THE QUASI-MONOTONE SCHEMES
FOR SCALAR CONSERVATION LAWS
PART II

BY

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The Quasi-Monotone Schemes for Scalar Conservation Laws. Part II

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Abstract. In this paper we introduce the quasi-monotone finite-element numerical methods for scalar conservation laws in one space dimension. This new class of fe schemes is a class of TVDM (total variation diminishing in the means) schemes that always converge to the entropy solution. They also satisfy maximum principles inherited from monotone schemes. However, unlike those schemes, their local order of accuracy is not necessarily bounded by one. Error estimates in the L^1_{loc} -norm are provided for both explicit and implicit qmfe schemes assuming only that the initial data is an L^1 -function of bounded total variation. Finally, a time-dependent-grid version of some of these schemes is introduced and analysed.

1. Introduction. In this paper we introduce and analyse quasi-monotone finite-element (qmfe) numerical methods for the scalar conservation law

$$\begin{aligned} (1.1a) \quad & \partial_t u + \partial_x f(u) = 0 && \text{in } (0,T) \times \mathbb{R}, \\ (1.1b) \quad & u(t=0) = u_0 && \text{on } \mathbb{R}, \end{aligned}$$

where f is assumed to be C^1 . These schemes are L^∞ -stable, TVB (total-variation-bounded), and, more important, they are entropy schemes; i.e., their approximate solution always converge to the entropy solution of (1.1). Moreover, assuming only that the initial data u_0 belongs to $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the space of bounded total

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variation functions, error estimates in the $L^\infty(L^1_{loc})$ -norm have been obtained. As far as we know, no other type of fe methods have been proven to have these stability and convergence properties. In the linear case many fe methods have been studied, and L^2 -stability results, as well as L^2 -type error estimates, have been obtained assuming enough regularity on the initial data u_0 . Let us mention, for example, the work of LeSaint and Raviart [7], in which the analysis is made by means of energy techniques and interpolation theory, and the one of Johnson and Pitkaranta [5], in which they use a mixture of fd and fe techniques. Unfortunately, these methods can hardly be extended to the nonlinear case. By other hand, very recently Johnson and Szepessy [6] considered the stream-line diffusion Petrov-Galerkin fe method applied to (1.1) where $f(u) = u^2/2$, and proved convergence to the entropy solution assuming that the method is L^∞ -stable. They used energy estimates, interpolation theory, and, most interesting, the theory of compensated compactness to obtain L^2 -like error estimates. However, the latter theory can only be applied (... at least for the moment!) when the nonlinear function f is convex. Also, there are no results available for the d -dimensional case, $d > 1$. Our method of analysis does not make use of either energy estimates or interpolation theory, but employs the finite-difference techniques used to study monotone schemes; see Part I and the bibliography therein. We write the approximate solution as $u_h = \bar{u}_h + \tilde{u}_h$, where $\bar{u}_h$ is a piecewise-constant function obtained as a local average of u_h ; the function $\tilde{u}_h$ is viewed as a parameter updated at each time-step. In this way, we can consider our Petrov-Galerkin method as a set of equations defining the parameter $\tilde{u}_h$ plus a single equation, that is essentially a fd scheme, defining "the means" $\bar{u}_h$. The theory of qmfd schemes is then applied to that "fd scheme" to get stability, convergence to the entropy solution and error estimates; all of this only in terms of the means $\bar{u}_h$. Finally, if the approximate solution is discontinuous, it is possible to obtain similar results for the whole approximate solution u_h .

by means of a non-constraining local control of the growth of $\tilde{u}_h$, that does not destroy the order of accuracy of the method in smooth monotone regions.

An outline of the paper is as follows. In Section 2 we display the class of Petrov-Galerkin methods that will be considered in this theory, and we establish a link between them and fd schemes. This will allow us to use for the qmfe schemes the same analysis techniques used for qmfd schemes. In Sections 3 and 4 we introduce and analyse explicit and implicit qmfe schemes, respectively. In Section 5 we discuss in detail an example of an explicit first order accurate qmfe scheme that produces less viscosity than the Godunov scheme, the G-1/2 scheme. Finally, in Section 6 we display an example of a locally high order accurate implicit qmfe scheme.

2. Preliminaries. In this Section we display the class of Petrov-Galerkin finite-element methods we consider in the theory of qmfe schemes, and we establish a link between these methods and the 3-point fd schemes. We also recall some notation.

