

NONLINEAR EFFECTS IN THE WAVE EQUATION
WITH A CUBIC RESTORING FORCE

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WITH A CUBIC RESTORING FORCE

by

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0. INTRODUCTION

This paper is devoted to the investigation of global properties of solutions of the following semilinear hyperbolic problem

$$\left. \begin{aligned} u_{tt} - u_{xx} + u^3 &= 0 && \text{for } t \in \mathbb{R}, 0 < x < 1 \\ u(t, 0) = u(t, 1) &= 0 && \text{for } t \in \mathbb{R} . \end{aligned} \right\} \quad (0.1)$$

It is well known ([13], Theorem VI.1.2.3) that for any $g \in C^1(\mathbb{R})$ such that

$$\forall u \in \mathbb{R}, \quad g(u)u \geq 0 \quad (0.2)$$

and any (u_0, v_0) in $H_0^1(0, 1) \times L^2(0, 1)$, there exists a unique function $u : \mathbb{R} \times]0, 1[\rightarrow \mathbb{R}$ such that

$$u \in C(\mathbb{R}, H_0^1(0, 1)) \cap C^1(\mathbb{R}, L^2(0, 1)) \quad (0.3)$$

$$u_t \in C^1(\mathbb{R}, H^{-1}(0, 1)) \quad (0.4)$$

$$u_{tt} = u_{xx} - g(u) \text{ in } C(\mathbb{R}, H^{-1}(0, 1)) \quad (0.5)$$

$$u(0) = u_0 \quad \text{and} \quad u_t(0) = v_0 . \quad (0.6)$$

When $g(u) = u^3$ this solves the initial value problem associated to (0.1).

When g is odd, i. e. ,

$$\forall u \in \mathbb{R}, \quad g(-u) = -g(u) \quad (0.7)$$

it is convenient, following [6], to introduce the function $\tilde{u} \in C(\mathbb{R}^2)$ defined by the conditions

$$\left. \begin{aligned} \tilde{u}(t, x) &= -\tilde{u}(t, -x), && \forall (t, x) \in \mathbb{R}^2 \\ \tilde{u}(t, x+2) &= \tilde{u}(t, x), && \forall (t, x) \in \mathbb{R}^2 \\ \tilde{u}(t, x) &= u(t, x), && \forall (t, x) \in \mathbb{R} \times]0, 1[\end{aligned} \right\} \quad (0.8)$$

1. \mathbb{R}^n is a vector space over \mathbb{R} with the usual addition and scalar multiplication.

2. \mathbb{R}^n is a vector space over \mathbb{C} with the usual addition and scalar multiplication.

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Then \tilde{u} satisfies the integration equation

$$\tilde{u}(t, x) = p(t+x) - p(t-x) - \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} g(\tilde{u}(s, y)) dy ds \quad (0.9)$$

where p is given by

$$p(s) = \frac{1}{2} \left\{ \tilde{u}_0(s) + \int_0^s \tilde{v}_0(\sigma) d\sigma \right\} + C, \quad (0.10)$$

\tilde{u}_0, \tilde{v}_0 being respectively the odd and 2-periodic extensions of u_0 and v_0 respectively, and C an arbitrary constant chosen at our convenience. Of course $p: \mathbb{R} \rightarrow \mathbb{R}$ is in $H_{loc}^1(\mathbb{R})$ and $p(s+2) = p(s)$. It will sometimes be convenient to choose C so that $\int_0^2 p(s) ds = 0$.

When $g(u) = cu^3$ with $c \geq 0$, (0.9) becomes

$$\tilde{u}(t, x) = p(t+x) - p(t-x) - \frac{c}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} \tilde{u}^3(s, y) dy ds. \quad (0.11)$$

In the paper we shall use formula (0.11) quite extensively, since it has very interesting properties, among which a kind of "smoothing" effect.

Apart from the initial value problem which is easily treated by standard methods, various questions connected to (0.1) have been studied in the literature. For example, in the pioneering work [18], P. RABINOWITZ established the existence of solutions to (0.1) which fulfill the additional conditions

$$\left. \begin{aligned} u(t+2, x) &= u(t, x), \quad \forall (t, x) \in \mathbb{R} \times]0, 1[\\ u &\text{ is not identically 0 on } \mathbb{R} \times]0, 1[\end{aligned} \right\} \quad (0.12)$$

Part of the proof has been simplified in [3] by BREZIS-CORON-NIRENBERG. However, it remains delicate to give a complete construction of such solutions in the regularity class (0.3). Also, it seems completely unknown what these solutions look like.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$(1.1.1) \quad \sin^2 \theta + \cos^2 \theta = 1 \quad \Rightarrow \quad \sin^2 \theta = 1 - \cos^2 \theta = (\sin \theta)^2$$

Therefore, we have

$$\sin^2 \theta = (\sin \theta)^2$$

$$(1.1.2) \quad \sin^2 \theta = (\sin \theta)^2 \quad \Rightarrow \quad \sin \theta = \pm \sqrt{\sin^2 \theta}$$

Since $\sin \theta$ is a real number, it must be either positive or negative. Therefore, we can write $\sin \theta = \pm \sqrt{\sin^2 \theta}$. This is the same as saying $\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$. This is the same as saying $\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$.

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$$(1.1.3) \quad \sin^2 \theta + \cos^2 \theta = 1 \quad \Rightarrow \quad \sin^2 \theta = 1 - \cos^2 \theta = (\sin \theta)^2$$

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$$\sin^2 \theta + \cos^2 \theta = 1 \quad \Rightarrow \quad \sin^2 \theta = 1 - \cos^2 \theta = (\sin \theta)^2$$

$$(1.1.4) \quad \sin^2 \theta + \cos^2 \theta = 1 \quad \Rightarrow \quad \sin^2 \theta = 1 - \cos^2 \theta = (\sin \theta)^2$$

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In [6], investigation of global behavior for arbitrary solutions of (0.1) was initiated by looking at oscillatory properties of the function $t \mapsto u(t, x_0)$ with $0 < x_0 < 1$. There it was proved that the solutions must oscillate "at least as fast as" the solutions of the linear wave equation. However, the study of asymptotics as $t \rightarrow +\infty$ is still essentially open from the theoretical point of view.

The object of this paper is two-fold. First, we present the most typical results of numerical experiments which were attempted to test some conjectures on global behavior of solutions: for three different kinds of initial data, the solutions to (0.1) have been computed on a large time interval by means of a finite-difference scheme. More precisely, the computation technique is the same as in [25] and will be discussed in the last section.

Secondly, we prove a few partial theoretical results either related to behavior as $t \rightarrow +\infty$, or to a detailed investigation of the oscillatory properties of solutions. All the theoretical results will be illustrated by a numerical study corresponding to at least one of the typical initial data mentioned above.

For simplicity, in all the numerical computations we assume $v_0 = 0$. This is usually admitted as physically meaningful and, as a matter of fact, is not so disturbing from the point of view of generality, as long as the problems remain open even this case. It is our hope that the results of this paper will be helpful in the future, either to distinguish more clearly the various phenomena which may happen and understand the underlying reasons for this complexity, or to find new ideas solving some of the mathematical questions involved.

Apart from pure curiosity, our work has been partly motivated and strongly encouraged by the existence of at least one well-known previous success in this direction: the discovery of quasi-periodic solutions for the KDV equation by P. D. LAX [16] about one decade ago, following the numerical experiments of M. D. KRUSKAL and N. J. ZABUSKY [28] (cf. also [9] and [17]).

Acknowledgments. A significant part of the research for this paper was supported by the Institute for Mathematics and its Applications at the University of Minnesota

1. The first part of the document discusses the importance of maintaining accurate records of all transactions.

2. It is essential to ensure that all entries are supported by appropriate documentation.

3. The second part of the document outlines the procedures for handling discrepancies and errors.

4. It is crucial to identify the cause of any errors and take corrective action promptly.

5. The third part of the document provides a detailed overview of the accounting cycle.

6. Each step of the cycle is explained in detail, including the necessary journal entries.

7. The fourth part of the document discusses the preparation of financial statements.

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13. The seventh part of the document provides a summary of the key points discussed.

14. It is hoped that this document will be helpful in understanding the accounting process.

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16. Accountants are responsible for providing accurate financial information to management and stakeholders.

17. The ninth part of the document covers the importance of ethics in accounting.

18. Accountants must adhere to a strict code of ethics to ensure the integrity of the profession.

19. The tenth part of the document discusses the future of accounting.

20. With the advancement of technology, accountants will need to stay updated on the latest trends.

21. The eleventh part of the document provides a list of resources for further study.

22. These resources include textbooks, articles, and online courses.

23. The twelfth part of the document discusses the importance of continuous learning.

24. Accountants should strive to improve their skills and knowledge throughout their careers.

25. The thirteenth part of the document covers the role of the auditor.

26. Auditors are responsible for providing an independent opinion on the financial statements.

27. The fourteenth part of the document discusses the importance of transparency.

28. Transparency is essential for building trust and confidence in the financial system.

as part of its year of concentration on continuum physics and partial differential equations (1984-85). The fourth author was supported for the entire year at the I.M.A., and the second as a visitor for the month of June 1985. It was then that serious progress on some theoretical aspects of this research led to the idea for this paper. The fourth author was also partially supported by NSF Grant DMS 8201639. Finally, the authors are indebted to P. JOLY, M. LEGENDRE and P.J. PASCUAL for their generous assistance concerning the computational aspects of this research.

1. ON THE POSSIBILITY OF DECAYING SOLUTIONS:

One important property of (0.1) is the conservation of the energy integral

$$\forall t \in \mathbb{R}, \quad \int_0^1 \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \frac{1}{4} u^4 \right\} (t, x) dx = \int_0^1 \left\{ \frac{1}{2} v_0^2(x) + \frac{1}{2} u_0'^2(x) + \frac{1}{4} u_0^4(x) \right\} dx . \quad (1.1)$$

This property implies in particular that no solution of (0.1) can tend strongly to 0 in $H_0^1(0, 1)$ as $t \rightarrow +\infty$ (respectively $-\infty$) except the trivial solution $u \equiv 0$. Indeed, multiplication of (0.1) by $u(t, x)$ yields the following identity, valid for all t ,

$$\begin{aligned} \int_t^{t+1} \int_0^1 u_s^2 dx ds &= \int_t^{t+1} \int_0^1 [u_x^2 + u^4] dx ds + \int_0^1 u(t+1, x) u_t(t+1, x) dx \\ &\quad - \int_0^1 u(t, x) u_t(t, x) dx . \end{aligned}$$

Therefore if, for instance, $u(t, \cdot) \rightarrow 0$ in $H_0^1(0, 1)$ as $t \rightarrow +\infty$, taking account of (1.1) we find first $\int_t^{t+1} \int_0^1 u_s^2 dx ds \rightarrow 0$, and then by integrating (1.1) in t over $[t, t+1]$ we conclude that $u_0 = v_0 = 0$.

On the other hand, (1.1) does not a priori preclude the possibility that

$$\sup_{x \in [0, 1]} |u(t, x)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty . \quad (1.2)$$

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data. The text also mentions that regular audits are necessary to identify any discrepancies or errors in the accounting process.

In addition, the document highlights the need for a clear and consistent chart of accounts. This helps in categorizing transactions correctly and facilitates the preparation of financial statements. It is also noted that the accounting system should be updated regularly to reflect changes in business operations and tax regulations.

The second part of the document focuses on the practical aspects of accounting, such as the use of double-entry bookkeeping. This method ensures that the accounting equation remains balanced at all times. It also discusses the importance of timely recording of transactions to avoid any lag in the financial data. The text provides examples of how to record various types of transactions, including sales, purchases, and adjustments.

Furthermore, the document addresses the role of the accounting department in providing valuable insights into the company's financial performance. By analyzing the data, accountants can identify trends, spot potential risks, and recommend strategies to improve profitability. This proactive approach is essential for long-term business success.

Finally, the document concludes by stressing the importance of ethical behavior in accounting. Accountants must adhere to professional standards and maintain the highest level of integrity in all their dealings. This not only protects the interests of the company but also builds trust with stakeholders.

Physically difficult to believe, the weak convergence (1.2) to 0 could be eliminated if we knew a priori that $u(t, \cdot)$ remains in a compact subset of $H_0^1(0, 1)$ for all $t \geq 0$. Rather paradoxically, this is precisely unknown even when $u_0 \in \mathcal{D}(0, 1[)$ and $v_0 = 0$, the basic reason being the absence of known higher order conserved integrals for (0.1). A first simple observation is that if $u(t, \cdot)$ happens to tend to 0 at infinity, the convergence cannot be very fast. More precisely, we have the following result.

Proposition 1.1. Let u be any solution of (0.1). Then if $u \not\equiv 0$ we have

$$\int_0^{+\infty} \left\{ \int_0^1 |u(t, x)|^6 dx \right\}^{1/2} dt = +\infty \quad (1.3)$$

Proof. Let $\mathcal{H} = H_0^1(0, 1) \times L^2(0, 1)$ and let us denote by $T(t) : \mathcal{H} \rightarrow \mathcal{H}$ the isometry group generated by the linear wave equation in \mathcal{H} . If we introduce

$$U(t) := (u(t), u_t(t)) \in C(\mathbb{R}, \mathcal{H})$$

and

$$F(t) := (0, -g(u(t))) \quad ,$$

then the variation of parameters formula applied to the system satisfied by (u, u_t) yields

$$U(t) = T(t) \left(U(0) + \int_0^t T(-s) F(s) ds \right) . \quad (1.4)$$

In particular, if we assume, by contradiction, $u \equiv 0$ and

$$\int_0^{+\infty} \left\{ \int_0^1 |u(t, x)|^6 dx \right\}^{1/2} < +\infty \quad , \quad (1.5)$$

then $F(t) \in L^1(0, +\infty, \mathcal{H})$ and (1.4) implies

$$U(t) - T(t)U_\infty \rightarrow 0 \text{ in } \mathcal{H} \text{ as } t \rightarrow +\infty \quad (1.6)$$

with

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$\frac{1}{2} \leq \frac{1}{2} + \frac{1}{2} \leq 1$

$$U_{\infty} := U(0) + \int_0^{+\infty} T(-s)F(s) ds .$$

In particular, $u(t, x)$ is asymptotic in the strong topology of $H_0^1(\Omega)$, as $t \rightarrow +\infty$, to a 2-periodic continuous function $u_{\infty} : \mathbb{R} \rightarrow H_0^1(0, 1)$. Then (1.5) immediately implies $u_{\infty} = 0$, and therefore $u(t, \cdot) \rightarrow 0$ strongly in $H_0^1(0, 1)$ as $t \rightarrow +\infty$, a contradiction with the preliminary remark which concludes the proof of Proposition 1.1. ◻

As a consequence of Proposition 1.1 we deduce

Corollary 1.2. If u is a nontrivial solution of (0.1), then for all $\varepsilon > 0$,

$$\limsup_{t \rightarrow +\infty} \left\{ t^{\frac{1}{3} + \varepsilon} \|u(t, \cdot)\|_6 \right\} = +\infty . \quad (1.7)$$

Using numerical methods, the solution u of (0.1) and (0.6) has been computed when $v_0 = 0$ and $u_0 = k\phi$ with $k \in \{1, 5, 10, 20, 30, 50\}$ and $\phi(x)$ is one of the functions

$$\sin \pi x , \quad \sin^2 \pi x , \quad x(1-x) .$$

In figures 1 and 2*, we show the evolution of $u(t, 1/2)$ in the first case with $k = 10$, $0 \leq t \leq 3$ and also $71.65 \leq t \leq 74.65$. All the other numerical experiments show the same behavior: there is no tendency for $|u(t, 1/2)|$ (a fortiori for $\max_{[0, 1]} |u(t, x)|$)

to decrease in a significant manner as t increases (cf. also the comments in Section 2, especially the comparison with $u_{tt} + \pi^2 u + u^3 = 0$). Another natural question to be investigated mathematically is whether or not the conditions $u_0 \in H^2(0, 1)$ and $v_0 \in H_0^1(0, 1)$ imply that

$$\sup_{t \geq 0} \|u_{xx}(t, x)\|_2 < +\infty . \quad (1.8)$$

* Figures and tables all appear in Section 5.

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Our numerical computations seem to indicate that (1.8) is satisfied when $v_0 = 0$ and $u_0(x) = 10 \sin \pi x$. Indeed, according to figure 5, the function

$$D(t) = \sup_{0 \leq s \leq t} \|u_{xx}(s, x)\|_2$$

seems to be constant for $45 \leq t \leq 90$. Therefore, at least in this case, the curve $t \rightarrow u(t, \cdot)$ would remain in a compact subset of $H_0^1(0, 1)$, a fact consistent with the absence of decay of $\|u(t, \cdot)\|_\infty$ as $t \rightarrow +\infty$.

