ON THE APPROXIMATION OF MISCIBLE DISPLACEMENT IN POROUS MEDIA BY A METHOD OF
CHARACTERISTICS COMBINED WITH A MIXED METHOD

by

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Abstract

A miscible displacement of one incompressible fluid by another is modeled by a coupled system of partial differential equations. The approximation of this system using a Galerkin method that makes use of a modified method of characteristics for the concentration equation combined with a mixed finite element method for the pressure equation has been analysed in [9].

In this work, we prove optimal error estimates in L^2 for this algorithm under milder restrictions in the time step than those required in the previous analysis.

In particular, in the two dimensional case our estimates are valid when the time and space discretization parameters are of the same order, which is a reasonable assumption in practice.

Introduction

We consider the miscible displacement of one incompressible fluid by another in a porous medium Ω over the time interval $[0, T] = J$. Conservation of mass for the fluid mixture and for one of the fluids lead to the following coupled system of equations for the pressure $p(x, t)$ and the concentration of one of the fluids $c(x, t)$ (see [16]).

$$a) \quad -\operatorname{div} \left(\frac{k}{\mu(c)} (\nabla p - \gamma(c) \nabla d) \right) = \operatorname{div} u = q \quad \text{in } \Omega \times J$$

$$b) \quad \phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \operatorname{div} (D(u) \nabla c) = (\tilde{c} - c) \tilde{q} \quad \text{in } \Omega \times J$$

(0.1)

$$c) \quad u \cdot \eta = 0 \quad \text{and} \quad (D(u) \nabla c) \cdot \eta = 0 \quad \text{on } \partial\Omega \times J$$

$$d) \quad c(x, 0) = c_0(x) \quad \text{in } \Omega$$

where $k(x)$ is the permeability of the porous rock, $\mu(c)$ the viscosity of the fluid mixture; $\gamma(x, c)$ and $d(x)$ the gravity coefficient and vertical coordinate; $u(x, t)$ the Darcy velocity of the mixture; $q(x, t)$ represents flow rate at wells, $\phi(x)$ is the porosity of the rock; $D(x, u)$ is the coefficient of molecular diffusion and mechanical dispersion of one fluid into the other; $\tilde{c}(x, t)$ is the injected concentration at injection wells and the resident concentration at production wells; $c_0(x)$ is the initial concentration and $\tilde{q} = \max(q, 0)$.

We assume that the medium is vertically homogeneous and take $\Omega \subset \mathbb{R}^2$.

In this paper we analyse the approximation to the solution of this system using a mixed finite element method for the pressure equation and a modified method of characteristics for the concentration equation.

The method of characteristics combined with finite element or finite difference methods was introduced and analysed by Douglas and Russell [7] for a single parabolic equation and by Pironneau [12] for the Navier-Stokes equations.

The mixed method for second order elliptic problems was studied first by Raviart and Thomas [14] and analysed later in several works, see for instance ([4], [6]) .

As the variable of physical interest is the concentration $c(x,t)$ and only the Darcy velocity u and not p appears in equation (0.1 b)), it is natural to use a mixed method to approximate directly u together with p . This avoids the differentiation of p and the multiplication by the coefficient $\frac{k}{\mu}$ that can be rough.

On the other hand the diffusion coefficient D is quite small in realistic displacements arising for instance in oil recovery process, and then the concentration equation is strongly convection dominated. Consequently, the derivatives of $c(x,t)$ in the characteristic direction are small in comparison with the derivatives with respect to t . In the Douglas-Russell algorithm the error estimates depend on second derivatives in the characteristic direction but not on second derivatives in t . Then the constant in front of Δt will be small and this allows to use big time steps.

The Douglas-Russell method was analysed for the miscible displacement problem combined with a standard Galerkin procedure by Russell [15] and combined with the mixed method by Ewing, Russell and Wheeler [9] . Their analysis needs the restrictive assumption $\Delta t = o(h)$ and in practice one would

like to take bigger time steps.

In this paper we prove optimal error estimates in L^2 under the milder assumption

$$\lim_{\Delta t_c \rightarrow 0} |\log h_c|^{-\frac{1}{2}} \frac{\Delta t_c^3}{h_p h_c} = 0$$

where Δt_c is the time step in the concentration equation, h_p the mesh size in the pressure equation and h_c the mesh size in the concentration equation. We will need also other less important assumptions that will be listed later.

To avoid technical boundary difficulties we assume that Ω is $[0,1]^2$ and that the problem (0.1) is periodic. Boundary conditions could be treated as indicated by Douglas and Russell in [7] .

Throughout the rest of this paper all functions will be assumed to be Ω - periodic.

For the sake of notation we will assume that the time step is the same for the pressure equation than for the concentration equation and that it is constant. Variable time steps can be considered and our proofs are also valid in this case.

