NONLINEAR EFFECTS IN THE WAVE EQUATION WITH A CUBIC RESTORING FORCE

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by

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0. INTRODUCTION

This paper is devoted to the investigation of global properties of solutions of the following semilinear hyperbolic problem

$$\left.\begin{aligned} &u_{tt} - u_{xx} + u^3 = 0 && \text{for } t \in \mathbb{R},\ 0 < x < 1 \\ &u(t,0) = u(t,1) = 0 && \text{for } t \in \mathbb{R}\ . \end{aligned}\right\} \tag{0.1}$$

It is well known ([13], Theorem VI.1.2.3) that for any $g \in C^1(\mathbb{R})$ such that

$$\forall\, u \in \mathbb{R}\ , \quad g(u)u \geqslant 0 \tag{0.2}$$

and any (u_0, v_0) in $H_0^1(0,1) \times L^2(0,1)$, there exists a unique function $u : \mathbb{R} \times]0,1[\longrightarrow \mathbb{R}$ such that

$$u \in C\big(\mathbb{R}, H_0^1(0,1)\big) \cap C^1\big(\mathbb{R}, L^2(0,1)\big) \tag{0.3}$$

$$u_t \in C^1\big(\mathbb{R}, H^{-1}(0,1)\big) \tag{0.4}$$

$$u_{tt} = u_{xx} - g(u) \ \text{ in } \ C\big(\mathbb{R}, H^{-1}(0,1)\big) \tag{0.5}$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = v_0\ . \tag{0.6}$$

When $g(u) = u^3$ this solves the initial value problem associated to (0.1).

When g is <u>odd</u>, i.e.,

$$\forall\, u \in \mathbb{R}, \quad g(-u) = -g(u) \tag{0.7}$$

it is convenient, following [6], to introduce the function $\tilde{u} \in C(\mathbb{R}^2)$ defined by the conditions

$$\left.\begin{aligned} &\tilde{u}(t,x) = -\tilde{u}(t,-x)\ , && \forall\,(t,x) \in \mathbb{R}^2 \\ &\tilde{u}(t,x+2) = \tilde{u}(t,x)\ , && \forall\,(t,x) \in \mathbb{R}^2 \\ &\tilde{u}(t,x) = u(t,x)\ , && \forall\,(t,x) \in \mathbb{R} \times]0,1[\end{aligned}\right\} \tag{0.8}$$

Then $\tilde{u}$ satisfies the integration equation

$$\tilde{u}(t,x) = p(t+x) - p(t-x) - \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} g(\tilde{u}(s,y))\,dyds \tag{0.9}$$

where p is given by

$$p(s) = \frac{1}{2}\left\{\tilde{u}_0(s) + \int_0^s \tilde{v}_0(\sigma)\,d\sigma\right\} + C \ , \tag{0.10}$$

$\tilde{u}_0$, $\tilde{v}_0$ being respectively the odd and 2-periodic extensions of u_0 and v_0 respectively, and C an abritrary constant chosen at our convenience. Of course $p : \mathbb{R} \to \mathbb{R}$ is in $H^1_{loc}(\mathbb{R})$ and $p(s+2) = p(s)$. It will sometimes be convenient to choose C so that $\int_0^2 p(s)ds = 0$.

When $g(u) = cu^3$ with $c \geqslant 0$, (0.9) becomes

$$\tilde{u}(t,x) = p(t+x) - p(t-x) - \frac{c}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} \tilde{u}^3(s,y)\,dyds \ . \tag{0.11}$$

In the paper we shall use formula (0.11) quite extensively, since it has very interesting properties, among which a kind of "smoothing" effect.

Apart from the initial value problem which is easily treated by standard methods, various questions connected to (0.1) have been studied in the literature. For example, in the pioneering work [18], P. RABINOWITZ established the existence of solutions to (0.1) which fulfill the additional conditions

$$\left.\begin{array}{l} u(t+2,x) = u(t,x) \ , \quad \forall\,(t,x) \in \mathbb{R} \times]0,1[\\ u \text{ is not identically } 0 \text{ on } \mathbb{R} \times]0,1[\ . \end{array}\right\} \tag{0.12}$$

Part of the proof has been simplified in [3] by BREZIS-CORON-NIRENBERG. However, it remains delicate to give a complete construction of such solutions in the regularity class (0.3). Also, it seems completely unknown what these solutions look like.

In [6], investigation of global behavior for arbitrary solutions of (0.1) was initiated by looking at oscillatory properties of the function $t \mapsto u(t, x_0)$ with $0 < x_0 < 1$. There it was proved that the solutions must oscillate "at least as fast as" the solutions of the linear wave equation. However, the study of asymptotics as $t \to +\infty$ is still essentially open from the theoretical point of view.

The object of this paper is two-fold. First, we present the most typical results of numerical experiments which were attempted to test some conjectures on global behavior of solutions: for three different kinds of initial data, the solutions to (0.1) have been computed on a large time interval by means of a finite-difference scheme. More precisely, the computation technique is the same as in [25] and will be discussed in the last section.

Secondly, we prove a few partial theoretical results either related to behavior as $t \to +\infty$, or to a detailed investigation of the oscillatory properties of solutions. All the theoretical results will be illustrated by a numerical study corresponding to at least one of the typical initial data mentioned above.

For simplicity, in all the numerical computations we assume $v_0 = 0$. This is usually admitted as physically meaningful and, as a matter of fact, is not so disturbing from the point of view of generality, as long as the problems remain open even this case. It is our hope that the results of this paper will be helpful in the future, either to distinguish more clearly the various phenomena which may happen and understand the underlying reasons for this complexity, or to find new ideas solving some of the mathematical questions involved.

Apart from pure curiosity, our work has been partly motivated and strongly encouraged by the existence of at least one well-known previous success in this direction: the discovery of quasi-periodic solutions for the KDV equation by P. D. LAX [16] about one decade ago, following the numerical experiments of M. D. KRUSKAL and N. J. ZABUSKY [28] (cf. also [9] and [17]).

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as part of its year of concentration on continuum physics and partial differential equations (1984-85). The fourth author was supported for the entire year at the I.M.A., and the second as a visitor for the month of June 1985. It was then that serious progress on some theoretical aspects of this research led to the idea for this paper. The fourth author was also partially supported by NSF Grant DMS 8201639. Finally, the authors are indebted to P. JOLY, M. LEGENDRE and P.J. PASCUAL for their generous assistance concerning the computational aspects of this research.

1. ON THE POSSIBILITY OF DECAYING SOLUTIONS:

One important property of (0.1) is the conservation of the energy integral

$$\forall\ t \in \mathbb{R}, \quad \int_0^1 \left\{\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{1}{4} u^4\right\}(t,x)\,dx = \int_0^1 \left\{\frac{1}{2} v_0^2(x) + \frac{1}{2} u_0'^2(x) + \frac{1}{4} u_0^4(x)\right\} dx . \tag{1.1}$$

This property implies in particular that no solution of (0.1) can tend strongly to 0 in $H_0^1(0,1)$ as $t \to +\infty$ (respectively $-\infty$) except the trivial solution $u \equiv 0$. Indeed, multiplication of (0.1) by $u(t,x)$ yields the following identity, valid for all t,

$$\int_t^{t+1}\int_0^1 u_s^2\,dxds = \int_t^{t+1}\int_0^1 \left[u_x^2 + u^4\right] dxds + \int_0^1 u(t+1,x)u_t(t+1,x)\,dx - \int_0^1 u(t,x)u_t(t,x)\,dx .$$

Therefore if, for instance, $u(t,\cdot) \to 0$ in $H_0^1(0,1)$ as $t \to +\infty$, taking account of (1.1) we find first $\int_t^{t+1}\int_0^1 u_s^2\,dxds \to 0$, and then by integrating (1.1) in t over $[t, t+1]$ we conclude that $u_0 = v_0 = 0$.

On the other hand, (1.1) does not a priori preclude the possibility that

$$\sup_{x\in[0,1]} |u(t,x)| \to 0 \quad \text{as} \quad t \to +\infty . \tag{1.2}$$

Physically difficult to believe, the weak convergence (1.2) to 0 could be eliminated if we knew a priori that $u(t, \cdot)$ remains in a compact subset of $H_0^1(0,1)$ for all $t \geqslant 0$. Rather paradoxically, this is precisely unknown even when $u_0 \in \mathcal{D}(]0,1[)$ and $v_0 = 0$, the basic reason being the absence of known higher order conserved integrals for (0.1). A first simple observation is that if $u(t, \cdot)$ happens to tend to 0 at infinity, the convergence cannot be very fast. More precisely, we have the following result.

Proposition 1.1. Let u be any solution of (0.1). Then if $u \not\equiv 0$ we have

$$\int_0^{+\infty} \left\{ \int_0^1 |u(t,x)|^6 \, dx \right\}^{1/2} dt = +\infty \tag{1.3}$$

Proof. Let $\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$ and let us denote by $T(t) : \mathcal{H} \to \mathcal{H}$ the isometry group generated by the linear wave equation in $\mathcal{H}$. If we introduce

$$U(t) := \big(u(t),\, u_t(t)\big) \in C(\mathbb{R}, \mathcal{H})$$

and

$$F(t) := \big(0,\, -g(u(t))\big) \quad ,$$

then the variation of parameters formula applied to the system satisfied by (u, u_t) yields

$$U(t) = T(t)\left(U(0) + \int_0^t T(-s)F(s)\, ds\right) . \tag{1.4}$$

In particular, if we assume, by contradiction, $u \not\equiv 0$ and

$$\int_0^{+\infty} \left\{ \int_0^1 |u(t,x)|^6 dx \right\}^{1/2} < +\infty \quad , \tag{1.5}$$

then $F(t) \in L^1(0, +\infty, \mathcal{H})$ and (1.4) implies

$$U(t) - T(t)U_\infty \to 0 \text{ in } \mathcal{H} \text{ as } t \to +\infty \tag{1.6}$$

with

$$U_\infty := U(0) + \int_0^{+\infty} T(-s)F(s)\,ds \ .$$

In particular, $u(t,x)$ is asymptotic in the strong topology of $H_0^1(\Omega)$, as $t \to +\infty$, to a 2-periodic continuous function $u_\infty : \mathbb{R} \to H_0^1(0,1)$. Then (1.5) immediately implies $u_\infty = 0$, and therefore $u(t,\cdot) \to 0$ strongly in $H_0^1(0,1)$ as $t \to +\infty$, a contradiction with the preliminary remark which concludes the proof of Proposition 1.1. ∎

As a consequence of Proposition 1.1 we deduce

Corollary 1.2. If u is a nontrivial solution of (0.1), then for all $\varepsilon > 0$,

$$\limsup_{t \to +\infty} \left\{ t^{\frac{1}{3}+\varepsilon} \|u(t,\cdot)\|_6 \right\} = +\infty \ . \tag{1.7}$$

Using numerical methods, the solution u of (0.1) and (0.6) has been computed when $v_0 = 0$ and $u_0 = k\phi$ with $k \in \{1, 5, 10, 20, 30, 50\}$ and $\phi(x)$ is one of the functions

$$\sin \pi x \ , \qquad \sin^2 \pi x \ , \qquad x(1-x) \ .$$

In figures 1 and 2*, we show the evolution of $u(t, 1/2)$ in the first case with $k = 10$, $0 \leqslant t \leqslant 3$ and also $71.65 \leqslant t \leqslant 74.65$. All the other numerical experiments show the same behavior: there is no tendency for $|u(t,1/2)|$ (a fortiori for $\max_{[0,1]} |u(t,x)|$) to decrease in a significant manner as t increases (cf. also the comments in Section 2, especially the comparison with $u_{tt} + \pi^2 u + u^3 = 0$). Another natural question to be investigated mathematically is whether or not the conditions $u_0 \in H^2(0,1)$ and $v_0 \in H_0^1(0,1)$ imply that

$$\sup_{t \geqslant 0} \|u_{xx}(t,x)\|_2 < +\infty \ . \tag{1.8}$$

* Figures and tables all appear in Section 5.

Our numerical computations seem to indicate that (1.8) is satisfied when $v_0 = 0$ and $u_0(x) = 10 \sin \pi x$. Indeed, according to figure 5, the function

$$D(t) = \sup_{0 \leqslant s \leqslant t} \|u_{xx}(s,x)\|_2$$

seems to be constant for $45 \leqslant t \leqslant 90$. Therefore, at least in this case, the curve $t \rightarrow u(t, \cdot)$ would remain in a compact subset of $H_0^1(0,1)$, a fact consistent with the absence of decay of $\|u(t,\cdot)\|_\infty$ as $t \rightarrow +\infty$.

2. THE ALMOST PERIODICITY CONJECTURE

Another natural conjecture concerning the solutions of (0.1) (which can be extended for $t \leqslant 0$ since the system is reversible) is that most solutions (and maybe all of them) are almost periodic with respect to t, either weakly or even strongly in $H_0^1(0,1)$. In either case, one should then observe a recurrent character of the function $t \mapsto u(t, \cdot)$ in the space $C([0,1])$. In this section, we try to evaluate the validity of this conjecture on the basis of heuristic arguments as well as numerical results. A mathematical study of this question would be interesting, but most probably extremely difficult even for this simple one-dimensional model.

2.1 Previous results of this type

In 1965, L. AMERIO established that all solutions of the linear wave equation with homogeneous boundary conditions in a bounded domain of $\mathbb{R}^n$, $n \geqslant 1$, are almost periodic in the energy space. The final argument (following partial results by previous authors) ultimately did not require any regularity of the domain and related the almost-periodicity property only to three circumstances: conservation of the energy, linearity of the equation, and precompactness of trajectories [1].

Later, in 1975, P.D. LAX established in [16] the quasi-periodic character for a large class of solutions of the KDV equation on a one-dimensional torus. The proof, unfortunately, relied heavily on the "completely integrable" character of the system and a very precise determination of invariant tori in the phase space.

