0, s_M], \mathbb{R})$ and (6.11) is satisfied. ■

Before proceeding to the next result we note that, for the special stored energy function (4.5), (iv) of Theorem 5.1 is equivalent to

$$\left(\frac{\varphi(s_1)}{s_1} \right)^{n-1} [h'(u_+(s_1)) - h'(u_-(s_1))] = (s_1^2 - 1) [\dot{\varphi}_+(s_1) - \dot{\varphi}_-(s_1)],$$

where

$$u_{\pm}(s_1) = \lim_{s \rightarrow s_1^{\pm}} \dot{\varphi}(s) \left[\frac{\varphi(s)}{s} \right]^{n-1}.$$

Proposition 6.4. Let $\varphi \in C^1([0, s_M], \mathbb{R}) \cap C^2([0, s_M), \mathbb{R})$ be the unique solution of (6.1) and (6.2) that satisfies $\varphi(0) = v$. Then either

$$\lambda_M := \frac{\varphi(s_M)}{s_M} = \dot{\varphi}(s_M) \quad (6.15)$$

or $\lambda_M > \dot{\varphi}(s_M)$. In the latter case there is an $s_J \in (0, s_M)$ such that

$$(\lambda_J - \dot{\varphi}(s_J))(1 - s_J^2) + \lambda_J^{n-1} [h'(\lambda_J^n) - h'(\varphi(s_J))] = 0, \quad (6.16)$$

where $\lambda_J := \varphi(s_J)/s_J$.

Proof. By Proposition 6.3 (specifically (6.4)₁) and the continuity of φ on $[0, s_M]$ $\dot{\varphi}(s_M) \leq \lambda_M$ and hence either (6.15) is satisfied or $\dot{\varphi}(s_M) < \lambda_M$.

If $\dot{\varphi}(s_M) < \lambda_M$ we define $\theta \in C^0((0, s_M], \mathbb{R})$ by

$$\theta(s) := \left[\frac{\varphi(s)}{s} - \dot{\varphi}(s) \right] (1 - s^2) + \left[\frac{\varphi(s)}{s} \right]^{n-1} \left[h' \left(\left[\frac{\varphi(s)}{s} \right]^n \right) - h'(\varphi(s)) \right].$$

Then, since $\dot{\varphi}(s) \rightarrow 0$, $\varphi(s) \rightarrow H$, and $\varphi(s)/s \rightarrow +\infty$ as $s \rightarrow 0^+$ we find, with the aid of (4.6)₂, that

$$\lim_{s \rightarrow 0^+} \theta(s) = +\infty. \quad (6.17)$$

Next, $\varphi(s_M) = \dot{\varphi}(s_M) \lambda_M^{n-1} < \lambda_M^n$ and therefore, by the mean value theorem, there is a $c \in (\varphi(s_M), \lambda_M^n)$ that satisfies

$$h'(\lambda_M^n) - h'(\varphi(s_M)) = h''(c) \lambda_M^{n-1} (\lambda_M - \dot{\varphi}(s_M)). \quad (6.18)$$

If we combine (6.11) and (6.18) we find that

$$\begin{aligned} \theta(s_M) &= [\lambda_M - \dot{\varphi}(s_M)] [1 - s_M^2 + \lambda_M^{2n-2} h''(c)] \\ &= [\lambda_M - \dot{\varphi}(s_M)] \lambda_M^{2n-2} [h''(c) - h''(\varphi(s_M))]. \end{aligned}$$

Since $\lambda_M > \dot{\varphi}(s_M)$, $c > \varphi(s_M)$, and $h''' < 0$ (cf. (4.6)₁) we conclude that $\theta(s_M) < 0$. This together with (6.17) and the intermediate value theorem yield (6.16). ■