We will use the following norms ,

$$\|f\| = \left(\int_{\Omega} |f|^2 dx \right)^{1/2}$$

$$\|f\|_{\infty} = \text{ess sup}_{\Omega} |f|$$

$$\|f\|_m = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^2 \right)^{1/2}, \quad m \geq 0$$

$$\|f\|_{W_\infty^m} = \max_{|\alpha| \leq m} \|D^\alpha f\|_\infty, \quad m \geq 0$$

We also will work with the following norms for functions defined in $\Omega \times [a, b]$ where $[a, b] \subset J$. If X is any functional space,

$$\|f\|_{H^m(a, b, X)} = \left(\sum_{j=0}^m \int_a^b \left\| \frac{\partial^j f}{\partial t^j}(\cdot, t) \right\|_X^2 dt \right)^{1/2}, \quad m \geq 0$$

$$\|f\|_{W_\infty^m(a, b, X)} = \max_{0 \leq j \leq m} \operatorname{ess\,sup}_{[a, b]} \left\| \frac{\partial^j f}{\partial t^j}(\cdot, t) \right\|_X, \quad m \geq 0$$

If $[a, b] = [0, T] = J$ we drop it from the notation. The letter K will denote a positive constant, not necessarily the same at each occurrence and ε will denote a positive small constant.

We make the following assumptions on the data,

$$0 < \frac{1}{K} \leq \frac{k(x)}{\mu(c)} \leq K$$

$$0 < \frac{1}{K} \leq \phi(x) \leq K$$

(0.2)

$$0 < \frac{1}{K} \leq D(x, u)$$

$$\begin{aligned} & \left| \frac{\partial(k/\mu)}{\partial c} (x,c) \right| + \left| \frac{\partial \gamma}{\partial c} (x,c) \right| + \left| \nabla \phi(x) \right| + \left| \frac{\partial D}{\partial u} (x,u) \right| + \\ & + \left| \tilde{q} (x,t) \right| + \left| \frac{\partial \tilde{q}}{\partial t} (x,t) \right| \leq K \end{aligned}$$

1. The Method of Characteristics

In this section we define a modified method of characteristics for the concentration equation following Douglas and Russell [7]. Their algorithm is based on a discretization in the characteristic direction for the hyperbolic part combined with finite elements or finite differences for the diffusion part. We will consider here the finite element approach.

For $h > 0$ and an integer $r > 1$ let $M_h^r \subset W_\infty^1$ the finite dimensional subspace

$$M_h^r = \{ \chi \in W_\infty^1(\Omega) / \chi|_T \in P_r \quad \forall \quad T \in Z_h \}$$

where $\{Z_h\}$ is a quasi regular family of triangulations of Ω and P_r is the space of polynomials of degree less than or equal to r .

Let τ be the unit vector in the direction (u, ϕ) and $\psi^2 = |u|^2 + \phi^2$, then the concentration equation can be written as

$$\psi \frac{\partial c}{\partial \tau} - \operatorname{div}(D \nabla c) = (c - \tilde{c}) \tilde{q}$$

Given a partition of J , $0 = t^0 < t^1 < \dots < t^N = T$, $t^n - t^{n-1} = \Delta t$ let us denote $c^n(x) = c(x, t^n)$. Then the derivative with respect to τ is

approximated by

$$\psi \frac{\partial c^n}{\partial \tau} \approx \phi \frac{(c^n(x) - c^{n-1}(x - \frac{u^n(x)}{\phi} \Delta t))}{\Delta t}$$

or if we set $\bar{x} = x - \frac{u^n}{\phi} \Delta t$ and $c(\bar{x}) = \bar{c}(x)$

$$\psi \frac{\partial c^n}{\partial \tau} \approx \phi \frac{c^n - \bar{c}^{n-1}}{\Delta t}$$

Our error analysis will make use of a projection of c (as in [7], [17]) defined in the following way

Let $\tilde{c}_h : J \rightarrow M_h^r$ be such that ,

$$\begin{aligned} (D(u) \nabla \tilde{c}_h, \nabla \chi) + (\tilde{c}_h, \chi) + (\tilde{q} \tilde{c}_h, \chi) = \\ (1.1) \\ = (D(u) \nabla c, \nabla \chi) + (c, \chi) + (\tilde{q} c, \chi) \quad \forall \chi \in M_h^r \end{aligned}$$

It can be shown as in [8] that,

$$\begin{aligned} a) \quad \|c - \tilde{c}_h\|_{L^\infty(L^2)} + h \|c - \tilde{c}_h\|_{L^\infty(H^1)} \leq Kh^m \|c\|_{L^\infty(H^m)}, \quad 2 \leq m \leq r+1 \\ (1.2) \quad b) \quad \left\| \frac{\partial}{\partial t} (c - \tilde{c}_h) \right\|_{L^2(L^2)} \leq Kh^m \|c\|_{H^1(H^m)}, \quad 2 \leq m \leq r+1 \\ c) \quad \|\tilde{c}_h\|_{L^\infty(W_\infty^1)} \leq K \|c\|_{L^\infty(W_\infty^1)} \end{aligned}$$

where the constant K depends on $\|u\|_{W_\infty^1(L^\infty)}^1$. For c) see [11], [13]

If the exact solution u were known the approximation c_h would be defined as

$$c_h^o = \tilde{c}_h^o$$

$$\left(\phi \frac{c_h^n - c_h^{n-1}}{\Delta t}, \chi \right) + (D(u^n) \nabla c_h^n, \nabla \chi) + (\tilde{q}^n c_h^n, \chi) = (\tilde{q}^n \tilde{c}^n, \chi)_{M_h^n}.$$

In practice u^n will be replaced by an approximation of the Darcy velocity.