In 1980, H. CABANNES and A. HARAUX [4] found a large family of almost periodic solutions for the equation of a string with fixed ends vibrating against a

straight, fixed obstacle. The solutions are generally not periodic (cf. Theorem 1.2, b) of [12]). More complicated almost periodic motions have been discovered later by H. CABANNES [5]. Here again, the success relies on the possibility of computing rather explicitly the solutions (cf. also [22]).

The physical content of equation (0.1) suggests a behavior rather similar to this last example. On the other hand, we are far from being able to compute any solution, or even to reduce the problem in some specific circumstances. These remarks suggest that an entirely new method needs to be developed.

2.2 The importance of diffusion

If the operator $u \mapsto -u_{xx}$ is replaced by its contribution on the first eigenspace in the sense of $H^1_0(0,1)$, we obtain the equation

$$\left.\begin{array}{ll} u_{tt} + \pi^2 u + u^3 = 0 , & t \in \mathbb{R},\ x \in]0,1[\\ u(t,0) = u(t,1) = 0 & t \in \mathbb{R} \end{array}\right\} \qquad (2.2.1)$$

which can be solved uniquely in the function space $C^1(\mathbb{R}, H^1_0(0,1))$ for all initial data $(u_0, v_0) \in [H^1_0(0,1)]^2$.

Classical results on O.D.E. (cf., e.g., [10]) show that for all $x_0 \in]0,1[$ the function $u(t, x_0)$ is periodic in t.

Since the period is a decreasing continuous function of the "local" energy $\frac{1}{2} v_0^2(x) + \frac{1}{2} u_0^2(x) + \frac{1}{2} u_0^4(x)$, if, for instance, $v_0 = 0$ and $u_0 \not\equiv 0$, the solution <u>cannot</u> be almost periodic: $\mathbb{R} \to H^1_0(\Omega)$. Numerically, we investigated the behavior of the function $t \mapsto \int_0^1 u(t,x)\,dx$ when

$$v_0 = 0 \quad \text{and} \quad u_0(x) = 10 \sin \pi x$$

for (0.1) (figure 3) and (2.2.1)(figure 4).

The observation of figure 4 seems to indicate that this integral tends to 0, or at least is far from being recurrent as $t \mapsto +\infty$. In the case of (2.2.1) it is not absurd to imagine that <u>all</u> solutions tend weakly to 0 in $L^2(0,1)$ as $t \to \pm\infty$ (but

in general the H_0^1 norm of $u(t, \cdot)$ will blow up at infinity). On the other hand, figure 3 suggests an almost periodic behavior.

Unfortunately, it has been impossible to follow the behavior pictured in figure 4 for very large values of $|t|$, since after a while, numerical effects obviously related to the highly oscillatory character of the solution are no longer negligible and the picture degenerates in a very irregular way.

In any case, the comparison between figures 3 and 4 points out in a very intuitive way the crucial role played by diffusion in the global behavior of solutions. Finally, the comparison with finite-dimensional systems [11] suggests that the problem may be quite hard.

2.3 On a result of P. RABINOWITZ

There is at least some mathematical evidence that difussion plays an important part in global behavior of solutions to (0.1), again by comparing with the "simpler" problem (2.2.1). As pointed out in §2.2, the only *time-periodic* solution of (2.2.1) is the trivial solution $u \equiv 0$ on $\mathbb{R} \times]0,1[$. On the other hand, P. RABINOWITZ [18, 19] established the existence of at least one solution of (0.1), (0.3) and (0.12). It turns out that this is a genuine *nonlinear property* since, unlike the linear case, some nontrivial solutions of (0.1) in the class (0.3) do not satisfy (0.12). More precisely, we can state the following result, which is related to an unpublished observation of H. BREZIS, communicated to the authors by P. RABINOWITZ [21].

Proposition 2.3.1. If u is a solution of (0.1) in the regularity class (0.3) - (0.4) such that

$$u(t+2, \cdot) = u(t, \cdot) \quad \text{for all } t \in \mathbb{R} , \tag{2.3.1}$$

then either $u \equiv 0$ or

$$\int_0^2 \int_0^1 |u(t,x)|^2 \, dxdt \geqslant 2 . \tag{2.3.2}$$

<u>Proof.</u> We apply formula (0.11) and we set

$$Q =]0,2[\times]0,1[$$

$$u_1(t,x) = p(t+x) - p(t-x)$$

$$u_2(t,x) = -\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+t-s} \tilde{u}^3(s,y)\,dyds .$$

As a consequence of (0.8) we obtain easily the following estimate

$$\|u_2\|_{L^\infty(Q)} \leq \frac{1}{2} \sup_{t\in[0,2]} \int_0^t \int_0^1 |u(s,y)|^3 dyds = \frac{1}{2}\|u^3\|_{L^1(Q)} . \tag{2.3.3}$$

On the other hand, let v be a solution of (0.3) - (0.5) and $w \in L^2(0,2;\ H_0^1(\Omega)) \cap H^1(0,2;\ L^2(\Omega))$; we have the identity

$$\frac{d}{dt}\left(\int_0^1 v_t w\,dx\right) = \int_0^1 (v_{tt}w + v_t w_t)\,dx = \int_0^1 \{v_t w_t - v_x w_x - g(v)w\}\,dx .$$

By integrating on $(0,2)$, we find that if v and w are such that $v_t(0,x) \equiv v_t(2,x)$ and $w(0,x) \equiv w(2,x)$, then

$$\int_0^2 \int_0^1 \{v_t w_t - v_x w_x - g(v)w\}\,dxdt = 0 . \tag{2.3.4}$$

It is clear that u_1 satisfies (0.3) - (0.5) with $g = 0$ and by choosing $u_1 = v$ and $u = w$ in (2.3.4) we obtain

$$\iint_Q \{(u_1)_t u_t - (u_1)_x u_x\}\,dxdt = 0 . \tag{2.3.5}$$

On the other hand, by choosing $g(u) = u^3$, $u = v$ and $u_1 = w$ in (2.3.4) we find

$$\iint_Q \{u_t(u_1)_t - u_x(u_1)_x + u^3 u_1\} dxdt = 0 \quad . \tag{2.3.6}$$

By subtracting (2.3.5) and (2.3.6) we get

$$\iint_Q u^3 u_1 \, dxdt = 0 \quad . \tag{2.3.7}$$

Therefore

$$\iint_Q u^4 dxdt = \iint_Q u^3 u_2 \, dxdt \ ,$$

and by (2.3.3) we obtain

$$\|u\|^4_{L^4(Q)} \leqslant \frac{1}{2}\|u^3\|^2_{L^1(Q)} \quad . \tag{2.3.8}$$

On the other hand, by Hölder's inequality,

$$\|u^3\|_{L^1(Q)} \leqslant \|u\|^3_{L^4(Q)} \left[\text{meas}(Q)\right]^{1/4} \ ,$$

and since meas(Q) = 2, (2.3.8) yields $\|u\|^4_{L^4(Q)} \leqslant \frac{1}{\sqrt{2}} \|u\|^6_{L^4(Q)}$; hence if $u \not\equiv 0$, we conclude that

$$\|u\|^2_{L^4(Q)} \geqslant \sqrt{2} \quad . \tag{2.3.9}$$

This is clearly equivalent to (2.3.2). ∎

Remark 2.3.2. In particular, if $v_0 = 0$, $u_0 \neq 0$ and $\|u_0\|_{H^1_0(0,1)}$ is small enough, the solution of (0.3) - (0.6) with $g(u) = u^3$ is not 2-periodic in t.

Remark 2.3.3. On the other hand, in [27], M. WILHEM established the existence of nontrivial periodic solutions with arbitrarily small energies and periods of the form 2n, $n \to +\infty$. A related result can also be found in [20].

2.4 Numerical experiments and the recurrence property

A common point among all the solutions that we computed numerically is the following: assuming that t remains in the region where the computation is presumably correct, the curve $u(t,x)$ is "very close" to $u_0(x)$ when t is taken in the union of small intervals regularly spaced on the half-line $t \geq 0$. This property suggests a recurrence phenomenon quite consistent with the almost-periodicity conjecture. In this paragraph we describe some observations which make the conjecture look quite reasonable.

Observation 2.4.1. Let us consider the special case $u_0(x) = 10 \sin \pi x$ (the coefficient 10 is chosen large so that we are far enough from linearity, and still small enough so that numerical effects due to the fast oscillations are not excessive). In the interval $0 \leq t \leq 100$, we look for all values t such that

$$\frac{\|u(t) - u_0\|_\infty}{\|u_0\|_\infty} =: \varepsilon(t) < 0.05.$$

The set ξ of such points is the union of 15 very short intervals quite regularly spaced on the interval $[0, 100]$. For example, the maximal distance of two successive intervals is about 13, while the mean-value of all such distances is of the order of 7.

For more precise information, see Table 1. The same calculation with 0.05 replaced by any $\varepsilon \geq 0.03$ leads to very similar conclusions. One observes similar results if the L^∞ norm is replaced by the H_0^1 norm.

Observation 2.4.2. Another way of following the evolution of u is to study the function $t \mapsto u(t, 1/2)$. Indeed, the solution u can be reconstructed from the values of $u(t, 1/2)$ by using the method of characteristics. As shown by the comparison of figures 1 and 2, the function $y(t) := u(t, 1/2)$ is such that

$\sup_{0 \leq t \leq 3} |y(t) - y(t+71.65)|$ is of the order of 2 while the oscillation of y is greater than 20. In addition, the mean-value of $|y(t) - y(t+71.65)|$ on $[0,3]$ is close to 0.66.

Remark 2.4.3. The above two observations suggest that after a certain amount of time, the solution recovers approximately its initial shape. This property is especially striking when following the solution on a large time interval, since in the meantime, one then observes drastic changes in the shape of $u(t, \cdot)$ as a function of x. For example, starting with a positive concave initial datum, we observe the appearance of successive pairs of zeroes which, from time to time, suddenly cancel each other to reconstruct a shape quite similar to u_0 (compare figure 8). This strange phenomenon was already observed a long time ago by M.D. KRUSKAL and N.J. ZABUSKY [28] for another kind of nonlinear wave equation (in fact, a KDV equation on the one-dimensional torus). The remarkable stability of the initial configuration under the action of a dynamical system which, otherwise, does not seem to obey any simple "conservation rule" with respect to the shape, seems to be the most convincing evidence of an almost periodic motion.

3. OSCILLATORY PROPERTIES

In [6], Theorem 2.1 it has been established that if g satisfies (0.7) and u is a solution of (0.3) - (0.5), then for any $x_0 \in]0,1[$ the function $u(t, x_0)$ has at least one zero on any time interval of length ≥ 2. At this level there is no major difference between $g = 0$ and $g \neq 0$, especially since the result is optimal with respect to the length of the interval if u_0, v_0 are arbitrary.

Another kind of oscillation result, of "global" type, can be proved without assumption (0.7): in [6], Theorem 3.1, it is shown in particular that if $u \geq 0$ on $J \times]0,1[$ with J some interval, then either $u \equiv 0$ or $|J| \leq 1$. For $g = 0$ this result is optimal since then $\sin \pi t \sin \pi x$ is a nonnegative solution on $]0,1[\times]0,1[$.

In this section we consider the special case where $v_0 = 0$ and u_0 is such that

$$u_0 \text{ is increasing on } [0, 1/2] \tag{3.1}$$

$$u_0(1-x) = u_0(x) \ , \quad \forall\, x \in [0,1] \quad . \tag{3.2}$$

Under these assumptions, we compare the oscillation properties in the linear and nonlinear cases.

3.1 A property specific to the linear case

In this paragraph, we assume $g = 0$. Then the solution of (0.3) - (0.6) with $v_0 = 0$ is given by the formula

$$u(t,x) = \frac{1}{2}\left\{\tilde{u}_0(t+x) - \tilde{u}_0(t-x)\right\}$$

with $\tilde{u}_0$ the odd and 2-periodic extension of u_0 on $\mathbb{R}$. The hypotheses (3.1) and (3.2) on u_0 imply

$$\tilde{u}_0 \text{ is increasing on } [-1/2, 1/2] \tag{3.1.1}$$

$$\tilde{u}_0(1-x) = \tilde{u}_0(x), \quad \forall x \in \mathbb{R} \quad . \tag{3.1.2}$$

In particular, for any $\ell \in \mathbb{R}$ the equation $\tilde{u}_0(x) = \ell$ has at most 2 solutions x_1 and $x_2 = 1 - x_1$ on $[-1/2, 3/2]$.

Since $\tilde{u}_0$ is continuous and increasing on $[-1/2, 1/2]$, it follows that

$$\forall x \in]0, 1/2[\;, \quad \forall t \in [0, 1/2[,$$

$$\tilde{u}_0(t+x) - \tilde{u}_0(t-x) > 0 \;.$$

In other words, we have

$$\forall x \in]0,1[\;, \quad \forall t \in [0, 1/2[\;, \quad u(t,x) > 0 \;, \tag{3.1.3}$$

$$\forall x \in [0, 1/2] \;, \quad u(1/2, x) = 0 \;. \tag{3.1.4}$$

More generally, for any $m \in \mathbb{Z}$ we have

$$\forall x \in]0,1[\;, \quad \forall t \in]m - 1/2, m + 1/2[\;, \quad (-1)^m u(t,x) > 0 \tag{3.1.5}$$

and

$$\forall x \in \mathbb{R} \;, \quad \forall m \in \mathbb{Z} \;, \quad \tilde{u}(m + 1/2, x) = 0 \quad . \tag{3.1.6}$$

This means that the solution vanishes simultaneously at all points x for $t \in \mathbb{Z} + \frac{1}{2}$. In Section 4 we shall see that the situation is quite different when $g(u) = u^3$.

