THE QUASI-MONOTONE SCHEMES
FOR SCALAR CONSERVATION LAWS.

PART III

By

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The Quasi-Monotone Schemes for Scalar Conservation Laws. Part III

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Abstract. In this paper we extend the definition and analysis of the quasi-monotone numerical schemes to the general case $d > 1$, where d is the number of space variables. These schemes are constructed using the simple but very important class of monotone schemes defined by two-point monotone fluxes. In order to enforce the compactness in $L^\infty(L^1_{loc})$ of our approximate solutions we restrict ourselves to the case of meshes that are a Cartesian product of one-dimensional partitions. We prove that the main stability and convergence results for one-dimensional quasi-monotone schemes (of the first type) hold also in the general case. As a by-product of this theory we develop the theory of relaxed quasi-monotone schemes. These schemes are L^∞ -stable, and they can be more accurate than the quasi-monotone ones. Due to this fact, the compactness in $L^\infty(L^1_{loc})$ is lost. However, if converging, they converge to the entropy solution.

1. Introduction. In this paper we define and analyse the quasi-monotone numerical schemes for the scalar conservation law

$$\begin{aligned} (1.1a) \quad & \partial_t u + \operatorname{div} f(u) = 0 && \text{in } (0, T) \times \mathbb{R}^d, \\ (1.1b) \quad & u(t=0) = u_0 && \text{on } \mathbb{R}^d, \end{aligned}$$

where f is assumed to be C^1 . These schemes are entropy schemes that are L^∞ -stable, and TVB (total-variation-bounded). (More precisely, they are TVD (total-variation-diminishing) in the finite-difference case, and TVDM (total-variation-diminishing in the means) in the finite-element one). They are constructed using monotone schemes defined by means of two-point monotone fluxes. These monotone schemes are important not

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only because they can be defined employing a single one-dimensional flux, but because they are entropy schemes; see Sanders [5]. We have to point out that this result has only been obtained in the case in which the mesh is a Cartesian product of one-dimensional partitions. In the case of an arbitrary triangulation the compactness in $L^\infty(0,T;L^1_{loc})$ of the sequence of approximate solutions may not be reached. This is why in this paper we shall restrict ourselves to that type of grid. However, we point out that quasi-monotone schemes for arbitrarily-shaped triangulations can also be defined. This will be done in a forthcoming paper. Parallel to the theory of qm schemes we shall develop, as a by-product, the theory of relaxed quasi-monotone (rqm) schemes. These schemes are L^∞ -stable, but they are not necessarily TVD. However, if they converge (i.e., if they are TVB), they do converge to the entropy solution. Moreover, by using these schemes local high order accuracy is easier to achieve than with qm schemes.

An outline of the paper is as follows. In Section 2 we recall some existence and regularity results for the entropy solution of the continuous problem (1.1). We also review briefly the main stability and convergence results for the class of monotone schemes described above. In Sections 3 and 4 we define and analyse quasi-monotone finite-difference (qmfd), and quasi-monotone finite-element (qmfe) schemes, respectively. Relaxed quasi-monotone schemes are also considered. For the sake of clarity we shall only consider the case $d = 2$. The generalization of the theory to the case $d > 2$ will be found to be obvious. In Section 5 we consider in detail an explicit qm scheme: the two-dimensional version of the G-1/2 scheme considered in Section 5 Part II. Finally, in Section 6 we make some remarks on the generalization to the two-dimensional case of the LRG implicit schemes introduced in Section 6 Part II.

2. Preliminaries. In this Section, after stating briefly uniqueness and regularity results for the entropy solution of problem (1.1), we consider finite difference monotone schemes constructed using

two-point monotone numerical fluxes. We review the main stability and convergence results.

2.1. Existence and regularity of the entropy solution. Kruzkov [4] proved that, if the initial function u_0 is in $L^\infty(\mathbb{R}^d)$, there exists a weak solution of (1.1) verifying for every real number c the following entropy inequality:

$$(2.1a) \quad \partial_t U_c + \operatorname{div} F_c \leq 0 \quad \text{in } (0, T) \times \mathbb{R}^d,$$

where the (Kruzkov) entropy U_c and its flux F_c are given by

$$(2.1b) \quad U_c(u) = |u - c|,$$

$$(2.1c) \quad F_c(u) = (f(u) - f(c)) \operatorname{sgn}(u - c).$$

Moreover, this weak solution is unique and is called the entropy solution of (1.1); it can also be characterised as the limit of the viscous solutions of the parabolic problem associated to (1.1) [4]. In this work we shall assume that u_0 belongs to the space $L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, where $BV(\mathbb{R}^d)$ denotes the space of functions whose gradients are bounded measures. In this case there exists a unique entropy solution in the space $L^\infty(0, T; L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d))$. See [4] for more general results.

2.2. Definition of monotone schemes constructed with two-point monotone fluxes. Since now on, we restrict ourselves to the two-dimensional case. We denote by f_x and f_y the components of the function f in the x and y -directions, respectively. Let $\{t^n\}_{n=0,1,\dots,N}$, $\{x_{i+1/2}\}_i \in \mathbb{Z}$, and $\{y_{j+1/2}\}_j \in \mathbb{Z}$ be partitions of $[0, T]$, $\mathbb{R}$ and $\mathbb{R}$, respectively. As usual, we set $\Delta t^n = t^{n+1} - t^n$, $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, $\Delta y_j = y_{j+1/2} - y_{j-1/2}$, and we denote by J^n and Q_{ij} the sets (t^n, t^{n+1}) and

$(x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2})$, respectively. Finally, we denote by K_{ij}^n the element $J^n \times Q_{ij}$, and set

$$\delta = \sup_{i,j,n} \text{diam}(K_{ij}^n).$$

The explicit monotone schemes that will be considered in this work are of the form

$$(2.2) \quad (u_{i,j}^{n+1} - u_{i,j}^n) / \Delta t^n + (f_x^{m,n}{}_{i+1/2,j} - f_x^{m,n}{}_{i-1/2,j}) / \Delta x_i + (f_y^{m,n}{}_{i,j+1/2} - f_y^{m,n}{}_{i,j-1/2}) / \Delta y_j = 0,$$

where the numerical fluxes f_x^m and f_y^m are two-point monotone consistent fluxes; see the definition in (2.5) Part I. We shall say that the flux $f^m = (f_x^m, f_y^m)$ is a two-point monotone flux. As in the one-dimensional case, if the function H determined by (2.2) and

$$(2.3) \quad u_{i,j}^{n+1} = H_{i,j}(u^n),$$

is a nondecreasing function in each of its arguments, we say that the scheme is monotone [2]. Note that if the one-dimensional schemes

$$(2.4a) \quad (u_i^{n+1} - u_i^n) / \Delta t^n + (f_x^{m,n}{}_{i+1/2} - f_x^{m,n}{}_{i-1/2}) / \Delta x_i = 0,$$

$$(2.4b) \quad (u_j^{n+1} - u_j^n) / \Delta t^n + (f_y^{m,n}{}_{j+1/2} - f_y^{m,n}{}_{j-1/2}) / \Delta y_j = 0,$$

are monotone for $cf l_x, cf l_y \in [0, cf l_0]$, where as usual

$$cf l_x = \sup_{i,n} \Delta t^n \Delta x_i^{-1} \|f_x'(u_0)\|_{L^\infty(\mathbb{R}^2)},$$

$$cf l_y = \sup_{j,n} \Delta t^n \Delta y_j^{-1} \|f_y'(u_0)\|_{L^\infty(\mathbb{R}^2)},$$

then, the scheme (2.2) is monotone for $cfl \in [0, cfl_0/2]$, where

$$cfl = \max \{ cfl_x, cfl_y \}.$$

This condition is, of course, sufficient and less restrictive cfl-type conditions can be found; see, for example, Sanders [5].