2. The Mixed Method for the Pressure Equation

The mixed method [4], [6], [14] consists in approximating u and p simultaneously using the following weak formulation

$$(a^{-1} u, v) - (\operatorname{div} v, p) = (\gamma(c) \nabla d, v), \quad v \in H(\operatorname{div}, \Omega)$$

$$(\operatorname{div} u, w) = (q, w), \quad w \in L^2(\Omega)$$

where $a = \frac{k}{\mu}$

Remark : Since p is defined up to a constant,

$L^2(\Omega)$ will mean $L^2(\Omega) / \mathbb{R}$.

Given a quasi uniform decomposition of Ω into triangles or rectangles of

diameter less than h , there exists a family of spaces $V_h^k \subset H(\text{div}, \Omega)$ and $W_h^k \subset L^2(\Omega)$ for $k \geq 0$ and a projection $\Pi_h : H(\text{div}, \Omega) \rightarrow V_h^k$ such that,

$$a) \quad \|v - \Pi_h v\| \leq K \|v\|_m h^m, \quad 1 \leq m \leq k+1$$

(2.1)

$$b) \quad \|\text{div}(v - \Pi_h v)\| \leq K \|\text{div} v\|_m h^m, \quad 1 \leq m \leq k$$

Moreover if $P_h : L^2 \rightarrow W_h^k$ is the L^2 -projection

$$c) \quad \|w - P_h w\|_{L^2} \leq K \|w\|_m h^m, \quad 1 \leq m \leq k$$

$$d) \quad (\text{div}(\Pi_h v - v), w) = 0, \quad w \in W_h^k$$

$$e) \quad \text{div} V_h^k = W_h^k$$

The spaces V_h^k, W_h^k can be the spaces of Raviart and Thomas [14] for $k \geq 0$ or the spaces of Brezzi, Douglas and Marini [4] for $k \geq 1$. If the Raviart-Thomas spaces are used (2.1 b)) and (2.1 c)) are valid also for $m = k+1$.

For the error analysis it will be useful to define the following projections of u and p .

Let $(\tilde{u}_h, \tilde{p}_h) : J \rightarrow V_h^k \times W_h^k$ such that ,

$$(2.2) \quad \begin{aligned} (a^{-1}(c) \tilde{u}_h, v) - (\operatorname{div} v, \tilde{p}_h) &= (\gamma(c) \nabla d, v) \quad v \in V_h^k \\ (\operatorname{div} \tilde{u}_h, w) &= (q, w) \quad w \in W_h^k \end{aligned}$$

where c is the exact solution of (0.1). Then it is known [2], [4], [5] that

$$(2.3) \quad \begin{aligned} a) \quad & \|u - \tilde{u}_h\|_{L^\infty(L^2)} \leq K \|u\|_{L^\infty(H^{k+1})} h^{k+1} \\ b) \quad & \|\operatorname{div}(u - \tilde{u}_h)\|_{L^\infty(L^2)} \leq K \|\operatorname{div} u\|_{L^\infty(H^m)} h^m \\ c) \quad & \|p - \tilde{p}_h\|_{L^\infty(L^2)} \leq K \|p\|_{L^\infty(H^m)} h^m \end{aligned}$$

where $m = k$ for the spaces of [4] and $m = k + 1$ for the spaces of [14].

Using inverse inequalities we can see that,

$$(2.4) \quad \|\tilde{u}_h\|_{L^\infty(L^\infty)} \leq K \{ \|u\|_{L^\infty(L^\infty)} + \|u\|_{L^\infty(H^1)} \}$$

Given an approximation c^* of the concentration c at time $t \in J$ the mixed method consists in finding $(u_h, p_h) \in V_h^k \times W_h^k$ such that

$$(2.5) \quad \begin{aligned} (a^{-1}(c^*) u_h, v) - (\operatorname{div} v, p_h) &= (\gamma(c^*) \nabla d, v), \quad v \in V_h^k \\ (\operatorname{div} u_h, w) &= (q, w) \quad w \in W_h^k \end{aligned}$$

In [5] it was proved that the solution exists and satisfies

$$(2.6) \quad \|u_h - \tilde{u}_h\|_{H(\text{div})} + \|p_h - \tilde{p}_h\| \leq K(1 + \|\tilde{u}_h\|_{L^\infty}) \|c - c^*\|$$

Now we present the discretization for the system (0.1) that combines the modified method of characteristics with the mixed method.

For simplicity we assume that the time step is the same for both equations and that it is constant. In practice it is usually convenient to use a longer time step for the pressure equation because the velocity changes less rapidly in time than the concentration.

To approximate the concentration at time t^n , we will use the approximation to the velocity at the previous time step. It is possible to do an extrapolation using earlier values of the velocity in order to get a better approximation of c as it was shown in [9].

The approximate solution $(c_h, u_h, p_h) : \{t^0, \dots, t^N\} \rightarrow M_h^r \times V_h^k \times W_h^k$ is defined by,

$$a) \quad c_h^0 = P_h c^0$$

where P_h is the L^2 - projection on M_h^r

$$b) \quad (a^{-1}(c_h^n) u_h^n, v) - (\text{div } v, p_h^n) = (\gamma(c_h^n) \nabla d, v) \quad v \in V_h^k \quad \text{for } n \geq 0$$

$$c) \quad (\text{div } u_h^n, w) = (q^n, w) \quad w \in W_h^k \quad \text{for } n \geq 0$$

$$d) \quad \left(\phi \frac{c_h^{n+1} - \hat{c}_h^n}{\Delta t}, x \right) + (D(u_h^n) \nabla c_h^{n+1}, \nabla x)$$

$$+ (\tilde{q}^n c_h^n, x) = (\tilde{q}^n \tilde{c}^n, x) \quad x \in M_h^r, \quad \text{for } n \geq 0$$

where

$$\hat{c}_h^n(x) = c_h^n(\hat{x}) = c_h^n\left(x - \frac{u_h^n}{\phi} \Delta t\right)$$

We will also make use of an analogue of $\hat{x}$ defined for the exact velocity. Given a function f on Ω we define.

$$\overset{v}{f}(x) = f(\overset{v}{x}) = f\left(x - \frac{u^n}{\phi} \Delta t\right)$$

where the n will be clear from the context.