2. THE ALMOST PERIODICITY CONJECTURE

Another natural conjecture concerning the solutions of (0.1) (which can be extended for $t \leq 0$ since the system is reversible) is that most solutions (and maybe all of them) are almost periodic with respect to t , either weakly or even strongly in $H_0^1(0, 1)$. In either case, one should then observe a recurrent character of the function $t \mapsto u(t, \cdot)$ in the space $C([0, 1])$. In this section, we try to evaluate the validity of this conjecture on the basis of heuristic arguments as well as numerical results. A mathematical study of this question would be interesting, but most probably extremely difficult even for this simple one-dimensional model.

2.1 Previous results of this type

In 1965, L. AMERIO established that all solutions of the linear wave equation with homogeneous boundary conditions in a bounded domain of \mathbb{R}^n , $n \geq 1$, are almost periodic in the energy space. The final argument (following partial results by previous authors) ultimately did not require any regularity of the domain and related the almost-periodicity property only to three circumstances: conservation of the energy, linearity of the equation, and precompactness of trajectories [1].

Later, in 1975, P.D. LAX established in [16] the quasi-periodic character for a large class of solutions of the KDV equation on a one-dimensional torus. The proof, unfortunately, relied heavily on the "completely integrable" character of the system and a very precise determination of invariant tori in the phase space.

In 1980, H. CABANNES and A. HARAUX [4] found a large family of almost periodic solutions for the equation of a string with fixed ends vibrating against a

straight, fixed obstacle. The solutions are generally not periodic (cf. Theorem 1.2, b) of [12]). More complicated almost periodic motions have been discovered later by H. CABANNES [5]. Here again, the success relies on the possibility of computing rather explicitly the solutions (cf. also [22]).

The physical content of equation (0.1) suggests a behavior rather similar to this last example. On the other hand, we are far from being able to compute any solution, or even to reduce the problem in some specific circumstances. These remarks suggest that an entirely new method needs to be developed.

2.2 The importance of diffusion

If the operator $u \mapsto -u_{xx}$ is replaced by its contribution on the first eigenspace in the sense of $H_0^1(0,1)$, we obtain the equation

$$\left. \begin{aligned} u_{tt} + \pi^2 u + u^3 &= 0, & t \in \mathbb{R}, x \in]0,1[\\ u(t,0) = u(t,1) &= 0 & t \in \mathbb{R} \end{aligned} \right\} \quad (2.2.1)$$

which can be solved uniquely in the function space $C^1(\mathbb{R}, H_0^1(0,1))$ for all initial data $(u_0, v_0) \in [H_0^1(0,1)]^2$.

Classical results on O.D.E. (cf., e.g., [10]) show that for all $x_0 \in]0,1[$ the function $u(t, x_0)$ is periodic in t .

Since the period is a decreasing continuous function of the "local" energy $\frac{1}{2} v_0^2(x) + \frac{1}{2} u_0^2(x) + \frac{1}{2} u_0^4(x)$, if, for instance, $v_0 = 0$ and $u_0 \neq 0$, the solution cannot be almost periodic: $\mathbb{R} \rightarrow H_0^1(\Omega)$. Numerically, we investigated the behavior of the function $t \mapsto \int_0^1 u(t, x) dx$ when

$$v_0 = 0 \quad \text{and} \quad u_0(x) = 10 \sin \pi x$$

for (0.1) (figure 3) and (2.2.1) (figure 4).

The observation of figure 4 seems to indicate that this integral tends to 0, or at least is far from being recurrent as $t \mapsto +\infty$. In the case of (2.2.1) it is not absurd to imagine that all solutions tend weakly to 0 in $L^2(0,1)$ as $t \rightarrow +\infty$ (but

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data. The text also mentions that regular audits are necessary to identify any discrepancies or errors in the accounting process.

In the second section, the author details the various methods used for data collection and analysis. It describes how primary data is gathered through direct observation and interviews, while secondary data is obtained from existing sources. The text highlights the importance of using statistical tools to analyze the collected data and draw meaningful conclusions.

The third part of the document focuses on the implementation of the research findings. It outlines the steps involved in developing a strategic plan based on the insights gained from the data analysis. The author stresses the need for clear communication and collaboration among all stakeholders to ensure the successful execution of the plan. Additionally, it discusses the importance of monitoring and evaluating the progress of the implementation to make necessary adjustments.

Finally, the document concludes with a summary of the key findings and recommendations. It reiterates the significance of data-driven decision-making and the role of effective communication in achieving organizational goals. The author encourages ongoing learning and improvement to stay competitive in a rapidly changing market.

in general the H_0^1 norm of $u(t, \cdot)$ will blow up at infinity). On the other hand, figure 3 suggests an almost periodic behavior.

Unfortunately, it has been impossible to follow the behavior pictured in figure 4 for very large values of $|t|$, since after a while, numerical effects obviously related to the highly oscillatory character of the solution are no longer negligible and the picture degenerates in a very irregular way.

In any case, the comparison between figures 3 and 4 points out in a very intuitive way the crucial role played by diffusion in the global behavior of solutions. Finally, the comparison with finite-dimensional systems [11] suggests that the problem may be quite hard.

2.3 On a result of P. RABINOWITZ

There is at least some mathematical evidence that diffusion plays an important part in global behavior of solutions to (0.1), again by comparing with the "simpler" problem (2.2.1). As pointed out in §2.2, the only *time-periodic* solution of (2.2.1) is the trivial solution $u \equiv 0$ on $\mathbb{R} \times]0, 1[$. On the other hand, P. RABINOWITZ [18, 19] established the existence of at least one solution of (0.1), (0.3) and (0.12). It turns out that this is a genuine *nonlinear property* since, unlike the linear case, some nontrivial solutions of (0.1) in the class (0.3) do not satisfy (0.12). More precisely, we can state the following result, which is related to an unpublished observation of H. BREZIS, communicated to the authors by P. RABINOWITZ [21].

Proposition 2.3.1. If u is a solution of (0.1) in the regularity class (0.3) - (0.4) such that

$$u(t+2, \cdot) = u(t, \cdot) \quad \text{for all } t \in \mathbb{R} \quad , \quad (2.3.1)$$

then either $u \equiv 0$ or

$$\int_0^2 \int_0^1 |u(t, x)|^2 dx dt \geq 2 \quad . \quad (2.3.2)$$

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that this is crucial for ensuring transparency and accountability in the organization's operations.

2. The second part of the document outlines the various methods and tools used to collect and analyze data. It highlights the need for consistent and reliable data collection processes to support effective decision-making.

3. The third part of the document focuses on the role of technology in data management and analysis. It discusses how modern software solutions can streamline data collection, storage, and reporting, thereby improving efficiency and accuracy.

4. The fourth part of the document addresses the challenges associated with data management, such as data quality, security, and integration. It provides strategies to overcome these challenges and ensure that the data is reliable and secure.

5. The fifth part of the document discusses the importance of data governance and the role of various stakeholders in ensuring that data is used ethically and in compliance with relevant regulations.

6. The sixth part of the document provides a detailed overview of the data collection and analysis process, from identifying the data sources to interpreting the results and drawing conclusions.

7. The seventh part of the document discusses the importance of data visualization in making complex data more understandable and actionable. It highlights the various tools and techniques used for data visualization.

8. The eighth part of the document provides a summary of the key findings and recommendations from the study. It emphasizes the need for continuous improvement and regular updates to the data management processes.

9. The ninth part of the document discusses the future of data management and the potential impact of emerging technologies like artificial intelligence and machine learning.

10. The tenth part of the document provides a final conclusion and a call to action for the organization to implement the recommended changes and improve its data management practices.

11. The eleventh part of the document provides a list of references and sources used in the study, ensuring that the information is credible and verifiable.

12. The twelfth part of the document provides a list of appendices and additional information that supports the main findings of the study.

Proof. We apply formula (0.11) and we set

$$Q =]0, 2[\times]0, 1[$$

$$u_1(t, x) = p(t+x) - p(t-x)$$

$$u_2(t, x) = -\frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} \tilde{u}^3(s, y) dy ds .$$

As a consequence of (0.8) we obtain easily the following estimate

$$\|u_2\|_{L^\infty(Q)} \leq \frac{1}{2} \sup_{t \in [0, 2]} \int_0^t \int_0^1 |u(s, y)|^3 dy ds = \frac{1}{2} \|u^3\|_{L^1(Q)} . \quad (2.3.3)$$

On the other hand, let v be a solution of (0.3) - (0.5) and $w \in L^2(0, 2; H_0^1(\Omega)) \cap H^1(0, 2; L^2(\Omega))$; we have the identity

$$\frac{d}{dt} \left(\int_0^1 v_t w dx \right) = \int_0^1 (v_{tt} w + v_t w_t) dx = \int_0^1 \{v_t w_t - v_x w_x - g(v)w\} dx .$$

By integrating on $(0, 2)$, we find that if v and w are such that $v_t(0, x) \equiv v_t(2, x)$ and $w(0, x) \equiv w(2, x)$, then

$$\int_0^2 \int_0^1 \{v_t w_t - v_x w_x - g(v)w\} dx dt = 0 . \quad (2.3.4)$$

It is clear that u_1 satisfies (0.3) - (0.5) with $g = 0$ and by choosing $u_1 = v$ and $u = w$ in (2.3.4) we obtain

$$\iint_Q \{(u_1)_t u_t - (u_1)_x u_x\} dx dt = 0 . \quad (2.3.5)$$

On the other hand, by choosing $g(u) = u^3$, $u = v$ and $u_1 = w$ in (2.3.4) we find

$$\iint_Q \{u_t(u_1)_t - u_x(u_1)_x + u^3 u_1\} dxdt = 0 . \quad (2.3.6)$$

By subtracting (2.3.5) and (2.3.6) we get

$$\iint_Q u^3 u_1 dxdt = 0 . \quad (2.3.7)$$

Therefore

$$\iint_Q u^4 dxdt = \iint_Q u^3 u_2 dxdt ,$$

and by (2.3.3) we obtain

$$\|u\|_{L^4(Q)}^4 \leq \frac{1}{2} \|u^3\|_{L^1(Q)}^2 . \quad (2.3.8)$$

On the other hand, by Hölder's inequality,

$$\|u^3\|_{L^1(Q)} \leq \|u\|_{L^4(Q)}^3 [\text{meas}(Q)]^{1/4} ,$$

and since $\text{meas}(Q) = 2$, (2.3.8) yields $\|u\|_{L^4(Q)}^4 \leq \frac{1}{\sqrt{2}} \|u\|_{L^4(Q)}^6$; hence if $u \neq 0$, we conclude that

$$\|u\|_{L^4(Q)}^2 \geq \sqrt{2} . \quad (2.3.9)$$

This is clearly equivalent to (2.3.2). □

Remark 2.3.2. In particular, if $v_0 = 0$, $u_0 \neq 0$ and $\|u_0\|_{H_0^1(0,1)}$ is small enough, the solution of (0.3) - (0.6) with $g(u) = u^3$ is not 2-periodic in t .

Remark 2.3.3. On the other hand, in [27], M. WILHEM established the existence of nontrivial periodic solutions with arbitrarily small energies and periods of the form $2n$, $n \rightarrow +\infty$. A related result can also be found in [20].

2.4 Numerical experiments and the recurrence property

A common point among all the solutions that we computed numerically is the following: assuming that t remains in the region where the computation is presumably correct, the curve $u(t, x)$ is "very close" to $u_0(x)$ when t is taken in the union of small intervals regularly spaced on the half-line $t \geq 0$. This property suggests a recurrence phenomenon quite consistent with the almost-periodicity conjecture. In this paragraph we describe some observations which make the conjecture look quite reasonable.

Observation 2.4.1. Let us consider the special case $u_0(x) = 10 \sin \pi x$ (the coefficient 10 is chosen large so that we are far enough from linearity, and still small enough so that numerical effects due to the fast oscillations are not excessive). In the interval $0 \leq t \leq 100$, we look for all values t such that

$$\frac{\|u(t) - u_0\|_\infty}{\|u_0\|_\infty} =: \varepsilon(t) < 0.05.$$

The set ξ of such points is the union of 15 very short intervals quite regularly spaced on the interval $[0, 100]$. For example, the maximal distance of two successive intervals is about 13, while the mean-value of all such distances is of the order of 7.

For more precise information, see Table 1. The same calculation with 0.05 replaced by any $\varepsilon \geq 0.03$ leads to very similar conclusions. One observes similar results if the L^∞ norm is replaced by the H_0^1 norm.

Observation 2.4.2. Another way of following the evolution of u is to study the function $t \mapsto u(t, 1/2)$. Indeed, the solution u can be reconstructed from the values of $u(t, 1/2)$ by using the method of characteristics. As shown by the comparison of figures 1 and 2, the function $y(t) := u(t, 1/2)$ is such that

$\sup_{0 \leq t \leq 3} |y(t) - y(t+71.65)|$ is of the order of 2 while the oscillation of y is greater than 20. In addition, the mean-value of $|y(t) - y(t+71.65)|$ on $[0, 3]$ is close to 0.66.

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Remark 2.4.3. The above two observations suggest that after a certain amount of time, the solution recovers approximately its initial shape. This property is especially striking when following the solution on a large time interval, since in the meantime, one then observes drastic changes in the shape of $u(t, \cdot)$ as a function of x . For example, starting with a positive concave initial datum, we observe the appearance of successive pairs of zeroes which, from time to time, suddenly cancel each other to reconstruct a shape quite similar to u_0 (compare figure 8). This strange phenomenon was already observed a long time ago by M.D. KRUSKAL and N.J. ZABUSKY [28] for another kind of nonlinear wave equation (in fact, a KDV equation on the one-dimensional torus). The remarkable stability of the initial configuration under the action of a dynamical system which, otherwise, does not seem to obey any simple "conservation rule" with respect to the shape, seems to be the most convincing evidence of an almost periodic motion.

3. OSCILLATORY PROPERTIES

In [6], Theorem 2.1 it has been established that if g satisfies (0.7) and u is a solution of (0.3) - (0.5), then for any $x_0 \in]0, 1[$ the function $u(t, x_0)$ has at least one zero on any time interval of length ≥ 2 . At this level there is no major difference between $g = 0$ and $g \neq 0$, especially since the result is optimal with respect to the length of the interval if u_0, v_0 are arbitrary.

Another kind of oscillation result, of "global" type, can be proved without assumption (0.7): in [6], Theorem 3.1, it is shown in particular that if $u \geq 0$ on $J \times]0, 1[$ with J some interval, then either $u \equiv 0$ or $|J| \leq 1$. For $g = 0$ this result is optimal since then $\sin \pi t \sin \pi x$ is a nonnegative solution on $]0, 1[\times]0, 1[$.

In this section we consider the special case where $v_0 = 0$ and u_0 is such that

$$u_0 \text{ is increasing on } [0, 1/2] \tag{3.1}$$

$$u_0(1-x) = u_0(x), \quad \forall x \in [0, 1] \tag{3.2}$$

Under these assumptions, we compare the oscillation properties in the linear and nonlinear cases.

3.1 A property specific to the linear case

In this paragraph, we assume $g = 0$. Then the solution of (0.3) - (0.6) with $v_0 = 0$ is given by the formula

$$u(t, x) = \frac{1}{2} \{ \tilde{u}_0(t+x) - \tilde{u}_0(t-x) \}$$

with \tilde{u}_0 the odd and 2-periodic extension of u_0 on \mathbb{R} . The hypotheses (3.1) and (3.2) on u_0 imply

$$\tilde{u}_0 \text{ is increasing on } [-1/2, 1/2] \quad (3.1.1)$$

$$\tilde{u}_0(1-x) = \tilde{u}_0(x), \quad \forall x \in \mathbb{R} \quad . \quad (3.1.2)$$

In particular, for any $\ell \in \mathbb{R}$ the equation $\tilde{u}_0(x) = \ell$ has at most 2 solutions x_1 and $x_2 = 1 - x_1$ on $[-1/2, 3/2]$.

Since \tilde{u}_0 is continuous and increasing on $[-1/2, 1/2]$, it follows that

$$\forall x \in]0, 1/2[, \quad \forall t \in [0, 1/2[,$$

$$\tilde{u}_0(t+x) - \tilde{u}_0(t-x) > 0 \quad .$$

In other words, we have

$$\forall x \in]0, 1[, \quad \forall t \in [0, 1/2[, \quad u(t, x) > 0 \quad , \quad (3.1.3)$$

$$\forall x \in [0, 1/2] , \quad u(1/2, x) = 0 \quad . \quad (3.1.4)$$

More generally, for any $m \in \mathbb{Z}$ we have

$$\forall x \in]0, 1[, \quad \forall t \in]m - 1/2, m + 1/2[, \quad (-1)^m u(t, x) > 0 \quad (3.1.5)$$

and

$$\forall x \in \mathbb{R} , \quad \forall m \in \mathbb{Z} , \quad \tilde{u}(m + 1/2, x) = 0 \quad . \quad (3.1.6)$$

This means that the solution vanishes simultaneously at all points x for $t \in \mathbb{Z} + \frac{1}{2}$.