If we combine Propositions 6.2, 6.3, and 6.4 we find, with the aid of Theorem 5.1, that there is at least one $\lambda \in \mathbb{R}^+$ where the conclusions of Theorem 6.1 are valid; i.e., where (5.1) has a cavitation solution. In order to complete the proof of Theorem 6.1 we need to prove

Proposition 6.5. Let $\lambda_*(v) : (0, \infty) \rightarrow (0, \infty)$ be given by

$$\lambda_*(v) := \begin{cases} \lambda_M(v) & \text{if } \lambda_M(v) = \dot{\varphi}(s_M(v); v) \\ \lambda_J(v) & \text{otherwise} \end{cases}$$

(cf. (6.15) and (6.16)). Then

$$\sup_{v \in (0, \infty)} \lambda_*(v) = +\infty.$$

Proof. If we differentiate $u(s) = \dot{\varphi}(s)(\varphi(s)/s)^{n-1}$ and use (6.1) we discover that

$$\dot{u} = \frac{\frac{n-1}{s} \left[\dot{\varphi} - \frac{\varphi}{s} \right] \left[\frac{\varphi}{s} \right]^{n-2} \left[s^2 \dot{\varphi} + \frac{\varphi}{s} - \dot{\varphi} \right]}{s^2 - 1 - \left[\frac{\varphi}{s} \right]^{2n-2} h''(u)}. \quad (6.19)$$

Thus by Proposition 6.3 (specifically (6.3) and (6.4)₁) $\dot{u} > 0$ on $(0, s_M)$ and hence (cf. (6.4)₂)

$$H < u(s) < \dot{\varphi}(s_M) \lambda_M^{n-1} \leq \lambda_M^n \quad (6.20)$$

for all $s \in (0, s_M)$.

Next, we claim that

$$\sup_{v \in (0, \infty)} \lambda_M(v) = +\infty.$$

If not, we combine (6.11) and (6.20) to conclude that $s_M(\cdot)$ is bounded on $(0, \infty)$ and hence that $\varphi(s_M(v); v) = \lambda_M(v) s_M(v)$ is uniformly bounded for $v \in (0, \infty)$. However, by Proposition 6.3 (specifically (6.3)₁), $\dot{\varphi} > 0$ and hence $\varphi(s_M(v); v) \geq \varphi(0; v) = v$, a contradiction.

Finally, if there are $v_i \rightarrow +\infty$ such that (6.16) is satisfied we claim that

$$\limsup_{i \rightarrow \infty} \lambda_J(v_i) = +\infty. \quad (6.21)$$

To show this we apply the mean value theorem to the function h' on the interval $[\psi(s_J), \lambda_J^n]$ to conclude, with the aid of (6.16), that

$$s_J^2 = 1 + \lambda_J^{2n-2} h''(c), \quad (6.22)$$

for some $c \in (H, \lambda_J^n)$. Thus if $\lambda_J(v_i)$ were bounded then $s_J(v_i)$ would be bounded and hence $\psi(s_J(v_i); v_i) = \lambda_J(v_i) s_J(v_i)$ would be bounded. However, $\dot{\psi} > 0$ and hence $\psi(s_J(v_i); v_i) \geq v_i$. Therefore (6.21) must be satisfied. ■

7. The Energy. Admissibility

The total energy, at time t , of a weak solution $\underline{u}$ of (5.1) is given by

$$E(t, \underline{u}, \Omega) := \int_{\Omega} [W(\nabla \underline{u}(x, t)) + \frac{1}{2} |\underline{u}_t(x, t)|^2] dx.$$

The total energy of the equilibrium solution $\bar{\underline{u}}(x, t) = \lambda x$ is therefore equal to $W(\lambda \underline{I}) \text{vol}(\Omega)$. In this section we show that the total energy of the cavitation solution constructed in Section 6 is no greater than the total energy of $\bar{\underline{u}}$. Thus, since the total energy is the appropriate entropy in this context, the cavitation solutions are admissible according to the usual entropy criterion.¹⁰

¹⁰Cf. Lax [12]. See Dafermos [5] for a discussion of admissibility criteria.