3. Error Estimates

In this section we prove optimal error estimates for the concentration. Optimal estimates for the velocity and the pressure will follow immediately from (2.6).

Our main tool will be the following lemma

Lemma 3.1 : Let $f \in W_{\infty}^1(\Omega)$, $g^1 \in [H^1(\Omega)]^2 \cap [C^0(\Omega)]^2$, $g^2 \in H(\text{div}, \Omega)$ and $\phi \in H^1(\Omega) \cap C^0(\Omega)$. Let $g = g^1 - g^2$ then ,

a) If $g^2 \in [H^1(\Omega)]^2 \cap [C^0(\Omega)]^2$ we have

$$\left| \int_{\Omega} \frac{f(x - \Delta t g^2) - f(x - \Delta t g^1)}{\Delta t} \phi(x) dx \right| \leq$$

$$\leq K \{ \|f\|_{\infty} \|g\|_{L^2} \|\nabla \phi\|_{L^2} + \|f\|_{\infty} \|\text{div } g\|_{L^2} \|\phi\|_{L^2}$$

$$+ \Delta t \| \nabla f \|_{\infty} \| \nabla g^1 \|_{L^2} \|g\|_{L^2} \|\phi\|_{\infty} + \Delta t \| \nabla f \|_{\infty} \| \nabla g \|_{L^2} \|g\|_{L^2} \|\phi\|_{\infty} \}$$

where we have denoted $\nabla g = \left(\frac{\partial g_i}{\partial x_j} \right)$

b) If $g^2 \in [H^1(T)]^2$ for every $T \in Z_h$ and g satisfies the following inverse inequalities,

$$\|g\|_{L^2(\partial T)} \leq \frac{K}{\frac{1}{2}h} \|g\|_{L^2(T)}, \quad \|g \cdot \eta\|_{L^2(\partial T)} \leq \frac{K}{\frac{1}{2}h} \|g\|_{L^2(T)}$$

and

$$\|\nabla g\|_{L^2(T)} \leq \frac{K}{h} \|g\|_{L^2(T)}$$

then,

$$\left| \int_{\Omega} \frac{f(x - \Delta t g^2) - f(x - \Delta t g^1)}{\Delta t} \phi(x) dx \right| \leq$$

$$\leq K \{ \|f\|_{\infty} \|g\|_{L^2} \|\nabla \phi\|_{L^2} + \|f\|_{\infty} \|\operatorname{div} g\|_{L^2} \|\phi\|_{L^2} +$$

$$+ \Delta t \|\nabla f\|_{\infty} \|\nabla g^1\|_{L^2} \|g\|_{L^2} \|\phi\|_{\infty} + \frac{\Delta t}{h} \|\nabla f\|_{\infty} \|g\|_{L^2}^2 \|\phi\|_{\infty}$$

Proof : By density it is enough to assume that $f \in C^1$. We have

$$(3.1) \quad \frac{f(x - \Delta t g^2) - f(x - \Delta t g^1)}{\Delta t} = \int_0^1 \nabla f(x - \Delta t g^1 + s g \Delta t) \cdot g \, ds$$

Denote $\bar{x}_s = x - \Delta t g^1 + s g \Delta t$ and $\bar{f}_s(x) = f(\bar{x}_s)$. We will write $\bar{f} = \bar{f}_s$ and $\bar{x} = \bar{x}_s$ if there is no confusion.

For $T \in Z_h$ and $x \in T$ we have,

$$(3.2) \quad \operatorname{div}(\bar{f} g) = \bar{f} \operatorname{div} g + \nabla \bar{f} \cdot g$$

but

$$\frac{\partial \bar{f}}{\partial x_j} = \sum_{i=1}^2 \frac{\partial f}{\partial x_i}(\bar{x}) \left(\delta_{ij} - \Delta t \frac{\partial g_i^1}{\partial x_j} + s \Delta t \frac{\partial g_i}{\partial x_j} \right)$$

then,

$$(3.3) \quad \nabla \bar{f} \cdot g = \nabla f(\bar{x}) \cdot g - \nabla f(\bar{x}) \cdot (\nabla g^1) g \Delta t + \nabla f(\bar{x}) \cdot (\nabla g) g \, s \Delta t$$

using (3.2) and (3.3) we get,

$$(3.4) \quad \nabla f(\bar{x}) \cdot g = \operatorname{div}(\bar{f}g) - \bar{f} \operatorname{div} g + \nabla f(\bar{x}) \cdot (\nabla g^1)g \Delta t + \nabla f(\bar{x}) \cdot (\nabla g)g s \Delta t$$

From (3.1) we have,

$$\int_{\Omega} \frac{f(x - \Delta t g^2) - f(x - \Delta t g^1)}{\Delta t} \phi(x) dx =$$

(3.5)

$$= \int_0^1 \int_{\Omega} \nabla f(\bar{x}_s) \cdot g(x) \phi(x) dx ds$$

then, it is enough to bound $\int_{\Omega} \nabla f(\bar{x}_s) \cdot g(x) \phi(x) dx$ independently of s .