The implicit monotone schemes that will be considered in this work are of the form

$$(2.5) \quad (u_{i,j}^n - u_{i,j}^{n-1}) / \Delta t^n + (f_{xs}^{m,n}{}_{i+1/2,j} - f_{xs}^{m,n}{}_{i-1/2,j}) / \Delta x_i + (f_{ys}^{m,n}{}_{i,j+1/2} - f_{ys}^{m,n}{}_{i,j-1/2}) / \Delta y_j = 0,$$

where

$$(2.6a) \quad f_{xs}^{m,n} = s f_x^{m,n} + (1-s) f_x^{m,n-1},$$

$$(2.6b) \quad f_{ys}^{m,n} = s f_y^{m,n} + (1-s) f_y^{m,n-1}, \text{ for } s \in (0,1],$$

and f_x^m and f_y^m are two-point monotone consistent fluxes. We shall use the notation $f_s^m = (f_{xs}^m, f_{ys}^m)$. Again, if the function H determined by (2.5) and (2.3) is a nondecreasing function in each of its arguments, we say that the scheme is monotone. In this case it can be easily proven that if the schemes (2.4) are monotone for cfl_x and $cfl_y \in [0, cfl_0]$, then, (2.5) is also monotone for $cfl \in [0, cfl_0/2(1-s)]$.

2.3. Stability and convergence properties. Let u_h denote the piecewise-constant extension of the discrete function $\{u_{i,j}^n\}$ defined by

$$u_h(t,x,y) = u_{i,j}^n \quad \text{for } (t,x,y) \in K_{i,j}^n.$$

Then, we have

$$\|u_h^n\|_{BV(\mathbb{R}^2)} = \sum_{i,j} |u_{i+1,j}^n - u_{i,j}^n| \Delta y_j + \sum_{i,j} |u_{i,j+1}^n - u_{i,j}^n| \Delta x_i.$$

For the class of schemes under consideration we have the following stability result.

PROPOSITION 2.1. Suppose that the scheme (2.2) is monotone for $cf1 \in [0, cf1_0]$. Then, for $cf1 \in [0, cf1_0]$,

$$(2.7) \quad u_{i,j}^{n+1} \in I(u_{i,j}^n, u_{i+1,j}^n, u_{i-1,j}^n, u_{i,j+1}^n, u_{i,j-1}^n),$$

and

$$(2.8) \quad \|u_h^{n+1}\|_{BV(\mathbb{R}^2)} \leq \|u_h^n\|_{BV(\mathbb{R}^2)};$$

i.e., the scheme is TVD. For $cf1 \in [0, cf1_0/(1-s)]$, the implicit scheme (2.5) is also TVD; moreover,

$$(2.9a) \quad u_{i,j}^{n+1} \in sI_{i,j}^1 + (1-s)I_{i,j}^2,$$

$$(2.9b) \quad I_{i,j}^1 = I(u_{a,b}^n, u_{c,d}^{n+1}, \text{ where } \bar{Q}_{c,d} \subset \bar{\Psi}_{i,j} \text{ and } \bar{Q}_{a,b} \subset \Psi_{i,j}),$$

$$(2.9c) \quad I_{i,j}^2 = I(u_{a,b}^n, \text{ where } \bar{Q}_{a,b} \subset \Psi_{i,j} \cup \bar{\Psi}_{i,j}),$$

where

$$(2.9d) \quad \Psi_{i,j} \text{ is equal to } \bar{Q}_{i,j}. \text{ Otherwise, it is equal to any union of sets } \bar{Q}_{a,b} \text{ (containing the set } \bar{Q}_{i,j}) \text{ such that if } \bar{Q}_{a,b} \subset \Psi_{i,j},$$

then, there exists another set $\bar{Q}_{l,m} \subset \Psi_{i,j}$ such that $\bar{Q}_{a,b}$ and $\bar{Q}_{l,m}$ share a common edge,

(2.9e) $\bar{\Psi}_{i,j}$ is the union of all sets $\bar{Q}_{a,b}$ not in $\Psi_{i,j}$ that share a common edge with some set $\bar{Q}_{l,m} \subset \Psi_{i,j}$.

A proof of (2.7) and (2.8) can be found in Sanders [5]. The maximum principle (2.9) is a generalization of its one-dimensional version (2.14) Part I, and as it is a new result we include here a proof for the case $s=1$. The case $s \in (0,1)$ follows in a similar way.

PROOF. First, set

$$\begin{aligned}\Theta_{x,a,b}^+ &= -\Delta t^n (f_x^m|_{a+1/2,b} - f_x(u_{a,b})) / (\Delta x_a (u_{a+1,b} - u_{a,b})), \\ \Theta_{x,a,b}^- &= +\Delta t^n (f_x^m|_{a-1/2,b} - f_x(u_{a,b})) / (\Delta x_a (u_{a-1,b} - u_{a,b})), \\ \Theta_{y,a,b}^+ &= -\Delta t^n (f_y^m|_{a,b+1/2} - f_y(u_{a,b})) / (\Delta y_j (u_{a,b+1} - u_{a,b})), \\ \Theta_{y,a,b}^- &= +\Delta t^n (f_y^m|_{a,b-1/2} - f_y(u_{a,b})) / (\Delta y_j (u_{a,b-1} - u_{a,b})),\end{aligned}$$

and note that these quantities are nonnegative. This is because the fluxes f_x^m and f_y^m are monotone. Now, set

$$D_{a,b} = (1 + \Theta_{x,a,b}^+ + \Theta_{x,a,b}^- + \Theta_{y,a,b}^+ + \Theta_{y,a,b}^-)^{-1},$$

and rewrite the fully implicit scheme (2.5) as follows:

$$\begin{aligned}(*1) \quad u_{a,b}^n &= D_{a,b}^n \{ u_{a,b}^{n-1} + \Theta_{x,a,b}^+ u_{a+1,b}^n + \Theta_{x,a,b}^- u_{a-1,b}^n \\ &\quad + \Theta_{y,a,b}^+ u_{a,b+1}^n + \Theta_{y,a,b}^- u_{a,b-1}^n \}.\end{aligned}$$

To prove (2.9) we proceed by contradiction. Assume then, that (2.9) is false. For example, assume that

$$(*2) \quad u^{n+1}_{i,j} > l^1_{i,j}.$$

The case "<" is treated in the same way. Let us denote by $P_{i,j}$ the point (x_i, y_j) . We say that the oriented segment $P_{a,b} \rightarrow P_{c,d}$ is an elementary oriented segment if either $a=c$ and $b=d+1$ or $b=d-1$, or $b=d$ and $a=c+1$ or $a=c-1$. We denote by Γ a path of elementary oriented segments starting from $P_{i,j}$. Now, consider the set of those paths $\Xi = \{\Gamma\}$ with the property that if $P_{a,b} \rightarrow P_{c,d}$ then,

$$(*3) \quad u^{n+1}_{a,b} < u^{n+1}_{c,d}.$$

The set Ξ is nonempty, because by (*1) with $(a,b)=(i,j)$, and (*2) one of the values of the set $\{ u^{n+1}_{i+1,j}, u^{n+1}_{i-1,j}, u^{n+1}_{i,j+1}, u^{n+1}_{i,j-1} \}$ is strictly greater than $u^{n+1}_{i,j}$. By other hand, a path Γ of Ξ cannot cross the boundary of $\Psi_{i,j}$, by (*2) and (*3). Also, Γ cannot intersect itself, so it consists of a finite number of elementary oriented segments. This implies that the end point of it must lie in the interior of $\Psi_{i,j}$, and by (*1), (*2), and (*3) this is a contradiction.

This $L^\infty \cap BV$ -stability result implies the existence of a subsequence converging to a weak solution of (1.1). In fact, this weak solution is the entropy solution. Let us define the following discrete entropy flux associated to the (Kruzkov) entropy U_c defined by (2.1b)

$$\begin{aligned} E_{S,C}^n_{i+1/2,j} &= E_{S,C}(f_x^m; u^{n+1}_{i+1,j}, u^{n+1}_{i,j}; u^n_{i+1,j}, u^n_{i,j}), \\ E_{S,C}^n_{i,j+1/2} &= E_{S,C}(f_y^m; u^{n+1}_{i,j+1}, u^{n+1}_{i,j}; u^n_{i,j+1}, u^n_{i,j}), \end{aligned}$$

where $E_{S,C}$ is defined by (2.15) Part I.