Our first result shows that the difference in energy between $\bar{u}$ and a (sufficiently smooth) solution of the form (3.3) is due to the discontinuities in $\dot{\psi}$. For simplicity we only consider the case where $\dot{\psi}$ has one discontinuity.

Proposition 7.1. Let

$$u(x, t) := \begin{cases} \frac{\psi(s)}{s} x, & t > 0 \\ \lambda x, & t = 0 \end{cases}, \quad s := |x|/t,$$

where $\psi \in PC^2([0, \infty), [0, \infty))$ has (at most) one discontinuity in $\dot{\psi}$ (at $s = \sigma$). Suppose that ψ satisfies (i)-(v) of Theorem 5.1,

$$\begin{aligned} s^{n-2} \hat{\phi}_2(\dot{\psi}(s), \frac{\psi(s)}{s}) &\in L^1((0, \sigma), \mathbb{R}), \\ \lim_{s \rightarrow 0^+} s^n \hat{\phi}(\dot{\psi}(s), \frac{\psi(s)}{s}) &= 0. \end{aligned} \tag{7.1}$$

Then

$$E(t, u, \mathfrak{P}_\rho) = E(t, \lambda x, \mathfrak{P}_\rho) + t^n \sigma^n J/n,$$

where

$$\begin{aligned} J &:= \hat{\phi}(\dot{\psi}_-, \lambda) - \hat{\phi}(\lambda, \lambda) + \frac{1}{2} [\hat{\phi}_1(\dot{\psi}_-, \lambda) + \hat{\phi}_1(\lambda, \lambda)] (\lambda - \dot{\psi}_-), \\ \dot{\psi}_- &= \lim_{s \rightarrow \sigma^-} \dot{\psi}(s). \end{aligned}$$

Remark. The above proposition reduces the problem of admissibility to showing that $J \leq 0$. This inequality is similar to an inequality obtained by James [10] in connection with the admissibility of solutions to one-dimensional elastodynamics with non-convex energy. James observes that such an inequality is a necessary condition for his elastodynamic solutions to be the zero

viscosity limit of a sequence of solutions to the dynamic equations of (instantaneous) viscoelasticity.

Proof of Proposition 7.1. We first note that

$$s^n \frac{d}{ds} \hat{\phi}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) = s^n \hat{\phi}_1 \ddot{\varphi} + (n-1)s^{n-2}(s\dot{\varphi} - \varphi) \hat{\phi}_2 \quad (7.2)$$

and hence we conclude, with the aid of (7.1)₁, and (v) of Theorem 5.1, that

$$s^n \frac{d}{ds} \hat{\phi}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) \in L^1((0, \sigma), \mathbb{R}). \quad (7.3)$$

If we integrate $s^n \frac{d}{ds} \hat{\phi}(s)$ by parts we find that

$$\int_{\epsilon}^{\sigma} s^n \frac{d}{ds} \hat{\phi}(s) ds = -n \int_{\epsilon}^{\sigma} s^{n-1} \hat{\phi}(s) ds + \sigma^n \hat{\phi}(\sigma) - \epsilon^n \hat{\phi}(\epsilon).$$

Thus, if we take the limit of the last equation as $\epsilon \rightarrow 0^+$ we discover, with the aid of (7.1)₂, (7.3), the fact that $\hat{\phi} \geq 0$, the Lebesgue dominated convergence theorem, and the monotone convergence theorem, that

$$n \int_0^{\sigma} s^{n-1} \hat{\phi}(s) ds = - \int_0^{\sigma} s^n \frac{d}{ds} \hat{\phi}(s) ds + \sigma^n \hat{\phi}(\sigma) < +\infty. \quad (7.4)$$