3.2 Corresponding results in the nonlinear case

Since the condition $v_0 = 0$ implies $u(-t,x) = u(t,x)$ we obtain that u cannot remain nonnegative on $]0, 1/2[\times]0,1[$. (This follows from the proof of Theorem 2.1 in [6] since g is not identically zero.) In addition, the symmetry property implies the following stronger result.

Proposition 3.2.1. Let $g \in C^1(\mathbb{R})$ satisfy (0.7) and assume that $g : \mathbb{R} \to \mathbb{R}$ is increasing. Then for any u_0 satisfying (3.1) - (3.2) we have

$$\forall\, x \in]0,1[\ , \qquad \operatorname{Inf}\{t \geq 0,\ u(t,x) < 0\} < 1/2 \ . \tag{3.2.1}$$

Proof. This is basically an application of the methods of [7], therefore we only give a brief sketch of the proof.

Property (3.2) implies that $\tilde{u}_0(1+x) = -\tilde{u}_0(x)$ and therefore for all $(t,x) \in \mathbb{R} \times \mathbb{R}$ we have

$$\tilde{u}(t, x+1) = -\tilde{u}(t,x) \ . \tag{3.2.2}$$

Let $x_0 \in]0,1[$ be such that $u(t,x_0) \geq 0$ for all $t \in [0, 1/2]$. Then in fact

$$u(t,x_0) \geq 0 \quad \text{for all } t \in [-1/2, 1/2]. \tag{3.2.3}$$

We introduce

$$w(t,x) = \tilde{u}(t,x) + \tilde{u}(t, 2x_0 - x) \ . \tag{3.2.4}$$

Then by the method of [7], we deduce from (3.2.3) that

$$w(t,x) \geq 0 \ , \qquad \forall\, (t,x) \text{ with } |t| + |x - x_0| \leq 1/2 \ .$$

Since on the other hand $w(t, x+1) = -w(t,x)$ we deduce

$$w(0, x_0 - 1/2) = w(0, x_0 + 1/2) = 0 \ . \tag{3.2.5}$$

From (3.2.5), by computing the value of $w(0, x_0 + 1/2)$ in terms of $w(t,x_0)\big|_{[-1/2,1/2]}$ and $\displaystyle\iint_{|t|+|x-x_0| \leq 1/2} g(w(s,y))\,ds\,dy$ we conclude, since g is increasing, that $w \equiv 0$ in $\{|t| + |x - x_0| \leq 1/2\}$. In particular $w(0,x_0) = 2u_0(x_0) = 0$ and this contradicts (3.1).

3.3 Fast oscillations for large initial data

In the linear case, (3.1.3)-(3.1.6) imply in particular that, assuming $v_0 = 0$ and (3.1) - (3.2), the frequency of the oscillations is independent of u_0. In the nonlinear case the situation is quite different. In particular, we have the following result which will be used in Section 4.

Proposition 3.3.1. Let $g(u) = cu^3$ for some $c > 0$, assume that $v_0 = 0$, u_0 being as previously specified. Then we have

$$\sup\left\{t > 0,\ u(s,x) \geqslant 0 \text{ for all } x \in [0,1] \text{ and } s \in [0,t]\right\}$$

$$=: \bar{t} < \frac{\pi}{\sqrt{2c}\int_0^1 u_0(x)\sin \pi x\,dx} \quad . \tag{3.3.1}$$

Proof. Let

$$z(t) := \int_0^1 u(t,x)\sin \pi x\,dx \quad . \tag{3.3.2}$$

We have $z \in C^2(\mathbb{R})$ and for all $t \in [0,\bar{t}]$,

$$z'' = \int_0^1 u_{tt}\sin \pi x\,dx = -\pi^2 z - c\int_0^1 u^3 \sin \pi x\,dx \leqslant -cz^3$$

with $z'(0) = 0$. Therefore on $[0,\bar{t}]$ we have

$$z(t) \geqslant 0 \tag{3.3.3}$$

$$z''(t) \leqslant -cz^3(t) \tag{3.3.4}$$

$$z'(t) \leqslant 0 \quad . \tag{3.3.5}$$

From (3.3.4) and (3.3.5) we deduce

$$\frac{d}{dt}\left(z'^2 + \frac{c}{2}z^4\right) = 2z'(z'' + cz^3) \geqslant 0$$

and therefore

$$z'(t) < -\sqrt{\frac{c}{2}\left[z_0^4 - z^4(t)\right]} \quad \text{for all } t \in [0,\bar{t}] \quad . \tag{3.3.6}$$

By letting

$$w(t) = z(t)/z_0 \,, \qquad \forall\, t \in [0,\bar{t}] \,,$$

we find

$$\frac{w'}{\sqrt{1-w^4}} \leqslant -\sqrt{c/2}\, z_0 \qquad \text{for all } t \in [0,\bar{t}] \ . \qquad (3.3.7)$$

Therefore we have

$$\frac{d}{dt}\left[\int_0^{w(t)} \frac{ds}{\sqrt{1-s^4}}\right] \leqslant -\sqrt{c/2}\, z_0$$

and by integrating on $[0,\bar{t}]$ we obtain

$$\int_0^1 \frac{ds}{\sqrt{1-s^4}} > \sqrt{c/2}\, z_0 \cdot \bar{t} \quad . \qquad (3.3.8)$$

Since

$$\int_0^1 \frac{ds}{\sqrt{1-s^4}} < \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \pi/2 \ ,$$

(3.3.8) implies (3.3.1). ∎

3.4 Oscillatory properties and the numerical experiments

Of course, all the theoretical results obtained in Section 3 are confirmed by the numerical experiments (cf., e.g., Table 1).

The various computations also confirm the fact that when $v_0 = 0$ and $u_0 = k\phi$, $k > 0$, the following estimates of the "crossing time" $\bar{t}$:

$$\bar{t} \sim 1/2 \quad \text{if } k \text{ is small}$$

$$\bar{t} \sim \frac{C(\phi)}{k} \quad \text{if } k \text{ is large [compare (3.3.1)]}$$

are rather satisfactory. These estimate of the crossing time will be intensively exploited in Section 4 below.

4. LOCATION OF THE "FIRST CROSSING"

It is interesting to compare the behaviors of the respective solutions of the <u>linear equation</u> and the nonlinear equation "without diffusion"(2.2.1), when $v_0 = 0$

and u_0 satisfies (3.1) - (3.2). We saw (formulas (3.1.3)-(3.1.6) that in the linear case, $u(t,x)$ vanishes for the first time at $t = 1/2$ for all $x \in]0,1[$. On the other hand, it is easy to check that the solution of (2.2.1) vanishes for the first time at $x = 1/2$ (this is indeed related to the fact that the period of the function $t \mapsto u(t,x_0)$ is a decreasing function of $u(x_0)$ in this situation). It is a natural question to ask whether for the full equation (0.1), the solution will always cross first at the "center" $x = 1/2$. In this section, we shall see that the answer is negative, and in fact the situation is already complicated even when $u_0(x) = k \sin \pi x$, $k > 0$: more precisely the answer is different for small values and large values of k.

4.1 Preliminaries

Throughout Section 4, we keep the following assumptions and notation. We assume $v_0 = 0$ and

$$u_0 = k\phi \ , \qquad k > 0 \tag{4.1.1}$$

$$\phi \in H_0^1(0,1) \tag{4.1.2}$$

$$\phi \text{ is increasing on } [0,1/2] \text{ and for all } x \in [0,1],\ \phi(1-x) = \phi(x) \ . \tag{4.1.3}$$

Let u be the solution of (0.3) - (0.6) with $g(u) = u^3$; then for all $t \in \mathbb{R}$, we have $u(t, 1-x) = u(t,x)$ for all $x \in [0,1]$. If we set

$$u = kv \ , \tag{4.1.4}$$

v is the solution of

$$\left.\begin{array}{l} v_{tt} - v_{xx} + k^2 v = 0 \\ v(t,0) = v(t,1) = 0 \\ v(0) = \phi \quad \text{and} \quad v_t(0) = 0 \ . \end{array}\right\} \tag{4.1.5}$$

Throughout this section, any function $\phi \in L^2(0,1)$ will be identified with its unique odd and 2-periodic extension over $\mathbb{R}$, each time when we write $\phi(s)$ for some $s \notin [0,1]$. By using this convention, formula (0.9) yields

$$v(t,x) = \frac{1}{2}\{\phi(x+t) + \phi(x-t)\} - \frac{k^2}{2}\int_0^t \int_{x-t+s}^{x+t-s} v^3(s,y)\,dy \tag{4.1.6}$$

$$v_x(t,x) = \frac{1}{2}\{\phi'(x+t) + \phi'(x-t)\} - \frac{k^2}{2}\int_0^t [v^3(s, x+t-s) - v^3(s, x-t+s)]ds \tag{4.1.7}$$

$$v_x(t,0) = \phi'(t) - k^2 \int_0^t v^3(s, t-s)\,ds\ . \tag{4.1.8}$$

Formula (4.1.6) is a pointwise equality between continuous functions and holds for $(t,x) \in \mathbb{R} \times \mathbb{R}$. Formula (4.1.7) holds for all $t \in \mathbb{R}$ as an equality between functions of $L^2_{loc}(\mathbb{R})$, while (4.1.8) holds for almost all $t \in \mathbb{R}$.

We shall also consider the free solution w given by

$$w(t,x) = \frac{1}{2}\{\phi(x+t) + \phi(x-t)\} \tag{4.1.9}$$

and the function h given by

$$h(t,x) = \int_0^t [v^3(s, x+t-s) - v^3(s, x-t+s)]\,ds\ . \tag{4.1.10}$$

Clearly, h is continuous on $\mathbb{R} \times \mathbb{R}$ and by (4.1.7), (4.1.9) and (4.1.10) we have:

$$v_x(t,x) - w_x(t,x) = -\frac{k^2}{2}\,h(t,x)\ . \tag{4.1.11}$$

Therefore, even if ϕ is only in $H^1_0(0,1)$, $v_x - w_x$ is continuous and v_x and w_x have the same singularities. We shall consider, for all $k > 0$, the "first crossing time" t_k defined as follows:

$$t_k = \sup\{t \geqslant 0,\ u(\varepsilon, x) \geqslant 0 \text{ for } (s,x) \in [0,t] \times [0,1]\}\ . \tag{4.1.12}$$

Obviously, t_k is also given by the formula

$$t_k = \sup\left\{t \geqslant 0,\ v(s,x) \geqslant 0 \text{ for } (s,x) \in [0,t] \times [0,1]\right\} \quad . \qquad (4.1.13)$$

(Note that v depends on k, cf. problem (4.1.5).)

4.2 <u>A partial result for large data</u>

In this paragraph, we assume that ϕ, in addition to (4.1.2)-(4.1.3), satisfies the following hypotheses:

$$\phi \in C^1([0,\eta]) \quad \text{for some } \eta > 0 \ . \qquad (4.2.1)$$

There exists $\delta > 0$, $M > 0$, $0 \leqslant \sigma \leqslant \nu$ such that $\alpha = 4+3\sigma-\nu > 0$ and

$$\forall\ x \in [0,\eta] \ , \quad \delta x^{\nu} \leqslant \phi'(x) \leqslant M x^{\sigma} \ . \qquad (4.2.2)$$

The first result of this paragraph is the following lemma.

<u>Lemma 4.2.1</u> Let ϕ be as above. Then

$$\forall\ t \in [0, \eta/2] \ , \qquad v(t,x) \in C^1([0, \eta/2]) \ ; \qquad (4.2.3)$$

$$t_k > 0 \quad . \qquad (4.2.4)$$

Furthermore, there exists $\varepsilon(\delta,\sigma,M) > 0$ such that $|x| + |t| \leqslant \eta$ and $k^2(|x|+|t|)^{\alpha} + k^5(|x|+|t|)^{2\alpha+\nu+2} \leqslant \varepsilon(\delta,\sigma,M)$ imply

$$v_x(t,x) \geqslant \frac{\delta}{4}(|x|+|t|)^{\nu} \quad . \qquad (4.2.5)$$

<u>Proof.</u> Since ϕ is C^1 on $[-\eta, \eta]$, (4.2.3) follows easily from (4.1.6).

Now let $\tau \in]0,\eta[$: if v were a classical solution of (4.1.5), a straightforward computation involving differentiation under the integral sign would show that

$$\psi(t) = \int_{-\tau+t}^{\tau-t} \left\{\frac{1}{2} v_t^2 + \frac{1}{2} v_x^2 + \frac{k^2}{4} v^4\right\}(t,x)\, dx$$

is nonincreasing on $[0,\tau]$. For the actual solution v, such a result is then easily deduced by a density argument. Hence for any $t \in [0,\tau]$, we find

$$\int_{-\tau+t}^{\tau-t} v_x^2(t,x)\,dx \leqslant \int_{-\tau}^{\tau} \left\{\phi'^2 + \frac{k^2}{2}\phi^4\right\} dx \ . \tag{4.2.6}$$

Let

$$m(s) = \int_{-s}^{s} \left\{\phi'^2 + \frac{k^2}{2}\phi^4\right\} dx \ . \tag{4.2.7}$$

By setting $s = \tau - t$ in(4.2.6) we find

$$\int_{-s}^{s} v_x^2(t,x)\,dx \leqslant m(t+s) \quad \text{for all } s \geqslant 0,\ t \geqslant 0 \text{ such that } t+s \leqslant \eta \ . \tag{4.2.8}$$

By symmetry, for $s > 0$ and $y \in \mathbb{R}$ such that $s + |y| < \eta$ we have

$$|v(s,y)| \leqslant \frac{1}{\sqrt{2}} |y|^{1/2} \left\{ \int_{-|y|}^{|y|} v_x^2(s,\sigma)\,d\sigma \right\}^{1/2} \ . \tag{4.2.9}$$

For $0 < s < t$ and $0 < x < \eta - t$, as a consequence of (4.2.8)-(4.2.9) we obtain

$$|v(s, x+t-s)|^2 + |v(s, x-t+s)|^2 \leqslant (x+t)m(x+t) \ . \tag{4.2.10}$$

Similarly, for $0 < s < t$ and $0 < x < \eta - t$ we find

$$|v(s, x+t-s) - v(s, x-t+s)| \leqslant [2(t-s)m(x+t)]^{1/2} \tag{4.2.11}$$

is an immediate consequence of (4.2.8).

