

STOCHASTIC GROWTH MODELS

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IMA Preprint Series # 288

January 1987

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Introduction and Summary

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Introduction and Summary

This paper is based on a talk given by the first author at the I.M.A. in February, 1986 but incorporates improvements discovered during six later repetitions. The second author should not be held responsible for the style of presentation of the results but should be given credit for discovering the results independently in the Fall of 1985. The discussion below is equal to the talk with most of the details of the proofs filled in, but we have tried to preserve the informal style of the talk and concentrate on the "main ideas" rather than giving complete details of the proofs. If we forget about definitions then the results can be summed up in a few words "Everything Durrett and Griffeath (1983) proved for one-dimensional nearest neighbor additive growth models is true for the corresponding class of finite range models, i.e., those which can be constructed from a percolation structure."

¹ This author was partially supported by an NSF grant and an AMS "mid-career" fellowship.

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We will describe the models we consider in a minute but even before we do this it is easy to see the main point of our generalization: the words nearest neighbor have been replaced by finite range. This generalization has two benefits. The first and most obvious is that it greatly increases the number of systems to which our results can be applied.

A second benefit is that we are able to improve what is known about the discrete time contact process (and other models) in Z^2 . To be precise results which Durrett and Griffeath (1982) could only prove for $p > p_c(Z)$ the critical value for the process on the integers Z can now be shown for $p > p_c(Z \times \{-L, \dots, L\})$ for any $L < \infty$. Presumably

$$\lim_{L \rightarrow \infty} p_c(Z \times \{-L, \dots, L\}) = p_c(Z^2).$$

(and then our results hold for all $p > p_c(Z^2)$) but we have no idea how to prove this, and in any case we are getting way ahead of ourselves. We will discuss the last topic in Section 6 but before this a number of other things must be done (e.g. defining $p_c(Z)$). In Section 1 we will describe the class of models for which we can prove our results. These "generalized percolation processes (g.p.p.s)" in Z^d are generalizations of oriented percolation in Z^{d+1} and have two special properties (additivity and duality) which make them easier to study than other discrete time growth models.

In Section 2 we will begin our study of g.p.p.'s by describing the questions we want to study, and the set up we will use to formulate our answers. The real work begins in Section 3 when we prove that "edge speeds characterize p_c ". This is one of two keys to developments that follow, the other being the renormalized bond construction described in Section 4. That construction, in the words of Durrett and Griffeath (1983), "was inspired by work of Russo and Kesten and allows us to reduce results concerning supercritical contact processes to corresponding results about 1-dependent oriented percolation with p arbitrarily close to 1".

Once one has the two results in the last paragraph one can, following the pattern of Durrett (1984), obtain a large number of results. In Sections 5 and

6 we will prove two of the most important of these: the complete convergence theorem (which is called complete because it describes the limit in distribution starting from any initial configuration) and the strong law for $|\xi_n^0|$, the number of particles at time n starting from a single particle at 0 . Exponential estimates and large deviations results like those in Sections 10-13 of Durrett (1984) could also be proved but no new ideas are needed so we will leave this as an exercise for the reader.

Given the dates of the papers with the two "keys" to the proof the reader may ask why he had to wait until 1986 for the results we have here. The answer is simple: the approach of Durrett (1980) relies on a "coupling" result which is a special feature of the nearest neighbor case (see Lemma 3.4 in Durrett (1980) or (6) in Section 3 below) and only recently did we have the idea to go around this step using the renormalized bond construction (Note: to close the circle, when we are done we can go back and prove that the coupling result is almost correct, see Section 6).

Having extolled the virtues of our results it is only fitting to close this introduction by listing their weaknesses. The first and most obvious is that we are able to prove our results only for generalized percolation processes and not for the more general class of monotone (or attractive) growth models. (If these terms are unfamiliar they will be defined in Section 1). Accomplishing that generalization will require a new idea and not just rearranging the old ones.

A second more technical defect is that we have only proved the result in discrete time. The reader will see the reason for this at the end of Section 4 when we use the green bonds to tie the blue paths together. This part of the argument can undoubtedly be done in continuous time but would require many more technical details, since continuous time paths can move arbitrarily fast while paths for a finite range discrete time system have a strict speed limit.

1. Description of the Models

In this section we will describe the various classes of models we will consider in this paper. In all cases the system will be a discrete time Markov

chain whose state at time n is $\xi_n \subset Z^d$ and which evolves according to the following rules

$$(i) \quad P(x \in \xi_{n+1} | \xi_n) = g(\xi_n(x + y_1), \dots, \xi(x + y_k)).$$

where $k < \infty$ $\{y_1, \dots, y_k\} \subset Z^d$ and we have used coordinate notation for the random set: $\xi_n(x) = 1$ if $x \in \xi_n$ and $\xi_n(x) = 0$ if $x \notin \xi_n$.

(ii) given ξ_n , the state at time $n + 1$ is decided by flipping independent coins, i.e. for any j and $x_1, \dots, x_j \in Z^d$

$$P(x_i \in \xi_{n+1} \text{ for } 1 \leq i \leq j | \xi_n) = \prod_{i=1}^j P(x_i \in \xi_{n+1} | \xi_n).$$

Systems which satisfy (i) and (ii) are what we would call discrete time particle systems but are often referred to in the physics literature as stochastic cellular automata. (See Kinzel (1985)). If we impose the additional condition

(iii) g is monotone i.e. if $x < y$ coordinatewise then $g(x) < g(y)$ then we say the process is monotone or "attractive"

and if we insist in addition that

(iv) there is no creation from nothing, i.e. $g(0) = 0$

then we have the class of processes mentioned in the title of the paper: stochastic growth models.

If one thinks (as we do) of the points in ξ_n as being occupied by a particle (think of an animal or better yet a plant) then assumption (iv) is clearly natural. Assumption (iii) is also reasonable. The probability of a birth should be an increasing function of the occupancy of the neighbors (unless severe overcrowding cause higher death rates). In any case, assumption (iii) is very useful (see Liggett (1985), Chapter III, Section 2) and for most of our results we will have to restrict our attention to an even smaller class of processes which are the discrete time analogues of the additive process of Harris (1978) and Griffeath (1979).

These generalized percolation processes are constructed from a "graphical representation". Specifically, we make Z^2 into a random graph in which the oriented bond $(x - y, n) \rightarrow (x, n + 1)$ is open (resp. closed) with probability $f(y)$ (resp. $1 - f(y)$); bonds ending at different sites are independent; and the system is translation invariant (so the joint distribution of bonds ending at a given site is always the same).

To construct the process from this graphical representation we let

$$\xi_n^A = \{y: \text{there is a path of open bonds from } (x, 0) \text{ to } (y, n) \text{ for some } x \in A\}.$$

The subscript and the superscript on ξ indicate that it is the state at time n when the initial state is A . To explain the name and the right hand side we observe that ξ_n^A is the set of wet sites at level n if we imagine there is a source of fluid at $(x, 0)$ for each $x \in A$ and the fluid can travel only through open bonds.

A few examples should help clarify the definitions.

Example 1: Oriented bond percolation. In this model $f(y) = p$ if $y \in S$ where S is a finite set and all the bonds are independently open or closed. The name comes from the fact that in the special case $S = \{0, 1\}$, ξ_n^0 is what results when we take the usual oriented bond percolation process in Z^2 , map $(x, y) \rightarrow (x, x + y)$, and look at $\{z: (z, n) \text{ can be reached from } (0, 0)\}$. For more on this see section 2 of Durrett (1984).

Example 2. Oriented site percolation. In this model $f(y) = p$ if $y \in S$ where S is a finite set (like the last model) but this time either all the bonds $(x - y, n) \rightarrow (x, n + 1)$ are open with probability p or all are closed with probability $1 - p$. Again the name comes from the fact that in the special case $S = \{0, 1\}$, ξ_n^0 is what results if we consider the points in Z^2 (called sites) to be the objects which are open or closed, define a path to be open if it contains no closed sites, map $(x, y) \rightarrow (x, x + y)$, and look at $\{z: (z, n) \text{ can be reached from } (0, 0)\}$.

Examples 1 and 2 are extreme cases and a large number of examples can be constructed by combining these two. In the next two examples we will consider what happens when g depends on two or three values of $\epsilon_n(x+y)$ to try to convince the reader that "many interesting examples but by no means all growth models are g.p.p."