In Section 4 we shall see that the situation is quite different when $g(u) = u^3$.

3.2 Corresponding results in the nonlinear case

Since the condition $v_0 = 0$ implies $u(-t, x) = u(t, x)$ we obtain that u cannot remain nonnegative on $]0, 1/2[\times]0, 1[$. (This follows from the proof of Theorem 2.1 in [6] since g is not identically zero.) In addition, the symmetry property implies the following stronger result.

Chapter 1: The Language of Mathematics

1.1. Sets and Elements

A set is a collection of objects, called elements or members, that can be distinguished from one another. Sets are denoted by capital letters, and elements by lowercase letters.

Example: Let S be the set of natural numbers less than 10. Then $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example: Let T be the set of vowels in the English alphabet. Then $T = \{a, e, i, o, u\}$.

Two sets are equal if and only if they contain exactly the same elements. The order of elements in a set does not matter.

Example: $\{1, 2, 3\} = \{3, 2, 1\}$ and $\{1, 2, 3\} \neq \{1, 2, 4\}$.

1.2. Operations on Sets

The union of two sets A and B , denoted $A \cup B$, is the set of all elements that are in A or in B or in both.

The intersection of two sets A and B , denoted $A \cap B$, is the set of all elements that are in both A and B .

Example: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$ and $A \cap B = \{3, 4, 5\}$.

The complement of a set A , denoted A^c , is the set of all elements that are not in A .

Example: Let $A = \{1, 2, 3\}$ and let the universal set be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $A^c = \{4, 5, 6, 7, 8, 9, 10\}$.

The difference of two sets A and B , denoted $A - B$, is the set of all elements that are in A but not in B .

Example: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. Then $A - B = \{1, 2\}$.

1.3. Venn Diagrams

Venn diagrams are used to represent sets and their relationships. Each set is represented by a region in a diagram, and the relationships between sets are shown by the overlap of these regions.

Example: A Venn diagram with two overlapping circles, A and B . The intersection of A and B is shaded, representing $A \cap B$.

Proposition 3.2.1. Let $g \in C^1(\mathbb{R})$ satisfy (0.7) and assume that $g: \mathbb{R} \rightarrow \mathbb{R}$ is increasing. Then for any u_0 satisfying (3.1) - (3.2) we have

$$\forall x \in]0, 1[, \quad \text{Inf}\{t \geq 0, u(t, x) < 0\} < 1/2 \quad , \quad (3.2.1)$$

Proof. This is basically an application of the methods of [7], therefore we only give a brief sketch of the proof.

Property (3.2) implies that $\tilde{u}_0(1+x) = -\tilde{u}_0(x)$ and therefore for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ we have

$$\tilde{u}(t, x+1) = -\tilde{u}(t, x) \quad . \quad (3.2.2)$$

Let $x_0 \in]0, 1[$ be such that $u(t, x_0) \geq 0$ for all $t \in [0, 1/2]$. Then in fact

$$u(t, x_0) \geq 0 \quad \text{for all } t \in [-1/2, 1/2]. \quad (3.2.3)$$

We introduce

$$w(t, x) = \tilde{u}(t, x) + \tilde{u}(t, 2x_0 - x) \quad . \quad (3.2.4)$$

Then by the method of [7], we deduce from (3.2.3) that

$$w(t, x) \geq 0 \quad , \quad \forall (t, x) \text{ with } |t| + |x - x_0| \leq 1/2 \quad .$$

Since on the other hand $w(t, x+1) = -w(t, x)$ we deduce

$$w(0, x_0 - 1/2) = w(0, x_0 + 1/2) = 0 \quad . \quad (3.2.5)$$

From (3.2.5), by computing the value of $w(0, x_0 + 1/2)$ in terms of $w(t, x_0) \Big|_{[-1/2, 1/2]}$ and $\iint_{|t| + |x - x_0| \leq 1/2} g(w(s, y)) ds dy$ we conclude, since g is increasing, that $w \equiv 0$ in $\{|t| + |x - x_0| \leq 1/2\}$. In particular $w(0, x_0) = 2u_0(x_0) = 0$ and this contradicts (3.1).

3.3 Fast oscillations for large initial data

In the linear case, (3.1.3)-(3.1.6) imply in particular that, assuming $v_0 = 0$ and (3.1) - (3.2), the frequency of the oscillations is independent of u_0 . In the nonlinear case the situation is quite different. In particular, we have the following result which will be used in Section 4.

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Proposition 3.3.1. Let $g(u) = cu^3$ for some $c > 0$, assume that $v_0 = 0$, u_0 being as previously specified. Then we have

$$\begin{aligned} & \sup\{t > 0, u(s, x) \geq 0 \text{ for all } x \in [0, 1] \text{ and } s \in [0, t]\} \\ & =: \bar{t} < \frac{\pi}{\sqrt{2c} \int_0^1 u_0(x) \sin \pi x dx} . \end{aligned} \quad (3.3.1)$$

Proof. Let

$$z(t) := \int_0^1 u(t, x) \sin \pi x dx . \quad (3.3.2)$$

We have $z \in C^2(\mathbb{R})$ and for all $t \in [0, \bar{t}]$,

$$z'' = \int_0^1 u_{tt} \sin \pi x dx = -\pi^2 z - c \int_0^1 u^3 \sin \pi x dx \leq -cz^3$$

with $z'(0) = 0$. Therefore on $[0, \bar{t}]$ we have

$$z(t) \geq 0 \quad (3.3.3)$$

$$z''(t) \leq -cz^3(t) \quad (3.3.4)$$

$$z'(t) \leq 0 . \quad (3.3.5)$$

From (3.3.4) and (3.3.5) we deduce

$$\frac{d}{dt} \left(z'^2 + \frac{c}{2} z^4 \right) = 2z'(z'' + cz^3) \geq 0$$

and therefore

$$z'(t) < -\sqrt{\frac{c}{2} [z_0^4 - z^4(t)]} \quad \text{for all } t \in [0, \bar{t}] . \quad (3.3.6)$$

By letting

$$w(t) = z(t)/z_0 , \quad \forall t \in [0, \bar{t}] ,$$

we find

$$\frac{w'}{\sqrt{1-w^4}} \leq -\sqrt{c/2} z_0 \quad \text{for all } t \in [0, \bar{t}] . \quad (3.3.7)$$

Therefore we have

$$\frac{d}{dt} \left[\int_0^{w(t)} \frac{ds}{\sqrt{1-s^4}} \right] \leq -\sqrt{c/2} z_0$$

and by integrating on $[0, \bar{t}]$ we obtain

$$\int_0^1 \frac{ds}{\sqrt{1-s^4}} > \sqrt{c/2} z_0 \cdot \bar{t} . \quad (3.3.8)$$

Since

$$\int_0^1 \frac{ds}{\sqrt{1-s^4}} < \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \pi/2 ,$$

(3.3.8) implies (3.3.1). ◻

3.4 Oscillatory properties and the numerical experiments

Of course, all the theoretical results obtained in Section 3 are confirmed by the numerical experiments (cf., e.g., Table 1).

The various computations also confirm the fact that when $v_0 = 0$ and $u_0 = k\phi$, $k > 0$, the following estimates of the "crossing time" \bar{t} :

$$\begin{aligned} \bar{t} &\sim 1/2 \quad \text{if } k \text{ is small} \\ \bar{t} &\sim \frac{C(\phi)}{k} \quad \text{if } k \text{ is large [compare (3.3.1)]} \end{aligned}$$

are rather satisfactory. These estimate of the crossing time will be intensively exploited in Section 4 below.

4. LOCATION OF THE "FIRST CROSSING"

It is interesting to compare the behaviors of the respective solutions of the linear equation and the nonlinear equation "without diffusion"(2.2.1), when $v_0 = 0$

and u_0 satisfies (3.1) - (3.2). We saw (formulas (3.1.3)-(3.1.6) that in the linear case, $u(t, x)$ vanishes for the first time at $t = 1/2$ for all $x \in]0, 1[$. On the other hand, it is easy to check that the solution of (2.2.1) vanishes for the first time at $x = 1/2$ (this is indeed related to the fact that the period of the function $t \mapsto u(t, x_0)$ is a decreasing function of $u(x_0)$ in this situation). It is a natural question to ask whether for the full equation (0.1), the solution will always cross first at the "center" $x = 1/2$. In this section, we shall see that the answer is negative, and in fact the situation is already complicated even when $u_0(x) = k \sin \pi x$, $k > 0$: more precisely the answer is different for small values and large values of k .

4.1 Preliminaries

Throughout Section 4, we keep the following assumptions and notation. We assume $v_0 = 0$ and

$$u_0 = k\phi, \quad k > 0 \tag{4.1.1}$$

$$\phi \in H_0^1(0, 1) \tag{4.1.2}$$

$$\phi \text{ is increasing on } [0, 1/2] \text{ and for all } x \in [0, 1], \phi(1-x) = \phi(x) \quad . \tag{4.1.3}$$

Let u be the solution of (0.3) - (0.6) with $g(u) = u^3$; then for all $t \in \mathbb{R}$, we have $u(t, 1-x) = u(t, x)$ for all $x \in [0, 1]$. If we set

$$u = kv \quad , \tag{4.1.4}$$

v is the solution of

$$\left. \begin{aligned} v_{tt} - v_{xx} + k^2 v &= 0 \\ v(t, 0) = v(t, 1) &= 0 \\ v(0) = \phi \quad \text{and} \quad v_t(0) &= 0 \quad . \end{aligned} \right\} \tag{4.1.5}$$

Throughout this section, any function $\phi \in L^2(0, 1)$ will be identified with its unique odd and 2-periodic extension over \mathbb{R} , each time when we write $\phi(s)$ for some $s \notin [0, 1]$. By using this convention, formula (0.9) yields

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

In the second section, the author outlines the process of reconciling bank statements with the company's internal records. This involves comparing the ending balance of the bank statement with the ending balance of the cash account in the general ledger. Any discrepancies should be investigated immediately to identify errors or unauthorized transactions.

The third section covers the preparation of financial statements. It details the steps for calculating net income, which is derived from the income statement. This calculation involves subtracting all expenses from total revenues. The resulting net income is then used to determine the company's profit for the period.

Finally, the document concludes with a summary of the key points discussed. It reiterates the importance of accuracy and consistency in financial reporting. The author encourages the reader to follow these guidelines to ensure the reliability of their financial data.

The following table shows the results of the reconciliation process for the month of January. The ending balance of the bank statement is \$1,200.00, and the ending balance of the cash account is \$1,150.00. The difference of \$50.00 is due to a bank charge for a service fee that was not recorded in the company's books.

Item	Amount
Ending Balance - Bank Statement	\$1,200.00
Ending Balance - Cash Account	\$1,150.00
Difference	\$50.00
Bank Charge - Service Fee	(\$50.00)
Adjusted Ending Balance - Cash Account	\$1,200.00

The net income for the month is calculated as follows: Total Revenue of \$5,000.00 minus Total Expenses of \$3,500.00, resulting in a net income of \$1,500.00.

The financial statements for the month of January are as follows:

Statement	Amount
Income Statement	\$1,500.00
Balance Sheet	\$1,200.00
Statement of Cash Flows	\$1,150.00

The author concludes by stating that these financial statements provide a clear and concise overview of the company's financial performance for the month. They are essential for management to make informed decisions about the company's future operations.

$$v(t, x) = \frac{1}{2} \{ \phi(x+t) + \phi(x-t) \} - \frac{k^2}{2} \int_0^t \int_{x-t+s}^{x+t-s} v^3(s, y) dy \quad (4.1.6)$$

$$v_x(t, x) = \frac{1}{2} \{ \phi'(x+t) + \phi'(x-t) \} - \frac{k^2}{2} \int_0^t [v^3(s, x+t-s) - v^3(s, x-t+s)] ds \quad (4.1.7)$$

$$v_x(t, 0) = \phi'(t) - k^2 \int_0^t v^3(s, t-s) ds . \quad (4.1.8)$$

Formula (4.1.6) is a pointwise equality between continuous functions and holds for $(t, x) \in \mathbb{R} \times \mathbb{R}$. Formula (4.1.7) holds for all $t \in \mathbb{R}$ as an equality between functions of $L_{loc}^2(\mathbb{R})$, while (4.1.8) holds for almost all $t \in \mathbb{R}$.

We shall also consider the free solution w given by

$$w(t, x) = \frac{1}{2} \{ \phi(x+t) + \phi(x-t) \} \quad (4.1.9)$$

and the function h given by

$$h(t, x) = \int_0^t [v^3(s, x+t-s) - v^3(s, x-t+s)] ds . \quad (4.1.10)$$

Clearly, h is continuous on $\mathbb{R} \times \mathbb{R}$ and by (4.1.7), (4.1.9) and (4.1.10) we have:

$$v_x(t, x) - w_x(t, x) = -\frac{k^2}{2} h(t, x) . \quad (4.1.11)$$

Therefore, even if ϕ is only in $H_0^1(0, 1)$, $v_x - w_x$ is continuous and v_x and w_x have the same singularities. We shall consider, for all $k > 0$, the "first crossing time" t_k defined as follows:

$$t_k = \sup \{ t \geq 0, u(\varepsilon, x) \geq 0 \text{ for } (s, x) \in [0, t] \times [0, 1] \} . \quad (4.1.12)$$

Obviously, t_k is also given by the formula

(1) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (2) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (3) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

(4) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (5) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (6) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

(7) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (8) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (9) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

(10) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (11) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (12) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

(13) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (14) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (15) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

(16) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (17) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (18) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$

(19) $\int_{-\infty}^{\infty} \delta(x) dx = 1$
 (20) $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$
 (21) $\int_{-\infty}^{\infty} \delta(x) dx = 1$

$$t_k = \sup \{t \geq 0, v(s, x) \geq 0 \text{ for } (s, x) \in [0, t] \times [0, 1]\} . \quad (4.1.13)$$

(Note that v depends on k , cf. problem (4.1.5).)

4.2 A partial result for large data

In this paragraph, we assume that ϕ , in addition to (4.1.2)-(4.1.3), satisfies the following hypotheses:

$$\phi \in C^1([0, \eta]) \text{ for some } \eta > 0 . \quad (4.2.1)$$

There exists $\delta > 0$, $M > 0$, $0 \leq \sigma \leq \nu$ such that $\alpha = 4 + 3\sigma - \nu > 0$ and

$$\forall x \in [0, \eta] , \quad \delta x^\nu \leq \phi'(x) \leq Mx^\sigma . \quad (4.2.2)$$

The first result of this paragraph is the following lemma.

Lemma 4.2.1 Let ϕ be as above. Then

$$\forall t \in [0, \eta/2] , \quad v(t, x) \in C^1([0, \eta/2]) ; \quad (4.2.3)$$

$$t_k > 0 . \quad (4.2.4)$$

Furthermore, there exists $\varepsilon(\delta, \sigma, M) > 0$ such that $|x| + |t| \leq \eta$ and $k^2(|x| + |t|)^\alpha + k^5(|x| + |t|)^{2\alpha + \nu + 2} \leq \varepsilon(\delta, \sigma, M)$ imply

$$v_x(t, x) \geq \frac{\delta}{4} (|x| + |t|)^\nu . \quad (4.2.5)$$

Proof. Since ϕ is C^1 on $[-\eta, \eta]$, (4.2.3) follows easily from (4.1.6).

Now let $\tau \in]0, \eta[$: if v were a classical solution of (4.1.5), a straightforward computation involving differentiation under the integral sign would show that

$$\psi(t) = \int_{-\tau+t}^{\tau-t} \left\{ \frac{1}{2} v_t^2 + \frac{1}{2} v_x^2 + \frac{k^2}{4} v^4 \right\} (t, x) dx$$

is nonincreasing on $[0, \tau]$. For the actual solution v , such a result is then easily deduced by a density argument. Hence for any $t \in [0, \tau]$, we find

$$\int_{-\tau+t}^{\tau-t} v_x^2(t, x) dx \leq \int_{-\tau}^{\tau} \left\{ \phi'^2 + \frac{k^2}{2} \phi^4 \right\} dx . \quad (4.2.6)$$

Let

$$m(s) = \int_{-s}^s \left\{ \phi'^2 + \frac{k^2}{2} \phi^4 \right\} dx . \quad (4.2.7)$$

By setting $s = \tau - t$ in(4.2.6) we find

$$\int_{-s}^s v_x^2(t, x) dx \leq m(t+s) \quad \text{for all } s \geq 0, t \geq 0 \text{ such that } t+s \leq \eta . \quad (4.2.8)$$

By symmetry, for $s > 0$ and $y \in \mathbb{R}$ such that $s+|y| < \eta$ we have

$$|v(s, y)| \leq \frac{1}{\sqrt{2}} |y|^{1/2} \left\{ \int_{-|y|}^{|y|} v_x^2(s, \sigma) d\sigma \right\}^{1/2} . \quad (4.2.9)$$

For $0 < s < t$ and $0 < x < \eta - t$, as a consequence of (4.2.8)-(4.2.9) we obtain

$$|v(s, x+t-s)|^2 + |v(s, x-t+s)|^2 \leq (x+t)m(x+t) . \quad (4.2.10)$$

Similarly, for $0 < s < t$ and $0 < x < \eta - t$ we find

$$|v(s, x+t-s) - v(s, x-t+s)| \leq [2(t-s)m(x+t)]^{1/2} \quad (4.2.11)$$

is an immediate consequence of (4.2.8).