From(4.2.10)-(4.2.11) we deduce in particular

$$|v^3(s, x+t-s) - v^3(s, x-t+s)| \leqslant \frac{3}{\sqrt{2}}(x+t)(t-s)^{1/2}[m(x+t)]^{3/2} \ . \tag{4.2.12}$$

Now (4.1.10) and (4.2.12) yield

$$h(t,x) \leqslant \sqrt{2}\,(x+t)^{5/2}[m(x+t)]^{3/2} \quad \text{for all } t,x \text{ such that } 0 < t < x+t < \eta \ . \tag{4.2.13}$$

On the other hand, as a consequence of (4.2.2)

$$2w_x(t,x) \geqslant \delta(x+t)^{\nu} \quad \text{for } 0 < t < x+t < \eta \quad . \tag{4.2.14}$$

Combining (4.1.11), (4.2.13) and (4.2.14) we obtain

$$2v_x(t,x) \geqslant \delta(x+t)^{\nu} - k^2(x+t)^{5/2}\left[m(x+t)\right]^{3/2} \text{ for } 0 < t < x+t < \eta \quad . \tag{4.2.15}$$

By using the right-hand side of (4.2.2), formula (4.2.7) shows on the other hand that

$$m(s) \leqslant C_1\left[s^{2\sigma+1} + k^2 s^{4\sigma+5}\right]$$

for all $s \in [0,\eta]$. As a consequence, for some $c \geqslant 0$ we have

$$2v_x(t,x) \geqslant (x+t)^{\nu}\left[\delta - ck^2(x+t)^{3\sigma+4-\nu} - ck^5(x+t)^{6\sigma+10-\nu}\right]$$
$$\text{for } 0 < t < x+t < \eta \quad . \tag{4.2.16}$$

Since $6\sigma+10-\nu = 2\alpha+2+\nu$, (4.2.5) is an easy consequence of (4.2.16).

It follows from (4.2.5) that $v > 0$ on $[0,\rho] \times [0,\rho]$ for some $\rho > 0$. On the other hand, since $\phi > 0$ on $[\rho, 1/2]$ we also have $v(t,x) > 0$ for $x \in [\rho, 1/2]$ and t small enough. Hence (4.2.4). ∎

Remark 4.2.2. In order that $t_k > 0$, it is not sufficient to assume (4.1.3). Indeed (4.2.4) is false if $\phi'(x_n) = 0$ for a sequence of positive numbers x_n tending to 0. More precisely, if $\phi'(y) = 0$ for some $y \in]0, 1/2[$ then it follows immediately from (4.1.8) that $t_k \leqslant y$. Hence if $\phi'(x_n) = 0$ and $x_n \to 0$ we deduce $t_k = 0$.

We are now in a position to state the main result of this paragraph.

Theorem 4.2.3. Let ϕ be as in the statement of Lemma 4.2.1 and assume in addition that $2+3\sigma-\nu > 0$. Then for k large enough, we have $v_x(t_k, 0) > 0$. Therefore the first crossing occurs away from the boundary.

Proof. Since $t_k \leqslant C(\phi)/k$, we have

$$k^2 t_k^{\alpha} + k^5 t_k^{2\alpha+\nu+2} \leqslant C'(k^{2-\alpha} + k^{3-2\alpha-\nu})$$

which tends to 0 as $k \to +\infty$ since by hypothesis we have $\alpha > 2$. Now (4.2.5) applied with $x = 0$ and $t = t_k$ gives the result. ∎

4.3 Generalities for small data

In this paragraph, we keep ϕ fixed and assume $k \leqslant 1$. Then the energy conservation property combined with (4.1.6) implies that v is uniformly bounded on $\mathbb{R} \times \mathbb{R}$, independently of k. Therefore from (4.1.6) - (4.1.7) we deduce

$$\forall\, T > 0,\ v \to w \text{ uniformly on } [0,T] \times \mathbb{R} \text{ as } k \to 0\ , \tag{4.3.1}$$

$$\forall\, T > 0,\ v_x - w_x \to 0 \text{ uniformly on } [0,T] \times \mathbb{R} \text{ as } k \to 0\ . \tag{4.3.2}$$

We assume that ϕ satisfies (4.1.2) - (4.1.3), (4.2.1) - (4.2.2) and

$$\phi \in C^1([0,\ 1/2[) \tag{4.3.3}$$

$$\forall\, x \in]0,\ 1/2[\ ,\quad \phi'(x) > 0\quad . \tag{4.3.4}$$

We introduce

$$\zeta = \bigcup_{m \in \mathbb{Z}} \left\{ (t,x) \in \mathbb{R} \times \mathbb{R},\ \left| x - \frac{1}{2} - m \right| = |t| \right\} \tag{4.3.5}$$

Under the hypotheses above, it is clear that w (hence v) is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \zeta$. We now have:

<u>Proposition 4.3.1.</u> Under the hypotheses above, it follows that

$$t_k \to 1/2 \quad \text{as} \quad k \to 0\quad . \tag{4.3.6}$$

<u>Proof.</u> As a consequence of Proposition 3.2.1 we know that $t_k < 1/2$. It is therefore sufficient to show that for any $\tau \in [0,\ 1/2[$, v is positive on $[0,\tau] \times]0,1[$ for k small enough. To that end we fix $\tau \in [0,\ 1/2[$ and we set $a = \frac{1}{2}(\frac{1}{2} - \tau)$. We now observe that as a consequence of (4.3.4), ϕ satisfies also (4.2.2) with $\eta = \tau + a$ (by perhaps modifying δ, M, σ and ν). Applying Lemma 4.2.1, it follows in particular from (4.2.5) that there exists $k_0 > 0$ such that for all $k \in]0, k_0]$:

$$\forall\, (t,x) \in [0,\tau] \times]0,a]\ ,\quad v(t,x) > 0\quad . \tag{4.3.7}$$

Also, as a consequence of (3.1.3) and (4.3.1) there exists $k_1 > 0$ such that for all $k \in]0, k_1]$:

$$\forall\ (t,x) \in [0,\tau] \times [a, 1/2]\ , \quad v(t,x) > 0\ . \tag{4.3.8}$$

Putting together (4.3.7) and (4.3.8) we conclude that $v > 0$ on $[0,\tau] \times]0,1[$ for all k such that $0 < k \leqslant \min\{k_0, k_1\}$. Since τ was arbitrary in $[0, 1/2[$, the proof is now complete. ∎

4.4 Crossing at the center in a special case

In this paragraph, we consider ϕ given by

$$\forall\ x \in [0,1], \quad \phi(x) = \min\{x, 1-x\}\ .$$

Obviously, ϕ satisfies (4.1.2) - (4.1.3) and (4.3.3) - (4.3.4). The free solution w is given, for all $t \in [0, 1/2]$, by the formulas

$$w(t,x) = x\ , \quad \forall\ x \in \left[0, \frac{1}{2} - t\right] \tag{4.4.1}$$

$$w(t,x) = \frac{1}{2} - t\ , \quad \forall\ x \in \left[\frac{1}{2} - t, 1/2\right]\ . \tag{4.4.2}$$

Our main result is the following.

THEOREM 4.4.1. For k small enough we have

$$\forall\ x \in]0, 1/2[\ , \quad v(t_k, x) > 0 \tag{4.4.3}$$

$$v_x(t_k, 0) > 0 \tag{4.4.4}$$

$$v(t_k, 1/2) = 0\ . \tag{4.4.5}$$

Hence crossing occurs at $x = 1/2$.

The proof of Theorem 4.4.1 relies on

Lemma 4.4.2. Let $h^0(t,x) = \int_0^t [w^3(s, x+t-s) - w^3(s, x-t+s)]\,ds$ for $(t,x) \in \mathbb{R} \times \mathbb{R}$. Then $h^0 \in C(\mathbb{R} \times \mathbb{R}) \cap C^1(\mathbb{R} \times \mathbb{R} \setminus \mathcal{C})$ with $\mathcal{C}$ as in (4.3.5) and

$$\forall\,(t,x) \in]0,\,1/2] \times [0,\,1/2[\,,\quad h^0(t,x) > 0 \tag{4.4.6}$$

$$\forall\, t \in]0,\,1/2]\,,\quad h^0_x(t,\,1/2) < 0\,. \tag{4.4.7}$$

<u>Proof.</u> Easy calculations show that

$$w(\sigma,z) \leqslant w(\sigma,y),\quad \forall\,\sigma \in]0,1/2[,\quad \forall\, y \in [0,\,\tfrac{1}{2}+\sigma],\quad \forall\, z \in [-1,y] \tag{4.4.8}$$

$$w(\sigma,z) < w(\sigma,y),\quad \forall\,\sigma \in]0,1/2[,\quad \forall\, y \in [0,\,\tfrac{1}{2}+\sigma],\quad \forall\, z \in]-1,y[\, \cap\,]-1,\,\tfrac{1}{2}-\sigma[\,. \tag{4.4.9}$$

Let $t \in [0,1/2]$. If $x \in [0,\,\frac{1}{2}-t]$, then by (4.4.7) we have $w(s,\,x-t+s) < w(s,\,x+t-s)$ for any $s \in]0,t[$ and therefore $h^0(t,x) > 0$. On the other hand, if $x \in]\frac{1}{2}-t,\,1/2[$, we distinguish two cases. For $s \in]\frac{1}{2}(x+t-\frac{1}{2}),t[$, by (4.4.8) $w(s,\,x-t+s) \leqslant w(s,\,x+t-s)$. For $s \in]0,\,\frac{1}{2}(x+t-\frac{1}{2})[$, we have $w(s,x+t-s) = w(s,\,1-(x+t-s))$, and by (4.4.9) we deduce $w(s,x-t+s) < w(s,x+t-s)$. Therefore in this case also we have $h^0(t,x) > 0$ and the proof of (4.4.6) is complete.

Now straightforward calculations show that $h^0 \in C^1(\mathbb{R} \times \mathbb{R}\setminus \mathcal{C})$ with

$$h^0_x(t,\,1/2) = 3\int_0^t \left\{(w^2 w_x)(s,\,\tfrac{1}{2}+t-s) - (w^2 w_x)(s,\,\tfrac{1}{2}-t+s)\right\}ds$$

$$= -6\int_0^t w^2(s,\,\tfrac{1}{2}-t+s)\,w_x(s,\,\tfrac{1}{2}-t+s)\,ds \quad \text{by symmetry.}$$

On the other hand we have

$w(s,\,\frac{1}{2}-t+s) > 0$ for $t \in]0,\,1/2]$ and $s \in]0,t[$

$w_x(s,\,\frac{1}{2}-t+s) \geqslant 0$ for $t \in [0,\,1/2]$ and $s \in]0,t[$

$w_x(s,\,\frac{1}{2}-t+s) > 0$ for $t \in]0,\,1/2]$ and $s \in]0,\,t/2[$.

Hence (4.4.7) is proved. ∎

Proof of Theorem 4.4.1. Let h be given by (4.1.10). From (4.1.11) we deduce

$$\frac{1}{k^2}\left[v_x(t,x) - w_x(t,x)\right] = -\frac{1}{2}h(t,x) \quad . \tag{4.4.10}$$

An easy calculation shows that as $k \to 0$, $h \to h^0$ uniformly on $[0,T] \times \mathbb{R}$ for any $T > 0$. Furthermore if we set $D = \left\{(t,x) \in \mathbb{R}^2,\ \left|t - \frac{1}{2}\right| + \left|x - \frac{1}{2}\right| \leq 1/4\right\}$ (so that $(1/2, 1/2) \in D$ and $\bar{D} \subset \mathbb{R}^2 \setminus \mathcal{C}$) we also have $h_x \to h_x^0$ uniformly on D.

By (4.4.7) and since $w_x \equiv 0$ in D, for k small enough we have $v_{xx} > 0$ near (1/2, 1/2). In particular, by (4.3.6) there exists $\delta > 0$ and $k_0 > 0$ such that

$$\forall\, k \in]0, k_0], \qquad \forall\, x \in \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right], \quad v_{xx}(t_k, x) > 0 \quad . \tag{4.4.11}$$

From (4.4.10), (4.4.5) and (4.3.6) we deduce that there exists $k_1 > 0$ such that for all $k \in]0, k_1[$ we have

$$-1/2 \leq v_x(t_k, x) - w_x(t_k, x) < 0 \quad \text{for } 0 \leq x \leq \frac{1}{2} - \delta \quad . \tag{4.4.12}$$

From the left-hand side of (4.4.12) and (4.4.1) we get

$$\forall\, k \in]0, k_1[, \quad \forall\, x \in [0, \frac{1}{2} - t_k[, \quad v_x(t_k, x) \geq \frac{1}{2} \quad . \tag{4.4.13}$$

Considering the right-hand side of (4.4.12) and (4.4.2) we find

$$\forall\, k \in]0, k_1[, \quad \forall\, x \in \left]\frac{1}{2} - t_k, \frac{1}{2} - \delta\right], \quad v_x(t_k, x) < 0 \quad . \tag{4.4.14}$$

As soon as $k < \min(k_0, k_1)$, (4.4.3) and (4.4.4) now follow from (4.4.11), (4.4.13) and (4.4.14) and the continuity of $v(t_k, \cdot)$ as a function of k. Then (4.4.3) is obvious and the proof is complete. ∎

4.5. The case $\phi(x) = \sin \pi x$

In this section we consider $\phi(x) = \sin \pi x$. Note that here the free solution w is simply

$$w(t,x) = \cos \pi t \sin \pi x \quad . \tag{4.5.1}$$

Since w is regular in both t and x, it follows that v is indeed a classical solution of (4.1.5) and that all the derivatives of formula (4.1.6) also hold in the classical sense.