Example 3: Two-site g.p.p. Consider systems in which

$$P(x \in \epsilon_{n+1} | \epsilon_n) = g(\epsilon_n(x+y_1), \epsilon_n(x+y_2))$$

where $y_1 \neq y_2$ are in Z . I claim that this model is a g.p.p. if and only if

$$g(0,0) = 0$$

$$g(0,1), g(1,0) < g(1,1) < g(0,1) + g(1,0).$$

To see this observe that if bonds $b_1 = (x - y_1, n) \rightarrow (x, n+1)$ and $b_2 = (x - y_2, n) \rightarrow (x, n+1)$ have

b_1	b_2	with probability
open	open	a
open	closed	b
closed	open	c
closed	closed	$1 - a + b + c$

then

$$g(1,1) = a + b + c$$

$$g(1,0) = a + b$$

$$g(0,1) = b + c$$

so the conditions above are necessary and sufficient to have $a, b, c > 0$

($a + b + c = g(1,1)$ so the sum is automatically < 1).

Example 4. A simple class of 3 site g.p.p. If we look at the general model on three sites then we get a bewildering number of conditions so to simplify things we will only consider what we call sum rules

$$P(x \in \xi_{n+1} | \xi_n) = f | \xi_n \cap \{x-1, x, x+1\} |$$

where $|A|$ = the number of points in A . (= the sum of the coordinates $\xi_n(x-1) + \xi_n(x) + \xi_n(x+1)$). Calculations similar to those in the last example show that these processes are g.p.p. if and only if

$$f_3 = a + 3b + 3c$$

$$f_2 = a + 3b + 2c$$

$$f_1 = a + 2b + c$$

for some $a, b, c > 0$ with $a + 3b + 3c < 1$. The last condition is automatic since $f_3 < 1$, and for the first three to hold we must have

$$c > 0: f_3 > f_2$$

$$b > 0: f_2 - f_1 > f_3 - f_2$$

$$a > 0: (f_1 - 0) - 2(f_2 - f_1) + (f_3 - f_2) > 0.$$

The inequalities above imply

$$0 < f_1 < f_2 < f_3$$

$$f_1 - 0 > f_2 - f_1 > f_3 - f_2 > 0$$

$$(f_1 - 0) - 2(f_2 - f_1) + (f_3 - f_2) > 0.$$

in contrast to the conditions for two site sum rules:

$$0 < f_1 < f_2$$

$$f_1 - 0 > f_2 - f_1.$$

We leave it to the reader to find the general result (or see Harris (1978)).

In closing the discussion of the models we would like to note that although we have arrived at our conditions from a desire to use the graphical representation, one can, after the fact, argue that the first two conditions, are not too bad biologically: increasing the number of occupied sites should increase the birth rate but each new individual should result in a smaller increase.

The third condition which says "the first difference is convex" is harder to defend but it is satisfied for three site bond percolation (where

$f_k = 1 - (1 - p)^k$. The number of unpleasant conditions we have to accept increases with the range but it is comforting to note that it is always an open set.

2. Basic Questions and Set Up

Having defined the models we want to study, the next thing to explain is what we want to prove about them. In the last section we mentioned the fact that we think of the points in ξ_n as occupied by plants or animals so it is natural to ask: Is $P(\xi_n^0 \neq \phi \text{ for all } n) > 0$? (i.e. does the species have positive probability of not dying out) and if the answer to the first question is yes, "what does ξ_n^0 look like on $\Omega_\infty = \{\xi_n^0 \neq \phi \text{ for all } n\}$?"

Most of the rest of the paper is devoted to answering the last two questions. We will not have much to say about the first but we will be able to give a fairly complete answer to the second question for all "supercritical" g.p.p. It will take a few minutes to explain what we mean by the word in quotation marks, so will postpone that for a moment and set the stage by describing what sorts of answers we have for the examples described in the last section. In oriented bond percolation (example 1) the fraction of open bonds increases to 1 as p does, so it is natural to let

$$p_c = \inf\{p: P(\xi_n^0 \neq \phi \text{ for all } n) > 0\}.$$

The first thing to be resolved is: "Is $p_c \in (0,1)$?" This question is answered by

Proposition 1. Let $|S|$ = the number of points in S .

$$\text{If } |S| > 2 \text{ then } \frac{1}{|S|} < p_c < \frac{8}{9}.$$

Proof. For the left side compare with a branching process. For the right see e.g. Durrett (1984), Section 10.

Computing p_c has turned out to be a difficult problem (see Durrett (1984), Section 6) but somewhat surprisingly it is possible to prove results valid for all $p > p_c$ without knowing what p_c is. This was done in Durrett (1984) for the case $S = \{0,1\}$ (or $= \{-1,1\}$) and will be done for general finite S below.

Having heard us say "supercritical" above the reader has probably noticed that the results above are only stated for $p > p_c$ and no mention is made of the critical case $p = p_c$. Presumably $P(\Omega_\infty) = 0$ at p_c so the asymptotic behavior of ξ_n^0 is trivial there, but this has turned out to be highly nontrivial to prove (and is an important open problem).

The situation for site percolation is the same as for bond percolation so we turn our attention now to Example 3: two site models. Suppose for simplicity that $\{y_1, y_2\} = \{0, 1\}$ and we have a sum rule, i.e. $g(1, 0) = g(0, 1) = p_1$, $g(1, 1) = p_2$. By results in Section 1 this process is a g.p.p. if and only if $p_1 < p_2 < 2p_1$ or, geometrically, (p_1, p_2) lies in the triangle with vertices $(0, 0)$, $(1, 1)$ and $(1/2, 1)$ (see Figure 2.1).

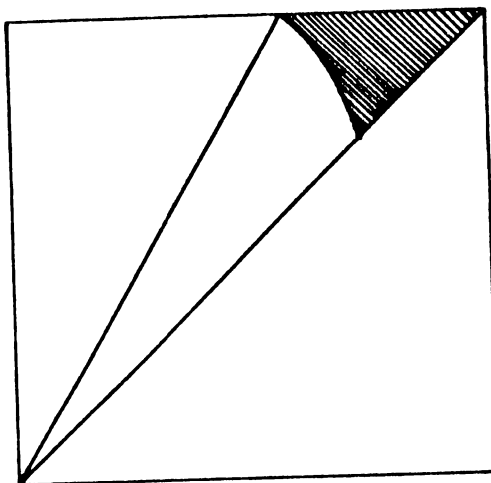


FIGURE 2.1.

The processes with $p_2 = 1$ and $p_1 = p$ are easy to understand. In this case if we draw a picture (see Figure 2.2) then it is easy to check that (for $0 < p < 1$) we have

0 0 0 ? 1 1 1 1 1 ? 0 0 0

0 0 0 1 1 1 1 1 1 0 0 0 0

? = 1 with prob p

? = 0 with prob $1-p$.

Figure 2.2.

(1) ξ_n^0 always equals $\{x: \ell_n^0 < x < r_n^0\}$, where $\ell_n^0 = \inf \xi_n^0$ and $r_n^0 = \sup \xi_n^0$.

(2) r_n^0 is a random walk which moves $x \rightarrow x + 1$ with probability p and $x \rightarrow x$ with probability $1 - p$.

(3) ℓ_n^0 is a random walk which moves $x \rightarrow x$ with probability p and $x \rightarrow x + 1$ with probability $1 - p$

and

(4) the increments $r_{n+1}^0 - r_n^0$ and $\ell_{n+1}^0 - \ell_n^0$ are independent on $\{\xi_n^0 \neq \emptyset\}$.

For (4) observe that if $\xi_n^0 = \{x\}$ then $\{x \in \xi_{n+1}^0\}$ and $\{x + 1 \in \xi_{n+1}^0\}$ are independent and these events are equal to $\{\ell_{n+1}^0 - \ell_n^0 = 0\}$ and $\{r_{n+1}^0 - r_n^0 = 1\}$. The case $|\xi_n^0| > 1$ is easier.

Combining the last four observation we see that the number of particles

$$Z_n = (1 + r_n^0 - \ell_n^0)1_{(\xi_n^0 \neq \emptyset)}$$

is a random walk starting from 1 and run until it hits 0. The mean of r_1^0 is p , the mean of ℓ_1^0 is $1 - p$, and $p > 1 - p$ if and only if $p > 1/2$, so from the last three facts it is easy to see that $p_c = 1/2$, i.e., if $p < 1/2$ then the increments in Z_n have negative mean and $P(\xi_n^0 \neq \emptyset \text{ for all } n) = 0$. On the other hand if $p > 1/2$ then $E r_1^0 - E \ell_1^0 = c > 0$ and we have $P(\Omega_\infty) > 0$.