2.1. A certain class of Petrov-Galerkin fe methods. Here, we define the class of fe-schemes we shall consider in this paper. Let us first recall some notation. As usual, the sets $\{t^n\}_{n=1,\dots,N}$ and $\{x_{i+1/2}\}_{i \in \mathbb{Z}}$ are partitions of $[0,T]$ and $\mathbb{R}$, respectively. We set $\Delta t^n = t^{n+1} - t^n$ and $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, and we denote by J^n and I_i the intervals (t^n, t^{n+1}) and $(x_{i-1/2}, x_{i+1/2})$, respectively. Finally, K_i^n will stand for the element $J^n \times I_i$, and δ for $\sup_{i,n} \{ \Delta t^n, \Delta x_i \}$.

Our Petrov-Galerkin methods are defined as follows:

- (2.1a) Choose the finite-dimensional space V_h to which the approximate solution u_h belongs. Note that if it is not

continuous accross the boundary ∂K of a generic element K , the definition of the trace of u_h must be included in the definition of V_h .

(2.1b) Choose the space of test functions W_h . Over this space we impose the following single, but very important, restriction : the characteristic function of the element K must belong to W_h , for every $K = K_i^n$.

(2.1c) The initial data u_{0h} is taken to be the L^2 -projection of u_0 on the space V_{0h} of traces of the functions of V_h on $\{t=0\} \times \mathbf{R}$. We shall denote this projection by P_h^2 .

(2.1d) The approximate solution u_h is determined by means of the following equations:

$$\begin{aligned} & - \iint_{K'} u_h \partial_t \varphi \, dt \, dx + \int_{\partial K'} u_h \varphi n_t \, d\mathcal{S} \\ & - \iint_{K'} f(u_h) \partial_x \varphi \, dt \, dx + \int_{\partial K'} f(u_h) \varphi n_x \, d\mathcal{S} = 0, \quad \varphi \in W_h, \end{aligned}$$

where (n_x, n_t) is the unit outward normal of $\partial K'$, and K' is the support of φ .

We assume, of course, that the spaces V_h and W_h are chosen in such a way that (2.1d) defines uniquely the approximate solution. We also want to point out that it is reasonable to take a discontinuous approximate solution, for the entropy solution may develop strong discontinuities in finite time even if the initial data u_0 is very smooth. In this case, the choice of the trace of u_h on ∂K is of crucial importance. From equations (2.1d) it is clear that the trace of u_h can be left undefined in a subset E of ∂K of zero-measure. Typically we have

$$E_i^n = \{ (t^{n+1}, x_{i+1/2}), (t^n, x_{i+1/2}), (t^n, x_{i-1/2}), (t^{n+1}, x_{i-1/2}) \}.$$

Also it is reasonable to assume that the actual trace of u_h on $\partial K \setminus E$ will depend only on its outer and inner traces. In what follows, for the sake of simplicity, we shall assume that $K = K'$.

2.2. The link with fd schemes. If in (2.1d) we take ϕ equal to the characteristic function of the element K the resulting equation can be rewritten as follows:

$$(2.2a) \quad (w_i^{n+1} - w_i^n) / \Delta t^n + (f^{fe,n}_{i+1/2} - f^{fe,n}_{i-1/2}) / \Delta x_i = 0,$$

where,

$$(2.2b) \quad w_i^n = (\Delta x_i)^{-1} \int_I u_h(t^n, x) dx, \quad I = I_i,$$

$$(2.2c) \quad f^{fe,n}_{i+1/2} = (\Delta t^n)^{-1} \int_J f(u_h(t, x_{i+1/2})) dt, \quad J = J^n.$$

Note that the time-flux w , as well as the space-flux f^{fe} , depend only on the trace of u_h on ∂K . If u_h is assumed to be piecewise-constant, say

$$u_h(t, x) = \bar{u}_h(t, x) = \bar{u}^n_i \quad \text{for } (t, x) \in K^n_i,$$

then,

$$\begin{aligned} w_i^n &= w(\bar{u}^n_i, \bar{u}^{n-1}_i), \\ f^{fe,n}_{i+1/2} &= f^{fe}(\bar{u}^n_{i+1}, \bar{u}^n_i), \end{aligned}$$

and the scheme (2.2) can be considered as a fd scheme defining the discrete approximate solution $\{\bar{u}^n_i\}$. In the general case we write u_h as

$\bar{u}_h + \tilde{u}_h$, where $\bar{u}_h$ is piecewise-constant, and we consider the function $\tilde{u}_h$ as a parameter updated using (2.1). Thus, (2.2) can also be considered as a fd scheme defining $\{\bar{u}_h\}$. In this way, we can use it to construct qmfd schemes using the techniques introduced in Part I. At these stage we are ready to define qmfe schemes. This will be done in the next two Sections.