Using (3.4) we write this integral over each T as the sum of four terms and we bound each one separately.

For the first term we integrate by parts over each T .

$$(3.6) \quad \int_{\Omega} \operatorname{div}(\bar{f}g) \phi dx = \sum_T \int_T \operatorname{div}(\bar{f}g) \phi dx =$$

$$= - \sum_T \int_T \bar{f} g \cdot \nabla \phi dx + \sum_T \int_{\partial T} \bar{f} g \cdot \eta_T \phi d\sigma$$

where η_T is the exterior normal on ∂T .

In the case a) g^2 has continuous components and then

$$\sum_T \int_{\partial T} \bar{f} g \cdot \eta_T \phi d\sigma = 0$$

then

$$\left| \int_{\Omega} \operatorname{div}(\bar{f}g) \phi \, dx \right| \leq \|f\|_{\infty} \|g\|_{L^2}^2 \|\nabla \phi\|_{L^2}$$

On the other hand, in the case b) we can bound the last term in (3.6) in the following way.

Since $g \in H(\operatorname{div}, \Omega)$, it has continuous normal components and so,

$$\sum_T \int_{\partial T} f(x - \Delta t g^1) g \cdot \eta_T \phi \, d\sigma = 0$$

because f , g^1 and ϕ are continuous.

then,

$$\left| \sum_T \int_{\partial T} f(\bar{x}) g \cdot \eta_T \phi \, d\sigma \right| = \left| \sum_T \int_{\partial T} (f(\bar{x}) - f(x - \Delta t g^1)) g \cdot \eta_T \phi \, d\sigma \right|$$

but,

$$|f(\bar{x}) - f(x - \Delta t g^1)| \leq \|\nabla f\|_{\infty} |g(x)| \Delta t$$

then,

$$\|f(\bar{x}) - f(x - \Delta t g^1)\|_{L^2(\partial T)} \leq \|\nabla f\|_{\infty} \|g\|_{L^2(\partial T)} \Delta t$$

and we can apply Schwartz inequality to get,

$$\left| \sum_T \int_{\partial T} f(\bar{x}) g \cdot \eta_T \phi \, d\sigma \right| \leq \sum_T \|\nabla f\|_{\infty} \Delta t \|g\|_{L^2(\partial T)} \|g \cdot \eta_T\|_{L^2(\partial T)} \|\phi\|_{\infty}$$

$$\leq K \frac{\Delta t}{h} \|\nabla f\|_{\infty} \|g\|_{L^2}^2 \|\phi\|_{\infty}$$

where we have used the hypothesis of b) for the last inequality.

To bound $\sum_T \int_T \bar{f} \operatorname{div} g \phi \, dx$ we apply Schwartz inequality and we get,

$$\left| \sum_T \int_T \bar{f} \operatorname{div} g \phi \, dx \right| \leq \|f\|_{\infty} \|\operatorname{div} g\|_{L^2} \|\phi\|_{L^2}$$

For the third term of (3.4) we have,

$$\begin{aligned} & \left| \sum_T \int_T \nabla f(\bar{x}) \cdot (\nabla g^1) g \, \Delta t \, \phi \, dx \right| \\ & \leq \sum_T \|\nabla f\|_{\infty} \|\nabla g^1\|_{L^2(T)} \|g\|_{L^2(T)} \|\phi\|_{\infty} \Delta t \\ & \leq \|\nabla f\|_{\infty} \|\nabla g^1\|_{L^2} \|g\|_{L^2} \|\phi\|_{\infty} \Delta t \end{aligned}$$

For the last term of (3.4) we have,

$$\begin{aligned} & \left| \sum_T \int_T \nabla f(\bar{x}) \cdot (\nabla g) g \, s \Delta t \, \phi \, dx \right| \\ (3.7) \quad & \leq \|\nabla f\|_{\infty} \left(\sum_T \|\nabla g\|_{L^2(T)}^2 \right)^{\frac{1}{2}} \|g\|_{L^2} \|\phi\|_{\infty} \Delta t \end{aligned}$$

in the case a) the last term in (3.7) is equal to

$$\|\nabla f\|_{\infty} \|\nabla g\|_{L^2} \|g\|_{L^2} \|\phi\|_{\infty} \Delta t$$

and in the case b) we use the inverse inequalities to get,

$$\left| \sum_T \int_T \nabla f(\bar{x}) \cdot (\nabla g) g \, s \, \Delta t \, \phi \, dx \right|$$

$$\leq K \frac{\Delta t}{h} \|\nabla f\|_{\infty} \|g\|_{L^2}^2 \|\phi\|_{\infty}$$

and combining all the estimates we get the lemma.

Theorem 3.2: Let $r, k \geq 1$. Assume that

$$u \in L^{\infty}(H^{k+1}) \cap H^1(L^2) \cap L^{\infty}(W_{\infty}^1) \cap W_{\infty}^1(L^{\infty})$$

$$c \in L^{\infty}(H^{r+1}) \cap L^{\infty}(W_{\infty}^1) \cap H^1(H^{r+1})$$

$$\frac{\partial^2 c}{\partial \tau^2} \in L^2(L^2)$$

and the conditions of (0.2) are satisfied.