For comparison with later results and an earlier conjecture, we would like the reader to observe that

$$P(\Omega_\infty) = 0 \quad \text{at } p_c,$$

and if $p > p_c$ then on Ω_∞

$$\frac{r_n^0}{n} \rightarrow p \quad \text{a.s.}$$

$$\frac{\ell_n^0}{n} \rightarrow 1 - p \quad \text{a.s.}$$

$$\frac{|\xi_n^0|}{n} \rightarrow 2p - 1 \quad \text{a.s.}$$

Having solved our problems when $p_2 = 1$ and $p_1 = p$ it is natural to make this a starting point for investigating the rest of the triangle $(0,0)$, $(1,1)$, $(1/2,1)$. Let

$$p_{1,c}(\theta) = \inf\{p: P_{p,\theta}(\Omega_\infty) > 0\}$$

where $P_{p,\theta}$ is the probability measure for the system with $p_1 = p$ and $p_2 = \theta$. As θ decreases from 1, $p_{1,c}(\theta)$ increases (i.e. if $\theta_1 < \theta_2$ $p_{1,c}(\theta_1) > p_{1,c}(\theta_2)$) at least as long as $(p_{1,c}(\theta), \theta)$ stays in the set $\{(p_1, p_2): p_1 \leq p_2\}$ of attractive interactions. [Life below the diagonal will be the subject of a later paper].

The models with $p_2 = \theta$, $p_1 = p_{1,c}(\theta)$ are "critical" since they lie on the boundary of $\{(p_1, p_2): P(\Omega_\infty) > 0\}$ and we will have nothing to say about them (except that they presumably also have $P(\Omega_\infty) = 0$). We will however be able to prove fairly complete results about the models which live strictly above the critical curve (i.e. in the shaded region in Figure 2.1).

There is a similar but more complicated picture (which we leave for the reader to draw) for three site sum rules (example 4) or for the general case in example 3. In each situation the parameter space is three dimensional, there is a two dimensional critical surface, and we will be able to prove results about the "supercritical" models in the interior of $\{p: P(\Omega_\infty) > 0\}$.

To prove results for these supercritical models it is convenient to embed them in a one parameter family like site or bond percolation so we will assume that $(x - y, n) \rightarrow (x, n + 1)$ is open with probability $f(y, p)$. In order to prove our results we will, of course, head to make some assumptions about $f(y, p)$ and (even though the first author failed to mention this in his talk) also make assumptions about how the joint distributions change as p increases.

The first and most obvious of these is

(H1) the joint distribution of the bonds $(x - y_i, n) \rightarrow (x, n + 1)$ is stochastically increasing in p ,

i.e., if $p' < p$ then the two systems can be constructed on the same space in such a way that if a bond is open in the p' system it is also open in the p system.

We will need a little more than this at two points below: we will need to know that the systems are strictly increasing in p . The technical assumptions required will be obvious when we get there so we will introduce them as they are needed. The reader can be assured that site and bond percolation and the models above with $p_2 = \theta$ and $p_1 = p$ will always be included.

3. Edge Speeds Characterize p_c .

The monotonicity assumption we just made allows us to define a critical value

$$p_c = \inf\{p: P\{\xi_n^0 \neq \phi \text{ for all } n\} > 0\}$$

which has the property that

$$\begin{aligned} \text{if } p < p_c & \text{ then } P(\Omega_\infty) = 0 \\ \text{if } p > p_c & \text{ then } P(\Omega_\infty) > 0. \end{aligned}$$

The key to being able to prove results for all $p > p_c$ without knowing what p_c is, is finding a way to characterize p_c . The answer is hinted at in the title of the section and described below. It, like everything else in this section, is from Durrett (1980).

The first step in our analysis is to define the "right edge" by

$$r_n = \sup(\xi_n^{(-\infty, 0]})$$

and embed r_n into a two parameter process by setting

$$r_{m,n} = \sup\{y: \text{there is an open path from } (x,m) \text{ to } (y + r_{m,n}) \text{ for some } x < r_m\}.$$

In words, $r_m + r_{m,n}$ is the rightmost site we can reach at time n if we pretend all the sites to the left of r_m are occupied at time m , so it is clear

that

$$r_m + r_{m,n} > r_n$$

$$r_{m,n} \stackrel{d}{=} r_{n-m} \text{ and is independent of } r_m.$$

Combining the last two observations with some ideas from the proof of Kingman's subadditive ergodic theorem one can show

(1) As $n \rightarrow \infty$ $r_n/n \rightarrow \alpha$ almost surely where

$$\alpha = \inf_{m>1} E r_m/m.$$

For a proof see Durrett (1980), 893-896 or for a better proof see Liggett (1985), Chapter VI, Section 2.

Looking at the last argument in the mirror gives

(1') Let $\ell_n = \inf\{\xi_n^0, \infty\}$. As $n \rightarrow \infty$ $\ell_n/n \rightarrow \beta$ almost surely where $\beta = \sup_{m>1} E \ell_m/m$.

From (1) and (1') it follows immediately that we have

(2) if $\alpha < \beta$ then $P(\xi_n^0 \neq \phi \text{ for all } n) = 0$.

Proof: Let $r_n^0 = \sup \xi_n^0$. Since $\xi_n^0 \in \xi_n^{(-\infty, 0]}$ (recall that the graphical representation defines the process simultaneously for all initial states) we have $r_n^0 \leq r_n$ and since $r_n/n \rightarrow \alpha$ it follows that

$$\limsup_{n \rightarrow \infty} r_n^0/n \leq \alpha.$$

and looking in the mirror again we see

$$\liminf_{n \rightarrow \infty} \ell_n^0/n \geq \beta.$$

Combining the last two observations it follows that if n is large then

$$\sup \xi_n^0 = r_n^0 < \ell_n^0 = \inf \xi_n^0$$

with high probability, but the only set A with $\sup A < \inf A$ is the empty set ($\sup \phi = -\infty < \inf \phi = +\infty$) so the proof is complete.

Remark. By observing that

$$r_{kN} \leq r_N + r_{N,2n} + \dots + r_{(k-1)N,N}$$

and that if $\alpha > \alpha$ and N is large the right hand side is a random walk with drift $< \alpha$. (and $E \exp(\theta r_N) < \infty$ for all $\theta > 0$) it is easy to see that if $\alpha > \alpha$ then there are constants $C, \gamma \in (0, \infty)$ so that

$$P(r_n > \alpha n) \leq C e^{-\gamma n}$$

(see Durrett (1984), Section 7 for details. We will need this fact in the next section).

Having seen that $\alpha < \beta$ implies that the process dies out it is natural to ask if there is a converse. This is true but much harder to prove.

Theorem 1. If (H1)-(H3) are satisfied then

$$\begin{aligned} p_c &= \inf\{p: \alpha(p) > \beta(p)\} \\ &= \sup\{p: \alpha(p) < \beta(p)\}. \end{aligned}$$

(Note: as we mentioned above we will have to make two technical assumptions to make this a correct statement but we will introduce them as they are needed in the proof. (H2) is given in the proof of (4) below, (H3) at the very end of Section 4.)

The first step in the proof of Theorem 1 is to prove the following fact which is a generalization of an observation due to Tom Liggett. (see Durrett (1980), Lemma 4.1).

(3) If $B \subset \{-1, -2, \dots\}$ is an infinite set and we let $r_n^B = \sup \xi_n^B$ then

$$E(r_n^{B \cup \{0\}} - r_n^B) > 1.$$

The proof is the same since all that it uses is the additivity property

$$\xi_n^{A \cup B} = \xi_n^A \cup \xi_n^B$$

of processes defined on graphical representations.

With this result in hand we can repeat the proof of Lemma 4.2 of Durrett (1980) to conclude that $p \rightarrow \alpha(p)$ is strictly increasing. If we let $R = \sup\{y: f(y,p) > 0\}$ and suppose

(H2) If $f(y,p) > 0$ for some $p > 0$ then

$$\frac{\partial f(y,p)}{\partial p} > C > 0,$$

then what we obtain is

$$(4) \quad E(r_n^{p+\delta} - r_n^p) > C\delta n.$$

Proof: Since this result is different from the one given in the talk we will supply a few details. As in Durrett (1980) if we let $\tau = \inf\{s > 0: r_s^{p+\delta} > r_s^p\}$ then using the Markov property and (3) we conclude

$$E(r_t^{p+\delta} - r_t^p) > P(\tau < t)$$

but this time

$$P(\tau > n) < (1 - [f(R,p+\delta) - f(R,p)])^n$$

since if the $p + \delta$ process jumps by R and the other one doesn't the $p + \delta$ process gets ahead by 1. Dividing the interval $[p,p+\delta]$ into M pieces, using the inequality above, and letting $M \rightarrow \infty$ gives (4). For more details see Durrett (1980), p. 901.