We end this Section by recalling the definition of the 3-point monotone schemes we shall use for the construction of the qmfe schemes. We shall consider the explicit scheme

$$(2.3a) \quad (\bar{u}_i^{n+1} - \bar{u}_i^n) / \Delta t^n + (f^{m,n}_{i+1/2} - f^{m,n}_{i-1/2}) / \Delta x_i = 0,$$

$$(2.3b) \quad \bar{u}_i^0 = \bar{u}_{0,i},$$

where f^m is a two-point monotone consistent flux; i.e.,

$$(2.4a) \quad f^m_{i+1/2} = f^m(\bar{u}_{i+1}, \bar{u}_i),$$

$$(2.4b) \quad f^m(u, w) \text{ is nonincreasing in } u \text{ and nondecreasing in } w,$$

$$(2.4c) \quad f^m(u, u) = f(u),$$

and also the implicit scheme

$$(2.5a) \quad (\bar{u}_i^n - \bar{u}_i^{n-1}) / \Delta t^n + (f_s^{m,n}_{i+1/2} - f_s^{m,n}_{i-1/2}) / \Delta x_i = 0,$$

$$(2.5b) \quad \bar{u}_i^0 = \bar{u}_{0,i},$$

where

$$(2.5c) \quad f_s^{m,n} = s f^{m,n} + (1-s) f^{m,n-1}, \quad s \in (0, 1].$$

3. Explicit, quasi-monotone finite-element schemes. In this Section we shall study the explicit qmfe schemes.

3.1. Definition and analysis. Consider a Petrov-Galerkin method of type (2.1). We say that such a method is explicit if

$$(3.1a) \quad u_h(t^n, x) = \lim_{\epsilon \downarrow 0} u_h(t^n + \epsilon),$$

$$(3.1b) \quad u_h(t) \text{ depends only on } u_h(t^n) \text{ for } t \in J^n.$$

Now, set

$$(3.1c) \quad \bar{u}_h^n(t, x) = \bar{u}^n_{I_i} = w^n_{I_i} = (\Delta x_i)^{-1} \int_{I_i} u_h(t^n, x) dx, \text{ for } (t, x) \in K^n_{I_i},$$

and note that the algebraic equations (2.1d) can also be rewritten as

$$(3.1d) \quad \begin{aligned} & - \int \int_K u_h \partial_t \phi dt dx + \int_{\partial K} u_h \phi n_t d\mathcal{X} - \\ & - \int \int_K f(u_h) \partial_x \phi dt dx + \int_{\partial K} f(u_h) (\phi - \bar{\phi}) n_x d\mathcal{X} \\ & + \Delta t^n (f^{fe, n}_{i+1/2} \bar{\phi}^n_{i+1/2} - f^{fe, n}_{i-1/2} \bar{\phi}^n_{i-1/2}) = 0, \phi \in W_h, \end{aligned}$$

where $\bar{\phi}^n_{I_i}$ is the mean value of ϕ on the set $(t^n, t^{n+1}) \times \{x_i\}$. Now, let f^{qm} be a qm flux constructed as in (3.1)–(3.2) Part I with an arbitrary two-point monotone flux f^m , see (2.4), depending only on the means $\bar{u}_h$. The flux f^{qm} can be taken as in (3.3) or (3.4) Part I using the flux f^{fe} defined in (2.2c) instead of f^h . When in (3.1d) f^{fe} is replaced by the qm flux f^{qm} the scheme (2.1)–(3.1) is said to be an explicit qmfe scheme.

To analyse this scheme is now a very simple matter. Taking ϕ equal to the characteristic function of $K^n_{I_i}$ in (3.1d) and adding the initial condition we obtain the scheme

$$\begin{aligned} & (\bar{u}^{n+1}_{I_i} - \bar{u}^n_{I_i}) / \Delta t^n + (f^{qm, n}_{i+1/2} - f^{qm, n}_{i-1/2}) / \Delta x_i = 0, \\ & \bar{u}^0_{I_i} = (\Delta x_i)^{-1} \int_{I_i} u_0(x) dx, \quad I_i = I_i. \end{aligned}$$

If we assume that we know how to calculate the parameter $\tilde{u}_h$ and we consider the scheme solely as a method to determine the means $\bar{u}_h$, it can be regarded as an explicit qmfd scheme. As a consequence, Lemma 3.1 and Theorem 3.2 of Part I hold and we have the following result.