PROPOSITION 2.2. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, for $cfl \in [0, cfl_0]$,

$$(2.16a) \quad (U_C(u^{n+1}_{i,j}) - U_C(u^n_{i,j})) / \Delta t^n + (E_{0,c}^n{}_{i+1/2,j} - E_{0,c}^n{}_{i-1/2,j}) / \Delta x_i \\ + (E_{0,c}^n{}_{i,j+1/2} - E_{0,c}^n{}_{i,j-1/2}) / \Delta y_j \leq 0,$$

and the sequence $\{u_h\}$ converges to the entropy solution of (1.1). Moreover,

$$(2.16b) \quad \|u(T) - u_h(T)\|_{L^1(\mathbf{R})} \leq \|u_0 - u_{0,h}\|_{L^1(\mathbf{R})} + CT^{1/2} \|u_0\|_{BV(\mathbf{R})} \delta^{1/2}.$$

If $cfl \in [0, cfl_0/(1-s)]$ then, the same result holds for the scheme (2.5) with $E_{s,c}^n$ replacing $E_{0,c}^n$ in (2.16a).

A proof for the cases $s = 0, 1$ can be found in Sanders [1]. The general case follows similarly. The following result will be needed.

COROLLARY 2.3. Let $J: \mathbf{R} \rightarrow \mathbf{R}$ be a nonnegative convex, Lipschitz function. Set $J_C(u) = J(u-c)$ and let G_C be an entropy flux associated with J_C ; i.e., $G'_C = f'J'_C$. Then, under the hypothesis of the preceding proposition, there exists a function $\underline{G}_C$ consistent with G_C such that, for every real number c , the discrete entropy inequality

$$(J_C(u^{n+1}_{i,j}) - J_C(u^n_{i,j})) / \Delta t^n + (\underline{G}_{xc}^n{}_{i+1/2,j} - \underline{G}_{xc}^n{}_{i-1/2,j}) / \Delta x_i \\ + (\underline{G}_{yc}^n{}_{i,j+1/2} - \underline{G}_{yc}^n{}_{i,j-1/2}) / \Delta y_j \leq 0$$

is verified by the solution of (2.2). The same result is valid for the scheme (2.5).

3. Quasi-monotone, finite-difference schemes. In this Section quasi-monotone finite-difference schemes are defined and analysed. Some examples are displayed and discussed.

3.1. Definition of qmfd-schemes. Let f^m be a two-point monotone flux. A qmfd-numerical flux f^{qm} is defined as follows :

$$(3.1) \quad f^{qm} = f^m + a^{qm},$$

where $a^{qm} = (a_x^{qm}, a_y^{qm})$, and

(3.2a) (Stability in the x-direction) There exist two discrete functions $v_{xx} = (v_{xx}^-, v_{xx}^+)$, and $v_{xy} = (v_{xy}^-, v_{xy}^+)$ such that

$$\begin{aligned} \text{i) } a_x^{qm}|_{i+1/2,j} &= v_{xx}^+|_{i+1/2,j} (f_x^m|_{i+3/2,j} - f_x^m|_{i+1/2,j}), \\ &= v_{xx}^-|_{i+1/2,j} (f_x^m|_{i+1/2,j} - f_x^m|_{i-1/2,j}); \end{aligned}$$

$$\text{ii) } (1 + v_{xx}^-|_{i+1/2,j} - v_{xx}^+|_{i-1/2,j}) \in [0,2];$$

$$\begin{aligned} \text{iii) } a_x^{qm}|_{i+1/2,j} &= v_{xy}^+|_{i+1/2,j} (f_x^m|_{i+1/2,j+1} - f_x^m|_{i+1/2,j}), \\ &= v_{xy}^-|_{i+1/2,j} (f_x^m|_{i+1/2,j} - f_x^m|_{i+1/2,j-1}); \end{aligned}$$

$$\text{iv) } (1 + v_{xy}^-|_{i+1/2,j+1} - v_{xy}^+|_{i+1/2,j}) \in [0,2];$$

(3.2b) (Stability in the y-direction) There exist two discrete functions $v_{yy} = (v_{yy}^-, v_{yy}^+)$, and $v_{yx} = (v_{yx}^-, v_{yx}^+)$ such that,

$$\begin{aligned} \text{i) } a_y^{qm}|_{i,j+1/2} &= v_{yy}^+|_{i,j+1/2} (f_y^m|_{i,j+3/2} - f_y^m|_{i,j+1/2}), \\ &= v_{yy}^-|_{i,j+1/2} (f_y^m|_{i,j+1/2} - f_y^m|_{i,j-1/2}); \end{aligned}$$

$$\text{ii)} \quad (1 + v_{yy}^-_{i,j+1/2} - v_{yy}^+_{i,j-1/2}) \in [0,2];$$

$$\begin{aligned} \text{iii)} \quad a_y^{qm}_{i,j+1/2} &= v_{yx}^+_{i,j+1/2}(f_y^m_{i+1,j+1/2} - f_y^m_{i,j+1/2}), \\ &= v_{yx}^-_{i,j+1/2}(f_y^m_{i,j+1/2} - f_y^m_{i-1,j+1/2}); \end{aligned}$$

$$\text{iv)} \quad (1 + v_{yx}^-_{i+1,j+1/2} - v_{yx}^+_{i,j+1/2}) \in [0,2];$$

$$(3.2c) \quad (\text{Entropy}) \quad \mathbf{a}^{qm} = O(h^\alpha) \text{ for some } \alpha \in (0,1].$$

These conditions are a generalization of conditions (3.2) Part I defining one-dimensional qm fluxes. It is not difficult to realise that conditions (3.2) state that each of the components of the qm flux $\mathbf{f}^{qm}$ is itself a one-dimensional qm flux in the x-direction as well as in the y-one! (If we relax the stability conditions (3.2a), and only require the x-component (resp., y-component) of the flux to be a one-dimensional qm flux in the x-direction (resp., y-direction) we obtain what we shall call a relaxed quasi-monotone (rqm) flux, $\mathbf{f}^{rqm} = \mathbf{f}^m + \mathbf{a}^{rqm}$). A flux of the form

$$(3.3) \quad \mathbf{f}_s^{qm,n} = s \mathbf{f}^{qm,n} + (1-s) \mathbf{f}^{qm,n-1}, \quad s \in [0,1],$$

is also called a qm flux. (The rqm flux $\mathbf{f}_s^{rqm}$ is defined in a similar way.) A scheme of the form

$$\begin{aligned} (3.4) \quad (u^n_{i,j} - u^{n-1}_{i,j})/\Delta t^n &+ (f_x^{h,n}_{i+1/2,j} - f_x^{h,n}_{i-1/2,j})/\Delta x_i \\ &+ (f_y^{h,n}_{i,j+1/2} - f_y^{h,n}_{i,j-1/2})/\Delta y_j = 0, \end{aligned}$$

where $\mathbf{f}^h = (f_x^h, f_y^h)$ is a qm flux (resp., a rqm flux) is called a qm scheme (resp., a rqm scheme). We shall prove later that, as in the one-dimensional case, any qm scheme is TVD thanks to the stability conditions (3.2a,b). These conditions also ensure that the scheme

verifies the same maximum principle than the corresponding monotone scheme. Finally, condition (3.2c) ensures convergence to the entropy solution. For qm schemes the situation is different because they are not necessarily TVD schemes. Thus, by relaxing the stability conditions (3.2a,b) we have lost the compactness in $L^\infty(0,T;L^1_{loc})$ of the sequence of approximate solutions. However, qm schemes verify the same maximum principle than the corresponding monotone scheme, and, thanks to (3.2c), if they converge, they converge to the entropy solution. Moreover, local order of accuracy is easier to obtain with this type of schemes, as we shall see later.