Looking in the mirror again we have

$$(4') \quad E(\ell_n^{p+\delta} - \ell_n^p) < -C\delta n$$

for the same constant. If we combine the last result with (4), let $p_\alpha = \sup\{p: \alpha(p) < \beta(p)\}$, and let $n \rightarrow \infty$ we get

$$(5) \quad \text{if } p > p_\alpha \text{ then } \alpha(p) - \beta(p) > C(p - p_\alpha) > 0.$$

The last result implies

$$\sup\{p: \alpha(p) < 0\} = \inf\{p: \alpha(p) > 0\}$$

so the "only" thing that remains is to show that if $\alpha(p) > \beta(p)$ then

$$P(\xi_n^0 \neq \phi \text{ for all } n) > 0.$$

In Durrett (1980) this was done for Example 3 by using a coupling result which is a special feature of that case:

$$(6) \quad \text{if } \ell_n = \inf \xi_n^{[0, \infty)} \text{ and } r_n = \sup \xi_n^{(-\infty, 0]} \text{ then}$$

$$\text{on } \{\ell_m < r_m \text{ for all } m < n\}$$

$$\xi_n^0 = \xi_n^Z \cap [\ell_n, r_n]$$

$$\text{and consequently } \ell_n^0 = \ell_n, r_n^0 = r_n, \xi_n^0 \neq \phi.$$

NOTE: this coupling property is NOT true in the three site case. See Durrett (1980), Section 6 for a discussion or draw a random picture.

With (6) it was easy to prove what we wanted to (see Durrett (1980) for details) but until recently it was not clear how to do without (6). A large part of the solution it turns out was in Durrett and Griffeath (1983) so we turn to describing those results now.

4. A Renormalized Bond Construction

To almost quote Durrett (1984), p. 1023. "In this section we will introduce a construction which will allow us to reduce questions about supercritical finite range g.p.p. to corresponding questions about a k-dependent nearest neighbor site percolation process with p arbitrarily close to 1." The argument given here like the last quote is a simple modification of the corresponding thing in Durrett (1984) so we will start by describing the argument in that special case: oriented bond percolation $S = \{-1, 1\}$; and then describe the changes which are necessary for finite range g.p.p.

The first thing to do is to define the site percolation process and its relationship to the original process. Let \mathcal{L} be the graph with vertices $V = \{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is even, } n > 0\}$ and oriented bonds connecting each $(m, n) \in V$ to $(m + 1, n + 1)$ and to $(m - 1, n + 1)$. Stealing a term from the physics literature we call \mathcal{L} the renormalized lattice. To explain the name

(and the idea behind the construction) the reader should imagine \mathcal{L} mapped into the upper half plane $\mathbb{R} \times [0, \infty)$ by $\phi(x,y) = (aLx, Ly)$ where a is a special constant and L is a large number to be chosen below.

We will define the site (m,n) in V to be open if a "good event" happens in the graphical representation near $z_{m,n} \in \mathbb{R} \times [0, \infty)$ and we will do this in such a way that

- (i) the random variables $\eta(v)$ $v \in V$ which indicate whether the sites are open or not are k -dependent (i.e. if the distance from x to y on the graph $> k$ they are independent).
- (ii) if L is large the probability $\eta(v) = 1$ is close to 1.
- (iii) if percolation occurs starting from 0 on the renormalized lattice then it does starting from some point near $z_{0,0}$ in the original percolation process.

It is by now well known that (i) and (ii) imply that if L is large then the probability of percolation is positive (for more on this see Durrett (1984), Section 10) so once (i)-(iii) are demonstrated we can conclude that if $\alpha(p) > \beta(p)$ then $P(\xi_n^0 \neq \phi \text{ for all } n) > 0$ completing the proof of Theorem 1 in Section 2. We will see below that the construction can be used to prove a number of other things about percolation processes so if the reader gets bored or confused by the details of the construction, he/she/it should skip ahead to the next two sections to see part of what it is good for: the complete convergence theorem and the strong law for ξ_n^0 .

Details of the construction (oriented percolation $S = \{-1, 1\}$).

We begin by describing the fundamental building blocks: the renormalized bonds which appear in the title of the section. Let A be the parallelogram with vertices $(-2\epsilon L, 0)$, $(2\epsilon L, 0)$, $((\alpha - 2\epsilon)L, L)$, and $((\alpha + 2\epsilon)L, L)$. From (1) in Section 3 we know that $r_L/L \rightarrow \alpha$ as $L \rightarrow \infty$, so if $\delta > 0$ and $L > L_0(\delta)$ then

$$P(r_L \in ((\alpha - \epsilon)L, (\alpha + \epsilon)L)) > 1 - \delta$$

When $r_L \in ((\alpha - \epsilon)L, (\alpha + \epsilon)L)$, we know that there is a path from $(-\infty, 0] \times \{0\}$ to $((\alpha - \epsilon)L, (\alpha + \epsilon)L) \times \{L\}$ but it might look like the dotted line in Figure 4.1. Our next task is to show that it looks like the solid line. i.e. it stays

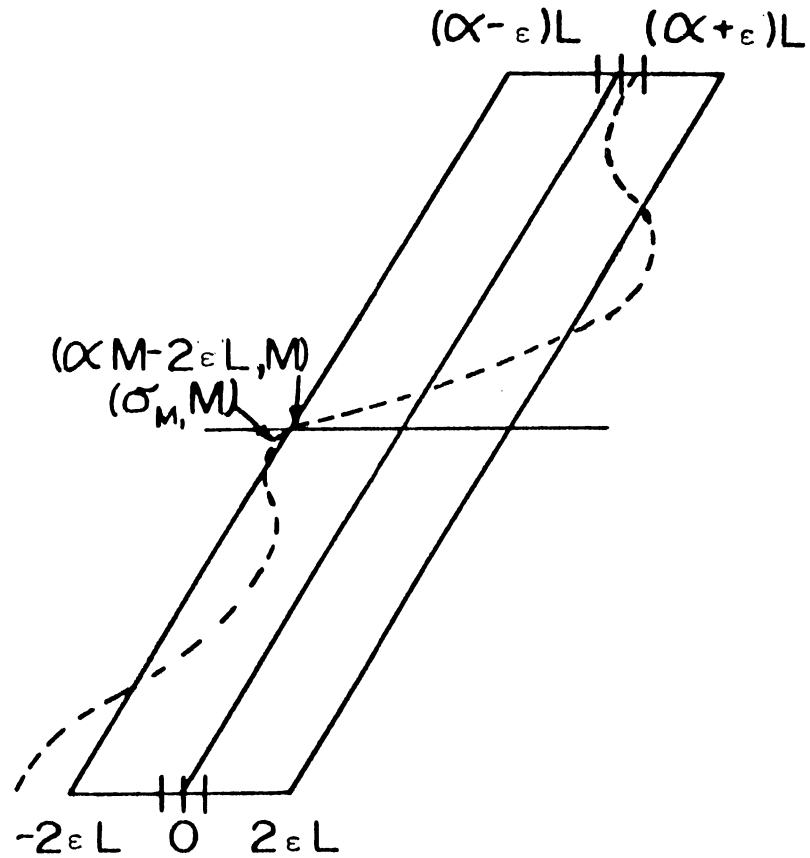


FIGURE 4.1.

in the parallelogram. To prove that it doesn't hit the right side we observe that if N is large $Er_N < (\alpha + \epsilon)N$ and

$$r_{kN} \leq r_N + r_{N,2N} + \dots + r_{(k-1)N,kN}$$

with the right hand side being a random walk, so the simple argument we used in the last section shows that there are constants $C, \gamma \in (0, \infty)$ such that

$$(1) \quad P(r_m > (\alpha + \epsilon)m) \leq Ce^{-\gamma m}$$

Now if $R = \sup\{y: f(y, p) > 0\}$ (which is independent of $p > 0$ by (H2)) the right edge can increase by at most R per jump so

$$\begin{aligned}
P(r_m \text{ exits right side of } A) &< \sum_{m=2\epsilon L/R}^{\infty} P(r_m > (\alpha + \epsilon)m) \\
&< Ce^{-\gamma L}
\end{aligned}$$

(where $C, \gamma \in (0, \infty)$ are new constants and will continue to change as we go along.)