The following result is a striking contrast to the crossing behavior for the previous examples.

Theorem 4.5.1. Let $\phi(x) = \sin \pi x$. Then for all sufficiently small $k > 0$ we have

$$v(t_k, x) > 0 , \quad \forall\, x \in]0,1[\tag{4.5.2}$$

$$v_x(t_k, 0) = v_x(t_k, 1) = 0 . \tag{4.5.3}$$

Remark 4.5.2. Clearly (4.5.3) follows immediately from (4.5.2) and the definition of t_k. Also, this theorem shows that the solution crosses first at the boundary. In other words, for t just slightly larger than t_k, $v(t,x)$ is positive everywhere on $]0,1[$ except close to $x = 0$ and $x = 1$.

Remark 4.5.3. The proof of Theorem 4.5.1 is very technical and relies on two computational lemmas.

Lemma 4.5.4. For all $k > 0$, $t \in [0, 1/2]$ and $x \in]0, 1/2]$, define

$$G(k,t,x) = (k \sin \pi x)^{-2}\left[\frac{v(t,x)}{\sin \pi x} - \frac{v_x(t,0)}{\pi}\right] . \tag{4.5.4}$$

Then as $k \to 0$, $G(k,t,x)$ converges uniformly on $[0, 1/2] \times]0, 1/2]$ to a continuous function $G(t,x)$ which is Lipschitz continuous in t, uniformly with respect to $x \in]0, 1/2]$.

Lemma 4.5.5. We have the formula

$$\forall\, x \in]0, 1/2], \quad G(1/2, x) = \frac{1}{48\pi} . \tag{4.5.5}$$

The proof of Lemma 4.5.5 will be postponed to the next section since it follows from a general computation valid for other initial data (cf. Theorem 4.6.5 and Corollary 4.6.11).

Proof of Lemma 4.5.4. We define

$$g(k,t,x) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} v^3(s,y)\,dyds \quad . \tag{4.5.6}$$

(Recall that v also depends on k.) From (4.1.6), (4.1.8) and (4.5.1) it follows immediately that we have

$$G(k,t,x) = -\frac{1}{\sin^2 \pi x}\left[\frac{g(k,t,x)}{\sin \pi x} - \frac{g_x(k,t,0)}{\pi}\right] \quad . \tag{4.5.7}$$

Because of the smoothness of the free solution w, it follows easily by induction that $g(k,t,x)$ and all its x-derivatives are continuous for $k \geqslant 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}$. In particular

$$\forall\, m \in \mathbb{N}, \quad \lim_{k\to 0} \frac{\partial^m g}{\partial x^m}(k,t,x) = \frac{\partial^m g}{\partial x^m}(0,t,x) \quad ,$$

$$\text{uniformly on compact subsets of } \mathbb{R}^2 \; . \tag{4.5.8}$$

Since v and w are smooth solutions we have

$$v(t,0) = w(t,0) = v_{xx}(t,0) = w_{xx}(t,0) = 0 \; ,$$

and this implies

$$\forall\, t \in \mathbb{R}, \quad \forall\, k \geqslant 0, \qquad g(k,t,0) = g_{xx}(k,t,0) = 0 \quad . \tag{4.5.9}$$

From (4.5.9) we deduce easily

$$g(k,t,x) = xg_x(k,t,0) + \int_0^x \int_0^y \int_0^z g_{xxx}(k,t,\sigma)\,d\sigma dydz \quad . \tag{4.5.10}$$

Therefore by (4.5.7) we find

$$G(k,t,x) = z_1(x) g_x(k,t,0) - z_2(x) \frac{1}{x^3} \int_0^x \int_0^y \int_0^z g_{xxx}(k,t,\sigma)\, d\sigma dy dz \quad , \tag{4.5.11}$$

where z_1, z_2 are continuous and given by

$$z_1(x) = \frac{1}{\pi \sin^3(\pi x)} [\sin \pi x - \pi x], \qquad z_2(x) = \frac{x^3}{\sin^3 \pi x} \quad .$$

As a consequence of (4.5.8), it is now clear that, as $k \to 0$, $G(k,t,x)$ converges uniformly on $[0, 1/2] \times]0, 1/2]$ to the function

$$G(t,x) = z_1(x) g_x(0,t,0) - z_2(x) \frac{1}{x^3} \int_0^x \int_0^y \int_0^z g_{xxx}(0,t,\sigma) d\sigma dy dz \quad . \tag{4.5.12}$$

Finally it is clear that G is bounded and Lipschitz continuous in $t \in [0, 1/2]$, uniformly with respect to $x \in]0, 1/2]$. ∎

Proof of Theorem 4.5.1. From Lemmas 4.5.4 - 4.5.5 it follows that for k small enough and t sufficiently close to $1/2$, we have with $0 < \alpha < 1/48\pi$ (α fixed),

$$G(k,t,x) \geqslant \alpha > 0, \quad \forall\, x \in]0, 1/2] \quad . \tag{4.5.13}$$

By choosing k perhaps smaller, Proposition 4.3.1 guarantees that we can choose $t = t_k$ in formula (4.5.13). Since $v_x(t_k, 0) \geqslant 0$ it follows from (4.5.4) and (4.5.13) that

$$\forall\, x \in]0, 1/2], \qquad v(t_k, x) \geqslant \alpha k^2 \sin^3(\pi x) \quad .$$

By symmetry, we obtain (4.5.2). The proof of Theorem 4.5.1 is now complete. ∎

4.6. Some results for general ϕ and applications

In this section, our goal is to understand the two previous examples as much as possible from a general point of view. In addition to the hypotheses of paragraph 4.3 (i.e., (4.1.2) - (4.1.3), (4.2.1) - (4.2.2) and (4.3.3) - (4.3.4)) we assume

$$\phi \text{ is } C^2 \text{ in a neighborhood of } x = 1/2 \text{ with } \phi''(1/2) < 0 \ . \tag{4.6.1}$$

It will be convenient to set

$$h(t,x) = \begin{cases} \dfrac{\phi(x+t) + \phi(x-t)}{2\phi'(t)} & \text{for } 0 < t < 1/2 \\[2ex] \dfrac{\phi'(x+\frac{1}{2})}{\phi''(1/2)} & \text{for } t = 1/2 \ . \end{cases} \tag{4.6.2}$$

By the hypotheses on ϕ, h is continuous on $]0, 1/2] \times \mathbb{R}$. We now introduce

$$H(k,t,x) = \frac{1}{k^2}\left[v(t,x) - h(t,x)v_x(t,0)\right] \ . \tag{4.6.3}$$

Note that if $\phi(x) = \sin \pi x$, then $h(t,x) = \frac{1}{\pi} \sin \pi x$ and therefore in this case

$$H(k,t,x) = \sin^3(\pi x) G(k,t,x) \tag{4.6.4}$$

with G given by (4.5.4).

It follows from (4.1.6) and (4.1.8) that in the general case

$$H(k,t,x) = h(t,x) \int_0^t v^3(s, t-s)\, ds - \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} v^3(s,y)\, dy ds \ . \tag{4.6.5}$$

We now set

$$H(t,x) = h(t,x) \int_0^t w^3(s, t-s)\, ds - \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} w^3(s,y)\, dy ds \ . \tag{4.6.6}$$

<u>Lemma 4.6.1</u>. As $k \to 0$ we have

$$H(k,t,x) \to H(t,x) \text{ uniformly on }]0, 1/2] \times \mathbb{R} \ , \tag{4.6.7}$$

$$H(k,t_k,x) \to H(1/2, x) \text{ uniformly on } \mathbb{R} \ . \tag{4.6.8}$$

<u>Proof.</u> (4.6.7) follows immediately from (4.3.1) and the formulas (4.6.5) - (4.6.6). On the other hand, since H(t, x) is Lipschitz continuous in t, uniformly with respect to $x \in \mathbb{R}$, (4.6.8) follows from (4.6.7) and (4.3.6). ∎

The following two propositions show the importance of $H(1/2, x)$ in determining the crossing behavior of the solution u for small values of k.

Proposition 4.6.2. Assume that there exists $x_0 \in]0,1[$ such that $H(1/2, x_0) < 0$. Then for k small enough we have $v_x(t_k, 0) > 0$. Hence the first crossing cannot occur at the boundary.

Proof. By (4.6.8), for k small enough we have $H(k, t_k, x_0) < 0$. Then (4.6.3) yields

$$h(t_k, x_0) v_x(t_k, 0) > v(t_k, x_0) \geqslant 0 .$$

Since (4.6.2) gives $h(t_k, x_0) > 0$, we deduce $v_x(t_k, 0) > 0$, and the proof is complete. ∎

Proposition 4.6.3. Assume that for all $x \in]0,1[$, $H(1/2, x) > 0$. Then for any $\varepsilon > 0$ there exists k_ε such that for all $k \in]0, k_\varepsilon[$ we have $v(t_k, x) > 0$, $\forall\, x \in [\varepsilon, 1-\varepsilon]$. Hence for k small, the first crossing occurs arbitrarily close to the boundary.

Proof. For any $\varepsilon > 0$, $\min_{x \in [\varepsilon, 1-\varepsilon]} H(1/2, x) > 0$. Therefore, by (4.6.8) we have for k small enough

$$\forall\, x \in [\varepsilon, 1-\varepsilon], \qquad H(k, t_k, x) > 0 .$$

By (4.6.3) we conclude that

$$\forall\, x \in [\varepsilon, 1-\varepsilon], \qquad v(t_k, x) > h(t_k, x) v_x(t_k, x) \geqslant 0 .$$ ∎

Remark 4.6.4. We have $H(k, t, 0) = 0$ for any $k > 0$ and $t \in \mathbb{R}$. Therefore the argument of Proposition 4.6.3 cannot be applied to show that first crossing occurs exactly at the boundary. In other words, while the arguments of this paragraph apply to a more general class of data than $\sin(\pi x)$, they are not as subtle as the analysis of paragraph 4.5.

The main result of this paragraph is the following explicit formula.

Theorem 4.6.5. We have

$$H(1/2,x) = \frac{1}{16}\left[\frac{\phi'(x+\frac{1}{2})}{\phi''(1/2)}\phi^3(1/2) - \int_0^x \phi^3(r+\frac{1}{2})\,dr\right]$$

$$+ \frac{3}{16}\left(\int_0^1 \phi^2(r)\,dr\right)\left[\frac{\phi'(x+\frac{1}{2})}{\phi''(1/2)}\phi(1/2) - \int_0^x \phi(r+\frac{1}{2})\,dr\right]. \quad (4.6.9)$$

Proof. We set

$$\theta(x) = \int_0^{1/2}\int_{x-\frac{1}{2}+s}^{x+\frac{1}{2}-s} w^3(s,y)\,dy \quad , \qquad (4.6.10)$$

$$\rho(x) = \int_0^{1/2}\left[w^3(s,\, x+\frac{1}{2}-s) - w^3(s,\, x-\frac{1}{2}+s)\right] ds \ . \qquad (4.6.11)$$

Note that $\rho(x) = \theta'(x)$ and by symmetry, we have

$$\rho(x) = \psi(x) + \psi(-x) \qquad (4.6.12)$$

with

$$\psi(x) = \int_0^{1/2} w^3(s,\, x+\frac{1}{2}-s)\,ds \quad . \qquad (4.6.13)$$

By (4.1.9) and (4.6.13) we have

$$\psi(x) = \frac{1}{8}\int_0^{1/2}\left[\phi(x+\frac{1}{2}) + \phi(x+\frac{1}{2}-2s)\right]^3 ds$$

$$= \frac{1}{16}\phi^3(x+\frac{1}{2}) + \frac{3}{16}\phi^2(x+\frac{1}{2})\int_{x-\frac{1}{2}}^{x+\frac{1}{2}}\phi(r)\,dr$$

$$+ \frac{3}{16}\phi(x+\frac{1}{2})\int_{x-\frac{1}{2}}^{x+\frac{1}{2}}\phi^2(r)\,dr + \frac{1}{16}\int_{x-\frac{1}{2}}^{x+\frac{1}{2}}\phi^3(r)\,dr \ .$$

Since ϕ and ϕ^3 are odd and ϕ^2 has period 1 it follows that

$$\psi(x) + \psi(-x) = \frac{1}{8}\phi^3(x+\frac{1}{2}) + \frac{3}{8}\phi(x+\frac{1}{2})\int_0^1 \phi^2(r)\,dr \;. \tag{4.6.14}$$

By (4.6.12) we obtain

$$\rho(0) = \frac{1}{8}\phi^3(1/2) + \frac{3}{8}\phi(1/2)\int_0^1 \phi^2(r)\,dr \;. \tag{4.6.15}$$

Since $\theta(0) = 0$, by integrating (4.6.14) we find

$$\theta(x) = \frac{1}{8}\int_0^x \phi^3(r+\frac{1}{2})\,dr + \frac{3}{8}\int_0^1 \phi^2(r)\,dr \int_0^x \phi(r+\frac{1}{2})\,dr \;. \tag{4.6.16}$$

Finally, from (4.6.6), (4.6.10), (4.6.11) and the symmetry we deduce

$$H(1/2, x) = \frac{1}{2}h(1/2, x)\rho(0) - \frac{1}{2}\theta(x) \;. \tag{4.6.17}$$

A combination of (4.6.2) and (4.6.15) - (4.6.17) finally yields (4.6.9). ∎

We now derive simple consequences of Theorem 4.6.5.

<u>Corollary 4.6.6</u>. Assume that $\phi'(0) = 0$. Then for k small enough, first crossing occurs away from the boundary.