THEOREM 3.1. Suppose that the scheme (2.3)-(2.4) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2]$, any explicit qmfe scheme verifies the maximum principle

$$(3.2) \quad \bar{u}_i^{n+1} \in I(\bar{u}_{i-1}^n, \bar{u}_i^n, \bar{u}_{i+1}^n)$$

and is TVDM (total variation diminishing in the means) ; i.e.,

$$(3.3) \quad \|\bar{u}_h^{n+1}\|_{BV(\mathbf{R})} \leq \|\bar{u}_h^n\|_{BV(\mathbf{R})}.$$

Moreover, the sequence $\{\bar{u}_h\}$ converges to the entropy solution of (1.1), and verifies the estimates

$$(3.4a) \quad \int_{\mathbf{R}} J(u(T) - \bar{u}_h(T)) \leq \int_{\mathbf{R}} J(u_0 - \bar{u}_{0,h}) + \|u_0\|_{BV(\mathbf{R})} (C_1 T^{1/2} \delta^{1/2} + C_2 T \delta^{\alpha/2}),$$

$$(3.4b) \quad \|u(T) - \bar{u}_h(T)\|_{L^1(\Omega)} \leq 2\|u_0 - \bar{u}_{0,h}\|_{L^1(\mathbf{R})} + C_3 T^{1/2} \|u_0\|_{BV(\mathbf{R})} \delta^{1/2} + C_4 T^{1/2} (\|u_0\|_{BV(\mathbf{R})} |\Omega|)^{1/2} \delta^{\alpha/2},$$

where J is an arbitrary nonnegative convex function with Lipschitz second derivative, and Ω is an arbitrary compact subset of $\mathbf{R}$.

We can say that qmfe schemes are schemes for the means $\bar{u}_h$ because this result is independent of the precise form of the flux f^e and

of the values of $\tilde{u}_h$. As an appropriate choice of $\tilde{u}_h$ near discontinuities may reduce the error, qmfe schemes provide freedom to do this. Note also that the local accuracy can be changed without altering stability properties. In particular, if the approximate solution is piecewise-polynomial, its degree can vary in time and in space without affecting the stability. This could possibly give rise to a p-version of the qmfe schemes, but we shall not discuss the matter in this work.

3.2. Convergence of u_h to the entropy solution. Now, let us consider the problem of the convergence of the sequence $\{u_h\}$ to the entropy solution of (1.1). Since

$$\|u - u_h\|_{L^\infty(L^1)} \leq \|u - \bar{u}_h\|_{L^\infty(L^1)} + \|\tilde{u}_h\|_{L^\infty(L^1)},$$

to obtain (3.4) with u_h instead of $\bar{u}_h$ it is enough to require that, as δ goes to zero, $\|\tilde{u}_h\|_{L^\infty(L^1)} = O(\delta^{\alpha/2})$. Note that $\tilde{u}_h$ is essentially an approximation of $h\partial_x u$, so that it is reasonable to expect that

$$\begin{aligned} \|\tilde{u}_h\|_{L^\infty(L^1)} &\approx C h \|\partial_x u\|_{L^\infty(L^1)} \\ &\approx C h \|u\|_{L^\infty(BV)} \\ &\approx C h \|u_0\|_{BV} \end{aligned}$$

In fact, we expect $\tilde{u}_h$ to "behave well" in the regions in which the entropy solution is smooth (... if the initial Petrov-Galerkin method has also this property!); however, $\tilde{u}_h$ can have an exaggerated growth near discontinuities, and so has to be modified. We shall denote by $\Lambda\pi_h^n(\tilde{u}_h)$ this modification, and the only restriction we shall impose to the family of operators $\{\Lambda\pi_h^n\}$ is the following

$$(3.5) \quad \|\Lambda\pi_h^n(\tilde{u}_h)\|_{L^1(\mathbf{R})} \leq C \delta.$$

Suppose, for the moment, that such a family exists for the scheme under consideration. Then, we modify it as follows:

$$(3.6a) \quad \text{Compute } u_{0,h} \text{ as follows : First, obtain } P_h^2(u_0) = \bar{u}_{0,h} + \tilde{u}_{0,h}; \\ \text{then, compute } \Lambda\pi_h^0(\tilde{u}_{0,h}), \text{ and set } u_{0,h} = \bar{u}_{0,h} + \Lambda\pi_h^0(\tilde{u}_{0,h});$$

$$(3.6b) \quad \text{Assuming } u_h^n \text{ known, compute } u_h^{n+1} \text{ as follows : First,} \\ \text{calculate } u_h^{n+1} \text{ using the qmfe scheme; then, compute} \\ \Lambda\pi_h^{n+1}(\tilde{u}_h^{n+1}) \text{ and set } u_h^{n+1} = \bar{u}_h^{n+1} + \Lambda\pi_h^{n+1}(\tilde{u}_h^{n+1}).$$

We have the following immediate result.