To estimate the probability that the path escapes from the left side of the box we use an observation due to Larry Gray which allows us to turn our upper bound into a lower bound. Let σ_t $0 \leq t \leq L$ be a path from $(-\infty, 0] \times \{0\}$ to $((\alpha - \epsilon)L, (\alpha + \epsilon)L) \times \{L\}$ in the graphical representation used to construct the g.p.p. Let $M = \sup\{m: (\sigma_m, m) \notin A\}$ Larry's simple but useful insight is (see Figure 4.1 again) that the line from (σ_M, M) to (σ_L, L) has slope $> (\alpha + \epsilon)$ so if $M = m$ then the right edge of the process starting from $(-\infty, -2\epsilon L + \alpha m]$ at time m must be $> (\alpha + \epsilon)(L - m)$ at time L and summing the estimate in (1) we conclude that

$$P(\sigma_m \text{ exits left side of } A) < Ce^{-\gamma L}$$

Combining the last three estimates shows that if $\epsilon > 0$ and $L > L_1(\delta)$ then with probability $> 1 - 3\delta$ there is a path lying in A . These events are the raw material for the construction that follows. The next step in carrying it out is to associate the sites in the renormalized lattice with translates of A in the percolation structure.

Drawing a picture (see Figure 4.2) motivates letting

$$Z_{m,n} = ((\alpha - 4\epsilon)m, n) \quad (m, n) \in \nu$$

be the points of the renormalized lattice, defining translates of A by

$$\begin{aligned}
A_{m,n} &= (z_{m,n} + (-4\epsilon L, 0)) + A \\
B_{m,n} &= (z_{m,n} + (4\epsilon L, 0)) - A
\end{aligned}$$

(where $x - A = \{x - y: y \in A\}$ etc.) and declaring that the site (m, n) is open if there are paths in $A_{m,n}$ and in $B_{m,n}$.

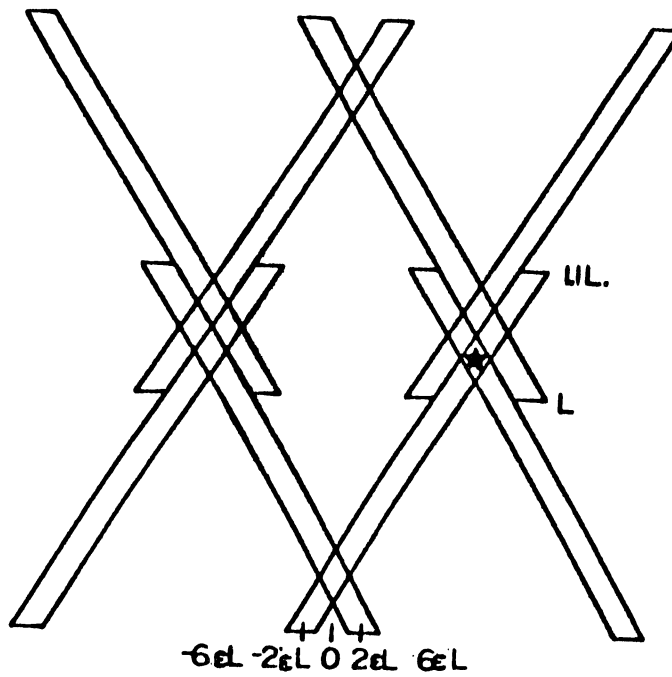


FIGURE 4.2.

From the definition it is clear that we have property (i) and the arguments above show that (ii) holds. To check (iii) we observe that in the case under consideration paths cannot "jump over each other" so the arrangement of the $A_{m,n}$ and $B_{m,n}$ guarantees that if say $(m-1, n-1)$ and (m, n) are open then there is a path from $z_{m-1, n-1} + (-6\epsilon L, -2\epsilon L)$ to $z_{m+1, n+1} + (2\epsilon L, 6\epsilon L)$ and to $z_{m-1, n+1} + (-6\epsilon L, -2\epsilon L)$ (see Figure 4.3). From this it follows easily that (iii) holds. (for more details consult Durrett (1984). p. 1025).

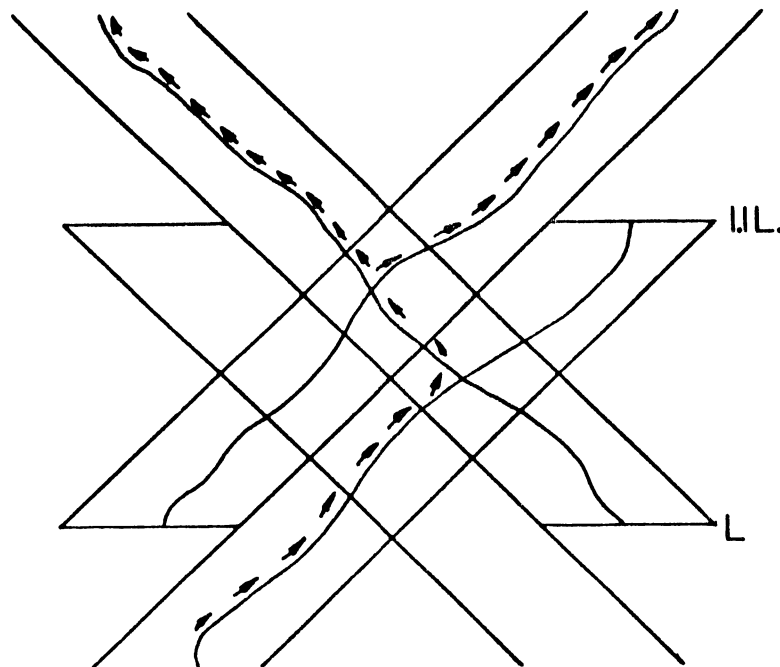


FIGURE 4.3.

Details of the construction (finite range g.p.p)

The last argument does not work outside the case $S = \{-1,1\}$ (or $S = \{0,1\}$) because paths which cross do not need to intersect but (here finally is our new idea) paths which cross will intersect with probability $> \eta > 0$ so if we use a zillion little paths to try to connect two of the (long) paths used in the construction then there will be a success with high probability. As the reader can probably guess carrying out this idea requires a little ingenuity and a large number of unpleasant details. To keep things as simple as possible we will first give the details for oriented bond percolation with $S = \{x: |x| \leq R\}$ and then treat the general case.

The first step in the argument is to make the tubes smaller. Let A' be the left half of A , i.e. the parallelogram with vertices $(-2\epsilon L, 0)$, $(0, 0)$, $((\alpha - 2\epsilon)L, L)$ and $(\alpha L, L)$. We keep the renormalized lattice the same

$$z_{m,n} = ((\alpha - 4\epsilon)Lm, Ln) \quad (m,n) \in V$$

define translates of A' as before by

$$A'_{m,n} = (z_{m,n} + (-4\epsilon L, 0)) + A'$$

$$B'_{m,n} = (z_{m,n} + (4\epsilon L, 0)) - A'.$$

Thinning the tubes creates space near each point of the renormalized lattice (see Figure 4.4) and into this space which has width $4\epsilon L$ we put $\lceil 4/\epsilon \rceil$ tubes of width $\epsilon^2 L$ and length $2L$ in the manner indicated in the picture and then (for reasons that will become clear in a minute) we remove every other one. Paths in the new smaller tubes will be used to connect the paths in the four large tubes.

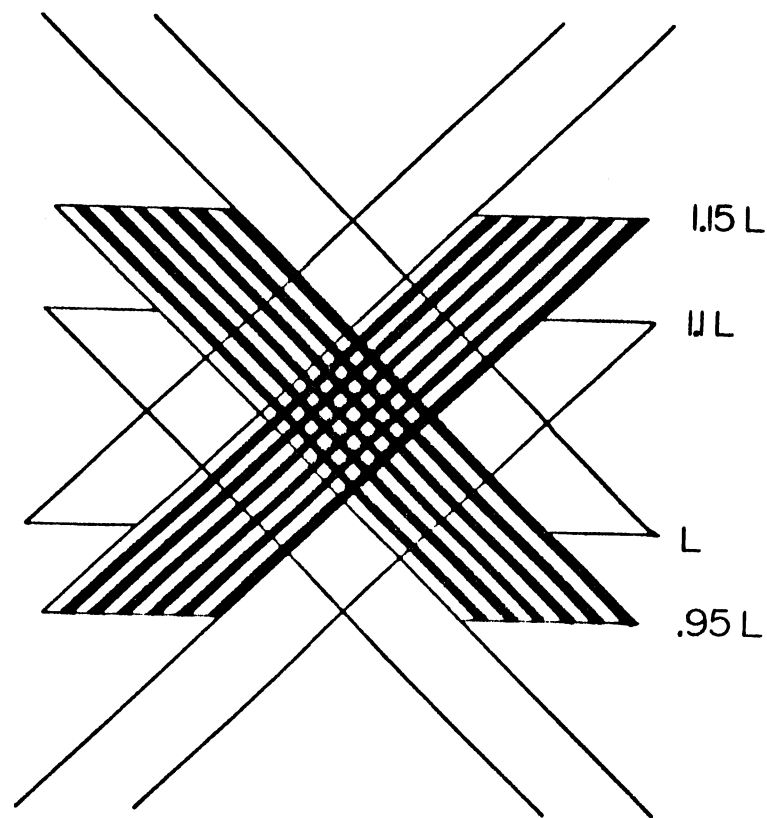


FIGURE 4.4.