From(4.2.10)-(4.2.11) we deduce in particular

$$|v^3(s, x+t-s) - v^3(s, x-t+s)| \leq \frac{3}{\sqrt{2}} (x+t)(t-s)^{1/2} [m(x+t)]^{3/2} . \quad (4.2.12)$$

Now (4.1.10) and (4.2.12) yield

$$h(t, x) \leq \sqrt{2} (x+t)^{5/2} [m(x+t)]^{3/2} \quad \text{for all } t, x \text{ such that } 0 < t < x+t < \eta . \quad (4.2.13)$$

$\frac{1}{2} \ln \frac{1}{2} = \frac{1}{2} \ln 2^{-1} = \frac{1}{2} (-1) \ln 2 = -\frac{1}{2} \ln 2$

$\frac{1}{2} \ln \frac{1}{4} = \frac{1}{2} \ln 2^{-2} = \frac{1}{2} (-2) \ln 2 = -\ln 2$

$\frac{1}{2} \ln \frac{1}{8} = \frac{1}{2} \ln 2^{-3} = \frac{1}{2} (-3) \ln 2 = -\frac{3}{2} \ln 2$

$\frac{1}{2} \ln \frac{1}{16} = \frac{1}{2} \ln 2^{-4} = \frac{1}{2} (-4) \ln 2 = -2 \ln 2$

$\frac{1}{2} \ln \frac{1}{32} = \frac{1}{2} \ln 2^{-5} = \frac{1}{2} (-5) \ln 2 = -\frac{5}{2} \ln 2$

$\frac{1}{2} \ln \frac{1}{64} = \frac{1}{2} \ln 2^{-6} = \frac{1}{2} (-6) \ln 2 = -3 \ln 2$

$\frac{1}{2} \ln \frac{1}{128} = \frac{1}{2} \ln 2^{-7} = \frac{1}{2} (-7) \ln 2 = -\frac{7}{2} \ln 2$

$\frac{1}{2} \ln \frac{1}{256} = \frac{1}{2} \ln 2^{-8} = \frac{1}{2} (-8) \ln 2 = -4 \ln 2$

$\frac{1}{2} \ln \frac{1}{512} = \frac{1}{2} \ln 2^{-9} = \frac{1}{2} (-9) \ln 2 = -\frac{9}{2} \ln 2$

$\frac{1}{2} \ln \frac{1}{1024} = \frac{1}{2} \ln 2^{-10} = \frac{1}{2} (-10) \ln 2 = -5 \ln 2$

On the other hand, as a consequence of (4.2.2)

$$2w_x(t, x) \geq \delta(x+t)^\nu \quad \text{for } 0 < t < x+t < \eta \quad . \quad (4.2.14)$$

Combining (4.1.11), (4.2.13) and (4.2.14) we obtain

$$2v_x(t, x) \geq \delta(x+t)^\nu - k^2(x+t)^{5/2} [m(x+t)]^{3/2} \quad \text{for } 0 < t < x+t < \eta \quad . \quad (4.2.15)$$

By using the right-hand side of (4.2.2), formula (4.2.7) shows on the other hand that

$$m(s) \leq C_1 \left[s^{2\sigma+1} + k^2 s^{4\sigma+5} \right]$$

for all $s \in [0, \eta]$. As a consequence, for some $c \geq 0$ we have

$$2v_x(t, x) \geq (x+t)^\nu \left[\delta - ck^2(x+t)^{3\sigma+4-\nu} - ck^5(x+t)^{6\sigma+10-\nu} \right] \\ \text{for } 0 < t < x+t < \eta \quad . \quad (4.2.16)$$

Since $6\sigma+10-\nu = 2\alpha+2+\nu$, (4.2.5) is an easy consequence of (4.2.16).

It follows from (4.2.5) that $v > 0$ on $[0, \rho] \times [0, \rho]$ for some $\rho > 0$. On the other hand, since $\phi > 0$ on $[\rho, 1/2]$ we also have $v(t, x) > 0$ for $x \in [\rho, 1/2]$ and t small enough. Hence (4.2.4). □

Remark 4.2.2. In order that $t_k > 0$, it is not sufficient to assume (4.1.3). Indeed (4.2.4) is false if $\phi'(x_n) = 0$ for a sequence of positive numbers x_n tending to 0. More precisely, if $\phi'(y) = 0$ for some $y \in]0, 1/2[$ then it follows immediately from (4.1.8) that $t_k \leq y$. Hence if $\phi'(x_n) = 0$ and $x_n \rightarrow 0$ we deduce $t_k = 0$.

We are now in a position to state the main result of this paragraph.

Theorem 4.2.3. Let ϕ be as in the statement of Lemma 4.2.1 and assume in addition that $2+3\sigma-\nu > 0$. Then for k large enough, we have $v_x(t_k, 0) > 0$. Therefore the first crossing occurs away from the boundary.

Proof. Since $t_k \leq C(\phi)/k$, we have

$$k^2 t_k^\alpha + k^5 t_k^{2\alpha+\nu+2} \leq C'(k^{2-\alpha} + k^{3-2\alpha-\nu})$$

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which tends to 0 as $k \rightarrow +\infty$ since by hypothesis we have $\alpha > 2$. Now (4.2.5) applied with $x = 0$ and $t = t_k$ gives the result. \square

4.3 Generalities for small data

In this paragraph, we keep ϕ fixed and assume $k \leq 1$. Then the energy conservation property combined with (4.1.6) implies that v is uniformly bounded on $\mathbb{R} \times \mathbb{R}$, independently of k . Therefore from (4.1.6) - (4.1.7) we deduce

$$\forall T > 0, v \rightarrow w \text{ uniformly on } [0, T] \times \mathbb{R} \text{ as } k \rightarrow 0, \quad (4.3.1)$$

$$\forall T > 0, v_x - w_x \rightarrow 0 \text{ uniformly on } [0, T] \times \mathbb{R} \text{ as } k \rightarrow 0. \quad (4.3.2)$$

We assume that ϕ satisfies (4.1.2) - (4.1.3), (4.2.1) - (4.2.2) and

$$\phi \in C^1([0, 1/2[) \quad (4.3.3)$$

$$\forall x \in]0, 1/2[, \phi'(x) > 0. \quad (4.3.4)$$

We introduce

$$\zeta = \bigcup_{m \in \mathbb{Z}} \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}, \left| x - \frac{1}{2} - m \right| = |t| \right\} \quad (4.3.5)$$

Under the hypotheses above, it is clear that w (hence v) is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \zeta$. We now have:

Proposition 4.3.1. Under the hypotheses above, it follows that

$$t_k \rightarrow 1/2 \text{ as } k \rightarrow 0. \quad (4.3.6)$$

Proof. As a consequence of Proposition 3.2.1 we know that $t_k < 1/2$. It is therefore sufficient to show that for any $\tau \in [0, 1/2[$, v is positive on $[0, \tau] \times]0, 1[$ for k small enough. To that end we fix $\tau \in [0, 1/2[$ and we set $a = \frac{1}{2}(\frac{1}{2} - \tau)$. We now observe that as a consequence of (4.3.4), ϕ satisfies also (4.2.2) with $\eta = \tau + a$ (by perhaps modifying δ, M, σ and ν). Applying Lemma 4.2.1, it follows in particular from (4.2.5) that there exists $k_0 > 0$ such that for all $k \in]0, k_0[$:

$$\forall (t, x) \in [0, \tau] \times]0, a[, v(t, x) > 0. \quad (4.3.7)$$

1. $\frac{1}{x^2} = x^{-2}$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

2. $\frac{1}{x^3} = x^{-3}$

$$\frac{d}{dx} x^{-3} = -3x^{-4}$$

3. $\frac{1}{x^4} = x^{-4}$

4. $\frac{1}{x^5} = x^{-5}$

$$\frac{d}{dx} x^{-5} = -5x^{-6}$$

5. $\frac{1}{x^6} = x^{-6}$

6. $\frac{1}{x^7} = x^{-7}$

7. $\frac{1}{x^8} = x^{-8}$

8. $\frac{1}{x^9} = x^{-9}$

9. $\frac{1}{x^{10}} = x^{-10}$

10. $\frac{1}{x^{11}} = x^{-11}$

$$\frac{d}{dx} x^{-11} = -11x^{-12}$$

11. $\frac{1}{x^{12}} = x^{-12}$

12. $\frac{1}{x^{13}} = x^{-13}$

13. $\frac{1}{x^{14}} = x^{-14}$

14. $\frac{1}{x^{15}} = x^{-15}$

15. $\frac{1}{x^{16}} = x^{-16}$

16. $\frac{1}{x^{17}} = x^{-17}$

17. $\frac{1}{x^{18}} = x^{-18}$

Also, as a consequence of (3.1.3) and (4.3.1) there exists $k_1 > 0$ such that for all $k \in]0, k_1[$:

$$\forall (t, x) \in [0, \tau] \times [a, 1/2] , \quad v(t, x) > 0 . \quad (4.3.8)$$

Putting together (4.3.7) and (4.3.8) we conclude that $v > 0$ on $[0, \tau] \times]0, 1[$ for all k such that $0 < k \leq \min\{k_0, k_1\}$. Since τ was arbitrary in $[0, 1/2[$, the proof is now complete. \square

4.4 Crossing at the center in a special case

In this paragraph, we consider ϕ given by

$$\forall x \in [0, 1], \quad \phi(x) = \min\{x, 1-x\} .$$

Obviously, ϕ satisfies (4.1.2) - (4.1.3) and (4.3.3) - (4.3.4). The free solution w is given, for all $t \in [0, 1/2]$, by the formulas

$$w(t, x) = x , \quad \forall x \in \left[0, \frac{1}{2} - t\right] \quad (4.4.1)$$

$$w(t, x) = \frac{1}{2} - t , \quad \forall x \in \left[\frac{1}{2} - t, 1/2\right] . \quad (4.4.2)$$

Our main result is the following.

THEOREM 4.4.1. For k small enough we have

$$\forall x \in]0, 1/2[, \quad v(t_k, x) > 0 \quad (4.4.3)$$

$$v_x(t_k, 0) > 0 \quad (4.4.4)$$

$$v(t_k, 1/2) = 0 . \quad (4.4.5)$$

Hence crossing occurs at $x = 1/2$.

The proof of Theorem 4.4.1 relies on

Lemma 4.4.2. Let $h^0(t, x) = \int_0^t [w^3(s, x+t-s) - w^3(s, x-t+s)] ds$ for $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Then $h^0 \in C(\mathbb{R} \times \mathbb{R}) \cap C^1(\mathbb{R} \times \mathbb{R} \setminus \zeta)$ with ζ as in (4.3.5) and

Handwritten text, likely bleed-through from the reverse side of the page. The text is extremely faint and illegible due to the quality of the scan. It appears to be organized into several paragraphs or sections, possibly containing technical or scientific information. Some faint words like "Figure" and "Table" might be discernible, but the rest is too light to read accurately.

$$\forall (t, x) \in]0, 1/2] \times [0, 1/2[, \quad h^0(t, x) > 0 \quad (4.4.6)$$

$$\forall t \in]0, 1/2] , \quad h_x^0(t, 1/2) < 0 . \quad (4.4.7)$$

Proof. Easy calculations show that

$$w(\sigma, z) \leq w(\sigma, y), \quad \forall \sigma \in]0, 1/2[, \quad \forall y \in [0, \frac{1}{2} + \sigma], \quad \forall z \in [-1, y] \quad (4.4.8)$$

$$w(\sigma, z) < w(\sigma, y), \quad \forall \sigma \in]0, 1/2[, \quad \forall y \in [0, \frac{1}{2} + \sigma], \quad \forall z \in]-1, y[\cap]-1, \frac{1}{2} - \sigma[. \quad (4.4.9)$$

Let $t \in [0, 1/2]$. If $x \in [0, \frac{1}{2} - t]$, then by (4.4.7) we have $w(s, x - t + s) < w(s, x + t - s)$ for any $s \in]0, t[$ and therefore $h^0(t, x) > 0$. On the other hand, if $x \in]\frac{1}{2} - t, 1/2[$, we distinguish two cases. For $s \in]\frac{1}{2}(x + t - \frac{1}{2}), t[$, by (4.4.8) $w(s, x - t + s) \leq w(s, x + t - s)$. For $s \in]0, \frac{1}{2}(x + t - \frac{1}{2})[$, we have $w(s, x + t - s) = w(s, 1 - (x + t - s))$, and by (4.4.9) we deduce $w(s, x - t + s) < w(s, x + t - s)$. Therefore in this case also we have $h^0(t, x) > 0$ and the proof of (4.4.6) is complete.

Now straightforward calculations show that $h^0 \in C^1(\mathbb{R} \times \mathbb{R} \setminus \{0\})$ with

$$\begin{aligned} h_x^0(t, 1/2) &= 3 \int_0^t \left\{ (w^2 w_x)(s, \frac{1}{2} + t - s) - (w^2 w_x)(s, \frac{1}{2} - t + s) \right\} ds \\ &= -6 \int_0^t w^2(s, \frac{1}{2} - t + s) w_x(s, \frac{1}{2} - t + s) ds \quad \text{by symmetry.} \end{aligned}$$

On the other hand we have

$$\begin{aligned} w(s, \frac{1}{2} - t + s) &> 0 \quad \text{for } t \in]0, 1/2] \text{ and } s \in]0, t[\\ w_x(s, \frac{1}{2} - t + s) &\geq 0 \quad \text{for } t \in [0, 1/2] \text{ and } s \in]0, t[\\ w_x(s, \frac{1}{2} - t + s) &> 0 \quad \text{for } t \in]0, 1/2] \text{ and } s \in]0, t/2[. \end{aligned}$$

Hence (4.4.7) is proved. ■

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Proof of Theorem 4.4.1. Let h be given by (4.1.10). From (4.1.11) we deduce

$$\frac{1}{k} \left[v_x(t, x) - w_x(t, x) \right] = -\frac{1}{2} h(t, x) . \quad (4.4.10)$$

An easy calculation shows that as $k \rightarrow 0$, $h \rightarrow h^0$ uniformly on $[0, T] \times \mathbb{R}$ for any $T > 0$. Furthermore if we set $D = \left\{ (t, x) \in \mathbb{R}^2, \left| t - \frac{1}{2} \right| + \left| x - \frac{1}{2} \right| \leq 1/4 \right\}$ (so that $(1/2, 1/2) \in D$ and $\bar{D} \subset \mathbb{R}^2 \setminus \xi$) we also have $h_x \rightarrow h_x^0$ uniformly on D .

By (4.4.7) and since $w_x \equiv 0$ in D , for k small enough we have $v_{xx} > 0$ near $(1/2, 1/2)$. In particular, by (4.3.6) there exists $\delta > 0$ and $k_0 > 0$ such that

$$\forall k \in]0, k_0], \quad \forall x \in \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta \right], \quad v_{xx}(t_k, x) > 0 . \quad (4.4.11)$$

From (4.4.10), (4.4.5) and (4.3.6) we deduce that there exists $k_1 > 0$ such that for all $k \in]0, k_1[$ we have

$$-1/2 \leq v_x(t_k, x) - w_x(t_k, x) < 0 \quad \text{for } 0 \leq x \leq \frac{1}{2} - \delta . \quad (4.4.12)$$

From the left-hand side of (4.4.12) and (4.4.1) we get

$$\forall k \in]0, k_1[, \quad \forall x \in \left[0, \frac{1}{2} - t_k \right], \quad v_x(t_k, x) \geq \frac{1}{2} . \quad (4.4.13)$$

Considering the right-hand side of (4.4.12) and (4.4.2) we find

$$\forall k \in]0, k_1[, \quad \forall x \in \left] \frac{1}{2} - t_k, \frac{1}{2} - \delta \right], \quad v_x(t_k, x) < 0 . \quad (4.4.14)$$

As soon as $k < \min(k_0, k_1)$, (4.4.3) and (4.4.4) now follow from (4.4.11), (4.4.13) and (4.4.14) and the continuity of $v(t_k, \cdot)$ as a function of k . Then (4.4.3) is obvious and the proof is complete. □

4.5. The case $\phi(x) = \sin \pi x$

In this section we consider $\phi(x) = \sin \pi x$. Note that here the free solution w is simply

$$w(t, x) = \cos \pi t \sin \pi x . \quad (4.5.1)$$

1. The first part of the document discusses the importance of maintaining accurate records of all transactions.

2. It is essential to ensure that all entries are supported by appropriate documentation and receipts.

3. Regular reconciliation of accounts is necessary to identify any discrepancies or errors.

4. The second part of the document outlines the procedures for handling customer inquiries and complaints.

5. All staff members should be trained to provide prompt and courteous service to our customers.

6. It is important to maintain a high level of transparency and communication with our clients.

7. The third part of the document details the financial reporting requirements for the quarter.

8. All financial statements must be prepared in accordance with the relevant accounting standards.

9. The fourth part of the document discusses the marketing and sales strategies for the upcoming period.

10. We will be implementing a series of promotional campaigns to attract new customers.

11. The fifth part of the document covers the human resources and training initiatives.

12. We will be investing in professional development courses for our staff members.

13. The sixth part of the document addresses the legal and compliance aspects of our operations.

14. It is crucial to stay up-to-date with the latest regulations and industry practices.

15. The seventh part of the document discusses the environmental and social responsibilities of our organization.

16. We are committed to reducing our carbon footprint and supporting local community initiatives.

17. The eighth part of the document covers the information technology and data security measures.

18. We will be upgrading our IT infrastructure to ensure the highest level of data protection.

19. The ninth part of the document discusses the overall performance and key metrics for the year.

20. We are pleased to report a steady increase in revenue and customer satisfaction.

Since w is regular in both t and x , it follows that v is indeed a classical solution of (4.1.5) and that all the derivatives of formula (4.1.6) also hold in the classical sense.