Let us consider the viscosity produced by the x-component of a qm flux in the x-direction

$$v_{xx}^{qm}{}_{i+1/2,j} = (f_x(u_{i+1,j}) - 2f_x^m{}_{i+1/2,j} + f_x(u_{i,j})) / (u_{i+1,j} - u_{i,j}).$$

This expression generalizes the definition of the one-dimensional viscosity (2.9) Part I. Taking (3.1) into account we can rewrite this expression as follows:

$$v_{xx}^{qm}{}_{i+1/2,j} = v_{xx}^m{}_{i+1/2,j} - 2 a_x^{qm}{}_{i+1/2,j} / (u_{i+1,j} - u_{i,j}).$$

It is then clear that a_x^{qm} controls the viscosity to be added or subtracted from the viscosity already produced by the monotone flux. It is also clear that if we want the viscosity v_{xx}^{qm} to be smaller than the viscosity v_{xx}^m we must impose the condition

$$\text{sgn}(a_x^{qm}{}_{i+1/2,j}) = \text{sgn}(u_{i+1,j} - u_{i,j}).$$

3.3. Some choices of a . The function a^{qm} can be chosen using a two-point monotone flux f^m and a high order accurate one f^h generalising the choices (3.3), (3.4) Part I of the function a in the case

$d = 1$. For example, we can take a_x^{qm} as follows:

$$(3.5a) \quad a_x^{qm}_{i+1/2,j} = \text{sgn}(u_{i+1,j} - u_{i,j}) \max \{ 0, \Theta_{i+1/2,j} \},$$

$$(3.5b) \quad \Theta_{i+1/2,j} = \min \{ |f_x^h_{i+1/2,j} - f_x^m_{i+1/2,j}|, \\ |f_x^m_{i+3/2,j} - f_x^m_{i+1/2,j}| s_{x,i+1,j}, c_{x,i+1,j} (\Delta x_{i+1})^\alpha, \\ |f_x^m_{i+1/2,j} - f_x^m_{i-1/2,j}| s_{x,i,j}, c_{x,i,j} (\Delta x_i)^\alpha, \\ |f_x^m_{i+1/2,j+1} - f_x^m_{i+1/2,j}| s_{y,i+1/2,j+1/2}, \\ |f_x^m_{i+1/2,j} - f_x^m_{i+1/2,j-1}| s_{y,i+1/2,j-1/2} \},$$

$$(3.5c) \quad s_{x,i,j} = \text{sgn}((u_{i+1,j} - u_{i,j})(u_{i,j} - u_{i-1,j})),$$

$$(3.5e) \quad s_{y,i+1/2,j+1/2} = \text{sgn}((u_{i+1,j+1} - u_{i,j+1})(u_{i+1,j} - u_{i,j})),$$

$$(3.5d) \quad c_{i,j} \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+.$$

Other choice is

$$(3.6a) \quad a_x^{qm}_{i+1/2,j} = \text{sgn}(u_{i+1,j} - u_{i,j}) \max \{ 0, \Theta_{i+1/2,j} \},$$

$$(3.6b) \quad \Theta_{i+1/2,j} = \min \{ |f_x^h_{i+1/2,j} - f_x^m_{i+1/2,j}|, \\ |f_x^m_{i+3/2,j} - f_x^m_{i+1/2,j}| s_{x,i+1,j}, c_{x,i+1,j} (\Delta x_{i+1})^\alpha, \\ |f_x^m_{i+1/2,j} - f_x^m_{i-1/2,j}| s_{x,i,j}, c_{x,i,j} (\Delta x_i)^\alpha, \\ 0.5 |f_x^m_{i+1/2,j+1} - f_x^m_{i+1/2,j}|, \\ 0.5 |f_x^m_{i+1/2,j} - f_x^m_{i+1/2,j-1}| \},$$

$$(3.6c) \quad s_{x,i,j} = \text{sgn}((u_{i+1,j} - u_{i,j})(u_{i,j} - u_{i-1,j})),$$

$$(3.6d) \quad c_{x,i,j} \in [0, K] \text{ for some fixed } K \in \mathbb{R}^+.$$

Finally, a third choice is the following

$$(3.7a) \quad a_x^{qm}_{i+1/2,j} = \text{sgn}(f_x^h_{i+1/2,j} - f_x^m_{i+1/2,j}) \max \{ 0, \Theta_{i+1/2,j} \},$$

$$(3.7b) \quad \Theta_{i+1/2,j} = \min \{ |f_x^h_{i+1/2,j} - f_x^m_{i+1/2,j}|, \\$$

$$\begin{aligned}
 & 0.5 | f_x^m_{i+3/2,j} - f_x^m_{i+1/2,j} |, cx_{i+1,j} (\Delta x_{i+1})^\alpha, \\
 & 0.5 | f_x^m_{i+1/2,j} - f_x^m_{i-1/2,j} |, cx_{i,j} (\Delta x_i)^\alpha, \\
 & 0.5 | f_x^m_{i+1/2,j+1} - f_x^m_{i+1/2,j} |, \\
 & 0.5 | f_x^m_{i+1/2,j} - f_x^m_{i+1/2,j-1} | \}, \\
 (3.7c) \quad & cx_{i,j} \in [0,K] \text{ for some fixed } K \in \mathbb{R}^+.
 \end{aligned}$$

If we want to define a rqm flux $f^{rqm} = f^m + a^{rqm}$, the function a can be defined in a similar way. For example, $a_x^{rqm}_{i+1/2,j}$ can be obtained from the expressions (3.5), (3.6) or (3.7) by simply dropping the terms containing either $f_x^m_{i+1/2,j+1}$ or $f_x^m_{i+1/2,j-1}$. We point out that in this case the viscosity v_{xx}^{qm} is greater or equal than the viscosity v_{xx}^{rqm} . This reflects the fact that qm schemes are more stable than the rqm ones. On the other hand, this also implies that with rqm schemes it is easier to recover the local accuracy associated to the flux f_x^h in smooth monotone regions. This fact will be illustrated in the examples at the end of this Section.

3.4. Stability and convergence Analysis. We first state the following stability result :

LEMMA 3.1. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2]$, any explicit qm scheme is TVD and verifies the maximum principle (2.7). (Under the same cfl -condition, any explicit rqm scheme also verifies the maximum principle (2.7)). On the other hand, if $cfl \in [0, cfl_0/2(1-s)]$ any implicit qm scheme is TVD and verifies the maximum principle (2.9). (Under the same cfl -condition, any implicit rqm scheme also verifies the maximum principle (2.9)).

PROOF. The proof of the maximum principles is straightforward and

will be omitted. We shall prove that the explicit qm scheme is TVD; the proof for the implicit case is similar and will not be displayed. In this proof we shall use the following notation:

$$\begin{aligned}\Delta_{i+1/2} w_j^n &= w_{i+1,j}^n - w_{i,j}^n, \\ \Delta_{j+1/2} w_i^n &= w_{i,j+1}^n - w_{i,j}^n.\end{aligned}$$

Consider the following explicit qm scheme

$$\begin{aligned}(u_{i,j}^{n+1} - u_{i,j}^n) / \Delta t^n &+ (f_x^{m,n}{}_{i+1/2,j} - f_x^{m,n}{}_{i-1/2,j}) / \Delta x_i \\ &+ (f_y^{m,n}{}_{i,j+1/2} - f_y^{m,n}{}_{i,j-1/2}) / \Delta y_j = 0,\end{aligned}$$

and rewrite it as follows

$$u_{i,j}^{n+1} = u_{i,j}^n - \Delta t^n (\Delta x_i)^{-1} \Delta_i (f_x^{qm,n}{}_j) - \Delta t^n (\Delta y_j)^{-1} \Delta_j (f_y^{qm,n}{}_i).$$

Next, apply the difference operator $\Delta_{i+1/2}$ to this expression to obtain

$$\begin{aligned} (*) \quad \Delta_{i+1/2} (u_j^{n+1}) &= \Delta_{i+1/2} (u_j^n) - \Delta_{i+1/2} (\Delta t^n (\Delta x)^{-1} \Delta (f_x^{qm,n}{}_j)) \\ &\quad - \Delta_{i+1/2} (\Delta t^n (\Delta y_j)^{-1} \Delta_j (f_y^{qm,n}{}_i)) \\ &= \Delta_{i+1/2} (u_j^n) - \Delta_{i+1/2} (\Delta t^n (\Delta x)^{-1} \Delta (f_x^{qm,n}{}_j)) \\ &\quad - \Delta t^n (\Delta y_j)^{-1} \Delta_j (\Delta_{i+1/2} (f_y^{qm,n}{}_i)).\end{aligned}$$

Now, note that

$$\begin{aligned}\Delta_i (f_x^{qm}{}_j) &= (1 + v_{xx}^-{}_{i+1/2,j} - v_{xx}^+{}_{i-1/2,j}) \Delta_i (f_x^m{}_j), \\ \Delta_{i+1/2} (f_y^{qm}{}_{j+1/2}) &= f_y^{qm}{}_{i+1,j+1/2} - f_y^{qm}{}_{i,j+1/2} \\ &= (1 + v_{yx}^-{}_{i+1,j+1/2} - v_{yx}^+{}_{i,j+1/2}) \Delta_{i+1/2} (f_y^m{}_{j+1/2}),\end{aligned}$$

by (3.2a)i) and (3.2b)iii), respectively. Finally, set

$$\begin{aligned}\Lambda_x^n_{i,j} &= \Delta t^n \Delta y_j (1 + v_{xx}^-_{i+1/2,j} - v_{xx}^+_{i-1/2,j}) / \Delta x_i, \\ \Lambda_{y,i+1/2,j+1/2} &= \Delta t^n (1 + v_{yx}^-_{i+1,j+1/2} - v_{yx}^+_{i,j+1/2}),\end{aligned}$$

and rewrite (*) times Δy_j as follows

$$\begin{aligned}\Delta y_j \Delta_{i+1/2}(u_j^{n+1}) &= \Delta y_j \Delta_{i+1/2}(u_j^n) - \Delta_{i+1/2}(\Lambda_{x,j} \Delta(f_x^{m,n})) \\ &\quad - \Delta_j(\Lambda_{y,i+1/2} \Delta_{i+1/2}(f_y^{m,n})).\end{aligned}$$