Tying paths together is delicate because of "conditioning problems" - i.e. picking a path by some algorithm makes the conditional distribution of bonds near the chosen path different from the original one. To avoid difficulties of this type we "save a little randomness to make connections at the end." To be precise:

- (a) We pick $p' < p$ with $\alpha(p') > 0$ (and observe that in the proof of $p_c < p_\alpha$ - which is what we're doing now! - this can be done without loss of generality.)
- (b) Construct the processes with parameters p and p' on the same space by assigning independent uniformly distributed random variables $U(b)$ to each bond $b = (x-y, n) \rightarrow (x, n+1)$ where $|y| < R$ and declaring b to be open for the p' (resp. p) system if $U(b) < p'$ (resp. $U(b) < p$).
- (c) Do the renormalized bond construction for the p' system (with the corresponding $\alpha(p') > 0$) and call one of the large or small tubes in the construction good if it has a path in the p' percolation structure from one end to the other which stays in the tube.

Since all the paths in the little tubes must pass within R of a path in the large tube and the number of little tubes is large then it is clear that if ϵ is small then the situation drawn in Figure 4.5 will occur with high probability i.e. there are paths in the small tubes which intersect and intersect the paths in the four large tubes.

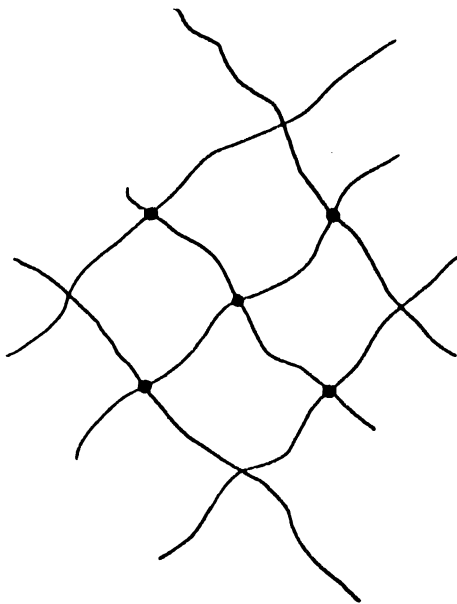


FIGURE 4.5.

To prove this we pick for each tube which was called good in (c), a path in the p' percolation structure with the desired properties and for ease of reference later, we will say that these paths are drawn in blue. Now each pair of blue paths σ, τ that we want to connect must come within a distance R of each other at some point, i.e. there are integers x, y and n with $\sigma_n = x$, $\tau_{n+1} = y$ and $|x - y| < R$.

Now if we condition on the value of $U(b) \wedge p'$ for all the bonds b then there is still probability $> (p - p') / (1 - p') > 0$ that $U((x, n) \rightarrow (y, n + 1)) < p$ and hence open in the p percolation structure. When present the "green bond" $(x, n) \rightarrow (y, n + 1)$ (its color intended to signify its unconditioned state) allows us to connect the blue paths. Since we have arranged for there to be lots of little tubes and we have separated them by removing every other one to make the connection events independent, it follows that if ϵ is small and L is large then all the desired connections happen with high probability.

Given the argument for oriented percolation on $S = \{-1,1\}$ the denouement should be clear at this point. We declare a site in the renormalized lattice to be open if there are paths in the four large tubes near it and the green bond construction above succeeds in connecting them as indicated in Figure 4.5. From the definition it is clear that we have property (i) listed in the first version of the proof and the arguments above show that (ii) holds. To check (iii) observe that above we have been careful to choose one path in each large tube and then connect these paths so, having worked harder to get here, the last step is now trivial.

With (i)-(iii) verified the rest follows as before and we have completed the proof for oriented bond percolation with $S = \{x: |x| < R\}$. In tackling the general g.p.p the first (trivial) extension to be considered is what happens for other oriented percolation processes, e.g. bond percolation process with $S = \{1,2,3,4\}$, $S = \{-2,2\}$, $S = \{-51,50\}$,... In the first case mentioned we just need to slant the construction: if $(m,n) \in V$ and $\ell = n - m$ then

$$z_{m,n} = (n - \ell)(\alpha - 4\epsilon) + \ell(\beta - 4\epsilon)$$

(for more details in the nearest neighbor case see Schonmann (1986)). In the second case (like $S = \{-1,1\}$) restricting to a sublattice gives a problem to which the results for solid intervals can be applied. Last but not least when S is not a solid interval but the group it generates is all of Z , a finite number of iterates allow us to reach all points in an interval and blah, blah, blah.

The generalization mentioned in the last paragraph are, like the extension of Markov Chain results from the case of a positive matrix to that of an irreducible, one, routine although somewhat tedious and hence are left as an exercise for an energetic reader. We turn now to the last important item of business: proving the result for a general g.p.p. Having discussed the asymmetric and non-interval cases above we will assume that model is symmetric and $f(y,p) > 0$ if and only if $|y| < R$.

Looking back at the proof now it is clear that special properties of oriented bond percolation were only used in the (p',p) property of the

construction above and for this the important point was

(H3). If we condition on the state of all bonds in the p' -system then for any x, n and $|y| < R$ there is always conditional probability $> \delta(p, p') > 0$ that $(x - y, n) \rightarrow (x, n + 1)$ is open.

This is our last hypothesis that "the models increase strictly with p and with it made it is trivial to complete the proof.

5. The Complete Convergence Theorem

In this section we will prove a result which allows us to determine the limiting distribution of ξ_n^A for any A when $p > p_c$. The first step in doing this is to describe the process which appears in the limit theorem.

Let $\tilde{\xi}_n$ be the process generated by the graphical representation in which the oriented bond $(x, n) \rightarrow (x - y, n + 1)$ is open (closed) with probability $f(y)$ (resp. $1 - f(y)$); bonds beginning at different sites are independent; and the joint distribution of the bonds $(x, n) \rightarrow (x - y, n + 1)$ $y \in Z$ is the same as that of $(x - y, n - 1) \rightarrow (x, n)$ in ξ_n . Comparing the last paragraph with the definition in Section 1 it should be clear that the new graphical representation can be obtained by reversing time (and the direction of the arrows) in the old one and a little more thought leads us to the following important conclusion

$$(1) \quad P(\xi_n^A \cap B \neq \phi) = P(A \cap \tilde{\xi}_n^B \neq \phi).$$

Proof: From the definition of ξ_n^A we see that $\{\xi_n^A \cap B \neq \phi\} = \{\text{there is an open path from } (x, 0) \text{ to } (y, n) \text{ for some } x \in A, y \in B \text{ and from the discussion above we see that the right hand side is equal to the probability of a path down from } (y, n) \text{ to } (x, 0) \text{ in the same percolation structure.}$

Taking $A = Z$ in (1) we see that $P(\xi_n^Z \cap B \neq \phi) = P(\tilde{\xi}_n^B \neq \phi)$ which decreases to a limit as $n \rightarrow \infty$ (since ϕ is an absorbing set for $\tilde{\xi}_n^B$). The inclusion-exclusion formula allows us to write all probabilities of the form

$$P(\xi_n^Z(x_1) = i_1, \dots, \xi_n^Z(x_k) = i_k)$$

where $\{x_1, \dots, x_k\} \subset Z$ and $i_1, \dots, i_k \in \{0,1\}$ in terms of $P(\xi_n^Z \cap B = \phi)$ so it follows that we have

$$(2) \quad \text{As } n \rightarrow \infty \quad \xi_n^Z \Rightarrow \text{to a limit } \xi_\infty^Z,$$

where \Rightarrow denotes weak convergence of probability measures on $\{0,1\}^Z$ (which in this setting is = convergence of finite dimensional distributions).