The following result is a striking contrast to the crossing behavior for the previous examples.

Theorem 4.5.1. Let $\phi(x) = \sin \pi x$. Then for all sufficiently small $k > 0$ we have

$$v(t_k, x) > 0, \quad \forall x \in]0, 1[\quad (4.5.2)$$

$$v_x(t_k, 0) = v_x(t_k, 1) = 0. \quad (4.5.3)$$

Remark 4.5.2. Clearly (4.5.3) follows immediately from (4.5.2) and the definition of t_k . Also, this theorem shows that the solution crosses first at the boundary. In other words, for t just slightly larger than t_k , $v(t, x)$ is positive everywhere on $]0, 1[$ except close to $x = 0$ and $x = 1$.

Remark 4.5.3. The proof of Theorem 4.5.1 is very technical and relies on two computational lemmas.

Lemma 4.5.4. For all $k > 0$, $t \in [0, 1/2]$ and $x \in]0, 1/2]$, define

$$G(k, t, x) = (k \sin \pi x)^{-2} \left[\frac{v(t, x)}{\sin \pi x} - \frac{v_x(t, 0)}{\pi} \right]. \quad (4.5.4)$$

Then as $k \rightarrow 0$, $G(k, t, x)$ converges uniformly on $[0, 1/2] \times]0, 1/2]$ to a continuous function $G(t, x)$ which is Lipschitz continuous in t , uniformly with respect to $x \in]0, 1/2]$.

Lemma 4.5.5. We have the formula

$$\forall x \in]0, 1/2], \quad G(1/2, x) = \frac{1}{48\pi}. \quad (4.5.5)$$

The proof of Lemma 4.5.5 will be postponed to the next section since it follows from a general computation valid for other initial data (cf. Theorem 4.6.5 and Corollary 4.6.11).

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Proof of Lemma 4.5.4. We define

$$g(k, t, x) = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} v^3(s, y) dy ds . \quad (4.5.6)$$

(Recall that v also depends on k .) From (4.1.6), (4.1.8) and (4.5.1) it follows immediately that we have

$$G(k, t, x) = -\frac{1}{2 \sin \pi x} \left[\frac{g(k, t, x)}{\sin \pi x} - \frac{g_x(k, t, 0)}{\pi} \right] . \quad (4.5.7)$$

Because of the smoothness of the free solution w , it follows easily by induction that $g(k, t, x)$ and all its x -derivatives are continuous for $k \geq 0$, $t \in \mathbb{R}$, $x \in \mathbb{R}$. In particular

$$\forall m \in \mathbb{N}, \quad \lim_{k \rightarrow 0} \frac{\partial^m g}{\partial x^m}(k, t, x) = \frac{\partial^m g}{\partial x^m}(0, t, x) ,$$

uniformly on compact subsets of \mathbb{R}^2 . (4.5.8)

Since v and w are smooth solutions we have

$$v(t, 0) = w(t, 0) = v_{xx}(t, 0) = w_{xx}(t, 0) = 0 ,$$

and this implies

$$\forall t \in \mathbb{R}, \quad \forall k \geq 0, \quad g(k, t, 0) = g_{xx}(k, t, 0) = 0 . \quad (4.5.9)$$

From (4.5.9) we deduce easily

$$g(k, t, x) = x g_x(k, t, 0) + \int_0^x \int_0^y \int_0^z g_{xxx}(k, t, \sigma) d\sigma dy dz . \quad (4.5.10)$$

Therefore by (4.5.7) we find

$$G(k, t, x) = z_1(x)g_x(k, t, 0) - z_2(x) \frac{1}{x} \int_0^x \int_0^y \int_0^z g_{xxx}(k, t, \sigma) d\sigma dy dz, \quad (4.5.11)$$

where z_1, z_2 are continuous and given by

$$z_1(x) = \frac{1}{\pi \sin^3(\pi x)} [\sin \pi x - \pi x], \quad z_2(x) = \frac{x^3}{\sin^3 \pi x}.$$

As a consequence of (4.5.8), it is now clear that, as $k \rightarrow 0$, $G(k, t, x)$ converges uniformly on $[0, 1/2] \times]0, 1/2]$ to the function

$$G(t, x) = z_1(x)g_x(0, t, 0) - z_2(x) \frac{1}{x} \int_0^x \int_0^y \int_0^z g_{xxx}(0, t, \sigma) d\sigma dy dz. \quad (4.5.12)$$

Finally it is clear that G is bounded and Lipschitz continuous in $t \in [0, 1/2]$, uniformly with respect to $x \in]0, 1/2]$. \square

Proof of Theorem 4.5.1. From Lemmas 4.5.4 - 4.5.5 it follows that for k small enough and t sufficiently close to $1/2$, we have with $0 < \alpha < 1/48\pi$ (α fixed),

$$G(k, t, x) \geq \alpha > 0, \quad \forall x \in]0, 1/2]. \quad (4.5.13)$$

By choosing k perhaps smaller, Proposition 4.3.1 guarantees that we can choose $t = t_k$ in formula (4.5.13). Since $v_x(t_k, 0) \geq 0$ it follows from (4.5.4) and (4.5.13) that

$$\forall x \in]0, 1/2], \quad v(t_k, x) \geq \alpha k^2 \sin^3(\pi x).$$

By symmetry, we obtain (4.5.2). The proof of Theorem 4.5.1 is now complete. \square

4.6. Some results for general ϕ and applications

In this section, our goal is to understand the two previous examples as much as possible from a general point of view. In addition to the hypotheses of paragraph 4.3 (i.e., (4.1.2) - (4.1.3), (4.2.1) - (4.2.2) and (4.3.3) - (4.3.4)) we assume

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that this is crucial for ensuring transparency and accountability in the organization's operations.

2. The second part of the document outlines the various methods and tools used to collect and analyze data. It highlights the need for consistent and reliable data collection processes to support effective decision-making.

3. The third part of the document focuses on the role of technology in data management and analysis. It discusses how modern software solutions can streamline data collection, storage, and reporting, thereby improving efficiency and accuracy.

4. The fourth part of the document addresses the challenges associated with data management, such as data quality, security, and privacy. It provides strategies to mitigate these risks and ensure that data is used responsibly and ethically.

5. The fifth part of the document discusses the importance of data governance and the role of leadership in establishing a strong data culture. It emphasizes that clear policies and standards are essential for successful data management.

6. The sixth part of the document explores the benefits of data-driven decision-making and how it can lead to improved performance and innovation. It provides examples of organizations that have successfully leveraged data to gain a competitive edge.

7. The seventh part of the document discusses the future of data management and the emerging trends in the field. It highlights the growing importance of artificial intelligence and machine learning in data analysis and the need for ongoing learning and adaptation.

8. The eighth part of the document provides a summary of the key points discussed and offers final thoughts on the importance of data in the modern business landscape. It encourages organizations to embrace data as a strategic asset and to invest in the necessary resources to maximize its value.

9. The ninth part of the document includes a list of references and resources for further reading. It provides a comprehensive overview of the literature and research in the field of data management and analysis.

10. The tenth part of the document is a conclusion that reiterates the main findings and offers a call to action for organizations to take a data-driven approach to their operations. It emphasizes that data is not just a byproduct of business but a key driver of success.

11. The eleventh part of the document is a list of appendices that provide additional information and data to support the main text. These appendices include detailed reports, charts, and tables that are too large to include in the main body of the document.

12. The twelfth part of the document is a list of footnotes that provide additional context and information for the references and other parts of the document. It includes details about the authors, publishers, and other relevant information for the cited works.

13. The thirteenth part of the document is a list of glossary terms that define key concepts and terminology used throughout the document. This helps to ensure that all readers have a clear understanding of the language and concepts being discussed.

14. The fourteenth part of the document is a list of index entries that provide a quick reference to the various topics and sections of the document. This allows readers to easily locate the information they are looking for and navigate the document more effectively.

$$\phi \text{ is } C^2 \text{ in a neighborhood of } x = 1/2 \text{ with } \phi''(1/2) < 0 . \quad (4.6.1)$$

It will be convenient to set

$$h(t, x) = \begin{cases} \frac{\phi(x+t) + \phi(x-t)}{2\phi'(t)} & \text{for } 0 < t < 1/2 \\ \frac{\phi'(x + \frac{1}{2})}{\phi''(1/2)} & \text{for } t = 1/2 . \end{cases} \quad (4.6.2)$$

By the hypotheses on ϕ , h is continuous on $]0, 1/2] \times \mathbb{R}$. We now introduce

$$H(k, t, x) = \frac{1}{2} \left[\frac{v(t, x)}{k} - h(t, x) v_x(t, 0) \right] . \quad (4.6.3)$$

Note that if $\phi(x) = \sin \pi x$, then $h(t, x) = \frac{1}{\pi} \sin \pi x$ and therefore in this case

$$H(k, t, x) = \sin^3(\pi x) G(k, t, x) \quad (4.6.4)$$

with G given by (4.5.4).

It follows from (4.1.6) and (4.1.8) that in the general case

$$H(k, t, x) = h(t, x) \int_0^t v^3(s, t-s) ds - \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} v^3(s, y) dy ds . \quad (4.6.5)$$

We now set

$$H(t, x) = h(t, x) \int_0^t w^3(s, t-s) ds - \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} w^3(s, y) dy ds . \quad (4.6.6)$$

Lemma 4.6.1. As $k \rightarrow 0$ we have

$$H(k, t, x) \rightarrow H(t, x) \text{ uniformly on }]0, 1/2] \times \mathbb{R} , \quad (4.6.7)$$

$$H(k, t_k, x) \rightarrow H(1/2, x) \text{ uniformly on } \mathbb{R} . \quad (4.6.8)$$

Proof. (4.6.7) follows immediately from (4.3.1) and the formulas (4.6.5) - (4.6.6). On the other hand, since $H(t, x)$ is Lipschitz continuous in t , uniformly with respect to $x \in \mathbb{R}$, (4.6.8) follows from (4.6.7) and (4.3.6). \square

The following two propositions show the importance of $H(1/2, x)$ in determining the crossing behavior of the solution u for small values of k .

Proposition 4.6.2. Assume that there exists $x_0 \in]0, 1[$ such that $H(1/2, x_0) < 0$. Then for k small enough we have $v_x(t_k, 0) > 0$. Hence the first crossing cannot occur at the boundary.

Proof. By (4.6.8), for k small enough we have $H(k, t_k, x_0) < 0$. Then (4.6.3) yields

$$h(t_k, x_0)v_x(t_k, 0) > v(t_k, x_0) \geq 0 .$$

Since (4.6.2) gives $h(t_k, x_0) > 0$, we deduce $v_x(t_k, 0) > 0$, and the proof is complete. ▣

Proposition 4.6.3. Assume that for all $x \in]0, 1[$, $H(1/2, x) > 0$. Then for any $\varepsilon > 0$ there exists k_ε such that for all $k \in]0, k_\varepsilon[$ we have $v(t_k, x) > 0$, $\forall x \in [\varepsilon, 1 - \varepsilon]$. Hence for k small, the first crossing occurs arbitrarily close to the boundary.

Proof. For any $\varepsilon > 0$, $\min_{x \in [\varepsilon, 1 - \varepsilon]} H(1/2, x) > 0$. Therefore, by (4.6.8) we have for k small enough

$$\forall x \in [\varepsilon, 1 - \varepsilon], \quad H(k, t_k, x) > 0 .$$

By (4.6.3) we conclude that

$$\forall x \in [\varepsilon, 1 - \varepsilon], \quad v(t_k, x) > h(t_k, x)v_x(t_k, x) \geq 0 . \quad \text{▣}$$

Remark 4.6.4. We have $H(k, t, 0) = 0$ for any $k > 0$ and $t \in \mathbb{R}$. Therefore the argument of Proposition 4.6.3 cannot be applied to show that first crossing occurs exactly at the boundary. In other words, while the arguments of this paragraph apply to a more general class of data than $\sin(\pi x)$, they are not as subtle as the analysis of paragraph 4.5.

The main result of this paragraph is the following explicit formula.

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that this is crucial for ensuring transparency and accountability in the organization's operations.

2. The second part of the document outlines the various methods and tools used to collect and analyze data. It highlights the use of advanced software and techniques to ensure that the data is reliable and valid.

3. The third part of the document discusses the importance of regular communication and reporting. It stresses that keeping stakeholders informed is essential for the success of the organization.

4. The fourth part of the document outlines the various risks and challenges that the organization may face. It provides a detailed analysis of these risks and offers strategies to mitigate them.

5. The fifth part of the document discusses the importance of continuous improvement and innovation. It emphasizes that the organization must constantly evolve and adapt to remain competitive in the market.

6. The sixth part of the document outlines the various roles and responsibilities of the organization's staff. It provides a clear framework for how each role contributes to the overall success of the organization.

7. The seventh part of the document discusses the importance of maintaining a strong corporate culture. It stresses that a positive and inclusive culture is essential for attracting and retaining top talent.

8. The eighth part of the document outlines the various financial and operational metrics that the organization uses to track its performance. It provides a detailed explanation of how these metrics are calculated and used.

9. The ninth part of the document discusses the importance of maintaining accurate financial records. It emphasizes that this is crucial for ensuring the organization's financial stability and growth.

10. The tenth part of the document outlines the various legal and regulatory requirements that the organization must comply with. It provides a detailed overview of these requirements and offers strategies to ensure compliance.

Theorem 4.6.5. We have

$$\begin{aligned} H(1/2, x) &= \frac{1}{16} \left[\frac{\phi'(x + \frac{1}{2})}{\phi''(1/2)} \phi^3(1/2) - \int_0^x \phi^3(r + \frac{1}{2}) dr \right] \\ &\quad + \frac{3}{16} \left(\int_0^1 \phi^2(r) dr \right) \left[\frac{\phi'(x + \frac{1}{2})}{\phi''(1/2)} \phi(1/2) - \int_0^x \phi(r + \frac{1}{2}) dr \right]. \end{aligned} \quad (4.6.9)$$

Proof. We set

$$\theta(x) = \int_0^{1/2} \int_{x - \frac{1}{2} + s}^{x + \frac{1}{2} - s} w^3(s, y) dy, \quad (4.6.10)$$

$$\rho(x) = \int_0^{1/2} \left[w^3(s, x + \frac{1}{2} - s) - w^3(s, x - \frac{1}{2} + s) \right] ds. \quad (4.6.11)$$

Note that $\rho(x) = \theta'(x)$ and by symmetry, we have

$$\rho(x) = \psi(x) + \psi(-x) \quad (4.6.12)$$

with

$$\psi(x) = \int_0^{1/2} w^3(s, x + \frac{1}{2} - s) ds. \quad (4.6.13)$$

By (4.1.9) and (4.6.13) we have

$$\begin{aligned} \psi(x) &= \frac{1}{8} \int_0^{1/2} \left[\phi(x + \frac{1}{2}) + \phi(x + \frac{1}{2} - 2s) \right]^3 ds \\ &= \frac{1}{16} \phi^3(x + \frac{1}{2}) + \frac{3}{16} \phi^2(x + \frac{1}{2}) \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} \phi(r) dr \\ &\quad + \frac{3}{16} \phi(x + \frac{1}{2}) \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} \phi^2(r) dr + \frac{1}{16} \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} \phi^3(r) dr. \end{aligned}$$

1. $\frac{1}{2} \ln 2$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

2. $\frac{1}{2} \ln 2$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

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$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

3. $\frac{1}{2} \ln 2$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{2}{1} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

4. $\frac{1}{2} \ln 2$

Since ϕ and ϕ^3 are odd and ϕ^2 has period 1 it follows that

$$\psi(x) + \psi(-x) = \frac{1}{8} \phi^3\left(x + \frac{1}{2}\right) + \frac{3}{8} \phi\left(x + \frac{1}{2}\right) \int_0^1 \phi^2(r) dr . \quad (4.6.14)$$

By (4.6.12) we obtain

$$\rho(0) = \frac{1}{8} \phi^3(1/2) + \frac{3}{8} \phi(1/2) \int_0^1 \phi^2(r) dr . \quad (4.6.15)$$

Since $\theta(0) = 0$, by integrating (4.6.14) we find

$$\theta(x) = \frac{1}{8} \int_0^x \phi^3\left(r + \frac{1}{2}\right) dr + \frac{3}{8} \int_0^1 \phi^2(r) dr \int_0^x \phi\left(r + \frac{1}{2}\right) dr . \quad (4.6.16)$$

Finally, from (4.6.6), (4.6.10), (4.6.11) and the symmetry we deduce

$$H(1/2, x) = \frac{1}{2} h(1/2, x) \rho(0) - \frac{1}{2} \theta(x) . \quad (4.6.17)$$

A combination of (4.6.2) and (4.6.15) - (4.6.17) finally yields (4.6.9). \square

We now derive simple consequences of Theorem 4.6.5.