<u>Proof</u>. We write (4.6.9) at $x = 1/2$. Since $\phi'(0) = \phi'(1) = 0$ we obtain

$$H(1/2, 1/2) = -\frac{1}{16}\int_0^{1/2} \phi^3(r+\frac{1}{2})\,dr - \frac{3}{16}\left(\int_0^{1/2} \phi(r+\frac{1}{2})\,dr\right)\left(\int_0^1 \phi^2(r)\,dr\right)$$

$$< 0 \;.$$

Hence by Proposition 4.6.2, the first crossing cannot occur at the boundary. ∎

<u>Remark 4.6.7.</u> Corollary 4.6.6 applies to $\phi(x) = \sin^2(\pi x)$, and the conclusion fits with the numerical results (Table 2).

Corollary 4.6.8. Assume that

$$\forall\, x \in]0, 1/2[, \qquad \frac{\phi'(x+\frac{1}{2})}{\phi''(1/2)} \geqslant x \ . \tag{4.6.9}$$

Then $H(1/2, x) > 0$, $\forall\, x \in]0,1[$, hence as k tends to 0, the first crossing occurs arbitrarily close to the boundary.

Proof. We have

$$\frac{\phi'(x+\frac{1}{2})}{\phi''(1/2)}\, \phi^3(1/2) - \int_0^x \phi^3(r+\tfrac{1}{2})\, dr \geqslant x\phi^3(1/2) - \int_0^x \phi^3(r+\tfrac{1}{2})\, dr$$

$$\geqslant \int_0^x \left[\phi^3(1/2) - \phi^3(r+\tfrac{1}{2})\right] dr \ , \qquad \forall\, x \in]0, 1/2[\quad .$$

Since for all $r \in]0, 1/2]$ we have $\phi^3(1/2) > \phi^3(r+\frac{1}{2})$, the first term in the right-hand side of (4.6.9) is positive. Similarly, the second term in the right-hand side of (4.6.9) is also positive and the result follows by applying Proposition 4.6.3. ∎

Remark 4.6.9. Corollary 4.6.8 applies to $\phi(x) = x(1-x)$ and its conclusion fits with the numerical results (Table 2).

Remark 4.6.10. In the extreme case where "$\phi''(1/2) = -\infty$," we would find $H(1/2, x) < 0$ for all $x \in]0,1[$. Therefore formula (4.6.9) also gives a heuristic explanation of the result obtained in Section 4.4.

To conclude this paragraph, we now check formula (4.5.5) of Lemma 4.5.5.

Corollary 4.6.11. When $\phi(x) = \sin(\pi x)$, we have

$$H(1/2, x) = \frac{1}{48\pi} \sin^3(\pi x) \ , \qquad \forall\, x \in]0,1[\quad . \tag{4.6.18}$$

Proof. By applying (4.6.9) and reducing the terms we obtain

$$H(1/2, x) = \frac{1}{16\pi} \sin(\pi x) - \frac{1}{16} \int_0^x \cos^3(\pi r)\, dr = \int_0^x \frac{1}{16} \left[\cos(\pi r) - \cos^3(\pi r)\right] dr$$

(cont'd)

$$= \frac{1}{16}\int_0^x \sin^2(\pi r)\cos(\pi r)\,dr = \frac{1}{48\pi}\int_0^x \frac{d}{dr}\left[\sin^3(\pi r)\right]dr = \frac{1}{48\pi}\sin^3(\pi r) \;. \quad \blacksquare$$

Proof of Lemma 4.5.5. Formula (4.5.5) is indeed an immediate consequence of (4.6.18) and (4.6.4). ∎

5. THE NUMERICAL RESULTS

5.1. Description of the scheme. Evaluation of the error.

We employed the explicit finite difference scheme described in KUO PEN-YU, L. VAZQUEZ [15] which is

$$\frac{u^{k+1}(x_j) - 2u^k(x_j) + u^{k-1}(x_j)}{(\delta t)^2} - \frac{u^k(x_{j+1}) - 2u^k(x_j) + u^k(x_{j-1})}{(\delta x)^2} + \frac{F(u^{k+1}(x_j)) - F(u^{k-1}(x_j))}{u^{k+1}(x_j) - u^{k-1}(x_j)} = 0 \qquad (5.1)$$

with $F(x) = x^4/4$. We imposed the boundary conditions $u^k(0) = u^k(1) = 0$. From the convexity of F it is easily seen that for any initial data u^0 and u^1, the discretized problem admits a unique solution, regardless of the size of δt and δx (compare L. VAZQUEZ [26]).

The convergence and stability of this scheme have been proved in [15]. Our boundary conditions require only an obvious change of the discrete Sobolev spaces considered in the proof. At each step, the implicit problem was solved by means of Newton's method. We always assumed an initial speed equal to 0, which corresponds in the continuous problem to a solution which is an even function of t. Therefore, to insure the maximum accuracy, u^{-1} was computed from the initial datum u^0 by (5.1) (with $k = 0$) on imposing the condition $u^{-1} = u^1$. Finally, the computations were performed in double precision with $\delta x = 0.02$ and $\delta t = 0.01$.

Several experiments have been done in order to estimate the error due to the method. We first considered the discretization of the d'Alembertian operator. We compared the solution of the discretized linear problem with the analytic solution of

the linear continuous problem. For the initial datum $10\sin(\pi x)$ and $0 \leqslant t \leqslant 100$ (i.e., 10,000 time steps) the mean error is 0.078 while the maximum error is 0.38 at time 99.51. We also computed the error coming from the approximation by Newton's method. Again with $u^0 = 10\sin(\pi x)$ the solution of (5.1) was computed up to $t = 100$ and then backwards to $t = 0$. After these 20,000 steps of computation, the departure from the initial datum did not exceed $1.05 \cdot 10^{-10}$. Therefore we can reasonable expect that the long term computations (up to $t = 100$) described in this paper are significant.

5.2. The results

5.2.1 Table 1. Here $u^0(x) = 10\sin(\pi x)$. We have been investigating the values of t in $[0, 100]$ for which $y(t) = \|u(t) - u^0\|_\infty / \|u^0\|_\infty$ is less than 0.05. We found fifteen sequences of consecutive times. In Table 1 we give the time of each sequence corresponding to the smallest value of $y(t)$. For completeness, we also tabulated the corresponding values of

$$z(t) = \frac{\|u(t) - u^0\|_{H_0^1(0,1)}}{\|u^0\|_{H_0^1(0,1)}} \quad \text{and} \quad w(t) = \frac{\|u(t) - u^0\|_4}{\|u^0\|_4} .$$

5.2.2. Table 2. We considered 3 cases:

$$u^0(x) = k\sin(\pi x) , \qquad u^0(x) = k\sin^2(\pi x) , \qquad u^0(x) = kx(1-x) .$$

In each case, we computed the solution for $k \in \{1, 5, 10, 20, 30, 50\}$. For each of these 18 examples, we indicate in Table 2 the value of the crossing time t_k. Of course what appears in the table is in fact the first time step θ_k greater than or equal to t_k. By identifying θ_k and t_k we give in Table 2 the location of the points in $]0,1[$ where $u(t_k, x) < 0$. Three different situations have been observed.

Invisible: $u(t_k, x) < 0$ for all $x \in]0,1[$.

Boundary: At time t_k, the solution is positive everywhere except near $x \in \{0, 1\}$.

Center: At time t_k the solution is 0 near $x = 0.5$ and positive elsewhere.

TABLE 1

t	y(t)	z(t)	w(t)
0	0	0	0
2.48	0.048	0.174	0.040
10.69	0.014	0.063	0.011
13.17	0.023	0.087	0.020
21.36	0.030	0.120	0.026
23.85	0.020	0.090	0.016
34.53	0.019	0.085	0.017
45.25	0.033	0.148	0.031
47.74	0.044	0.192	0.037
58.46	0.017	0.064	0.013
69.16	0.021	0.103	0.019
71.64	0.020	0.084	0.016
82.36	0.046	0.161	0.041
86.45	0.029	0.097	0.022
99.64	0.024	0.091	0.019

TABLE 2

k	Initial Datum					
	$k \sin(\pi x)$		$k \sin^2(\pi x)$		$kx(1-x)$	
	t_k	Location	t_k	Location	t_k	Location
1	0.49	invisible	0.45	center	0.50	invisible
5	0.32	center	0.29	center	0.48	boundary
10	0.18	center	0.18	center	0.42	boundary
20	0.10	center	0.10	center	0.33	invisible
35	0.07	center	0.07	center	0.24	center
50	0.05	center	0.05	center	0.15	center

5.2.3. The Figures

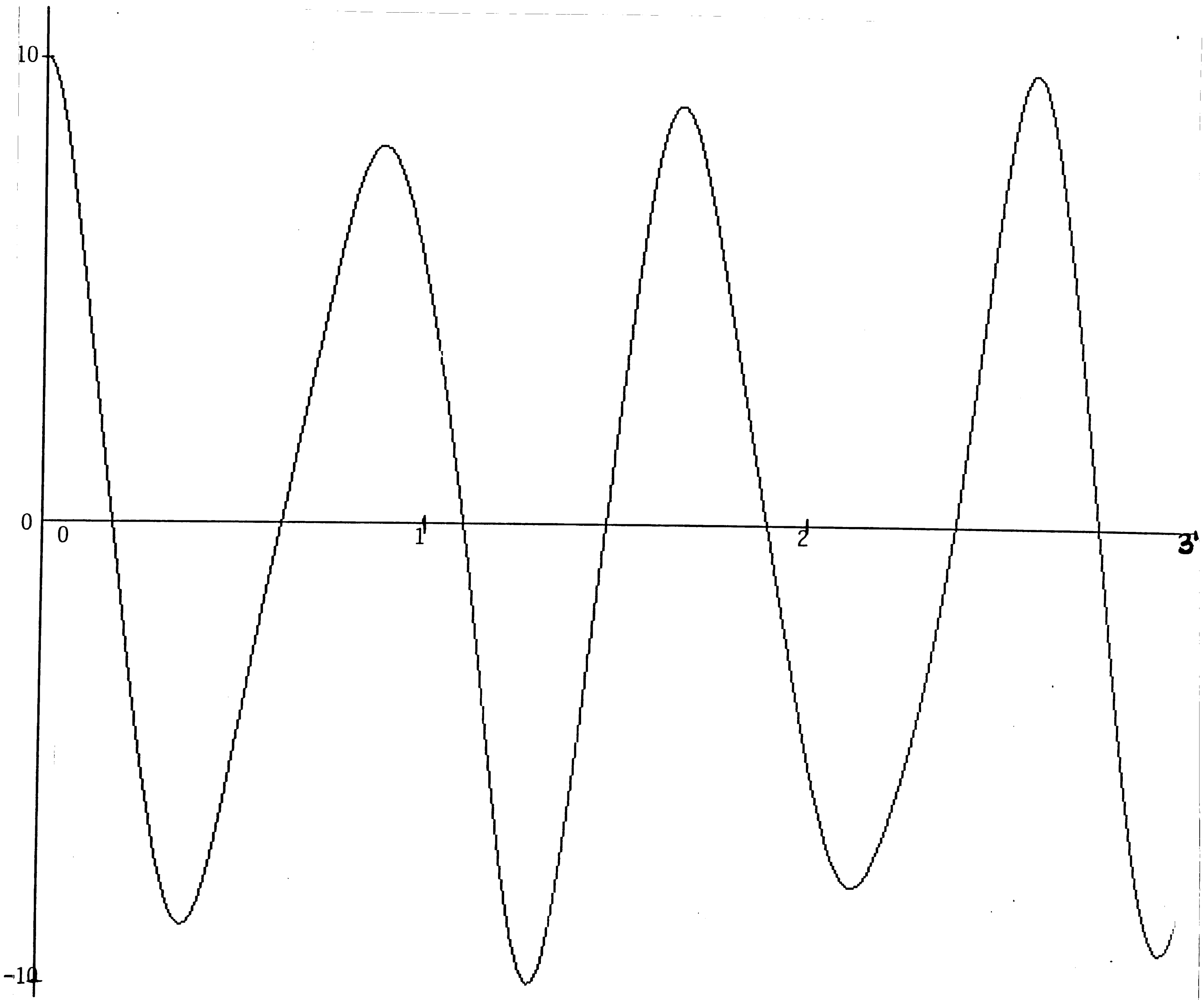

Fig. 1. The initial datum $u^0(x) = 10\sin(\pi x)$. Evolution of $u(t, 1/2)$ for $0 \leq 1 \leq 3$.