COROLLARY 3.2. Consider the explicit qmfe scheme (3.6). Then, under the assumptions of theorem 3.1, and supposing that (3.5) holds, the scheme under consideration is a TVDM-entropy scheme that verifies the maximum principle (3.2). Moreover, the estimates (3.4) are verified with $u_h(T)$ replacing $\bar{u}_h(T)$.

In the next Section we show that when the approximate solution u_h is discontinuous the operators $\Lambda\pi_h^n$ can be taken as local projections.

3.3. The local projections $\Lambda\pi$. Let us consider a qmfe scheme for which the approximate solution is discontinuous accross the boundaries ∂K . For this type of qmfe schemes a family of operators $\{\Lambda\pi_h^n\}$ verifying (3.5) does exist, and can be defined to be a family of local projections. To see this we proceed as follows. First, let us denote by $\tilde{V}(K)$ the space to which the restriction of $\tilde{u}_h$ to the interior of K , $\tilde{u}_h|_K$, belongs. Now, let us denote by $\tilde{V}'(K)$ the convex subset of $\tilde{V}(K)$ for

which we have:

$$(3.7a) \quad |\tilde{u}_h(t,x)| \leq \sum_{m=r,\dots,s} C_m \Theta_{i+m+1/2}^n \text{ for } (t,x) \text{ in } K_i^n.$$

$$(3.7b) \quad \Theta_{i+1/2}^n = |\bar{u}_{i+1}^n - \bar{u}_i^n|,$$

$$(3.7c) \quad C_m \geq 0,$$

where r and s (resp., C_m) are arbitrary but fixed natural (resp., real) numbers. Then, we require that

$$(3.8a) \quad \Lambda \Pi_h^n : \tilde{V}(K) \rightarrow \tilde{V}'(K) \text{ for every } K = K_i^n,$$

$$(3.8b) \quad (\Lambda \Pi_h^n)^2 = \Lambda \Pi_h^n.$$

This means that we require $\Lambda \Pi_h^n$ to be a local projection. Note that as u_h is discontinuous accross ∂K , the result of projecting $\tilde{u}_h|_K$ on $\tilde{V}'(K)$ does not have any influence on the values of u_h outside of K . Also, note that the very form of the projection has not been specified, and so there is absolute freedom in the choice of it. Finally, it is easy to verify that any family of operators verifying (3.7)-(3.8) satisfy the inequality (3.5); in fact, we have

$$\begin{aligned} \|\Lambda \Pi_h^n(\tilde{u}_h)\|_{L^1(\mathbf{R})} &\leq [\sum_{m=r,\dots,s} C_m] \|\bar{u}_h^n\|_{BV(\mathbf{R})} h, \\ &\leq [\sum_{m=r,\dots,s} C_m] \|u_0\|_{BV(\mathbf{R})} \delta. \end{aligned}$$

A very important particular case is when the set $\tilde{V}'(K)$ is taken as follows:

$$(3.9a) \quad \tilde{V}'(K_i^n) = \{ \tilde{u}_h \in \tilde{V}(K_i^n) : \tilde{u}_h(t,x) \in Y_i^n \text{ for } (t,x) \in K_i^n \},$$

$$(3.9b) \quad Y_i^n = I(\bar{u}_{i-1}^n - \bar{u}_i^n, 0, \bar{u}_{i+1}^n - \bar{u}_i^n).$$

In this case the following maximum principle is verified

$$(3.10) \quad u_h(t,x) \in I(\bar{u}^{n_{i-1}}, \bar{u}^{n_i}, \bar{u}^{n_{i+1}}) \text{ for } (t,x) \in K_i^n,$$

where we are assuming that $u_h = \bar{u}_h + \Delta \Pi(\tilde{u}_h)$. This implies that, if $u_{0,h}(x) \in [a,b]$ for $x \in \mathbf{R}$, the same is true for u_h . This is a very important property because in many physical problems only values inside a fixed interval have physical meaning.

We finish this Section by pointing out that if in (3.7) the constants C_m are not too small we should have $\Delta \Pi_h^n(\tilde{u}_h) = \tilde{u}_h$ in the smooth monotone regions of the entropy solution, and so, the projection is really effective in a very small region. Ideally, the measure of this region should be $O(\delta)$ as δ goes to zero!