Corollary 4.6.6. Assume that $\phi'(0) = 0$. Then for k small enough, first crossing occurs away from the boundary.

Proof. We write (4.6.9) at $x = 1/2$. Since $\phi'(0) = \phi'(1) = 0$ we obtain

$$H(1/2, 1/2) = -\frac{1}{16} \int_0^{1/2} \phi^3\left(r + \frac{1}{2}\right) dr - \frac{3}{16} \left(\int_0^{1/2} \phi\left(r + \frac{1}{2}\right) dr \right) \left(\int_0^1 \phi^2(r) dr \right) < 0 .$$

Hence by Proposition 4.6.2, the first crossing cannot occur at the boundary. \square

Remark 4.6.7. Corollary 4.6.6 applies to $\phi(x) = \sin^2(\pi x)$, and the conclusion fits with the numerical results (Table 2).

1. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

2. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

3. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

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9. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

10. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

11. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

12. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

13. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

14. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

15. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Corollary 4.6.8. Assume that

$$\forall x \in]0, 1/2[, \quad \frac{\phi'(x + \frac{1}{2})}{\phi''(1/2)} \geq x . \quad (4.6.9)$$

Then $H(1/2, x) > 0$, $\forall x \in]0, 1[$, hence as k tends to 0, the first crossing occurs arbitrarily close to the boundary.

Proof. We have

$$\begin{aligned} \frac{\phi'(x + \frac{1}{2})}{\phi''(1/2)} \phi^3(1/2) - \int_0^x \phi^3(r + \frac{1}{2}) dr &\geq x \phi^3(1/2) - \int_0^x \phi^3(r + \frac{1}{2}) dr \\ &\geq \int_0^x [\phi^3(1/2) - \phi^3(r + \frac{1}{2})] dr , \quad \forall x \in]0, 1/2[. \end{aligned}$$

Since for all $r \in]0, 1/2]$ we have $\phi^3(1/2) > \phi^3(r + \frac{1}{2})$, the first term in the right-hand side of (4.6.9) is positive. Similarly, the second term in the right-hand side of (4.6.9) is also positive and the result follows by applying Proposition 4.6.3. ▣

Remark 4.6.9. Corollary 4.6.8 applies to $\phi(x) = x(1-x)$ and its conclusion fits with the numerical results (Table 2).

Remark 4.6.10. In the extreme case where " $\phi''(1/2) = -\infty$," we would find $H(1/2, x) < 0$ for all $x \in]0, 1[$. Therefore formula (4.6.9) also gives a heuristic explanation of the result obtained in Section 4.4.

To conclude this paragraph, we now check formula (4.5.5) of Lemma 4.5.5.

Corollary 4.6.11. When $\phi(x) = \sin(\pi x)$, we have

$$H(1/2, x) = \frac{1}{48\pi} \sin^3(\pi x) , \quad \forall x \in]0, 1[. \quad (4.6.18)$$

Proof. By applying (4.6.9) and reducing the terms we obtain

$$H(1/2, x) = \frac{1}{16\pi} \sin(\pi x) - \frac{1}{16} \int_0^x \cos^3(\pi r) dr = \int_0^x \frac{1}{16} [\cos(\pi r) - \cos^3(\pi r)] dr$$

(cont'd)

1. The first part of the document discusses the importance of maintaining accurate records of all transactions.

2. It is essential to ensure that all entries are supported by appropriate documentation and receipts.

3. Regular reconciliation of accounts is necessary to identify any discrepancies or errors in a timely manner.

4. Maintaining clear and concise records will facilitate the preparation of financial statements and tax returns.

5. It is also important to review the records periodically to ensure their accuracy and completeness.

6. Proper record-keeping practices will help in the identification of trends and the detection of potential fraud.

7. The document further emphasizes the need for confidentiality and security of the recorded information.

8. Finally, it is recommended that all records be stored in a secure and accessible location for future reference.

9. The document concludes by stating that adherence to these guidelines will ensure the integrity and reliability of the financial records.

10. It is the responsibility of the management to implement and maintain these record-keeping procedures.

11. The document provides a detailed outline of the record-keeping process and the required documentation.

12. It is expected that these measures will contribute to the overall financial health and transparency of the organization.

13. The document is intended to serve as a guide for all employees involved in financial record-keeping.

14. It is the policy of the organization to maintain the highest standards of accuracy and integrity in all financial records.

15. The document is subject to periodic review and updates to reflect changes in regulations and best practices.

$$= \frac{1}{16} \int_0^x \sin^2(\pi r) \cos(\pi r) dr = \frac{1}{48\pi} \int_0^x \frac{d}{dr} [\sin^3(\pi r)] dr = \frac{1}{48\pi} \sin^3(\pi x) . \quad \blacksquare$$

Proof of Lemma 4.5.5. Formula (4.5.5) is indeed an immediate consequence of (4.6.18) and (4.6.4). \(\blacksquare\)

5. THE NUMERICAL RESULTS

5.1. Description of the scheme. Evaluation of the error.

We employed the explicit finite difference scheme described in KUO PEN-YU, L. VAZQUEZ [15] which is

$$\frac{u^{k+1}(x_j) - 2u^k(x_j) + u^{k-1}(x_j)}{(\delta t)^2} - \frac{u^k(x_{j+1}) - 2u^k(x_j) + u^k(x_{j-1}))}{(\delta x)^2} + \frac{F(u^{k+1}(x_j)) - F(u^{k-1}(x_j))}{u^{k+1}(x_j) - u^{k-1}(x_j)} = 0 \quad (5.1)$$

with $F(x) = x^4/4$. We imposed the boundary conditions $u^k(0) = u^k(1) = 0$. From the convexity of F it is easily seen that for any initial data u^0 and u^1 , the discretized problem admits a unique solution, regardless of the size of δt and δx (compare L. VAZQUEZ [26]).

The convergence and stability of this scheme have been proved in [15]. Our boundary conditions require only an obvious change of the discrete Sobolev spaces considered in the proof. At each step, the implicit problem was solved by means of Newton's method. We always assumed an initial speed equal to 0, which corresponds in the continuous problem to a solution which is an even function of t . Therefore, to insure the maximum accuracy, u^{-1} was computed from the initial datum u^0 by (5.1) (with $k = 0$) on imposing the condition $u^{-1} = u^1$. Finally, the computations were performed in double precision with $\delta x = 0.02$ and $\delta t = 0.01$.

Several experiments have been done in order to estimate the error due to the method. We first considered the discretization of the d'Alembertian operator. We compared the solution of the discretized linear problem with the analytic solution of

the linear continuous problem. For the initial datum $10 \sin(\pi x)$ and $0 \leq t \leq 100$ (i. e., 10,000 time steps) the mean error is 0.078 while the maximum error is 0.38 at time 99.51. We also computed the error coming from the approximation by Newton's method. Again with $u^0 = 10 \sin(\pi x)$ the solution of (5.1) was computed up to $t = 100$ and then backwards to $t = 0$. After these 20,000 steps of computation, the departure from the initial datum did not exceed $1.05 \cdot 10^{-10}$. Therefore we can reasonable expect that the long term computations (up to $t = 100$) described in this paper are significant.

5.2. The results

5.2.1 Table 1. Here $u^0(x) = 10 \sin(\pi x)$. We have been investigating the values of t in $[0, 100]$ for which $y(t) = \|u(t) - u^0\|_{\infty} / \|u^0\|_{\infty}$ is less than 0.05. We found fifteen sequences of consecutive times. In Table 1 we give the time of each sequence corresponding to the smallest value of $y(t)$. For completeness, we also tabulated the corresponding values of

$$z(t) = \frac{\|u(t) - u^0\|_{H_0^1(0,1)}}{\|u^0\|_{H_0^1(0,1)}} \quad \text{and} \quad w(t) = \frac{\|u(t) - u^0\|_4}{\|u^0\|_4} .$$

5.2.2. Table 2. We considered 3 cases:

$$u^0(x) = k \sin(\pi x) , \quad u^0(x) = k \sin^2(\pi x) , \quad u^0(x) = kx(1-x) .$$

In each case, we computed the solution for $k \in \{1, 5, 10, 20, 30, 50\}$. For each of these 18 examples, we indicate in Table 2 the value of the crossing time t_k . Of course what appears in the table is in fact the first time step θ_k greater than or equal to t_k . By identifying θ_k and t_k we give in Table 2 the location of the points in $]0, 1[$ where $u(t_k, x) < 0$. Three different situations have been observed.

Invisible: $u(t_k, x) < 0$ for all $x \in]0, 1[$.

Boundary: At time t_k , the solution is positive everywhere except near $x \in \{0, 1\}$.

Center: At time t_k the solution is 0 near $x = 0.5$ and positive elsewhere.

TABLE 1

t	y(t)	z(t)	w(t)
0	0	0	0
2.48	0.048	0.174	0.040
10.69	0.014	0.063	0.011
13.17	0.023	0.087	0.020
21.36	0.030	0.120	0.026
23.85	0.020	0.090	0.016
34.53	0.019	0.085	0.017
45.25	0.033	0.148	0.031
47.74	0.044	0.192	0.037
58.46	0.017	0.064	0.013
69.16	0.021	0.103	0.019
71.64	0.020	0.084	0.016
82.36	0.046	0.161	0.041
86.45	0.029	0.097	0.022
99.64	0.024	0.091	0.019

TABLE 2

k	Initial Datum					
	$k \sin(\pi x)$		$k \sin^2(\pi x)$		$kx(1-x)$	
	t_k	Location	t_k	Location	t_k	Location
1	0.49	invisible	0.45	center	0.50	invisible
5	0.32	center	0.29	center	0.48	boundary
10	0.18	center	0.18	center	0.42	boundary
20	0.10	center	0.10	center	0.33	invisible
35	0.07	center	0.07	center	0.24	center
50	0.05	center	0.05	center	0.15	center

Year	Q1	Q2	Q3	Q4
2010	1.2	1.5	1.8	2.1
2011	1.5	1.8	2.1	2.4
2012	1.8	2.1	2.4	2.7
2013	2.1	2.4	2.7	3.0
2014	2.4	2.7	3.0	3.3
2015	2.7	3.0	3.3	3.6
2016	3.0	3.3	3.6	3.9
2017	3.3	3.6	3.9	4.2
2018	3.6	3.9	4.2	4.5
2019	3.9	4.2	4.5	4.8
2020	4.2	4.5	4.8	5.1
2021	4.5	4.8	5.1	5.4
2022	4.8	5.1	5.4	5.7
2023	5.1	5.4	5.7	6.0
2024	5.4	5.7	6.0	6.3
2025	5.7	6.0	6.3	6.6
2026	6.0	6.3	6.6	6.9
2027	6.3	6.6	6.9	7.2
2028	6.6	6.9	7.2	7.5
2029	6.9	7.2	7.5	7.8
2030	7.2	7.5	7.8	8.1

Year	Q1	Q2	Q3	Q4
2010	1.2	1.5	1.8	2.1
2011	1.5	1.8	2.1	2.4
2012	1.8	2.1	2.4	2.7
2013	2.1	2.4	2.7	3.0
2014	2.4	2.7	3.0	3.3
2015	2.7	3.0	3.3	3.6
2016	3.0	3.3	3.6	3.9
2017	3.3	3.6	3.9	4.2
2018	3.6	3.9	4.2	4.5
2019	3.9	4.2	4.5	4.8
2020	4.2	4.5	4.8	5.1
2021	4.5	4.8	5.1	5.4
2022	4.8	5.1	5.4	5.7
2023	5.1	5.4	5.7	6.0
2024	5.4	5.7	6.0	6.3
2025	5.7	6.0	6.3	6.6
2026	6.0	6.3	6.6	6.9
2027	6.3	6.6	6.9	7.2
2028	6.6	6.9	7.2	7.5
2029	6.9	7.2	7.5	7.8
2030	7.2	7.5	7.8	8.1

5.2.3. The Figures

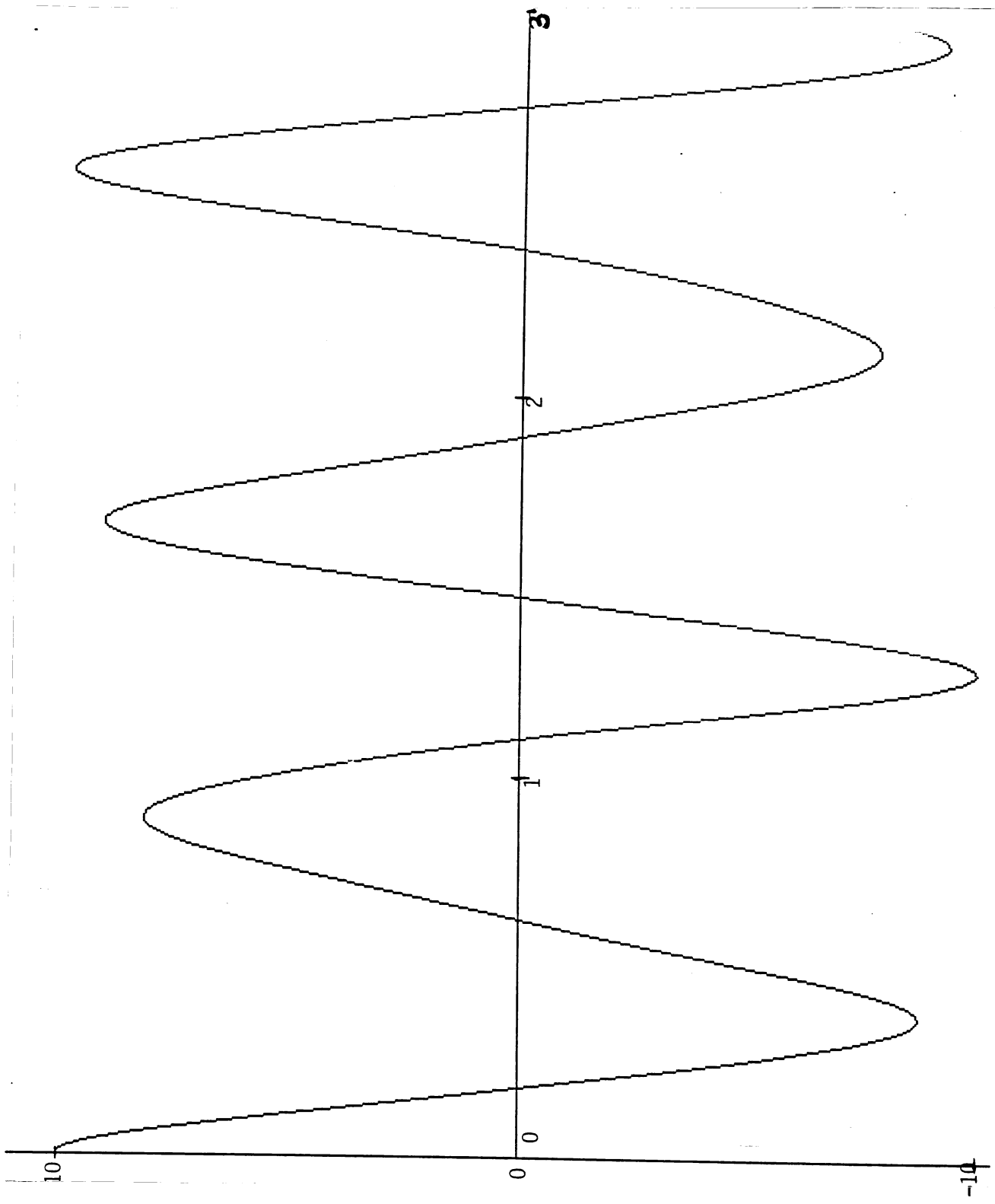


Fig. 1. The initial datum $u^0(x) = 10 \sin(\pi x)$. Evolution of $u(t, 1/2)$ for $0 \leq t \leq 3$.