Now, we only have to follow word by word the argument used by Sanders [5] to obtain

$$\sum_{i,j} \Delta y_j |\Delta_{i+1/2}(u_j^{n+1})| \leq \sum_{i,j} \Delta y_j |\Delta_{i+1/2}(u_j^n)|.$$

We have used strongly properties (3.2a)ii) and (3.2b)iv), as well as the condition $cfl \in [0, cfl_0/2]$. Using an analogous procedure we prove that

$$\sum_{i,j} \Delta x_i |\Delta_{j+1/2}(u_i^{n+1})| \leq \sum_{i,j} \Delta x_i |\Delta_{j+1/2}(u_i^n)|,$$

and this completes the proof.

THEOREM 3.2. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2]$, any explicit qm scheme is an entropy scheme. Moreover,

$$(3.8a) \quad \int_{\mathbf{R}} J(u(T) - u_h(T)) \leq \int_{\mathbf{R}} J(u_0 - u_{0,h}) + \|u_0\|_{BV(\mathbf{R})} (C_1 T^{1/2} \delta^{1/2} + C_2 T \delta^\alpha),$$

and

$$(3.8b) \quad \|u(T) - u_h(T)\|_{L^1(\Omega)} \leq 2 \|u_0 - u_{0,h}\|_{L^1(\mathbf{R})} + C_3 T^{1/2} \|u_0\|_{BV(\mathbf{R})} \delta^{1/2}$$

$$+C_4 T^{1/2} (\|u_0\|_{BV(R)} |\Omega|)^{1/2} \delta^{\alpha/2},$$

where J is any nonnegative convex function with Lischitz second derivative, and Ω is an arbitrary compact set. (Under the same cfl-condition, any explicit qm scheme that converges (i.e., that is TVB) also verifies the error estimates (3.8)). On the other hand, if $cfl \in [0, cfl_0/2(1-s)]$ any implicit qm scheme is an entropy scheme and verifies the estimates (3.8). (Under the same cfl-condition, any implicit qm scheme that converges also verifies (3.8)).

The proof of this result is analogous to the one of the one-dimensional case; see Section 5 Part I. We end this Subsection by pointing out that time-dependent grid versions of the fully implicit qm scheme can be defined by a straightforward generalization of the procedure used in the one dimensional case; see Subsection 4.6 Part I. As in that case, the resulting scheme can be proven to be an entropy mass-conserving, positive, TVD scheme.

3.5. Schemes with time-qm fluxes. In Part I we defined a qm scheme of second type to be an (implicit) scheme for which the time and space fluxes are qm fluxes. The generalization to d-dimensions is obvious. However, in this case it is easy to verify that we cannot prove that these schemes are TVD (or even TVB!) using the argument of the proof of Lemma 3.1. This is why we did not want to include these schemes in the class of qm schemes.

LEMMA 3.3. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, c_0 cfl_0/2(1-s)]$, any (implicit) scheme with time and space qm fluxes verifies the maximum principle (2.9). (Under the same cfl-condition, the same is true for the corresponding rqm schemes).

THEOREM 3.4. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, c_0 cfl_0 / 2(1-s)]$, any (implicit) converging (i.e., TVB) scheme with time and space fluxes is an entropy scheme, and verifies the error estimates (3.8). (Under the same cfl -condition, the same is true for the corresponding qm schemes).

3.6. Some Examples.

i) *The centered scheme*. This is an example of an explicit qmfd scheme. For simplicity, let us assume that the discretization parameters are constant. Then, this scheme can be written as follows:

$$u_{ij}^{n+1} = u_{ij}^n - \Delta t (f_x^{C,n}{}_{i+1/2,j} - f_x^{C,n}{}_{i-1/2,j}) / \Delta x \\ - \Delta t (f_y^{C,n}{}_{i+1/2,j} - f_y^{C,n}{}_{i-1/2,j}) / \Delta y,$$

where

$$f_x^C{}_{i+1/2} = (f_x(u_{i+1,j}) + f_x(u_{ij})) / 2, \\ f_y^C{}_{i+1/2} = (f_y(u_{i+1,j}) + f_y(u_{ij})) / 2.$$

As in the one-dimensional case, the viscosity produced by this flux is identically zero, and so even if the scheme were stable it would never converge to the entropy solution of (1.1). However, any qm scheme using $f^h = f^C$ is an entropy scheme. For example, let us take as monotone flux the Godunov flux f^G . The corresponding qmfd-scheme is only first order accurate, but it is stable under half the cfl -condition under which the Godunov scheme is stable. Also, as in the one-dimensional case, it can be proved that

$$\text{sgn}(f_x^C{}_{i+1/2,j} - f_x^G{}_{i+1/2,j}) = \text{sgn}(u_{i+1,j} - u_{ij}), \\ \text{sgn}(f_y^C{}_{i,j+1/2} - f_y^G{}_{i,j+1/2}) = \text{sgn}(u_{i,j+1} - u_{ij});$$

see Subsection 3.4 Part I. This shows that it is an entropy scheme

whose viscosities v_{xx} and v_{yy} are smaller than the ones of the Godunov scheme!

ii) **The Crank-Nicholson qm scheme**. This is an example of an (non-fully) implicit scheme. The Crank-Nicholson scheme is written as follows:

$$u_{ij}^{n+1} = u_{ij}^n - \Delta t (f_x^{CN,n}{}_{i+1/2,j} - f_x^{CN,n}{}_{i-1/2,j}) / \Delta x - \Delta t (f_y^{CN,n}{}_{i+1/2,j} - f_y^{CN,n}{}_{i-1/2,j}) / \Delta y,$$

where

$$f_x^{CN,n}{}_{i+1/2,j} = [(f_x(u^{n+1}_{i+1,j}) + f_x(u^{n+1}_{ij})) / 2 + (f_x(u^n_{i+1,j}) + f_x(u^n_{ij})) / 2] / 2,$$

$$f_y^{CN,n}{}_{i,j+1/2} = [(f_y(u^{n+1}_{i,j+1}) + f_y(u^{n+1}_{ij})) / 2 + (f_y(u^n_{i,j+1}) + f_y(u^n_{ij})) / 2] / 2.$$

This scheme is second order accurate in space and time. We consider now the qm scheme obtained with the (Crank-Nicholson-)Godunov flux:

$$f_x^{CNG,n}{}_{i+1/2,j} = [f_x^{G,n+1}{}_{i+1/2,j} + f_x^{G,n}{}_{i+1/2,j}] / 2,$$

$$f_y^{CNG,n}{}_{i,j+1/2} = [f_y^{G,n+1}{}_{i,j+1/2} + f_y^{G,n}{}_{i,j+1/2}] / 2,$$

where f^G is the Godunov flux. It is not difficult to verify that the scheme GCN is a monotone scheme twice the CFL-condition under which the Godunov scheme is monotone. As a consequence, the qm scheme under consideration is an entropy scheme under the same CFL-condition for which the Godunov scheme is monotone. On the other hand, from the expressions

$$\begin{aligned} |f_x^{\text{CNG},n}_{i+1/2,j} - f_x^{\text{CN},n}_{i+1/2,j}| &= \Delta x |\partial_x f_x(u^n_{ij})|/2 + O(\delta^2), \\ |f_x^{\text{CNG},n}_{i+3/2,j} - f_x^{\text{CNG},n}_{i+1/2,j}| &= \Delta x |\partial_x f_x(u^n_{ij})| + O(\delta^2), \\ |f_x^{\text{CNG},n}_{i+1/2,j} - f_x^{\text{CNG},n}_{i-1/2,j}| &= \Delta x |\partial_x f_x(u^n_{ij})| + O(\delta^2), \end{aligned}$$