4. Implicit, quasi-monotone, finite-element schemes. In this Section we define and analyse implicit qmfe schemes. A time-dependent-grid version of some of these schemes is also considered. Stability and convergence results are given.

4.1. Definition of Implicit Qmfe Schemes. Consider a Petrov-Galerkin method of type (2.1). We say that such a method is implicit if it is not explicit; see Subsection 3.1. To define implicit qmfe schemes we proceed as in the explicit case. Set,

$$(4.1a) \quad \bar{u}_h(t,x) = \bar{u}^{n_i} = w_i^n = (\Delta x_i)^{-1} \int_I u_h(t^n, x) dx, \text{ for } (t,x) \in K_i^n,$$

and rewrite (2.1d) as follows

$$(4.1b) \quad \begin{aligned} & -\int \int_K u_h \partial_t \phi \, dt \, dx + \int_{\partial K} u_h \phi n_t \, d\mathcal{H}^1 \\ & -\int \int_K f(u_h) \partial_x \phi \, dt \, dx + \int_{\partial K} f(u_h) (\phi - \bar{\phi}) n_x \, d\mathcal{H}^1 \\ & + \Delta t^n (f^{e,n}_{i+1/2} \bar{\phi}^{n_{i+1/2}} - f^{e,n}_{i-1/2} \bar{\phi}^{n_{i-1/2}}) = 0, \quad \phi \in W_h. \end{aligned}$$

As for the case of fd schemes, we extend the definition of a qm flux as follows: the flux f_s^{qm} is a qm flux if it is of the form

$$f_s^{fe,n} = s f^{qm,n} + (1-s) f^{qm,n-1}, \quad s \in (0,1],$$

where f^{qm} is a qm flux constructed as in the explicit case. When in (4.1b) f^{fe} is replaced by a qm flux the scheme (2.1)-(4.1) is said to be an implicit qmfe scheme of the first type.

Next, we define implicit schemes of second type. Set,

$$(4.2a) \quad \bar{u}_h(t,x) = \bar{u}^{n_i} = (\Delta x_i)^{-1} \int_I u_h(t^n + p\Delta t^n, x) dx, \text{ for } (t,x) \in K_i^n,$$

where $p \in [0,1]$ is a fixed number, and rewrite (2.1d) as follows

$$(4.2b) \quad \begin{aligned} & - \iint_K v_h \partial_t \phi \, dt \, dx + \int_{\partial K} v_h \phi n_t \, d\gamma \\ & + \Delta x_i (w_i^{n+1} \bar{\phi}^{n_i} - w_i^n \bar{\phi}^{n-1_i}) \\ & - \iint_K f(v_h) \partial_x \phi \, dt \, dx + \int_{\partial K} f(v_h) (\phi - \bar{\phi}) n_x \, d\gamma \\ & + \Delta t^n (f^{fe,n}_{i+1/2} \bar{\phi}^{n_{i+1/2}} - f^{fe,n}_{i-1/2} \bar{\phi}^{n_{i-1/2}}) = 0, \quad \phi \in W_h, \end{aligned}$$

where $\bar{\phi}^{n_i}$ is the mean value of ϕ on the set $\{t^n\} \times (x_{i-1/2}, x_{i+1/2})$. Let flux f_s^{qm} be a qm flux, and let the flux g^{qm} be a qm time flux constructed with the means $\bar{u}_h$ as in (4.5)-(4.6) Part I. Note that the function b can be taken as in (4.7) or (4.8) Part I. When in (4.2b) f^{fe} is replaced by a qm space flux the scheme, and w by a qm time flux, the scheme (2.1)-(4.2) is said to be an implicit qmfe scheme of the second type.

4.2. Stability and Convergence Analysis. For these schemes we have the following convergence result.

THEOREM 4.1. Suppose that the scheme (2.3)-(2.4) is monotone for $cfl \in [0, cfl_0]$. Then, if $cfl \in [0, cfl_0/2(1-s)]$, any qmfe scheme of first type is a TVDM scheme verifying the maximum principle

$$(4.3) \quad \bar{u}_i^{n+1} \in I(\bar{u}^{n+1}_j, \bar{u}^n_{j+1}, \dots, \bar{u}^n_{k-1}, \bar{u}^{n+1}_k), \quad j < i < k.$$

Moreover, the sequence $\{\bar{u}_h\}$ converges to the entropy solution of (1.1), and satisfies the error estimates (3.4). The same result is true for any implicit qmfe scheme of the second type under the condition $cfl \in [0, c_0 cfl_0/2(1-s)]$.

