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OF A SECOND ORDER-TENSOR

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1. Introduction. The main purpose of this note is to provide a basis-free derivation of the expression for the derivative of the determinant of an arbitrary second-order tensor. The modern treatments in the literature known to the authors (see, e.g., [1,§9] or [2,§3]) assume that the tensor is invertible. The older deductions based on components did not rest on invertibility, but they did take the tensor to be symmetric (see [3,§38] for an account of these investigations). Incidentally, our method supplies expressions for the derivatives of all of the principal invariants.

After disposing of a few mathematical preliminaries in Section 2, we give an independent and concise treatment of the three-dimensional case in Section 3. The general n -dimensional case is worked out in Section 4.

2. Mathematical preliminaries.^{*} A function $\alpha: \text{Lin} \rightarrow \mathbb{R}$ is said to be differentiable at $\underline{A}$ if there exists a linear transformation

$$D\alpha(\underline{A}): \text{Lin} \rightarrow \mathbb{R}$$

such that

$$\alpha(\underline{A} + \underline{I}) - \alpha(\underline{A}) = D\alpha(\underline{A})[\underline{I}] + o(\underline{I}) \text{ as } \underline{I} \rightarrow \underline{0}.$$

$D\alpha(\underline{A})$ is called the derivative of α at $\underline{A}$. By the representation theorem for linear functionals, there exists a unique second-order tensor $\frac{\partial \alpha}{\partial \underline{A}}$ such that

$$D\alpha(\underline{A})[\underline{I}] = \frac{\partial \alpha}{\partial \underline{A}} \cdot \underline{I} = \text{tr} \left[\left(\frac{\partial \alpha}{\partial \underline{A}} \right)^T \underline{I} \right].$$

^{*} We use the notation and terminology of Gurtin's text [2], except that the underlying vector space may be of any finite dimension unless explicitly stated otherwise.

In components,

$$\left[\frac{\partial \alpha}{\partial \underline{A}} \right]_{ij} = \frac{\partial \alpha}{\partial A_{ij}}.$$

We also will refer to $\frac{\partial \alpha}{\partial \underline{A}}$ as the derivative of α at $\underline{A}$, but the different notation should prevent any confusion.

Truesdell and Noll [1,§9] have shown that the derivatives of the moments

$$\tau_k(\underline{A}) = \text{tr}(\underline{A}^k), \quad k = 1, 2, \dots \quad (2.1)$$

are given by

$$\frac{\partial \tau_k}{\partial \underline{A}} = k(\underline{A}^{k-1})^T, \quad k = 1, 2, \dots \quad (2.2)$$

Of course,

$$\underline{A}^0 = \underline{I},$$

where $\underline{I}$ is the identity tensor.

3. The 3-dimensional case. The results given below are contained, of course, as a special case in those of the next section. However, it seems to us that their importance in continuum mechanics and the ease with which they are obtained warrant their independent presentation.

The principal invariants of a second-order tensor $\underline{A}$, $i_1(\underline{A})$, $i_2(\underline{A})$, and $i_3(\underline{A})$, are the coefficients in the characteristic polynomial

$$\det(\underline{A} - \lambda \underline{I}) = -\lambda^3 + i_1(\underline{A})\lambda^2 - i_2(\underline{A})\lambda + i_3(\underline{A}).$$

By direct calculation,

$$i_1(\underline{A}) = \tau_1(\underline{A}), \quad i_2(\underline{A}) = -\frac{1}{2} [\tau_2(\underline{A}) - \tau_1(\underline{A})^2], \quad (3.1)$$

and

$$i_3(\underline{A}) = \det \underline{A} ,$$

where τ_1 and τ_2 are defined by (2.1).