Now, by (3.4), (3.6), (3.7), and (7.4)

$$\begin{aligned} E &= \int_{\mathcal{B}_{\rho}} [\hat{\phi}(\dot{\varphi}(s), \frac{\varphi(s)}{s}) + \frac{1}{2}(\varphi(s) - s\dot{\varphi}(s))^2] dx \\ &= t^n \int_0^{\rho/t} [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] s^{n-1} ds < +\infty, \end{aligned}$$

provided $t \in (0, T]$. Since $\varphi(s) = \lambda s$ for $s \geq \sigma$,

$$E = \int_0^{\rho^n - \sigma^n t^n} \hat{\phi}(\lambda, \lambda) / n + t^n \int_0^{\sigma} [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] s^{n-1} ds. \quad (7.5)$$

We now consider the integral on the right-hand side of (7.5).

If we integrate by parts and apply (7.4) we arrive at

$$\begin{aligned} n \int_0^\sigma [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] s^{n-1} ds &= - \int_0^\sigma \frac{d}{ds} [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] s^n ds \\ &+ \hat{\phi}(\dot{\varphi}_-(\sigma), \lambda) \sigma^n + \frac{1}{2}(\lambda \sigma - \sigma \dot{\varphi}_-(\sigma))^2 \sigma^n. \end{aligned} \quad (7.6)$$

Then if we combine (7.2) and (iii) of Theorem 5.1 we find that

$$\begin{aligned} s^n \frac{d}{ds} [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] &= s^n \hat{\phi}_1 \ddot{\varphi} + (n-1) s^{n-2} \hat{\phi}_1 (s\dot{\varphi} - \varphi) + s^{n-1} (s\dot{\varphi} - \varphi) \frac{d}{ds} \hat{\phi}_1 \\ &= \frac{d}{ds} [s^{n-1} \hat{\phi}_1 (s\dot{\varphi} - \varphi)], \end{aligned}$$

which together with (7.6) and (v) of Theorem 5.1 yield

$$\int_0^\sigma s^n \frac{d}{ds} [\hat{\phi} + \frac{1}{2}(\varphi - s\dot{\varphi})^2] ds = \sigma^n \hat{\phi}_1 (\dot{\varphi}_-(\sigma), \lambda) [\dot{\varphi}_-(\sigma) - \lambda]. \quad (7.7)$$

Finally, (7.5)-(7.7), (iv) of Theorem 5.1, and the identity

$$E(t, \lambda \tilde{x}_\rho, \mathfrak{P}_\rho) = \rho^n \mathfrak{s}_n \hat{\phi}(\lambda, \lambda) / n$$

imply

$$n(E(\underline{u}) - E(\lambda \tilde{x}_\rho)) / t^n \sigma^n \mathfrak{s}_n = \hat{\phi}(\dot{\varphi}_-, \lambda) - \hat{\phi}(\lambda, \lambda) + \frac{1}{2} [\hat{\phi}_1(\dot{\varphi}_-, \lambda) + \hat{\phi}_1(\lambda, \lambda)] (\lambda - \dot{\varphi}_-),$$

which is the desired result. ■

Theorem 7.2. Suppose that the stored energy satisfies
(4.5)-(4.7) and that $n \geq 3$. Let $\underline{u}$ be the weak solution of (5.1) that is given by Theorem 6.1. Then

$$E(t, \underline{u}, \mathfrak{P}_\rho) \leq E(t, \lambda \tilde{x}_\rho, \mathfrak{P}_\rho)$$

with equality if and only if $\dot{\varphi}(s_M) = \varphi(s_M) / s_M$ (cf. Proposition 6.3).

Here

$$\lambda = \begin{cases} \lambda_M := \frac{\dot{\varphi}(s_M)}{s_M} & \text{if } \lambda_M = \dot{\varphi}(s_M) \\ \lambda_J := \frac{\dot{\varphi}(s_J)}{s_J} & \text{if } \lambda_M > \dot{\varphi}(s_M) \end{cases} \quad (7.8)$$

(cf. Proposition 6.4).