Let us finish this Subsection by pointing out that is a family of operators $\{\Lambda \Pi_h^n\}$ verifying the condition (3.5) can be found, (see Subsection 3.3), we can modify the implicit qmfe under consideration as in (3.6). For such a scheme we have the following immediate result.

COROLLARY 4.2. Consider the implicit qmfe scheme (3.6). Then, under the assumptions of theorem 4.1, and supposing that (3.5) holds, the scheme under consideration is a TVDM-entropy scheme that verifies the maximum principle (3.4). Moreover, the estimates (3.4) are verified with $u_h(T)$ replacing $\bar{u}_h(T)$.

4.3. Time-dependent-grid qmfe schemes. We shall define a time-dependent-grid version of fully implicit (i.e., $s = 1$) qmfe schemes of the first type. We follow the algorithm introduced in [4] for fd schemes.

Suppose that $u_h(t^{n-1})$ is known, and we want to compute $u_h(t^n)$. First, we must obtain the new partition $P_h^n = \{x^n_{i+1/2}\}$ on which $u_h(t^n)$

will be defined. Suppose that we are able to obtain P_h^n by some means; see Subsection 4.6 Part I. Once this has been done we project the approximate solution $u_h(t^{n-1})$, defined on the partition P_h^{n-1} , to the corresponding space associated to the new partition P_h^n using the L^2 -projection Π_h^n . In this way, we obtain the functions

$$(4.4a) \quad \bar{v}_h(t^{n-1}) = \Pi_h^n(\bar{u}_h(t^{n-1})),$$

$$(4.4b) \quad \tilde{v}_h(t^{n-1}) = \Pi_h^n(\tilde{u}_h(t^{n-1})).$$

Finally, we set $\bar{u}_h(t^n) = \bar{v}_h(t^{n-1})$, $\tilde{u}_h(t^n) = \tilde{v}_h(t^{n-1})$ and we solve for $u_h(t^n)$ as usual.

As for the fd case, we have

$$(4.5a) \quad \int_{\mathbf{R}} \Pi_h^n(\bar{u}_h(t^n)) = \int_{\mathbf{R}} \bar{u}_h(t^n),$$

$$(4.5b) \quad (\inf)\sup(\Pi_h^n(\bar{u}_h(t^n))) \leq (\inf)\sup(\bar{u}_h(t^n)),$$

$$(4.5c) \quad \|\Pi_h^n(\bar{u}_h(t^n))\|_{BV(\mathbf{R})} \leq \|\bar{u}_h(t^n)\|_{BV(\mathbf{R})},$$

i.e., mass is conserved and the scheme remains TVDM and positive in the means. It is not difficult to show that the sequence $\{\bar{u}_h\}$ converges to the entropy solution of (1.1); see [4]. Let us point out that if we want to obtain the convergence of the u_h we can use that operators $\Lambda\Pi_h^n$ as in the fixed-grid case.

5. The G-1/2 scheme. The G-1/2 scheme, introduced by the author [3], is a qmfe scheme obtained using the Godunov scheme and a first order accurate fe scheme devised by Chavent and Salzano [2], that we shall call the P^0P^1 -scheme. First, we shall describe this scheme and discuss its main properties. Then, we shall consider the G-1/2 scheme. This scheme is an example of a first order accurate entropy fe scheme

that produces less viscosity than the Godunov scheme.

5.1. Definition of the P^0P^1 -scheme. This is a Galerkin scheme (i.e., the trial and test functions are the same) of type (2.1). It is defined as follows:

(5.1a) In each element K_i^n the approximate solution u_h is constant in time and linear in space (or, in other words, P^0 in time and P^1 in space!); i.e.,

$$\begin{aligned} u_h(t, x) &= \bar{u}_i^n \phi^0((x-x_i)/\Delta x_i) + \tilde{u}_i^n \phi^1((x-x_i)/\Delta x_i), \quad (t, x) \in K_i^n, \\ \phi^0(s) &= 1, \quad s \in (-1/2, 1/2), \\ \phi^1(s) &= 2s\phi^0(s), \quad s \in (-1/2, 1/2). \end{aligned}$$

Note that since

$$\begin{aligned} \bar{u}_i^n &= (u_h(t^n, x_{i+1/2}^-) + u_h(t^n, x_{i-1/2}^+)) / 2, \\ \tilde{u}_i^n &= (u_h(t^n, x_{i+1/2}^-) - u_h(t^n, x_{i-1/2}^+)) / 2, \end{aligned}$$

the value $\bar{u}_i^n$ can be considered as an approximation of $u(t^n, x_i)$ and $\tilde{u}_i^n$ as one of $0.5h \partial_x u(t^n, x_i)$.