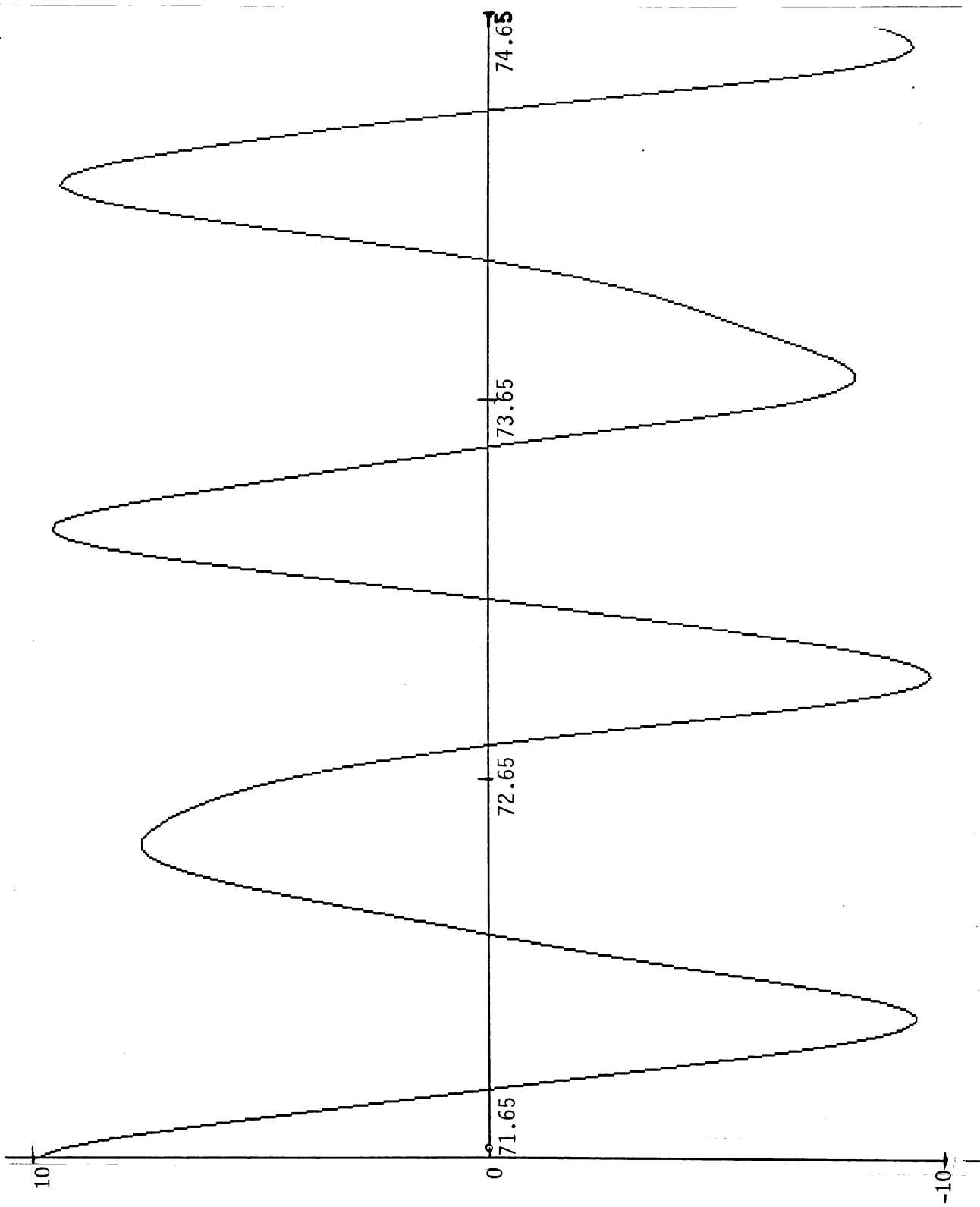


Fig. 2. Same as Fig. 1 but for $71.65 \leq t \leq 74.65$.

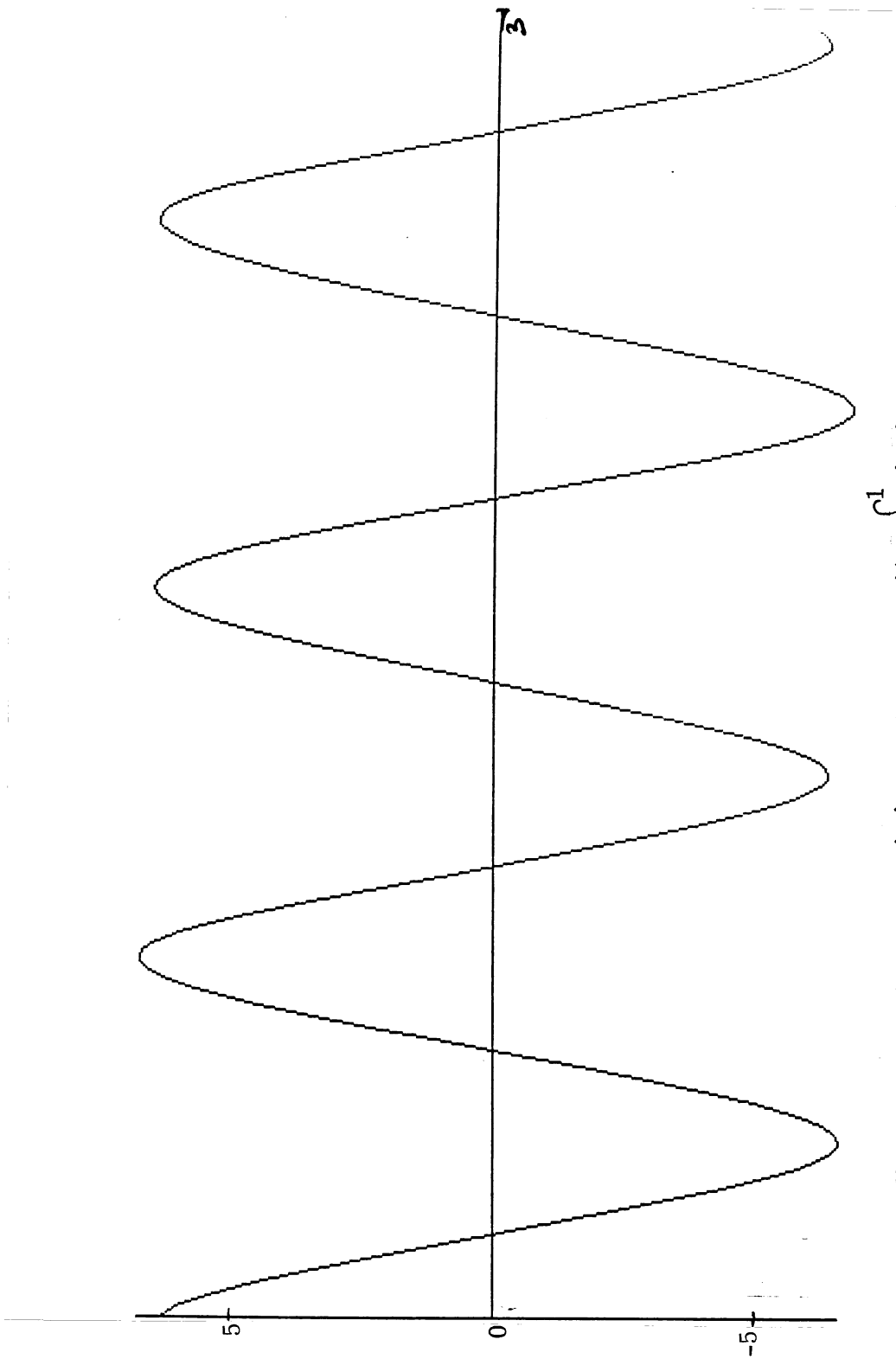


Fig. 3. Initial datum $10 \sin(\pi x)$. Evolution of $m(t) = \int_0^1 u(t, x) dx$ for $0 \leq t \leq 3$.

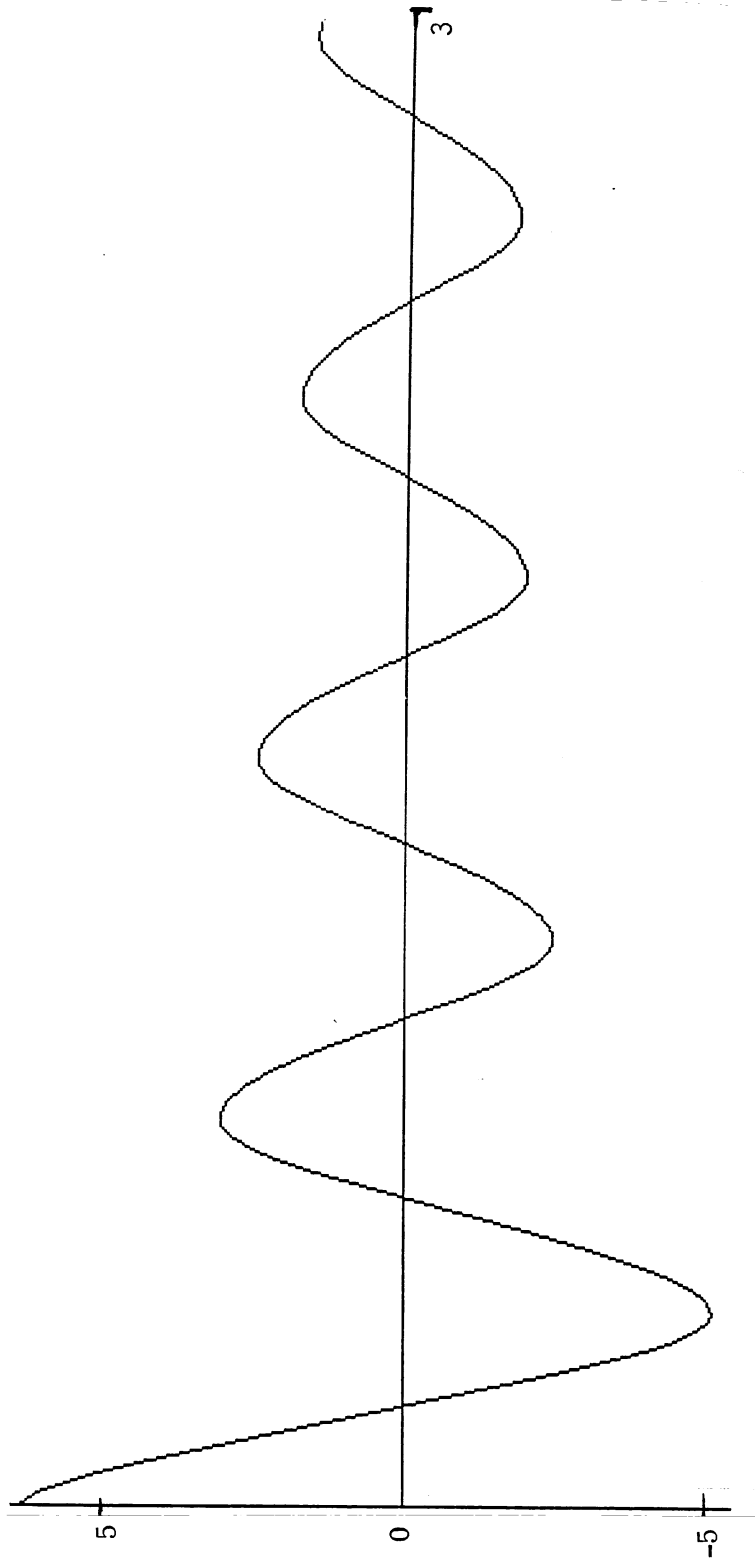


Fig. 4. Same as Fig. 3 but here u is the solution of $u_{tt} + \pi^2 u + u^3 = 0$.

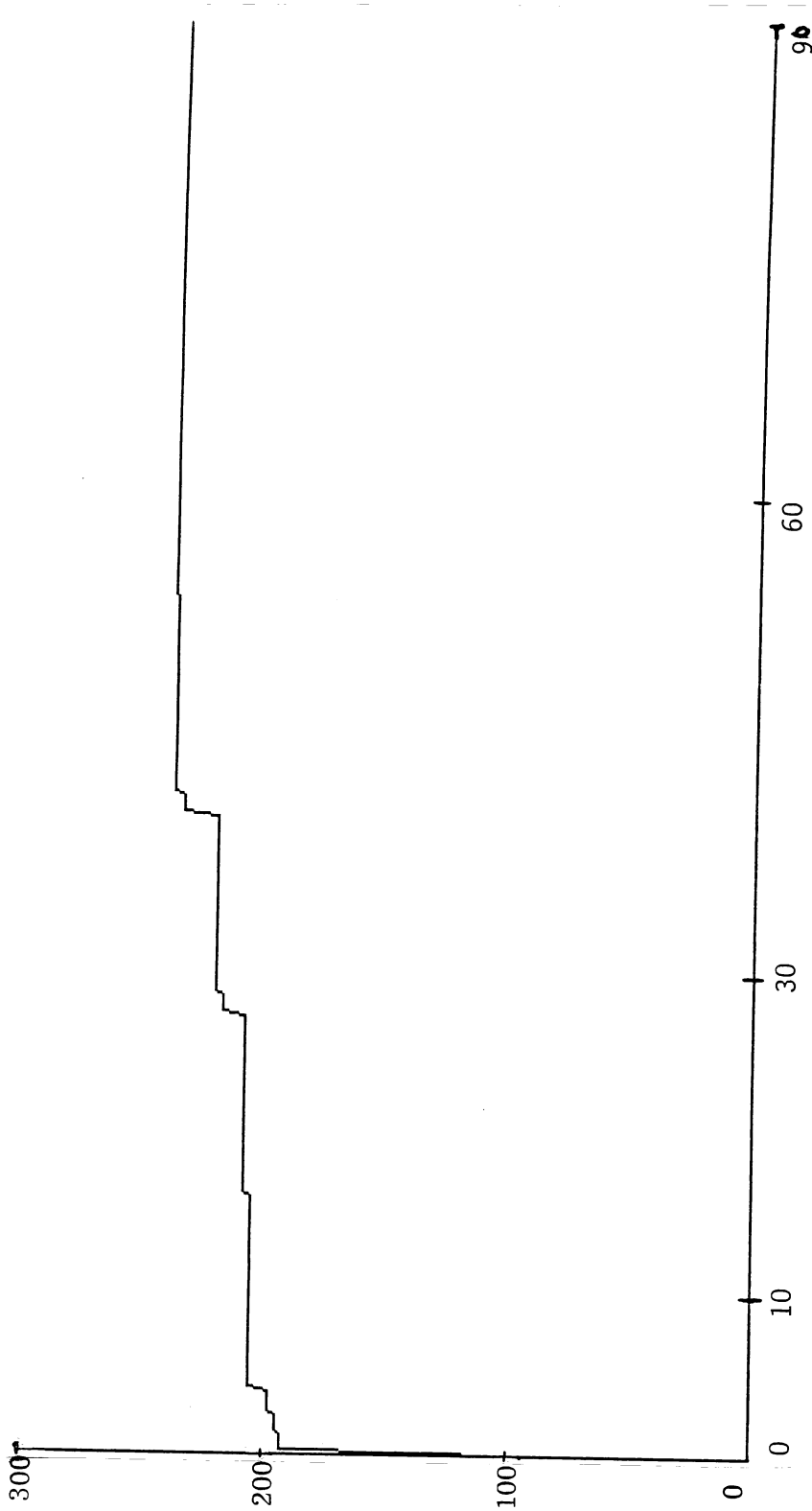


Fig. 5. Initial datum $10 \sin(\pi x)$. Evolution of $D(t) = \sup_{s \in [0, t]} \|u_{xx}(s, \cdot)\|_{L^2(0, 1)}^2$ for $0 \leq t \leq 90$.