Proof. By (4.5)

$$\begin{aligned} s^{n-1} \hat{\phi}(\dot{\varphi}, \varphi/s) &= \frac{1}{2} \dot{\varphi}^2 s^{n-1} + \frac{n-1}{2} \varphi^2 s^{n-3} + h(u(s)) s^{n-1}, \\ s^{n-2} \hat{\phi}_2(\dot{\varphi}, \varphi/s) &= \varphi s^{n-3} + h'(u(s)) \dot{\varphi} s^{n-2}. \end{aligned}$$

Since $n \geq 3$ and $u(s) \in [H, \lambda^n]$ (cf. (6.20)) it follows that (7.1) is satisfied. Thus we apply Proposition 7.1 to conclude that

$$\begin{aligned} E(t, \underline{u}, \mathfrak{F}_\rho) &= E(t, \lambda \underline{x}, \mathfrak{F}_\rho) + t^n \sigma^n \delta_n J/n, \\ J &= \hat{\phi}(\dot{\varphi}_-, \lambda) - \hat{\phi}(\lambda, \lambda) + \frac{1}{2} [\hat{\phi}_1(\dot{\varphi}_-, \lambda) + \hat{\phi}_1(\lambda, \lambda)] (\lambda - \dot{\varphi}_-), \\ \dot{\varphi}_- &= \lim_{s \rightarrow \sigma} \dot{\varphi}(s), \end{aligned} \quad (7.9)$$

where, by Proposition 6.4 (note that $T := \rho/\sigma$),

$$\sigma = \begin{cases} s_M & \text{if } \dot{\varphi}(s_M) = \lambda_M \\ s_J & \text{otherwise} \end{cases}. \quad (7.10)$$

We note that (7.8)-(7.10) yield $J = 0$ and hence

$$E(t, \underline{u}, \mathfrak{F}_\rho) = E(t, \lambda_M \underline{x}, \mathfrak{F}_\rho) \quad \text{provided} \quad \lambda_M = \dot{\varphi}(s_M).$$

Thus we suppose that $\lambda_M > \dot{\varphi}(s_M)$ and continue with the proof.

Define $\psi(\tau) := \hat{\phi}(\tau, \lambda_J)$, $\alpha := \dot{\varphi}_-(s_J)$,

$$\theta(\tau) := \psi(\tau) - \psi(\lambda_J) + \frac{1}{2} [\psi'(\tau) + \psi'(\lambda_J)] (\lambda_J - \tau),$$

and note that $\alpha \in (0, \lambda_J)$. Therefore, by (7.9), it suffices to show

that

$$\theta(\tau) < 0 \text{ for all } \tau \in (0, \lambda_J).$$

Now, $\theta(\lambda_J) = \theta'(\lambda_J) = 0$ and

$$\theta''(\tau) = \frac{1}{2} \gamma'''(\tau)(\lambda_J - \tau)$$

so that the desired result will follow provided the stored energy satisfies the constitutive inequality

$$\hat{\phi}_{111}(\tau, \lambda) < 0 \text{ for all } 0 < \tau < \lambda < +\infty.$$

This inequality follows directly from (4.5) and (4.6)₁. ■

Remark. The shock, should one exist, also satisfies the Lax shock admissibility criterion (cf. Lax [11] and Dafermos [5]). A simple computation shows that the acoustic tensor for (1.2) is given by (cf., e.g., Dafermos [5, p. 30])

$$\hat{Q}(\hat{F}, \hat{n}) = \hat{I} + (\det \hat{F})^2 h''(\det \hat{F}) \hat{F}^{-T} \hat{n} \otimes \hat{F}^{-T} \hat{n}$$

and hence the characteristic speeds (the square root of the eigenvalues of $\hat{Q}$) of (5.1)₁ associated with the radial shock we have constructed are

$$s_- = [1 + \lambda^{2n-2} h''(\alpha \lambda^{n-1})]^{1/2},$$

$$s_+ = [1 + \lambda^{2n-2} h''(\lambda^n)]^{1/2},$$

where $\lambda := \dot{\varphi}(s_J)/s_J$, $\alpha := \dot{\varphi}(s_J)$, and s_J is given by Proposition 6.4.