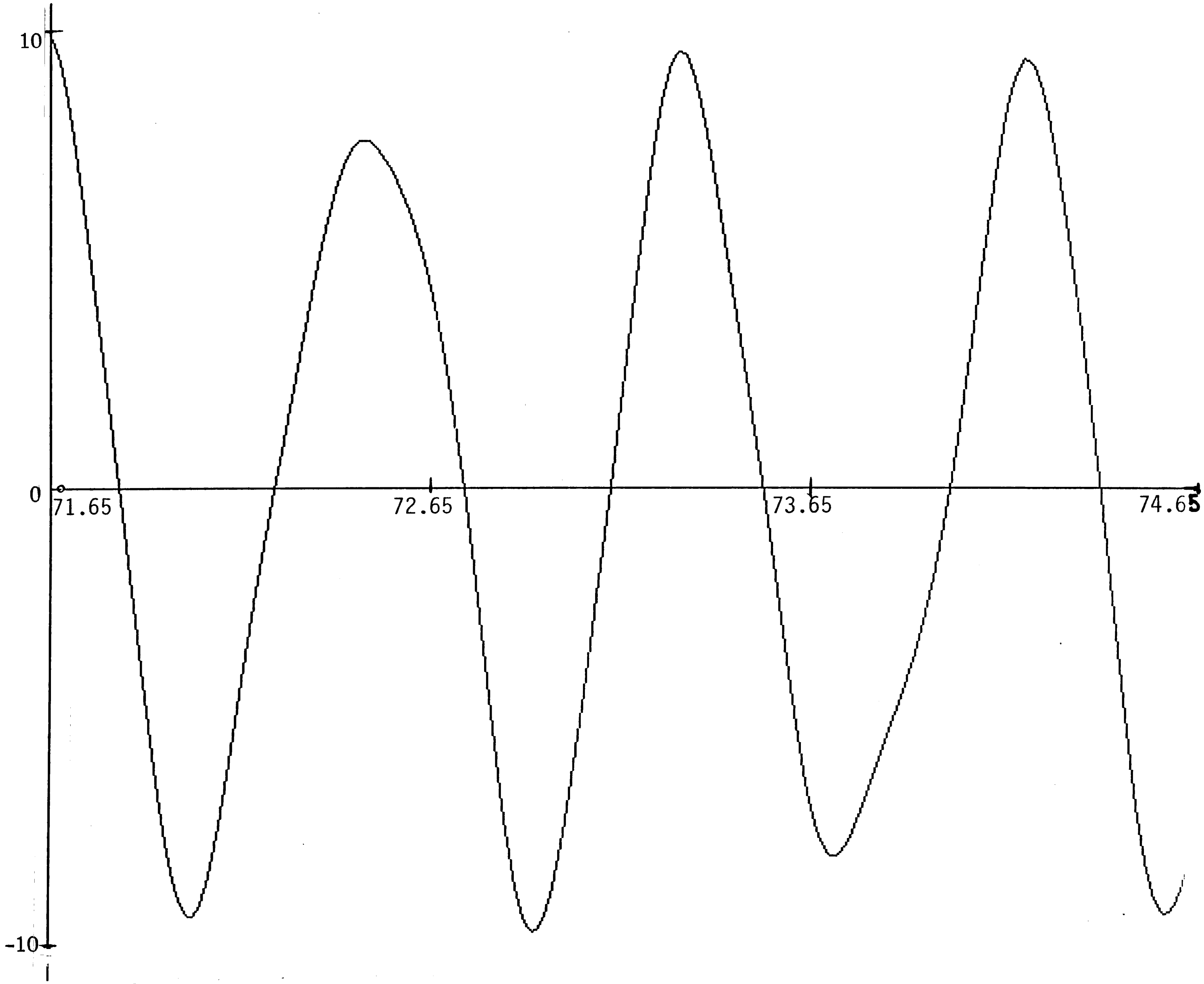

Fig. 2. Same as Fig. 1 but for $71.65 \leqslant t \leqslant 74.65$.

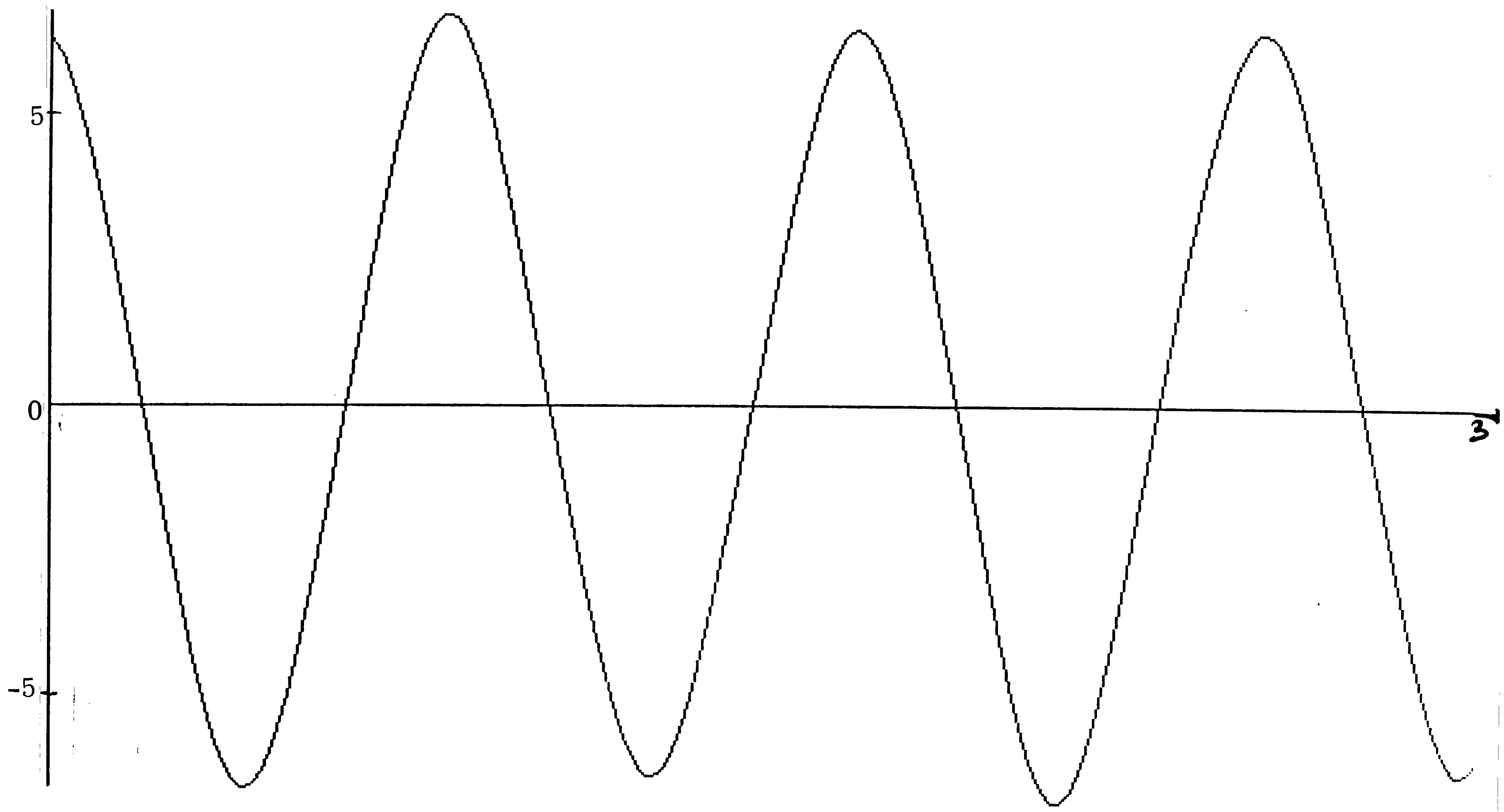

Fig. 3. Initial datum $10\sin(\pi x)$. Evolution of $m(t) = \int_0^1 u(t,x)\,dx$ for $0 \leqslant t \leqslant 3$.

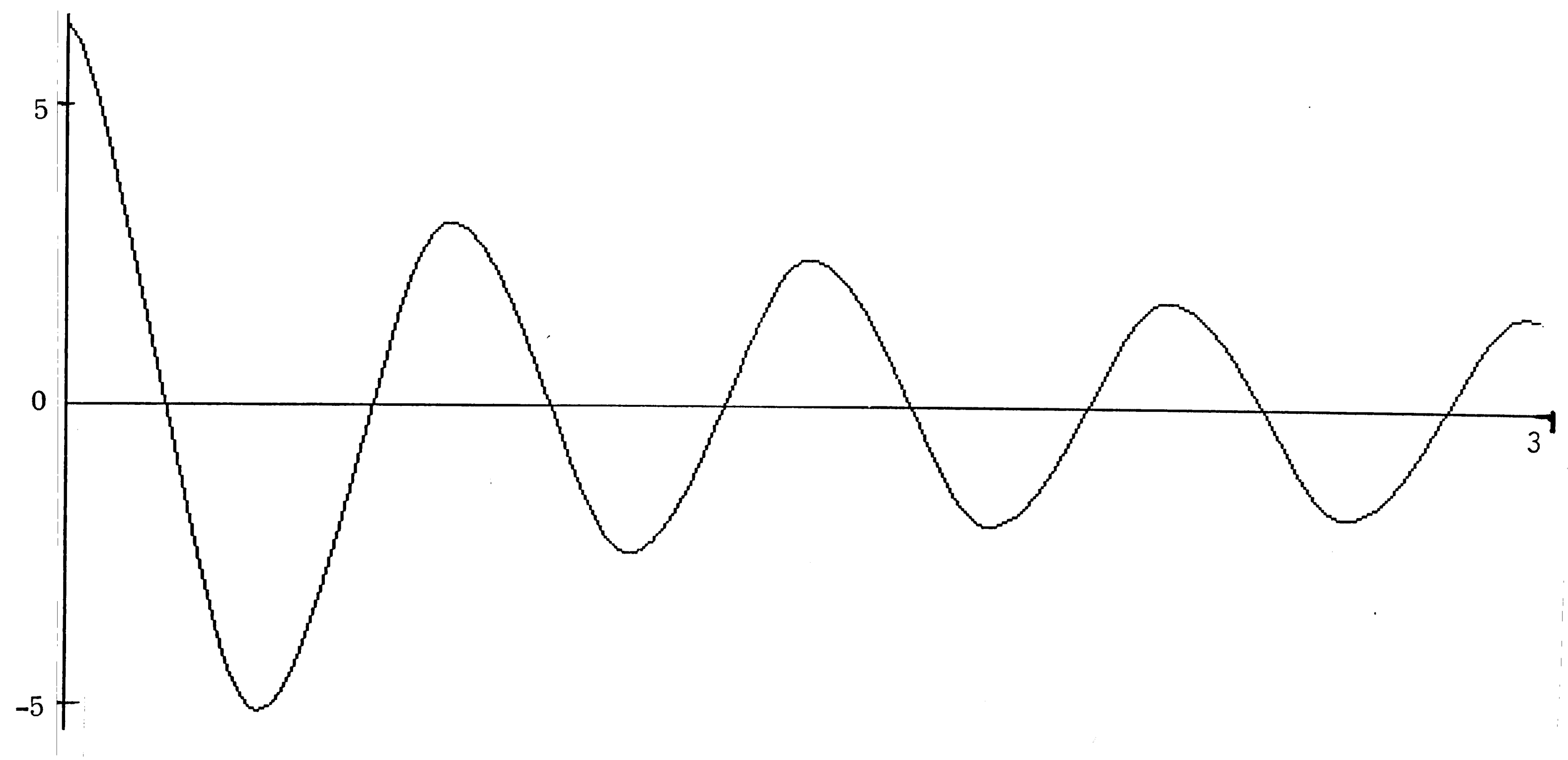

Fig. 4. Same as Fig. 3 but here u is the solution of $u_{tt} + \pi^2 u + u^3 = 0$.

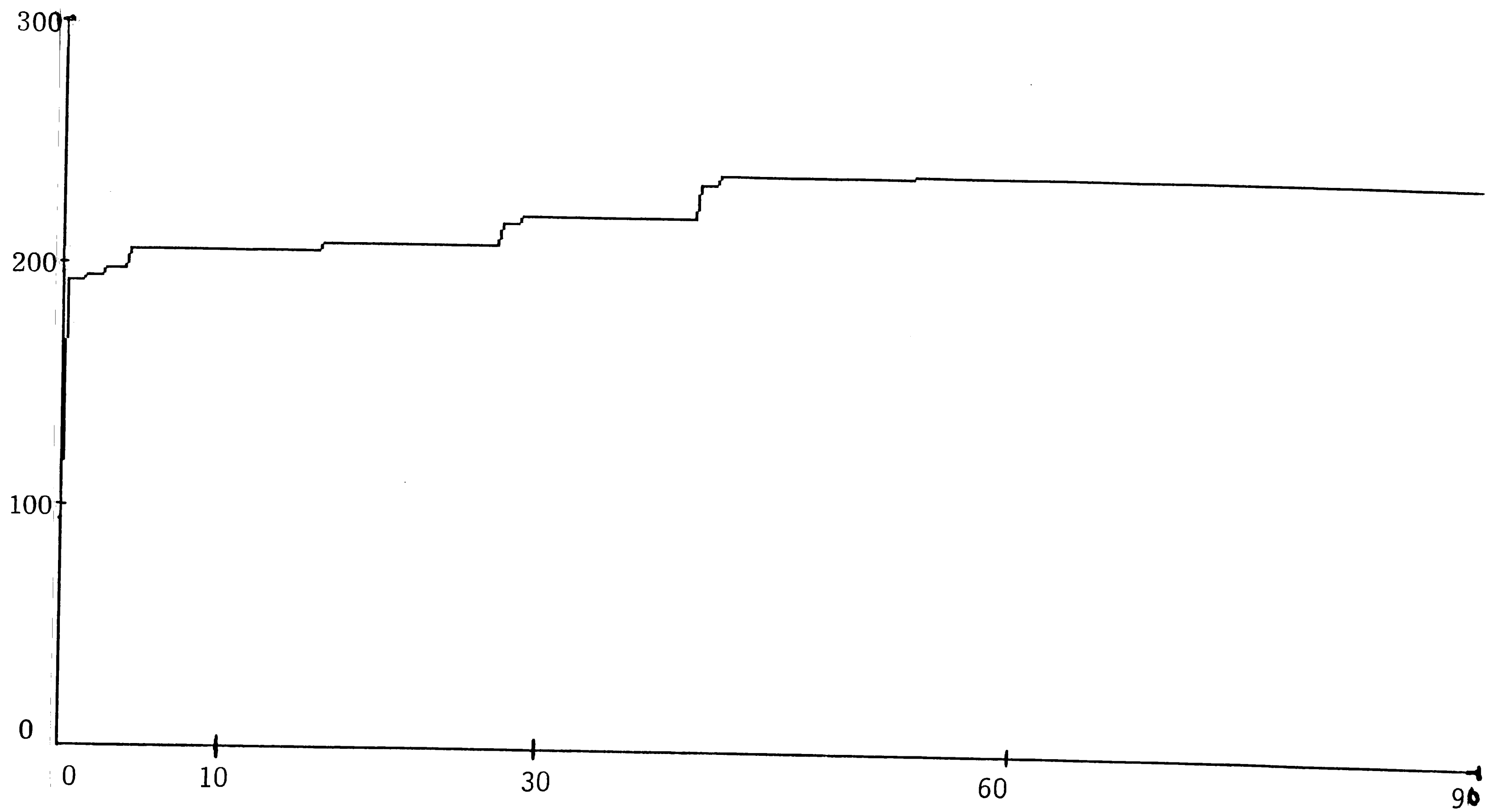

Fig. 5. Initial datum $10\sin(\pi x)$. Evolution of $D(t) = \sup_{s\in[0,t]} \|u_{xx}(s,\cdot)\|_{L^2(0,1)}$ for $0 \leqslant t \leqslant 90$.