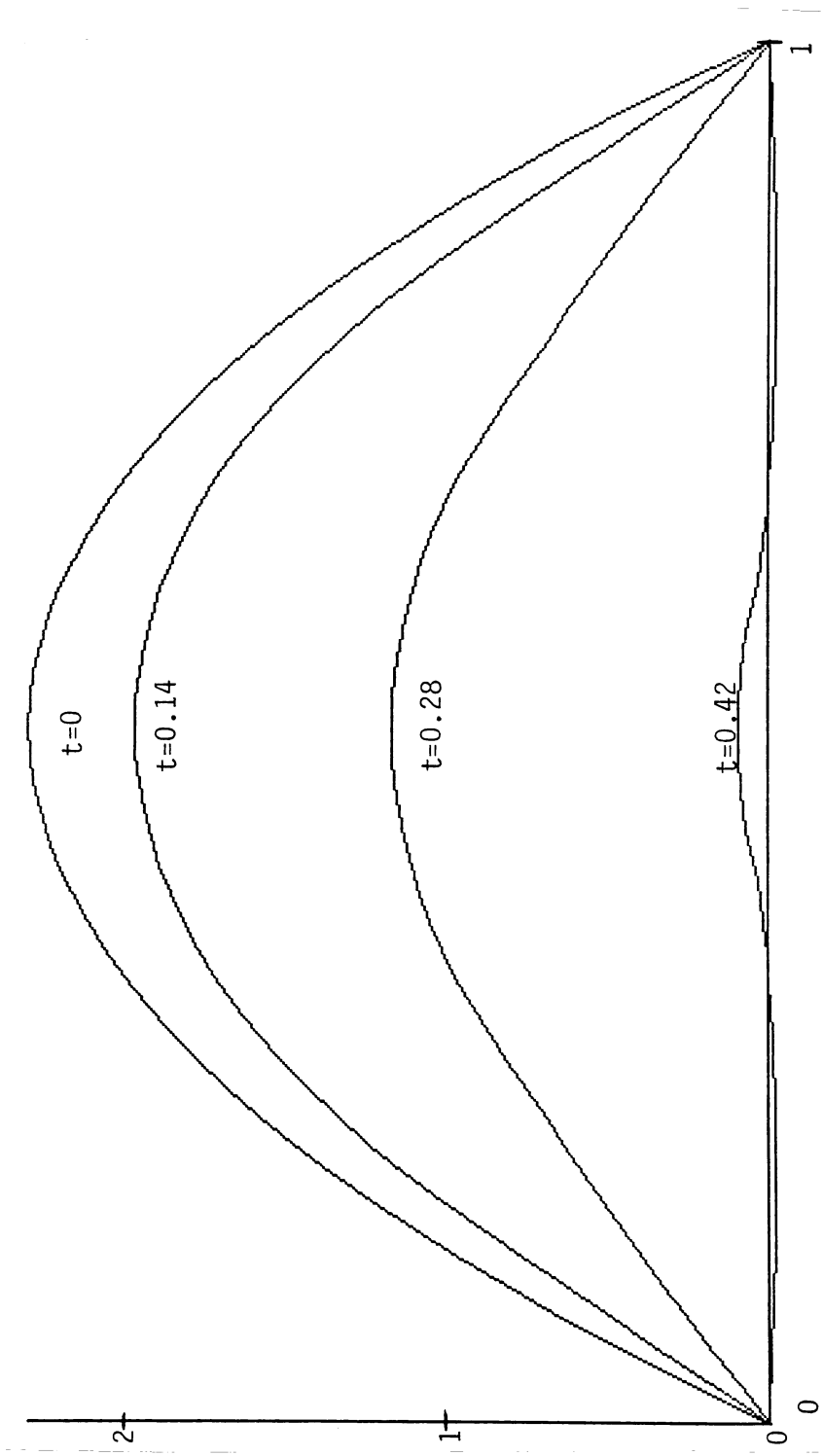


Fig. 6. Initial datum $10x(1-x)$. The shape of $u(t)$ for $t \in \{0, 0.14, 0.28, 0.42\}$.

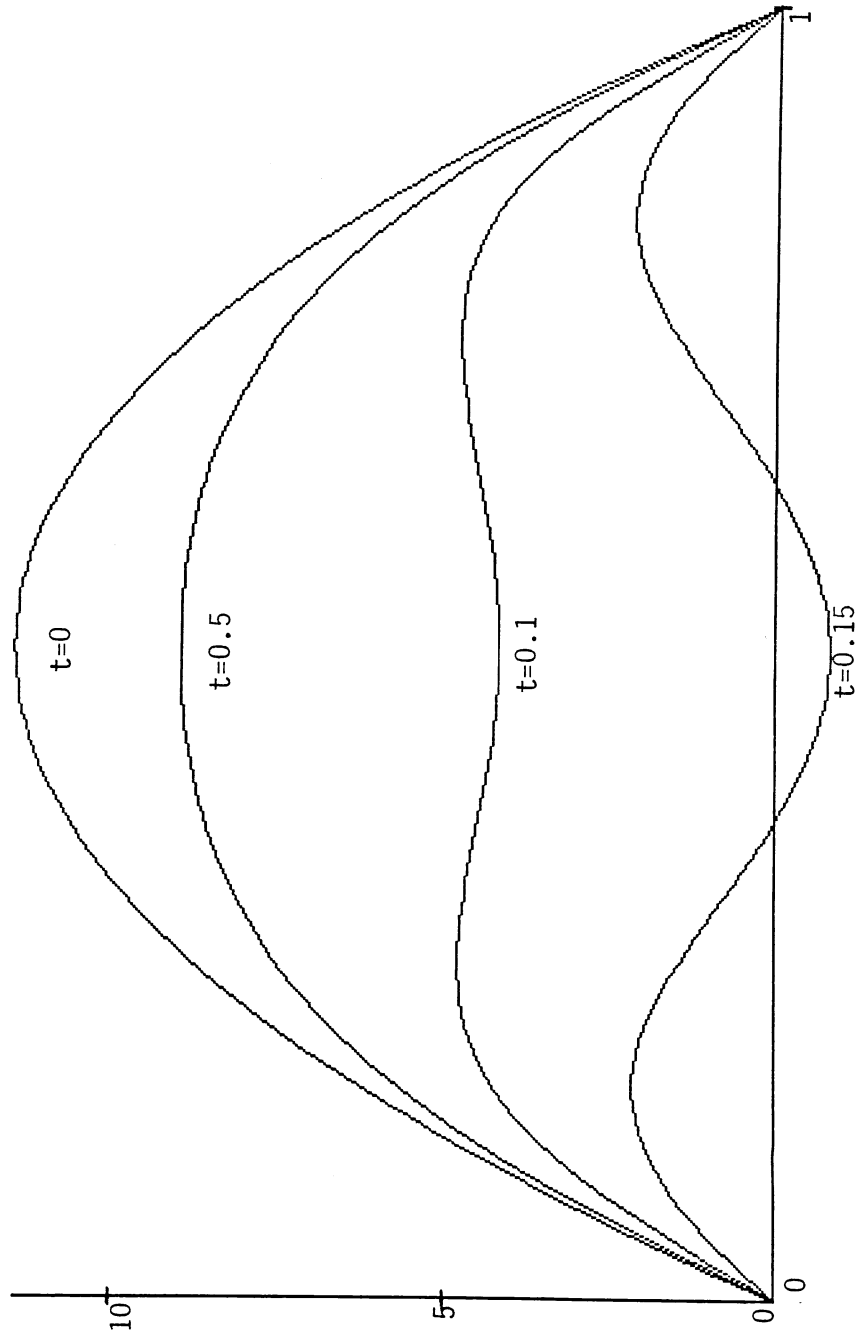


Fig. 7. Initial datum $50x(1-x)$. The shape of $u(t)$ for $t \in \{0, 0.05, 0.1, 0.15\}$.

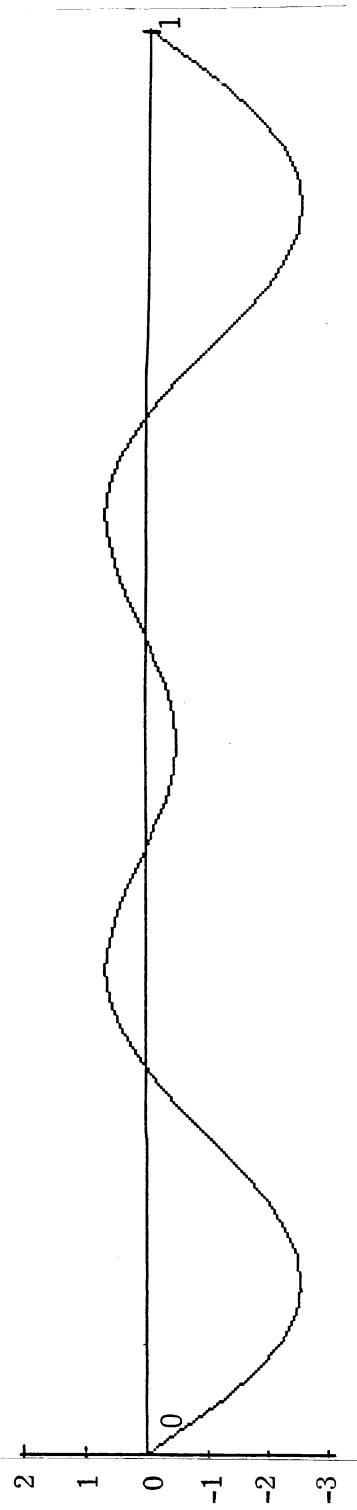


Fig. 8. Initial datum $10 \sin(\pi x)$. The shape of $u(t)$ for $t = 52.90$.

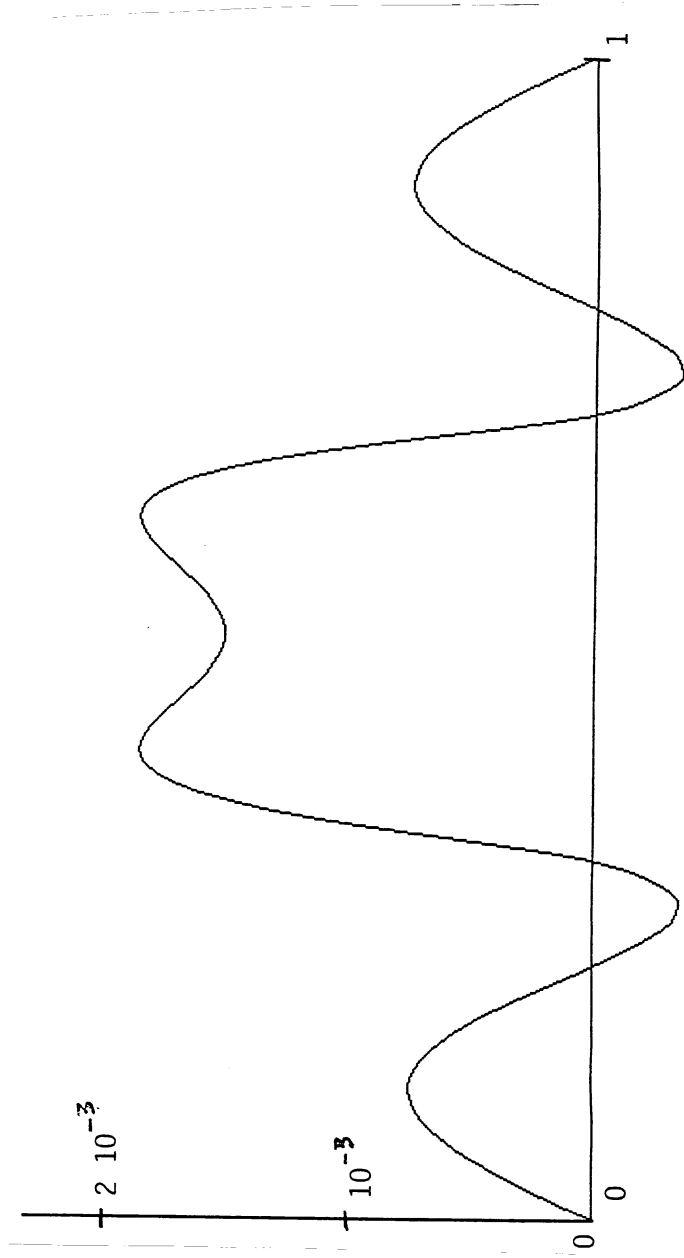


Fig. 9. See Section 5.3 below.

5.3. Further observations

In order to understand the cases where the location of the first crossing was "invisible" (cf. Table 2), we made further computations with smaller time and space steps. We chose $\delta t = 0.001$ and $\delta x = 0.005$. We observed the following phenomena.

When $u^0(x) = \sin(\pi x)$, the first crossing occurs near the boundary (crossing time $t_k = 0.486$).

When $u^0(x) = kx(1-x)$,

- if $k = 1$, the first crossing occurs near the boundary ($t_k = 0.499$);
- if $k = 20$, the first crossing occurs near the center ($t_k = 0.321$);
- The special configuration in the latter case led us to consider the case $k = 19$. Then first crossing occurs neither near the boundary nor near the center. Figure 9 represents the corresponding solution u at time $t_k = 0.330$.

BIBLIOGRAPHY

- [1] L. AMERIO, G. PROUSE, Abstract almost periodic functions and functional equations, Van Nostrand, New York (1971).
- [2] L. AMERIO, G. PROUSE, Study of the motion of a string vibrating against an obstacle, *Ac. Mat. Applic.* 8 (1975), 563-585.
- [3] H. BREZIS, J. M. CORON, L. NIRENBERG, Free vibrations for a nonlinear wave and a theorem of P. Rabinowitz, *C.P.A.M.* 33 (1980), 667-689.
- [4] H. CABANNES, A. HARAUX, Mouvements presque-périodiques d'une corde vibrante en présence d'un obstacle fixe, rectiligne ou ponctuel, *Int. J. Nonlinear Mechanics* 16 (1981), 449-458.
- [5] H. CABANNES, Mouvement d'une corde vibrante en présence d'un obstacle rectiligne, *J. Meca. theorique et appl.* 3 (1984), 397-414.
- [6] T. CAZENAVE, A. HARAUX, Propriétés oscillatoires des solutions de certaines équations des ondes semi-linéaires, *C.R.A.S. Paris* 298 (1984), 449-452.
- [7] T. CAZENAVE, A. HARAUX, Oscillatory phenomena associated to semi-linear wave equations in one spatial dimension, *Trans. A.M.S.*, in press.
- [8] T. CAZENAVE, A. HARAUX, On the nature of free oscillations associated with some semilinear wave equations, in "Nonlinear partial differential equations and their applications, College de France Seminar," vol. 7 (H. Brezis and J.L. Lions, eds.), *Research Notes in Math. No. 122*, Pittman (1984), 59-79.
- [9] C.S. GARDNER, J. GREENE, M.D. KRUSKAL, R.M. MIURA, Korteweg - DeVries equation and generalizations, methods for exact solutions, *Comm. Pure. Appl. Math.* 28 (1974), 97-133.
- [10] J.K. HALE, Ordinary differential equations, Wiley-Interscience, New York (1969).

- [11] A. HARAUX, Remarks on Hamiltonian systems, Chinese J. Math. 11 (1983), 5-32.
- [12] A. HARAUX, H. CABANNES, Almost periodic motion of a string vibrating against a straight, fixed obstacle, Nonlinear Analysis, T.M.A. 7 (1983), 129-141.
- [13] A. HARAUX, Semi-linear wave equations in bounded domains, to appear in "Mathematical reports," (J. Dieudonne, ed.).
- [14] K. KREITH, Oscillation theory, Lecture Notes in Math. No. 324, Springer (1973).
- [15] KUO PEN-YU, L. VAZQUEZ, A numerical scheme for nonlinear Klein-Gordon equations. J. Appl. Science 1 (1983).
- [16] P.D. LAX, Periodic solutions of the Korteweg-DeVries equation, Comm. Pure Appl. Math. 28 (1975), 141-188.
- [17] S.P. NOVIKOV, The periodic problem for the Korteweg-DeVries equation, Funct. Anal. Prilozh. 8 (1974), 54-66.
- [18] P.H. RABINOWITZ, Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure. Appl. Math. 20 (1967), 145-205.
- [19] P.H. RABINOWITZ, Free vibrations for a semi-linear wave equation, Comm. Pure Appl. Math. 31 (1978), 31-68.
- [20] P.H. RABINOWITZ, Subharmonic solutions of a forced wave equation, Proc. Conf. in Honor of Prof. Hartman (1982)
- [21] P.H. RABINOWITZ, Personal communication.
- [22] C. REDER, Etude qualitative d'un problème hyperbolique avec contrainte unilatérale. Thèse de 3eme cycle, Université de Bordeaux (1979).
- [23] M. SCHATZMANN, A hyperbolic problem of second order with unilateral constraints: the vibrating string with a concave obstacle, J. Math. Anal. Appl. 73 (1980), 138-191.

- [24] I. SEGAL, Nonlinear semi-groups, *Ann. of Math.* 2 (1963), 339-364.
- [25] W.A. STRAUSS, L. VAZQUEZ, Numerical solution of a nonlinear Klein-Gordon equation, *J. Comp. Phys.* 28 (1976), 271-278.
- [26] L. VAZQUEZ, Long time behavior in numerical solutions of certain dynamical systems, *An. Fis.*, to appear.
- [27] M. WILHEM, Subharmonic oscillations of a semi-linear wave equation, *Nonlinear Analysis, T.M.A.* 9 (1985), 503-514.
- [28] N.J. ZABUSKY, M.D. KRUSKAL, Interactions of solitons in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Letters* 15 (1965), 240-243.

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