Having defined the limit we can now state our convergence result

Theorem 2. If $\alpha(p) > 0 > \beta(p)$ then as $n \rightarrow \infty$

$$\xi_n^A \Rightarrow \delta_\phi P(\tau^A < \infty) + \xi_\infty^Z P(\tau^A = \infty).$$

where $\tau^A = \inf\{m > 0: \xi_m^A = \phi\}$,

δ_ϕ = the point mass on the empty set ϕ ,

and we use ξ_∞^Z to denote the limit distribution starting from $\xi_0^Z = Z$.

The first part of the right hand side is easy to see: on $\{\tau^A < \infty\}$ we have $\xi_n^A = \phi$ for $n > \tau^A$. The second part is much harder to prove: it says that if ξ_n^A does not die out (i.e. $\tau^A = \infty$) and n is large then ξ_n^A looks like ξ_n^Z with high probability on any (fixed) finite set. (We will prove a sharper version of this in the next section).

An immediate consequence of Theorem 2 is that all stationary distributions have the form $\theta \delta_\phi + (1 - \theta) \xi_\infty^Z$ for some $\theta \in [0,1]$. When confronted with the last observation the reader should ask: is $\xi_\infty^Z \neq \delta_\phi$ for $p > p_c$? The density of particles in ξ_∞^Z can be read off from the duality equation (1):

$$P(x \in \xi_\infty^Z) = P(\tilde{\xi}_n^0 \neq \phi \text{ for all } n).$$

so if we let

$$p_e = \inf\{p: \xi_\infty^Z \neq \delta_\phi\}$$

(where e is equilibrium) then it is clear that we have

$$p_e = \tilde{p}_c = \inf\{p: P(\tilde{\xi}_n^0 \neq \phi \text{ for all } n) > 0\},$$

and the question becomes " $p_c = \tilde{p}_c$ " ? The answer as we will see in the proof is: Yes.

The first step in proving Theorem 2 is to observe that if ξ_t^A and $\tilde{\xi}_t^B$ are independent

$$(3) \quad P(\xi_{2t}^A \cap B \neq \phi) = P(\xi_t^A \cap \tilde{\xi}_t^B \neq \phi).$$

(the first event being the probability of a path from $(x,0)$ to $(y,2t)$ for some $x \in A, y \in B$ while the second = {there are $x \in A, y \in B,$ and $z \in Z$ so that $(x,0) \rightarrow (z,t) \rightarrow (y,2t)$ }). Now

$$P(\xi_t^A \cap \tilde{\xi}_t^B \neq \phi) = P(\xi_t^A \neq \phi, \tilde{\xi}_t^B \neq \phi) - P(\xi_t^A \neq \phi, \tilde{\xi}_t^B \neq \phi, \xi_t^A \cap \tilde{\xi}_t^B = \phi)$$

and the first term = $P(\xi_t^A \neq \phi)P(\tilde{\xi}_t^B \neq \phi)$ which converges to

$P(\tau^A = \infty)P(\xi_\infty^Z \cap B \neq \phi)$ as $t \rightarrow \infty$ so to prove the theorem it suffices to show

$$(4) \quad P(\xi_t^A \neq \phi, \tilde{\xi}_t^B \neq \phi, \xi_t^A \cap \tilde{\xi}_t^B = \phi) \rightarrow 0.$$

The proof of (4) requires one new bit of inspiration (followed by quite a bit of perspiration) so we will start by stating the new idea: "when the renormalized bond construction works it produces points at a positive density of sites between the left and right edges so if $\beta(p) < 0 < \alpha(p)$ and the time is large t then ξ_t^A and $\tilde{\xi}_t^B$ will intersect with high probability (if both are nonempty)."

With this idea in mind the rest is routine following the proof of similar results in Durrett (1984) so we will just give an outline.

The sentences in quotation marks below were the ones I said during my talk. In between them I have tried to supply enough details so that the reader (with the help of the paper cited above) can fill in the rest. As in the last section the summary becomes tedious or confusing the reader can safely skip to the beginning of the next section where we will start to consider $|\xi_n^0|$ = the number of occupied sites at time n .

1. "After a geometric number of trials either (a) $\varepsilon_n^A = \phi$ or (b) the renormalized bond construction works and on the renormalized lattice ε_n^A dominates oriented percolation with p close to 1."

The proof is a "restart argument" following Durrett (1980), 903-904 and/or Durrett (1984), 1031-1032. The proof is based on a simple idea: "if at first you don't succeed try, try again" but requires a depressing number of definitions to carry out.

Let $x_0 = \sup A$. If $M > 6\varepsilon L$ and we are very lucky then (i) $[-6\varepsilon L + x_0, 6\varepsilon L + x_0] \subset \varepsilon_M^{x_0}$ and (ii) we get a path to ∞ on the renormalized lattice when we try the renormalized bond construction translated by x_0 . These are the two things we dream about and they have positive probability of happening on the first try.

When they don't then we have to go to work: if (i) does not occur and $\varepsilon_M^A = \phi$ then we are happy since all we have to do is show that things are OK when $\varepsilon_n^A \neq \phi$. If (i) does not occur and $\varepsilon_M^A \neq \phi$ we let $x_1 = \sup \varepsilon_M^A$ and look M units of time later to see if $[-6\varepsilon L + x_1, 6\varepsilon L + x_1] \subset \varepsilon_{(x_1, M)}^{1, M}$ where the superscript indicates we are looking at the process starting from $\{x_1\}$ at time M . Each time we repeat the last step we have a positive probability of success so after a geometric number of failures we get to try the renormalized bond construction.

As the reader has probably already anticipated the renormalized bond construction may fail but if it does we try again: we wait until the process on the renormalized lattice dies out (and then wait $1.1L$ units of time more for "good luck", i.e. so that the death of the construction does not adversely effect the future development of the graphical representation) and then start again to look for an interval of length $12\varepsilon L$ to try the construction again.

2. " $\tilde{\alpha}(p) = \beta(p)$ and $\tilde{\beta}(p) = \alpha(p)$ so the same construction can be used on the dual." As observed on p. 9 of Durrett and Griffeath (1983)

$$\begin{aligned}
P(r_m > k) &= P(\text{there is a path from } (-\infty, 0] \times \{0\} \text{ to } [k, \infty) \times \{m\}) \\
&= P(\text{there is a dual path from } [k, \infty) \times \{m\} \text{ to } (-\infty, 0] \times \{0\}) \\
&= P(\text{there is a dual path from } [0, \infty) \times \{m\} \text{ to } (-\infty, -k] \times \{0\}) \\
&= P(\tilde{r}_m < -k).
\end{aligned}$$

The last identity shows $r_m^d = -\tilde{r}_m$ from which it follows immediately that $\tilde{\beta}(p) = \alpha(p)$. From this the rest of the statement in quotes follows immediately and using results from Section 4 shows $p_c = \tilde{p}_c$

3. "When the renormalized bond construction works for the process and its dual then ξ_n^A and $\tilde{\xi}_n^B$ intersect with high probability."

In this case a picture is worth (and probably replaces) a thousand words. (see Figure 5.1). The squiggly line above A and below B indicates that with high probability we have to wait at most 100 years (i.e. a time independent of t) before the renormalized bond construction works and the rest of the picture is meant to suggest that when it does then the process on the renormalized lattice dominates (k-dependent) oriented site percolation with p close to 1. Results of Section 10-11 of Durrett (1984) show that if the probability of a site being open is close to 1 then the set of occupied sites for oriented percolation on the renormalized lattice

$$\liminf_{k \rightarrow \infty} |\mathcal{L}_k^0|/k > 1 - \delta \quad \text{on} \quad \{\mathcal{L}_k^0 \neq \emptyset \text{ for all } k\}$$

almost surely. Since $\mathcal{L}_k^0 \subset \{-k, \dots, k\}$ fills up a fraction $(1-\delta)$ of the available space. The last observation implies that if we pick δ much smaller than the minimum of α and β , and ξ_n^A and ξ_n^B are $\neq \emptyset$ then there will be a large number of pairs (x_i, y_i) with $x_i \in \xi_t^A$, $y_i \in \tilde{\xi}_t^B$ and $|x_i - y_i| < L$ and running things for $2L$ more units of time we conclude

$$P(\xi_n^A \neq \emptyset, \tilde{\xi}_n^B \neq \emptyset, \xi_n^A \cap \tilde{\xi}_n^B = \emptyset) \rightarrow 0$$