and

$$\begin{aligned} |f_x^{\text{CNG},n}_{i+1/2,j+1} - f_x^{\text{CNG},n}_{i+1/2,j}| &= \Delta y |\partial_y f_x(u^n_{ij})| + O(\delta^2), \\ |f_x^{\text{CNG},n}_{i+1/2,j} - f_x^{\text{CNG},n}_{i+1/2,j-1}| &= \Delta y |\partial_y f_x(u^n_{ij})| + O(\delta^2), \end{aligned}$$

it is clear that, if in smooth monotone parts of the entropy solution we want to have $f_x^{\text{qm}} = f_x^{\text{CN}}$, we must respect the condition

$$\Delta y |\partial_y f_x| > \Delta x |\partial_x f_x|/2,$$

or equivalently

$$\Delta y |\partial_y u| > \Delta x |\partial_x u|/2.$$

Roughly speaking, this means that the variation of the solution in the element Q along the y -direction must be greater than the half of its variation along the x -direction. As a similar condition is obtained by analysing the f_y -flux, we shall be able to recover the second order accuracy of the Crank-Nicholson scheme only if

$$2\Delta x |\partial_x u| > \Delta y |\partial_y u| > \Delta x |\partial_x u|/2.$$

This is in strong agreement with the result of Goodman and Leveque [3]. However, we do not have this restriction if we consider only a rqm scheme, for in this case the second set of expressions is not taken into account. Thus, for the rqm versions of the considered scheme, we can recover locally the second order accuracy of the Crank-Nicholson scheme

on smooth monotone parts of the entropy solution!

4. Quasi-monotone, finite-element schemes. In this Section qmfe schemes are defined and analysed.

4.1. Definition of qmfe schemes. As in the one-dimensional case, see Subsection 2.1 Part II, we shall consider the Petrov-Galerkin methods defined as follows:

- (4.1a) Choose the finite-dimensional space V_h to which the approximate solution u_h belongs.
- (4.1b) Choose the space of test functions W_h . The characteristic function of the element K must belong to W_h , for every $K = K_{i,j}^n$.
- (4.1c) The initial data u_{0h} is taken to be the L^2 -projection of u_0 on the space V_{0h} of traces of the functions of V_h on $\{t=0\} \times \mathbb{R}^2$.
- (4.1d) The approximate solution u_h is determined by means of the following equations:

$$\begin{aligned} & - \iint_{K'} u_h \partial_t \varphi \, dt \, dx \, dy + \int_{\partial K'} u_h \varphi n_t \, d\mathcal{H} \\ & - \iint_{K'} f(u_h) \cdot \text{grad} \varphi \, dt \, dx \, dy + \int_{\partial K'} f(u_h) \cdot n_{\partial Q} \varphi \, d\mathcal{H} = 0, \varphi \in W_h, \end{aligned}$$

where $(n_{\partial Q}, n_t)$ is the unit outward normal on $\partial K'$, and K' is the support of φ .

We assume, of course, that the spaces V_h and W_h are chosen in such a way that (4.1d) defines uniquely the approximate solution. For simplicity, we shall assume that $K' = K$.

Now, take $K = J^n \times Q_{i,j}$ and rewrite equations (4.1d) as follows:

$$\begin{aligned}
 (4.2a) \quad & - \int_K u_h \partial_t \phi \, dt \, dx \, dy \\
 & + \int_Q u_h(t^{n+1}) (\phi(t^{n+1}) - \bar{\phi}^{n+1}) \, dx \, dy + |Q| \bar{\phi}^{n+1} w^{n+1}_{i,j} \\
 & - \int_Q u_h(t^n) (\phi(t^n) - \bar{\phi}^n) \, dx \, dy - |Q| \bar{\phi}^n w^n_{i,j} \\
 & - \int_K f(u_h) \cdot \text{grad} \phi \, dt \, dx \, dy \\
 & + \sum_{A \in \partial Q} \int_{J^n \times A} f(u_h) \cdot n_{A,Q} (\phi - \bar{\phi}_A) \, d\mathcal{H}^1 \\
 & + \sum_{A \in \partial Q} \Delta t^n |A| \bar{\phi}_A f^{fe,n}_{A \cdot n_{A,Q}} = 0, \quad \phi \in W_h,
 \end{aligned}$$

where A is a generic face of Q , $\bar{\phi}_A$ denotes the mean value of ϕ on $J^n \times A$, $\bar{\phi}^n$ the mean value of ϕ on $\{t^n\} \times Q$, and

$$(4.2b) \quad w^n_{i,j} = \int_Q u_h(t^n, x, y) \, dx \, dy / |Q|, \quad Q = Q_{i,j},$$

$$(4.2c) \quad f^{fe,n}_A = \int_A \int_J f(u_h(t, x, y)) \, dt \, d\mathcal{H}^1 / (\Delta t^n |A|).$$

As in Part II, we write the approximate solution u_h as $\bar{u}_h + \tilde{u}_h$, where the function $\bar{u}_h$ is piecewise-constant. We shall define $\bar{u}_h$ by one of the following expressions:

$$(4.3a) \quad \bar{u}_h(t, x, y) = \bar{u}^n_{i,j} = w^n_{i,j}, \quad \text{for } (t, x, y) \in K^n_{i,j},$$

$$(4.3b) \quad \bar{u}_h(t, x, y) = \bar{u}^{n+1}_{i,j} = w^{n+1}_{i,j}, \quad \text{for } (t, x, y) \in K^n_{i,j}.$$

Now, let f^{qm} be a qm flux constructed using the flux f^{fe} defined by (4.2c) and a two-point monotone flux f^m (or f^m_s) that depends only on the means $\bar{u}_h$. A numerical scheme (4.1)-(4.2)-(4.3) in which f^{fe} is replaced by the qm flux f^{qm} is called a qmfe scheme. (Relaxed qmfe

schemes are obtained if f^{fe} is replaced by a qm flux f^{qm}). We shall say that the qmfe scheme is explicit (resp., implicit) if the scheme resulting by setting $\tilde{u}_h \equiv 0$ is explicit (resp., implicit).

4.2. Stability and Convergence Analysis. Proceeding as in the one-dimensional case, it is easy to obtain the following results.

LEMMA 4.1. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2]$, any explicit qmfe scheme is TVDM and verifies the maximum principle (2.7) (with the means $\bar{u}_{i,j}$ replacing the point-values $u_{i,j}$). (Under the same cfl -condition, any explicit qmfe scheme also verifies the maximum principle (2.7)). On the other hand, if $cfl \in [0, cfl_0/2(1-s)]$ any implicit qmfe scheme is TVDM and verifies the maximum principle (2.9) (with the means $\bar{u}_{i,j}$ replacing the point-values $u_{i,j}$). (Under the same cfl -condition, any implicit qmfe scheme also verifies the maximum principle (2.9)).

THEOREM 4.2. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2]$, any explicit qmfe scheme is an entropy scheme. Moreover,

$$(4.4a) \quad \int_{\mathbf{R}} J(u(T) - \bar{u}_h(T)) \leq \int_{\mathbf{R}} J(u_0 - \bar{u}_{0,h}) + \|u_0\|_{BV(\mathbf{R})} (C_1 T^{1/2} \delta^{1/2} + C_2 T \delta^\alpha),$$

and

$$(4.4b) \quad \|u(T) - \bar{u}_h(T)\|_{L^1(\Omega)} \leq 2 \|u_0 - \bar{u}_{0,h}\|_{L^1(\mathbf{R})} + C_3 T^{1/2} \|u_0\|_{BV(\mathbf{R})} \delta^{1/2} + C_4 T^{1/2} (\|u_0\|_{BV(\mathbf{R})} |\Omega|)^{1/2} \delta^{\alpha/2},$$

where J is any nonnegative convex function with Lischitz second derivative, and Ω is an arbitrary compact set. (Under the same cfl -condition, any explicit qmfe scheme that converges (i.e., that is

TVB) also verifies the error estimates (4.4)). On the other hand, if $cfl \in [0, cfl_0/2(1-s)]$ any implicit qmfe scheme is an entropy scheme and verifies the estimates (4.4). (Under the same cfl-condition, any implicit qmfe scheme that converges also verifies (4.4)).