A readily differentiated expression for i_3 can be obtained from the Cayley-Hamilton theorem, which states that

$$-\underline{A}^3 + i_1(\underline{A})\underline{A}^2 - i_2(\underline{A})\underline{A} + i_3(\underline{A})\underline{I} = \underline{0} .$$

On taking the trace, we get

$$i_3(\underline{A}) = \frac{1}{3} [\tau_3(\underline{A}) - i_1(\underline{A})\tau_2(\underline{A}) + i_2(\underline{A})\tau_1(\underline{A})] . \quad (3.2)$$

We can use (2.2)* to differentiate (3.1) and (3.2). The outcome is that the derivatives of the principal invariants of a second-order tensor on a 3-dimensional vector space are given by

$$\left. \begin{aligned} \frac{\partial i_1}{\partial \underline{A}} &= \underline{I} , \\ \frac{\partial i_2}{\partial \underline{A}} &= - [\underline{A} - i_1(\underline{A})\underline{I}]^T , \\ \frac{\partial i_3}{\partial \underline{A}} &= [\underline{A}^2 - i_1(\underline{A})\underline{A} + i_2(\underline{A})\underline{I}]^T . \end{aligned} \right\} \quad (3.3)$$

Of course, if $\underline{A}$ is invertible, then the Cayley-Hamilton theorem can be used to reduce (3.3)₃ to

* The particular cases of (2.2) needed here can be easily worked out directly; one needs only the derivatives of the trace, the square, and the cube of a tensor.

$$\frac{\partial i_3}{\partial \tilde{A}} = i_3(\tilde{A})(\tilde{A}^{-1})^T,$$

which is the conclusion reached in the earlier basis-free derivations ([1,§9], [2,§3]) alluded to in Section 1.

4. The n -dimensional case. When the dimension of the underlying vector space is n , the characteristic polynomial is

$$\det(\tilde{A} - \lambda I) = \sum_{i=0}^n i_i(\tilde{A})(-\lambda)^{n-i},$$

where

$$i_0(\tilde{A}) = 1.$$

Evaluation at $\lambda=0$ yields

$$i_n(\tilde{A}) = \det \tilde{A}.$$

The principal invariants are given in terms of the moments, (2.1), by

$$i_k(\tilde{A}) = \frac{(-1)^{k-1}}{k} \sum_{i=0}^{k-1} (-1)^i i_i(\tilde{A}) \tau_{k-i}(\tilde{A}), \quad k = 1, 2, \dots, n \quad (4.1)$$

(see, e.g., [4, Chpt. IV, §5]).

Since the principal invariants are linear combinations of products of traces of powers, there is no question about the existence of their derivatives.

Differentiation of (4.1) with the aid of (2.2) yields

$$\frac{\partial i_k}{\partial \tilde{A}} = \frac{(-1)^{k-1}}{k} \sum_{i=0}^{k-1} (-1)^i [\tau_{k-i}(\tilde{A}) \frac{\partial i_i}{\partial \tilde{A}} + (k-i) i_i(\tilde{A}) (\tilde{A}^{k-i-1})^T], \quad k=1, 2, \dots, n. \quad (4.2)$$

For $k=1, 2$, and 3 , (4.2) and (4.1) give formulas for $\frac{\partial i_k}{\partial \tilde{A}}$ that agree in form with

(3.3).^{*} Therefore, it is natural to conjecture that in general

$$\frac{\partial \iota_k}{\partial \tilde{A}} = \left[\sum_{i=0}^{k-1} \iota_i(\tilde{A})(-\tilde{A})^{k-1-i} \right]^T, \quad k=1,2,\dots, n. \quad (4.3)$$

The validity of (4.3) is easily established by mathematical induction as follows. We have already observed that (4.3) holds when $k=1$. Suppose then that it holds for $k=2,3,\dots, \ell$.

With this assumption, (4.2) and (4.1) imply

$$\frac{\partial \iota_{\ell+1}}{\partial \tilde{A}} = \left[\sum_{i=0}^{\ell} \iota_i(\tilde{A})(-\tilde{A})^{\ell-i} \right]^T,$$

which is (4.3) for $\ell+1$. Consequently the derivative of the k^{th} principal invariant of a second-order tensor is given by (4.3).

Of course, when $\tilde{A}$ is invertible, (4.3) and the Cayley-Hamilton theorem yield the familiar result that

$$\frac{\partial \iota_n}{\partial \tilde{A}} = \iota_n(\tilde{A})(\tilde{A}^{-1})^T.$$

* Note that this procedure provides an expression for $\frac{\partial \iota_k}{\partial \tilde{A}}$ (the general formula being (4.3)) which is independent of n , the dimension of the underlying vector space; of course, $k \leq n$.

- [1] C. Truesdell and W. Noll, The non-linear field theories of mechanics, Handbuch der Physik III/3 (1965).
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