If we denote h_p and h_c the parameters associated with the finite element spaces for the pressure and the concentration respectively and the following relations between the parameters are satisfied,

$$\frac{\Delta t^3}{h_p h_c} |\log h_c|^{\frac{1}{2}} = o(1), \quad \frac{\Delta t^2}{h_c} |\log h_c|^{\frac{1}{2}} = o(1)$$

(3.8)

$$h_p^k |\log h_c|^{\frac{1}{2}} = o(1), \quad \Delta t |\log h_c|^{\frac{1}{2}} = o(1), \quad h_p^{k+1} = o(h_c)$$

$$\text{and } h_c^{2r+1} = o(h_p)$$

then,

$$(3.9) \quad \max_{0 \leq n \leq N} \|c^n - c_h^n\| \leq K (h_c^{r+1} + h_p^{k+1} + \Delta t)$$

for $h_c, h_p, \Delta t$ sufficiently small.

Remark: If we use different time steps Δt_c and Δt_p for each equation, the condition (3.8) has to be understood with $\Delta t = \Delta t_c$ and the estimate (3.9) has to be replaced by,

$$\max_{0 \leq n \leq N} \|c^n - c_h^n\| \leq K (h_c^{r+1} + h_p^{k+1} + \Delta t_c + \Delta t_p)$$

A better order of approximation in Δt_p can be obtained using extrapolation argument (see [9]).

It is important to notice that the constant in front of Δt_c depends on $\left\| \frac{\partial^2 c}{\partial \tau^2} \right\|$ but not on $\left\| \frac{\partial^2 c}{\partial t^2} \right\|$.

Proof of the Theorem: Call $\eta = c - \tilde{c}_h$ and $\xi = c_h - \tilde{c}_h$.

Using (1.2) we see that it is enough to estimate ξ . Subtracting (1.1) from (2.7,d)) we get the following equation for the error.

$$\begin{aligned} & \left(\phi \frac{\xi^{n+1} - \xi^n}{\Delta t}, \chi \right) + (D(u_h^n) \nabla \xi^{n+1}, \nabla \chi) = \\ & = \left(\phi \frac{\partial c}{\partial t} + u^{n+1} \cdot \nabla c^{n+1} - \phi \frac{c^{n+1} - c^n}{\Delta t}, \chi \right) \end{aligned}$$

(3.10)

$$+ ([D(u^{n+1}) - D(u_h^n)] \nabla \tilde{c}_h^{n+1}, \nabla \chi) - (\tilde{q}^{n+1} \xi^{n+1}, \chi)$$

$$\begin{aligned}
 & - (\eta^{n+1} , \chi) + (\phi \frac{\eta^{n+1} - \eta^n}{\Delta t} , \chi) + (\phi \frac{\eta^n - v_{\eta}^n}{\Delta t} , \chi) \\
 & + (\phi \frac{\hat{c}_h^n - v_{\hat{c}_h}^n}{\Delta t} , \chi) + (\phi \frac{\hat{\xi}^n - v_{\hat{\xi}}^n}{\Delta t} , \chi) , \quad \chi \in M_h^r
 \end{aligned}$$

Now we choose $\chi = \xi^{n+1}$ in (3.10). Using the inequality $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$ to bound the left hand side (as in [7]) and calling $T_1, \dots, T_8$ the terms on the right hand side we get,

$$\begin{aligned}
 & \frac{1}{2\Delta t} [(\phi \xi^{n+1} , \xi^{n+1}) - (\phi \xi^n , \xi^n)] + (D(u_h^n) \nabla \xi^{n+1} , \nabla \xi^{n+1}) \\
 & \leq K \|\xi^n\|^2 + T_1 + \dots + T_8
 \end{aligned}$$

To estimate T_1 we write it as

$$T_1 = (\phi \frac{\partial c^{n+1}}{\partial t} + u^{n+1} \cdot \nabla c^{n+1} - \phi \frac{c^{n+1} - \bar{c}^n}{\Delta t} , \xi^{n+1}) + (\phi \frac{\bar{c}^n - v_{\bar{c}}^n}{\Delta t} , \xi^{n+1})$$

where $\bar{c}^n = c^n (x - \frac{u^{n+1}}{\phi} \Delta t)$. Then, the first term is bounded by (see [7])

$$K \|u\|_{L^\infty(L^\infty)}^4 \left\| \frac{\partial^2 c}{\partial \tau^2} \right\|_{L^2(t^n, t^{n+1}, L^2)}^2 \Delta t + \|\xi^{n+1}\|^2$$

and the second term is bounded by

$$K \left\| \frac{\bar{c}^n - v_{\bar{c}}^n}{\Delta t} \right\| \|\xi^{n+1}\| \leq K \|\nabla c^n\|_\infty \|u^{n+1} - u^n\| \|\xi^{n+1}\|$$

but ,

$$(3.12) \quad \|u^{n+1} - u^n\| \leq \Delta t \left\| \frac{\partial u}{\partial t} \right\|_{L^2}$$

then,

$$\begin{aligned} |T_1| &\leq K \|u\|_{L^\infty(L^\infty)}^4 \left\| \frac{\partial^2 c}{\partial \tau^2} \right\|_{L^2(t^n, t^{n+1}, L^2)}^2 \Delta t + K \|c\|_{L^\infty(W_\infty^1)}^2 \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(L^2)}^2 \Delta t^2 \\ &\quad + \|\xi^{n+1}\|^2 \end{aligned}$$