4.3. Convergence of u_h to the entropy solution. To enforce the convergence of the sequence $\{u_h\}$ to the entropy solution we can modify the qmfe under consideration by using the projections $\Lambda \Pi_h^n$, exactly as in the one-dimensional case. So, let us assume that these operators exist and verify the following inequality:

$$(4.5) \quad \|\Lambda \Pi_h^n(\tilde{u}_h)\|_{L^1(\mathbb{R}^2)} \leq C \delta.$$

Then, we modify our qmfe scheme as follows:

(4.6a) Compute $u_{0,h}$ as follows : First, obtain $P_h^2(u_0) = \bar{u}_{0,h} + \tilde{u}_{0,h}$; then, compute $\Lambda \Pi_h^0(\tilde{u}_{0,h})$, and set $u_{0,h} = \bar{u}_{0,h} + \Lambda \Pi_h^0(\tilde{u}_{0,h})$;

(4.6b) Assuming u_h^n known, compute u_h^{n+1} as follows : First, calculate u_h^{n+1} using the qmfe scheme; then, compute $\Lambda \Pi_h^{n+1}(\tilde{u}_h^{n+1})$ and set $u_h^{n+1} = \bar{u}_h^{n+1} + \Lambda \Pi_h^{n+1}(\tilde{u}_h^{n+1})$.

We have the following immediate result.

COROLLARY 4.2. Consider the qmfe scheme (4.1)-(4.2)-(4.3)-(4.6). Then, under the assumptions of theorem 4.2, and supposing that (4.5) holds, the scheme under consideration verifies Lemma 4.1 and Theorem 4.2. Moreover, the estimates (4.4) are verified with $u_h(T)$ replacing $\bar{u}_h(T)$.

4.4. The local projections $\Lambda\Pi$. As in the one-dimensional case, when the approximate solution is discontinuous accross the boundaries ∂K a family of operators $\{\Lambda\Pi_h^n\}$ verifying (4.5) does exist, and can be defined to be a family of local projections. Let us denote by $\tilde{V}(K)$ the space to which the restriction of $\tilde{u}_h$ to the interior of K , $\tilde{u}_h|_K$, belongs. Now, let us denote by $\tilde{V}'(K)$ the convex subset of $\tilde{V}(K)$ for which we have:

$$(4.7a) \quad |\tilde{u}_h(t,x)| \leq \sum_{l,m=r,\dots,s} C X_{m,l} \Theta_{i+m+1/2,j+l}^n + \sum_{l,m=r,\dots,s} C Y_{m,l} \Theta_{i+m,j+l+1/2}^n \text{ for } (t,x) \text{ in } K_{i,j}^n,$$

$$(4.7b) \quad \Theta_{i+1/2,j}^n = |\bar{u}_{i+1,j}^n - \bar{u}_{i,j}^n|,$$

$$(4.7c) \quad \Theta_{i,j+1/2}^n = |\bar{u}_{i,j+1}^n - \bar{u}_{i,j}^n|,$$

$$(4.7d) \quad C X_{m,l}, C Y_{m,l} \geq 0,$$

where r and s (resp., $C_{m,l}$) are arbitrary but fixed natural (resp., real) numbers. Then, we require that

$$(4.8a) \quad \Lambda\Pi_h^n : \tilde{V}(K) \rightarrow \tilde{V}'(K) \text{ for every } K = K_{i,j}^n,$$

$$(4.8b) \quad (\Lambda\Pi_h^n)^2 = \Lambda\Pi_h^n.$$

This means defines a family of local projections verifying (4.5). A very important particular case, that is the generalization of the projections defined by (3.8)-(3.9) Part II, is when the set $\tilde{V}'(K)$ is taken as follows:

$$(4.9a) \quad \tilde{V}'(K_{i,j}^n) = \{ \tilde{u}_h \in \tilde{V}(K_{i,j}^n) : \tilde{u}_h(t,x) \in Y_{i,j}^n \text{ for } (t,x) \in K_{i,j}^n \},$$

$$(4.9b) \quad Y_{i,j}^n = I(0, \bar{u}_{i-1,j}^n - \bar{u}_{i,j}^n, \bar{u}_{i+1,j}^n - \bar{u}_{i,j}^n, \bar{u}_{i,j-1}^n - \bar{u}_{i,j}^n, \bar{u}_{i,j+1}^n - \bar{u}_{i,j}^n).$$

In this case the following maximum principle is verified

$$(4.10) \quad u_h(t,x) \in I(\bar{u}_{i,j-1}^n, \bar{u}_{i,j+1}^n, \bar{u}_{i-1,j}^n, \bar{u}_{i,j}^n, \bar{u}_{i+1,j}^n), (t,x) \in K_{i,j}^n,$$

where we are assuming that $u_h = \bar{u}_h + \Delta\pi(\tilde{u}_h)$. This implies that, if $u_{0,h}(x,y) \in [a,b]$ for $(x,y) \in \mathbf{R}^2$, the same is true for u_h .

We end this Subsection by pointing out that time-dependent grid versions of the fully implicit qm scheme can be defined by a straightforward generalization of the procedure used in the one dimensional case; see Subsection 4.3 Part II. As in that case, the resulting scheme can be proven to be an entropy mass-conserving, positive (in the means!), TVDM scheme.

4.5. Schemes with time-qm fluxes. In Subsection 4.1 Part II we defined a qmfe scheme of second type to be an (implicit) scheme (4.2a) for which the time flux w and the space flux f^e have been replaced by qm fluxes. It is easy to generalize these scheme to d -dimensions (see Subsection 4.1 Part II), but we do not include these schemes in the class of qmfe schemes because these schemes are not necessarily TVDM. However, for these schemes we have the following results.

LEMMA 4.3. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, c_0 cfl_0 / 2(1-s)]$, any (implicit) scheme with time and space qm fluxes verifies the maximum principle (2.9) (with the means $\bar{u}_{i,j}$ replacing the point-values $u_{i,j}$). (Under the same cfl -condition, the same is true for the corresponding rqm schemes).

THEOREM 4.4. Suppose that the scheme (2.2) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, c_0 cfl_0 / 2(1-s)]$, any (implicit) converging (i.e., TVB) scheme with time and space fluxes defines a sequence $\{\bar{u}_h\}$ that converges to the entropy solution, and verifies the error estimates (4.4). Moreover, if the scheme is modified as in (4.6), then the scheme is an entropy scheme, and the error estimate (4.4) are verified with $u_h(T)$

replacing $\bar{u}_h(T)$ (Under the same cfl-condition, the same is true for the corresponding rqm schemes).

5. The two-dimensional G-1/2 scheme . The one-dimensional G-1/2 scheme was considered in Part I. Here we generalise the scheme to the two-dimensional case. For the sake of clarity we shall take a uniform triangulation. We assume that the reader is already familiar with the one-dimensional case.

5.1 Definition of the two-dimensional P^0Q^1 scheme . This Galerkin discontinuous method is defined as follows:

(5.1a) In each element $K_{ij}^n = J^n \times Q_{ij}$ the approximate solution u_h is constant in time and bilinear in space (or, P^0 in time and Q^1 in space!); i.e.

$$\begin{aligned} u_h(t, x, y) = & u_1^n{}_{i-1/2, j-1/2} \phi^1(s_x, s_y) + \\ & u_2^n{}_{i+1/2, j-1/2} \phi^2(s_x, s_y) + \\ & u_3^n{}_{i+1/2, j+1/2} \phi^3(s_x, s_y) + \\ & u_4^n{}_{i-1/2, j+1/2} \phi^4(s_x, s_y) \quad , \quad (t, x, y) \in K_{ij}^n. \end{aligned}$$

where

$$\begin{aligned} \phi^1(s_x, s_y) &= (1-s_x)(1-s_y)/4, \\ \phi^2(s_x, s_y) &= (1+s_x)(1-s_y)/4, \\ \phi^3(s_x, s_y) &= (1+s_x)(1+s_y)/4, \\ \phi^4(s_x, s_y) &= (1-s_x)(1+s_y)/4, \end{aligned}$$

and

$$s_x = 2(x - x_i)/\Delta x, \quad s_y = 2(y - y_j)/\Delta y.$$

(5.1b) The trace of u_h in ∂K is chosen as follows :

$$\begin{aligned} u_h(t^n, x, y) &= \lim_{\epsilon \downarrow 0} u_h(t^n + \epsilon, x, y), \quad (x, y) \in Q, \\ u_h(t, x_{i+1/2}, y) &= \xi_{i+1/2}^n(y), \quad y \in (y_{j-1/2}, y_{j+1/2}), \quad t \in J^n, \\ u_h(t, x, y_{j+1/2}) &= \xi_{j+1/2}^n(x), \quad x \in (x_{i-1/2}, x_{i+1/2}), \quad t \in J^n, \end{aligned}$$

where $\xi_{i+1/2}^n(y)$ is defined using the Godunov flux f_x^G in the following manner :

$$f_x(\xi_{i+1/2}^n(y)) = f_x^G(u_h(t, (x^+)_{i+1/2}, y), u_h(t, (x^-)_{i+1/2}, y)),$$

and $\xi_{j+1/2}^n(x)$ is defined by :

$$f_y(\xi_{j+1/2}^n(y)) = f_y^G(u_h(t, (x, y^+)_{j+1/2}), u_h(t, (x, y^-)_{j+1/2})).$$