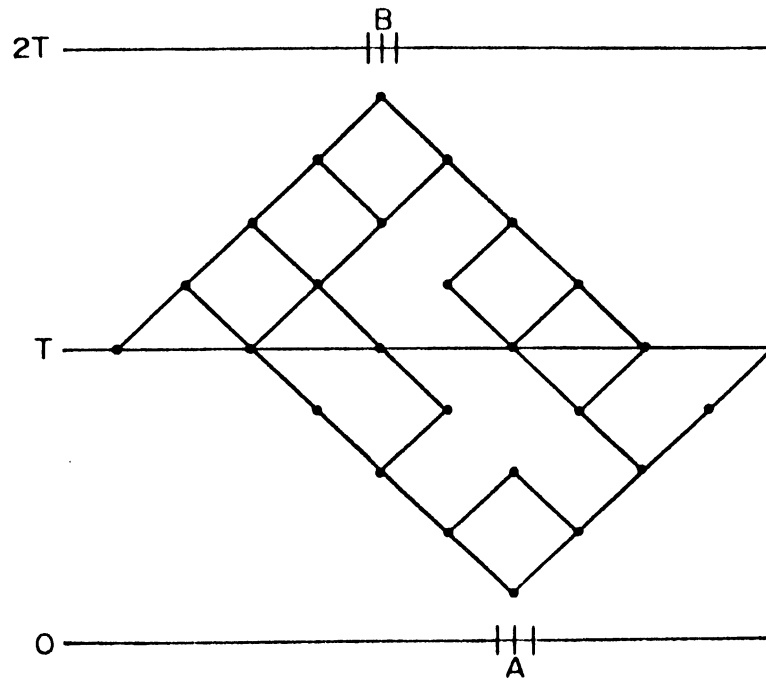


FIGURE 5.1.

6. Limit Laws for ξ_n^0

In this section we will take a closer look at the behavior of the system starting from a single particle at 0. The first step is to state a result which follows from the construction in the last section.

- (1) If $\varepsilon > 0$ then there are constants $C, \gamma \in (0, \infty)$ so that if $(\beta + \varepsilon)n < x < (\alpha - \varepsilon)n$ then

$$P(\xi_n^0 \neq \phi, \xi_n^0(x) \neq \xi_n^Z(x)) < Ce^{-\gamma n}.$$

Proof: As enunciated on p. 1031 of Durrett (1984) the proof is based on two simple ideas.

(i) If you have a sequence of independent events with probability p then K , then the number of failures before the first success has $P(K = n) = p(1 - p)^n$ $n = 0, 1, 2, \dots$ and

(ii) If X_i is a sequence of independent random variables with $P(X_i > m) < c \exp(-\gamma m)$ so that (X_1, \dots, X_k) is independent of $\{K = k\}$ then

$$P(X_1 + \dots + X_k > m) < C' \exp(-\gamma' m).$$

where C', γ' are new constants $\epsilon (0, \infty)$. To use these ideas to prove (1) you have to check that when the renormalized bond construction fails it only lasts an amount of time T with $P(T > t) \leq C \exp(-\gamma t)$ but this is true, see Durrett (1984), p. 1031-1032.

From (1) it follows immediately that we have

(2) For any $\epsilon > 0$, on $\Omega_\infty = \{\xi_n^0 \neq \phi \text{ for all } n\}$ we have

$$\{x: \xi_n^0(x) = \xi_n^Z(x)\} \supset [(\beta + \epsilon)n, (\alpha - \epsilon)n] \cap Z$$

for all n sufficiently large. With the coupling result ((6) in section 3) recaptured one can repeat arguments from Section 13 of Durrett (1984) now to show

Theorem 3. On Ω_∞ we have

$$|\xi_n^0|/n \rightarrow (\alpha - \beta) \text{ almost surely as } n \rightarrow \infty.$$

where $\rho = P(z \in Z)$ and α, β are the by now familiar limits of $r_n/n, \ell_n/n$.

The last result has a simple explanation: the distance between the left and right particles is $\sim (\alpha - \beta)n$ and this interval is filled with particles at density ρ .

7. Results for $d > 1$

Last but not least we come to the original motivation for doing this paper: to improve what is known in higher dimensions. We begin by "recalling" the results proved by Durrett and Griffeath (1982). We have put the word recalling in quotation marks because those results were proved for a class of models in continuous time (called permanent one-sided growth processes there) and we will have to ask the reader to believe that the analogous results are true in discrete time. First some notation:

$$H_n = \bigcup_{m \leq n} \xi_m^0 = \text{sites } \underline{\text{hit}} \text{ by time } n$$

$$K_n = \{x: \xi_n^0(x) = \xi_n^Z(x)\} = \text{sites } \underline{\text{coupled}} \text{ at time } n.$$

For a variety of reasons it is convenient to enlarge the last two sets by replacing each point x by a cube of side 1 centered at that point:

$$\overline{H}_n = \bigcup_{x \in H_n} x + [-\frac{1}{2}, \frac{1}{2}]^d$$

$$\overline{K}_n = \bigcup_{x \in K_n} x + [-\frac{1}{2}, \frac{1}{2}]^d$$

(the notation is meant to suggest closure).

With this notation introduced we are now (almost) ready to state the result of Durrett and Griffeath (1982). For simplicity and concreteness we will restrict our attention to oriented bond percolation in Z^3 : i.e. the process defined from the percolation structure in which all bonds are independent and $(x, n) \rightarrow (x + y, n + 1)$ is open with probability p if (and only if) $|y| = 1$.

(1) Suppose p is large enough so that the process restricted to $Z \times \{0\}$ has positive probability of survival for all time. There is a (non random) convex set U so that on $\Omega_\infty = \{\xi_n^0 \neq \phi \text{ for all } n\}$ we have

$$\frac{1}{n} (\overline{H}_n \cap \overline{K}_n) \rightarrow U \text{ a.s. on } \Omega_\infty$$

as $n \rightarrow \infty$ i.e. for any $\epsilon > 0$ and $\omega \in \Omega_\infty$

$$(1 - \epsilon)nU \subset (\overline{H}_n \cap \overline{K}_n) \subset (1 + \epsilon)nU$$

for all n sufficiently large.

Roughly speaking (1) says that ξ_n^0 looks like $\xi_\infty^Z \cap nU$ on Ω_∞ or even rougher it is a "blob in equilibrium" in the terminology of Durrett and Griffeath (1982). The statement of (1) is made contorted by the fact that $K_n \supset \{x: \xi_n(x) = 0\}$ for trivial reasons so we have to intersect with H_n to get the interesting part. The strength of this is the fact that the theorem says "almost everywhere we have hit we are in equilibrium" and has as a consequence the complete convergence theorem

(2) For any A , as $n \rightarrow \infty$

$$\xi_n^A \Rightarrow P(\tau^A < \infty) \delta_\phi + P(\tau^A = \infty) \xi_\infty^Z.$$

So much for the virtues of (1). Its shortcoming is obvious: the result is only for $p > p_c(Z)$ the critical value for the process on Z and not for $p > p_c(Z^2)$. The next result, our last theorem improves this but does not yet complete the story. Let p_c^L be the critical value for oriented percolation in $Z^Z \times \{-L, \dots, L\}$. It is easy to see that as $L \rightarrow \infty$ p_c^L decreases to a limit we call p_c^∞ and it is natural (if somewhat optimistic) to conjecture that $p_c^\infty = p_c(Z^3)$. In any case the next result improves on (1) but is not the last word. The reader should note that the complete convergence theorem is again a consequence.

Theorem 4. If $p > p_c^\infty$ then there is a nonrandom convex set U so that on $\Omega_\infty = \{\xi_n^0 \neq \phi \text{ for all } n\}$ we have

$$\frac{1}{n} (H_n^- \cap K_n^-) \rightarrow U \text{ a.s.}$$

as $n \rightarrow \infty$.

This result can be proved by using an abstract theorem (see Durrett and Griffeath (1982), p. 529) which was designed five years ago for the application we are making today: all we have to do is check that the three conditions of the theorem hold and then Theorem 3 follows. If we let $\tau = \inf\{n: \xi_n^0 = \phi\}$ then what we need to show is that there are constants $\delta, C, \gamma \in (0, \infty)$ so that

$$(a) \quad P(n < \tau < \infty) \leq C e^{-\gamma n}$$

$$(b) \quad P(x \in H_n, \tau = \infty) \leq C e^{-\gamma n} \text{ if } |x| < \delta n$$

$$(c) \quad P(x \in K_n, \tau = \infty) \leq C e^{-\gamma n} \text{ if } |x| < \delta n.$$

Checking (a), (b), and (c) is neither trivial nor pleasant but following the argument on p. 545-550 in Durrett and Griffeath (1982) and using the renormalized bond construction one can do this. Details of the proof of this will be the subject of a future publication.

Acknowledgements

RHS would like to thank the Rutgers mathematics department for the warm hospitality it showed him during his stay there.

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