For T_2 we have,

$$|T_2| \leq \|D(u^{n+1}) - D(u_h^n)\| \|\nabla \tilde{c}_h^{n+1}\|_\infty \|\xi^{n+1}\|_1$$

Since D is Lipschitz we have,

$$\|D(u^{n+1}) - D(u_h^n)\| \leq K \|u^{n+1} - u_h^n\| \leq K \{\|u^{n+1} - u^n\| + \|u^n - u_h^n\|\}$$

then, by (2.3a) , (2.4), (2.6) and (3.12) we have

$$|T_2| \leq K \left\{ \left\| \frac{\partial u}{\partial t} \right\|_{L^2} \Delta t + \{\|\xi^n\| + \|\eta^n\| + \|u\|_{L^\infty(H^{k+1})}^{h_p^{k+1}}\} \right\} \cdot \|\nabla \tilde{c}_h^{n+1}\|_\infty \|\xi^{n+1}\|_1$$

where the constant K depends on the bound for $\tilde{u}_h$ in (2.4).

Using (1.2 c)) we get ,

$$|T_2| \leq K \left\{ \left\| \frac{\partial u}{\partial t} \right\|_{L^2(L^2)}^2 \Delta t^2 + \|u\|_{L^2(H^{k+1})}^2 h_p^{2(k+1)} + \|\xi^n\|^2 + \|\eta^n\|^2 \right\} \|c\|_{L^\infty(W_\infty^1)}^2 \\ + \epsilon \|\xi^{n+1}\|_1^2 \quad \}$$

Obviously,

$$|T_3| \leq K \|\xi^{n+1}\|^2$$

and

$$|T_4| \leq \|\eta^{n+1}\|^2 + \|\xi^{n+1}\|^2$$

Now,

$$|T_5| \leq K \left\| \frac{\eta^{n+1} - \eta^n}{\Delta t} \right\| \|\xi^{n+1}\|$$

but,

$$\left\| \frac{\eta^{n+1} - \eta^n}{\Delta t} \right\| \leq \Delta t^{-\frac{1}{2}} \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(t^n, t^{n+1}, L^2)} \leq K \Delta t^{-\frac{1}{2}} h_c^{r+1} \|c\|_{H^1(t^n, t^{n+1}, H^{r+1})}$$

then,

$$|T_5| \leq K \Delta t^{-1} h_c^{2(r+1)} \|c\|_{H^1(t^n, t^{n+1}, H^{r+1})}^2 + K \|\xi^{n+1}\|^2$$

To bound T_6 we follow [7] and we get, $|T_6| \leq K \|\eta^n\|^2 + \epsilon \|\xi^{n+1}\|_1^2$

To bound T_7 we use (1.2 c)) and we get,

$$\left\| \frac{\hat{c}_h^n - \tilde{c}_h^n}{\Delta t} \right\| \leq K \|c\|_{L(W_\infty^1)} \|u^n - u_h^n\|$$

and consequently,

$$|T_7| \leq K \|c\|_{L(W_\infty^1)}^2 \{ \|\xi^n\|^2 + \|n^n\|^2 + h_p^{2(k+1)} \|u\|_{k+1}^2 \} + \|\xi^{n+1}\|^2$$

In order to bound T_8 we need to introduce some notation. Let $I_h u^n$ be the interpolation of u^n by piecewise continuous polynomials of degree k associated with the triangulation Z_{h_p} and set $\bar{\xi}(x) = \xi(x - \frac{I_h u^n}{\phi} \Delta t)$. Then, we write T_8 in the following way

$$(3.13) \quad T_8 = \left(\phi \frac{\hat{\xi}^n - \bar{\xi}^n}{\Delta t}, \xi^{n+1} \right) + \left(\phi \frac{\xi^{-n}}{\Delta t} \frac{v^n}{\xi}, \xi^{n+1} \right)$$

To bound the second term of (3.13) we use part a) of Lemma 3.1 with,

$$f = \xi^n, \quad g^1 = \frac{u^n}{\phi}, \quad g^2 = \frac{I_h u^n}{\phi} \quad \text{and} \quad \phi = \xi^{n+1}$$

and we get

$$\left| \left(\phi \frac{\xi^{-n}}{\Delta t} \frac{v^n}{\xi}, \xi^{n+1} \right) \right| \leq K \{ \|\xi^n\|_\infty \|u^n - I_h u^n\|_{H(\text{div})} \|\xi^{n+1}\|_1$$

$$(3.14) \quad + \Delta t \|\nabla \xi^n\|_\infty \|v^n\| \|u^n - I_h u^n\| \|\xi^{n+1}\|_\infty$$

$$+ \Delta t \|\nabla \xi^n\|_\infty \|\nabla(u^n - I_h u^n)\| \|u^n - I_h u^n\| \|\xi^{n+1}\|_\infty \}$$

where we have use the notation $\|v\|_{H(\text{div})}^2 = \|v\|_{L^2}^2 + \|\text{div } v\|_{L^2}^2$