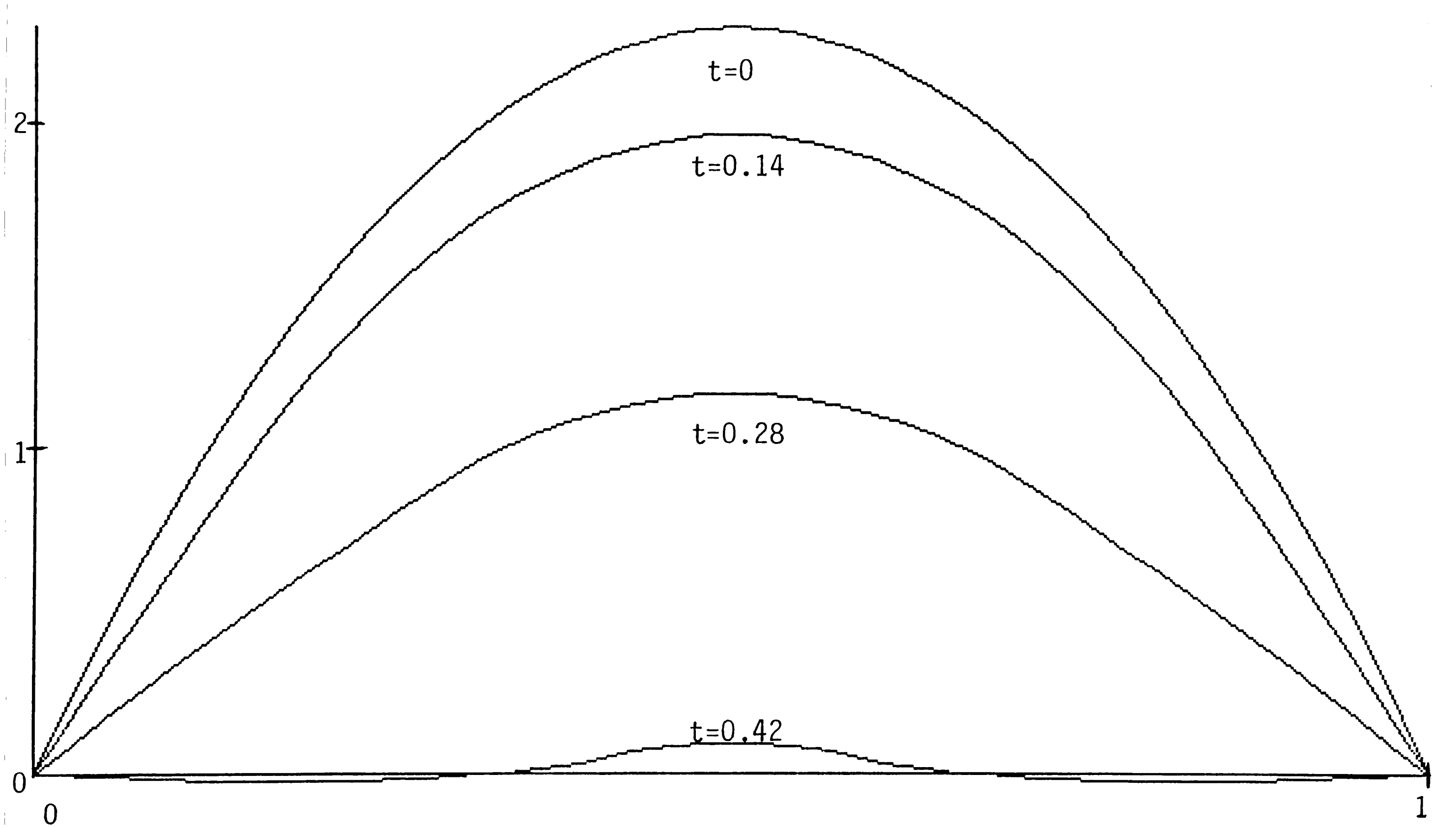

Fig. 6. Initial datum $10\,x(1-x)$. The shape of $u(t)$ for $t \in \{0, 0.14, 0.28, 0.42\}$.

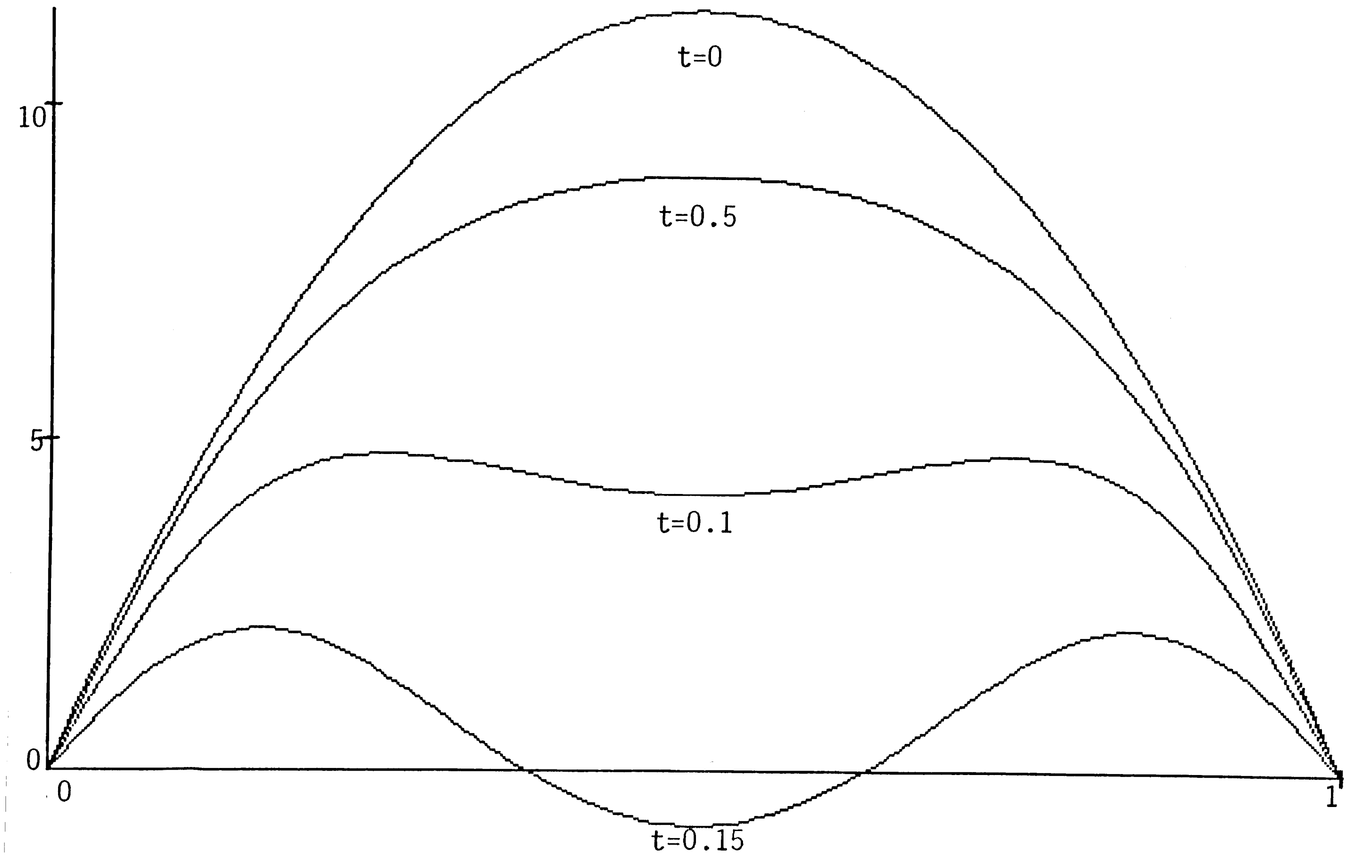

Fig. 7. Initial datum $50\,x(1-x)$. The shape of $u(t)$ for $t \in \{0, 0.05, 0.1, 0.15\}$.

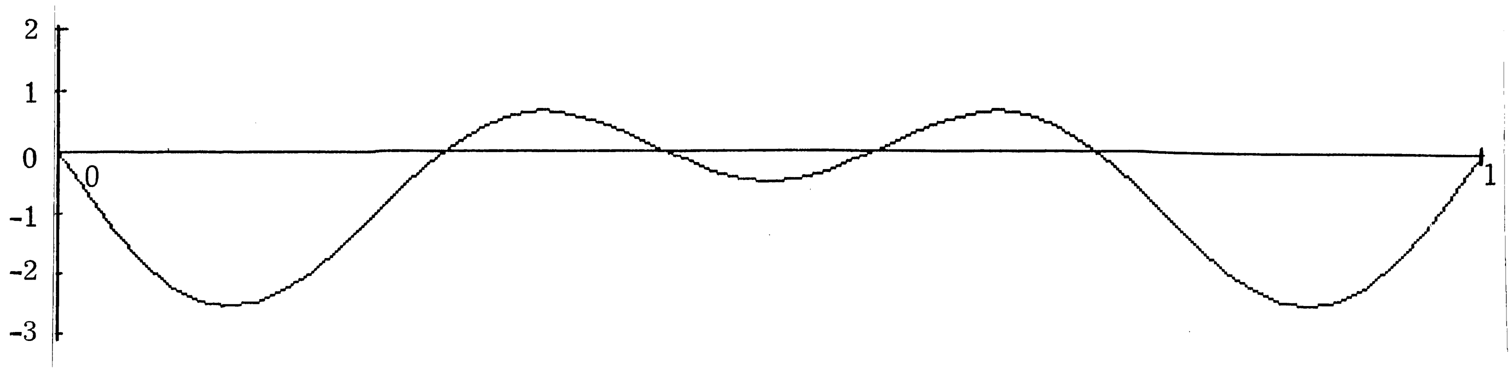

Fig. 8. Initial datum $10\sin(\pi x)$. The shape of $u(t)$ for $t = 52.90$.

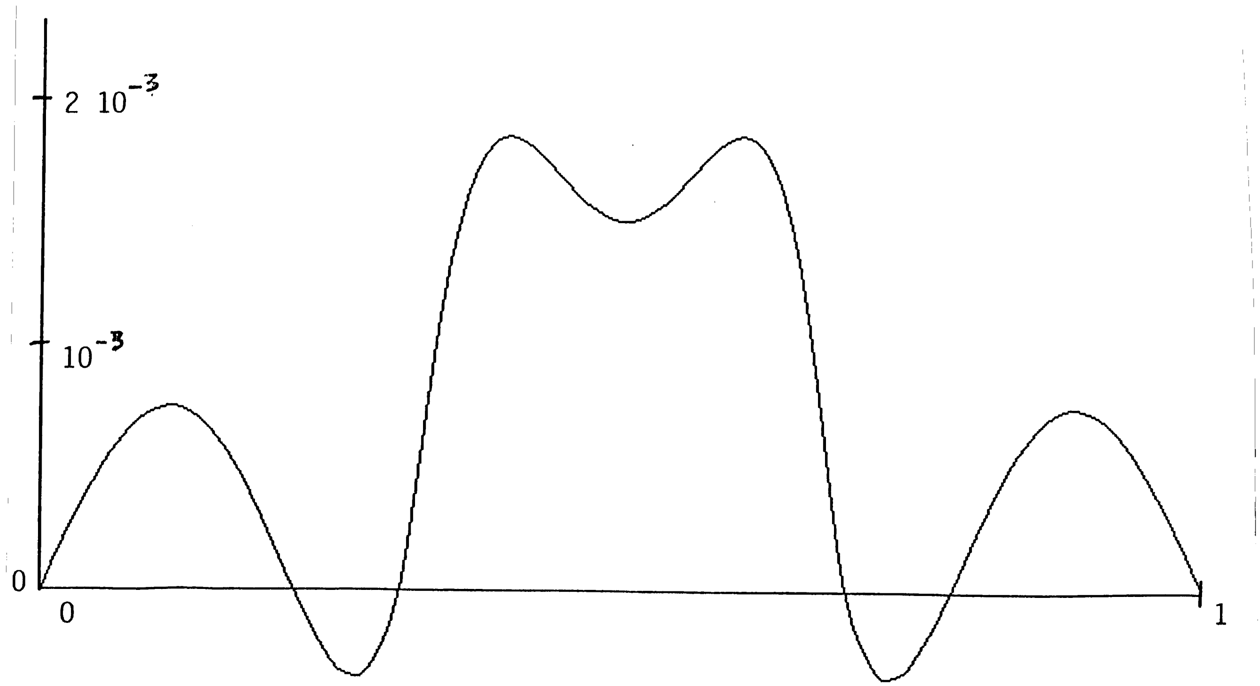

Fig. 9. See Section 5.3 below.

5.3. Further observations

In order to understand the cases where the location of the first crossing was "invisible" (cf. Table 2), we made further computations with smaller time and space steps. We chose $\delta t = 0.001$ and $\delta x = 0.005$. We observed the following phenomena.

When $u^0(x) = \sin(\pi x)$, the first crossing occurs near the boundary (crossing time $t_k = 0.486$).

When $u^0(x) = kx(1-x)$,

- if $k = 1$, the first crossing occurs near the boundary ($t_k = 0.499$);
- if $k = 20$, the first crossing occurs near the center ($t_k = 0.321$);
- The special configuration in the latter case led us to consider the case $k = 19$. Then first crossing occurs neither near the boundary nor near the center. Figure 9 represents the corresponding solution u at time $t_k = 0.330$.

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