Therefore all we need show is that $s_J \in (s_+, s_-)$. This relation can be obtained by applying the mean value theorem to (6.16) (cf. (6.22)) and using the constitutive inequality (4.6)₁.

8. Discussion

We have not resolved a number of important problems that are related to our analysis. The first is whether finite elasticity can be used, in practice, to give an accurate prediction of material failure. Except for specimens composed of a single perfect crystal one is tempted to dismiss the use of elasticity since one expects that microscopic inhomogeneities (e.g., lattice defects, impurities, grain boundaries) will be significant. However, there may be a useful relationship between the load at which failure occurs and the load predicted from an experimentally determined "average" constitutive relation.

Another problem is the smoothness of our solutions. One method of addressing this problem is to analyze the flow of the system

$$\begin{aligned}\dot{x} &= xyz^{n-1}S(x, y, z), \\ \dot{y} &= (n-1)z^{n-3}(x^n - yz^n)[(x^n - yz^n) + yz^{n+2}]/h''(y), \\ \dot{z} &= x^n S(x, y, z),\end{aligned}$$

where

$$S(x, y, z) = x^{2n-2} + (z^{2n-2} - z^{2n})/h''(y).$$

Here we have let $x = \varphi$, $y = u$, $z = s$, and changed independent variable to eliminate singularities.

The results in Section 6 imply that each trajectory that starts on the half-line $y = H$, $z = 0$, $x > 0$ intersects the surface $S = 0$. If the points of intersection are bounded away from the curve¹¹ (on

¹¹One might prove this by constructing a surface, which the flow cannot penetrate, that separates this curve from the curve of initial data.

S) $x^n = yz^n$ then the solutions will contain shocks that dissipate energy. In addition, a simple computation then shows that each trajectory intersects the surface (cf. (6.16))

$$\left[\left(\frac{x}{z} \right)^n - y \right] (1-z^2) + \left(\frac{x}{z} \right)^{2n-2} \left[h' \left(\left[\frac{x}{z} \right]^n \right) - h'(y) \right] = 0$$

transversely and hence that the intersection point depends continuously upon the initial data $x(0) = y(0) = v$. It then follows that cavitation solutions exist for all λ sufficiently large.

A more general investigation of the admissibility of our solutions is also needed. It is clear that none of these solutions yield maximal dissipation of energy (cf. Dafermos [3]) since one can superimpose an arbitrarily large number of these solutions (centered at distinct points and times) in order to increase the total energy dissipated. However, the admissibility of our solutions according to other criteria, for example, the viscosity criterion (cf., e.g., Dafermos [5]) and the viscosity-capillarity criterion (cf., e.g., Slemrod [16] and Shearer [14]), is of interest.

Finally, we note that the precise constitutive assumptions one needs to get dynamic cavitation are not clear. If one expects such solutions for arbitrarily large λ then our analysis indicates that one must impose the constitutive inequality¹² $\hat{\phi}_{111}(\tau, \lambda) < 0$ (does this have any physical significance?). Given this inequality we

¹²This inequality is crucial to our existence proof (see Proposition 6.4). It was also used to show that a shock, if it exists, dissipates energy (see Theorem 7.2). Of course the inequality need only be satisfied for $0 < \tau < \lambda < \Lambda$ (provided Λ is sufficiently large) to get dynamic cavitation on a bounded interval of λ .

conjecture that dynamic cavitation will occur in any material in which static cavitation occurs. (cf. Ball [11], Sivaloganathan [15], and Stuart [17]). A comparison of the critical loads at which each occurs is also of interest.

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Southern Illinois University
Carbondale

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