(5.1c) The degrees of freedom of the initial data are given by:

$$[u]_{i,j}^0 = \int_Q ([\psi], u_0) \, dx \, dy / |Q|,$$

where

$$([u]_{i,j}^0)^t = (u_1^n_{i-1/2, j-1/2}, u_2^n_{i+1/2, j-1/2}, u_3^n_{i+1/2, j+1/2}, u_4^n_{i-1/2, j+1/2}),$$

and

$$[\psi]^t = ((1-3s_x)(1-3s_y), (1+3s_x)(1-3s_y), (1+3s_x)(1+3s_y), (1-3s_x)(1+3s_y));$$

(5.1d) For $n=0, \dots, N-1$ the degrees of freedom of u_h^{n+1} are defined by

$$\begin{aligned} & ([u]_{ij}^{n+1} - [u]_{ij}^n) / \Delta t \\ & - \int_Q f(u_h) \cdot \text{grad} [\psi] \, dx / |Q| \\ & + \sum_{A \in \partial Q} \int_A f(u_h) \cdot \mathbf{n}_{A,Q} [\psi - \bar{\psi}_A] \, dx / |Q| \\ & + \sum_{A \in \partial Q} |A| [\bar{\psi}_A] f^{fe,n}_A \cdot \mathbf{n}_{A,K} / |Q| = 0, \end{aligned}$$

where (with the obvious notation)

$$\begin{aligned} [\bar{\psi}_{i-1/2,j}]^t &= (4, -2, -2, 4), \\ [\bar{\psi}_{i,j-1/2}]^t &= (4, 4, -2, -2), \\ [\bar{\psi}_{i+1/2,j}]^t &= (-2, 4, 4, -2), \\ [\bar{\psi}_{i,j+1/2}]^t &= (-2, -2, 4, 4), \end{aligned}$$

and

$$(5.2a) \quad f_x^{fe,n}_{i+1/2,j} = \int_A f_x(\xi^n_{i+1/2}(y)) \, dy / \Delta y, \quad A = (y_{j-1/2}, y_{j+1/2}),$$

$$(5.2b) \quad f_y^{fe,n}_{i,j+1/2} = \int_A f_y(\xi^n_{j+1/2}(x)) \, dx / \Delta x, \quad A = (x_{i-1/2}, x_{i+1/2}).$$

To evaluate the integrals we shall use quadrature rules. For example, we can use the four-point Gauss quadrature formula to integrate over $Q_{i,j}$ and the two-point Gauss formula to integrate over the faces of $Q_{i,j}$. Other possibility is to use Simpson formulas instead. A third possibility is to use quadrature formulas that are exact when the function f is linear; with this criterion, some of the integrals could be approximated by using only the trapezoidal rule. In the linear case all these choices are equivalent.

Note that if the approximate function is piecewise constant this scheme is nothing but the two-dimensional Godunov scheme. We recall that in the case $d = 1$ this scheme is at most first order accurate, L^2 -stable in the linear case under the condition $cf_1 = O(h^{1/2})$, and non-convergent in some cases when the function f is non-convex, even under the condition $cf_1 = O(h^{1/2})$. This is essentially due to the fact that the approximate solution was assumed to be constant in time on each interval (t^n, t^{n+1}) . See Section 5 Part I. As this is also the main assumption for the two-dimensional P^0Q^1 scheme, we expect similar stability and convergence properties.

5.2. Definition of the two-dimensional G-1/2 scheme. The G-1/2 is an explicit qm scheme constructed using the 2-dimensional Godunov scheme and the P^0Q^1 scheme. Thus, to define it we only need to define the space flux $f^{G-1/2}$, and the family of projections $\{\Lambda \Pi_h^n\}$.

i. **The flux $f^{G-1/2}$.** It is defined by (3.1)-(3.2)-(3.5) using the Godunov flux as the monotone flux depending only on the means, and the flux f^e given by (5.2). We recall that the piecewise-constant part of our approximate solution is given by

$$(5.3) \quad \bar{u}_h(t, x, y) = \bar{u}^n_{i,j} = e^t[u]^n_{i,j} \text{ for } (t, x, y) \in K^n_{i,j},$$

where $e^t = (1, 1, 1, 1)/4$.

ii. **The projections $\Lambda \Pi$.** The family of projections we shall consider is a generalization of the family of projections used in the one-dimensional case. This projection was introduced in [1]. By definition, the restriction of the approximate solution u_h to the element Q belongs to the space $Q^1(Q)$. Let us denote by $\tilde{Q}^1(Q)$ the space of functions of $Q^1(Q)$ with zero mean. It is then obvious that $\tilde{u}_h|_Q \in \tilde{Q}^1(Q)$. We define the

projection $\Lambda \Pi_h^n|_Q$, $Q = Q_{i,j}$, as a L^2 -type projection from $\tilde{Q}^1(Q_{i,j})$ to $\tilde{Q}^{1,n}(Q_{i,j})$, where

$$\tilde{Q}^{1,n}(Q_{i,j}) = \{ \tilde{w}_h \in \tilde{Q}^1(Q_{i,j}) : \tilde{w}_h(P) \in I^n(P) - \bar{u}_{i,j}^n, \\ P \text{ a vertex of } Q \},$$

and

$$I^n(x_{i+1/2}, y_{j+1/2}) = I(\bar{u}_{i,j}^n, \bar{u}_{i+1,j}^n, \bar{u}_{i+1,j+1}^n, \bar{u}_{i,j+1}^n).$$

In terms of the degrees of freedom of $u_h|_Q$ this projection can be described as follows. Let us denote by P_k , $k=1,2,3,4$ the vertex of the quadrilateral $Q_{i,j}$, and by $[u]^t = (u_1, u_2, u_3, u_4)$ the degrees of freedom of $u_h|_Q$. By $[u^*]$ we shall denote the degrees of freedom of its projection. Finally, set

$$C_{i,j}^n = \{ [u] \in R^4 : u_1 + u_2 + u_3 + u_4 = \bar{u}_{i,j}^n \}, \\ H_{i,j}^n = I^n(P_1) \times I^n(P_2) \times I^n(P_3) \times I^n(P_4).$$

Then, $[u]$ is the unique L^2 -projection of $[u]^n$ on the convex set $C_{i,j}^n \cap H_{i,j}^n$. For more details, see [1].

With this projection the maximum principle (4.10) is verified!

6. Generalizing the LRG qm implicit schemes to the two-dimensional case. At this stage it should be clear to the reader how to generalize to the two-dimensional case the LRG qm schemes considered in some detail in Subsection 6.3 of Part II. We have to point out that those schemes are schemes with time qm fluxes, and so compactness in $L^\infty(0,T;L^1_{loc})$ is not ensured anymore; see Subsection

4.5. To obtain a TVD property we are forced to redefine the means as in (4.3). However, in this case we will be able to recover locally the accuracy of the resulting implicit qmfe scheme only if the parameter Δt is not too large with respect to Δx and Δy ; see the discussion in Subsection 4.2 Part I. Finally, it is important to note that even if consider the LRG schemes with time qm fluxes, we shall be able to recover locally the accuracy of the LRG schemes only if we have the following condition :

$$2\Delta x |\partial_x u| > \Delta y |\partial_y u| > \Delta x |\partial_x u| / 2,$$

see the Subsection 3.6. Nevertheless, this "local accuracy" is the local accuracy associated only to the "means" of the approximate solution. Finally, let us point out that this difficulty does not arise if instead of taking qm space fluxes we take only qm ones! See Subsection 3.6.

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