For the first term of (3.14) we use the inequality (see [1])

$$(3.15) \quad \|\xi^n\|_\infty \leq K |\log h_c|^{\frac{1}{2}} \|\xi^n\|_1$$

and then,

$$\begin{aligned} \|\xi^n\|_\infty \|u^n - I_h u^n\|_{H(\text{div})} \|\xi^{n+1}\|_1 &\leq K |\log h_c|^{\frac{1}{2}} h_p^k \|u^n\|_{k+1} \|\xi^n\|_1 \|\xi^{n+1}\|_1 \\ &\leq \varepsilon (\|\xi^n\|_1^2 + \|\xi^{n+1}\|_1^2) \end{aligned}$$

for h_p, h_c small enough, because of (3.8)

The second term of (3.14) is bounded in the following way,

$$(3.16) \quad \Delta t \|\nabla \xi^n\|_\infty \|u^n\| \|u^n - I_h u^n\| \|\xi^{n+1}\|_\infty \leq K \frac{\Delta t}{h_c} \|u^n\| h_p^{k+1} \|u^n\|_{k+1} |\log h_c|^{\frac{1}{2}} \|\xi^n\|_1 \|\xi^{n+1}\|_1$$

where we have used (3.15) to bound $\|\xi^{n+1}\|_\infty$ and an inverse inequality to bound $\|\nabla \xi^n\|_\infty$.

Using (3.8) we see that the right hand side of (3.16) is bounded by

$$\varepsilon (\| \xi^n \|_1^2 + \| \xi^{n+1} \|_1^2)$$

when Δt , h_c and h_p are small enough.

Finally, the third term of (3.14) is bounded in an analogous way as the second one.

To bound the first term of (3.13) we use part b) of the lemma and we get,

$$\begin{aligned} (3.17) \quad & \left| \left(\phi \frac{\xi^n - \xi^{n+1}}{\Delta t}, \xi^{n+1} \right) \right| \leq K \left\{ \| \xi^n \|_\infty \| u_h^n - I_h u^n \|_{H(\text{div})} \| \xi^{n+1} \|_1 \right. \\ & + \Delta t \| \nabla \xi^n \|_\infty \| \nabla I_h u^n \| \| I_h u^n - u_h^n \| \| \xi^{n+1} \|_\infty \\ & \left. + \frac{\Delta t}{h_p} \| \nabla \xi^n \|_\infty \| I_h u^n - u_h^n \|^2 \| \xi^{n+1} \|_\infty \right\} \end{aligned}$$

The first term of (3.17) can be bounded as the first term of (3.14) if we assume the inductive hypothesis

$$(3.18) \quad \left| \log h_c \right|^{\frac{1}{2}} \| u_h^n - u^n \|_{H(\text{div})} = o(1)$$

The second summand in (3.17) is bounded by

$$\frac{\Delta t}{h_c} \left| \log h_c \right|^{\frac{1}{2}} \| u^n \|_1 \| I_h u^n - u_h^n \| \| \xi^n \|_1 \| \xi^{n+1} \|_1$$

and this is bounded by

$$\varepsilon (\| \xi^n \|_1^2 + \| \xi^{n+1} \|_1^2)$$

if we assume the inductive hypothesis.

$$(3.19) \quad \frac{\Delta t}{h_c} |\log h_c|^{\frac{1}{2}} \| u_h^n - u^n \| = o(1)$$

Finally, the last term in (3.17) is bounded by

$$\frac{\Delta t}{h_p h_c} |\log h_c|^{\frac{1}{2}} \| I_h u^n - u_h^n \|^2 \| \xi^n \|_1 \| \xi^{n+1} \|_1$$

and this is bounded by

$$\varepsilon (\| \xi^n \|_1^2 + \| \xi_1^{n+1} \|^2)$$

if the following inductive hypothesis is satisfied

$$(3.20) \quad \frac{\Delta t}{h_p h_c} |\log h_c|^{\frac{1}{2}} \| u_h^n - u^n \|^2 = o(1)$$

The inductive hypothesis (3.18), (3.19) and (3.20) are verified if,

$$a) \quad |\log h_c|^{\frac{1}{2}} (\Delta t + h_c^{r+1} + h_p^{k+1}) = o(1)$$

$$b) \quad \frac{\Delta t}{h_c} |\log h_c|^{\frac{1}{2}} (\Delta t + h_c^{r+1} + h_p^{k+1}) = o(1)$$

$$c) \quad \frac{\Delta t}{h_p h_c} |\log h_c|^{\frac{1}{2}} (\Delta t^2 + h_c^{2(r+1)} + h_p^{2(k+1)}) = o(1)$$

but the conditions (3.21) follows from our hypothesis (3.8) .

Collecting all the estimates and applying a Gronwall's lemma we get the theorem

Remark 3.22 : We have proved the theorem for the case $k \geq 1$. When $k = 0$ the proof can be done in the same way if we assume that $u \in L^\infty(H^{1+s})$ for some $s > 0$. In fact, we can easily see that this regularity is enough in order to bound the first term on the right hand side of (3.14). The rest of the theorem follows as in the case $k \geq 1$

Remark 3.23: Extensions of this method to three dimensions can be studied. In this case the pressure equation can be approximated using the spaces of [3] or [10] . The error analysis can be carried out in the same way assuming different relations between the parameters. Following the proof one finds out that the main assumptions needed are the following

$$\frac{\Delta t}{h_c} |\log h_c|^{\frac{1}{2}} = o(1)$$

and

$$\frac{\Delta t^3}{h_p h_c^2} |\log h_c| = o(1)$$

In practice one would like to take $\Delta t = O(h_c)$ and therefore it would be

interesting to improve the error analysis in the three dimensional case.

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