(5.1b) The trace of u_h in ∂K is chosen as follows :

$$\begin{aligned} u_h(t^n, x) &= \lim_{\epsilon \downarrow 0} u_h(t^{n+\epsilon}, x), \quad x \in I_i, \\ u_h(t, x_{i+1/2}) &= \xi_{i+1/2}^n, \quad t \in J^n, \end{aligned}$$

where $\xi_{i+1/2}^n$ is defined by

$$f(\xi_{i+1/2}^n) = f^G(\bar{u}_{i+1}^n - \tilde{u}_{i+1}^n, \bar{u}_i^n + \tilde{u}_i^n),$$

where f^G denotes the Godunov flux; see (2.11) Part I.

- (5.1c) The degrees of freedom of the initial data are computed as follows :

$$\begin{aligned}\bar{u}^0_i &= \int_I u_0(s) ds / \Delta x_i, \\ \tilde{u}^0_i &= 6 \int_I (s-x_i) u_0(s) ds / \Delta x_i^2,\end{aligned}$$

- (5.1d) For $n=0, \dots, N-1$, the degrees of freedom of u_h^{n+1} are obtained as the solution of

$$\begin{aligned}(\bar{u}_i^{n+1} - \bar{u}_i^n) / \Delta t^n + (f^{fe,n}_{i+1/2} - f^{fe,n}_{i-1/2}) / \Delta x_i &= 0, \\ (\tilde{u}_i^{n+1} - \tilde{u}_i^n) / \Delta t^n + \{ 3(f^{fe,n}_{i+1/2} + f^{fe,n}_{i-1/2}) \\ &\quad - 6 \iint_K f(u_h(t,x)) dt dx / (\Delta t^n \Delta x_i) \} / \Delta x_i = 0,\end{aligned}$$

where

$$(5.2a) \quad f^{fe}_{i+1/2} = f^G(\bar{u}^n_{i+1} - \tilde{u}^n_{i+1}, \bar{u}^n_i + \tilde{u}^n_i),$$

and the integral over $K = K^n_i$ is approximated using Simpson's quadrature formula

$$\begin{aligned}(5.2b) \quad \iint_K f(u_h(t,x)) dt dx / (\Delta t^n \Delta x_i) &\approx \\ & (f(\bar{u}^n_i - \tilde{u}^n_i) + 4f(\bar{u}^n_i) + f(\bar{u}^n_i + \tilde{u}^n_i)) / 6.\end{aligned}$$

5.2. The linear case. In the linear case $f(u)=u$ we have

$$f^{fe}_{i+1/2} = \bar{u}_i + \tilde{u}_i.$$

$$6 \iint_K f(u_h(t,x)) dt dx / (\Delta t^n \Delta x_i) \approx \bar{u}^n_i,$$

and for a uniform grid the scheme reads as follows :

(5.3a) Compute $u_{0,h}$ by (5.1c) ;

(5.3b) For $n=0, \dots, N-1$, compute the degrees of freedom of u_h^{n+1} as follows :

$$\begin{aligned} \bar{u}_i^{n+1} &= (1-cfl) \bar{u}_i^n + cfl \bar{u}_{i-1}^n - cfl \tilde{u}_i^n + cfl \tilde{u}_{i-1}^n, \\ \tilde{u}_i^{n+1} &= 3cfl \bar{u}_i^n - 3cfl \bar{u}_{i-1}^n + (1-3cfl) \tilde{u}_i^n - 3cfl \tilde{u}_{i-1}^n. \end{aligned}$$

The number cfl stands for $\Delta t/h$, of course. Note that for $cfl = 1$

$$\begin{aligned} \bar{u}_i^{n+1} &= \bar{u}_{i-1}^n - [\tilde{u}_i^n - \tilde{u}_{i-1}^n], \\ \tilde{u}_i^{n+1} &= \tilde{u}_{i-1}^n + [3(\bar{u}_i^n - \bar{u}_{i-1}^n) - 2\tilde{u}_i^n - 4\tilde{u}_{i-1}^n]. \end{aligned}$$

From this we see that the scheme does not "follow the characteristics" because the expressions between brackets are not identically zero. It is also easy to see that the constant solution $u_h \equiv c$ is unstable ! In fact, the scheme (5.3) is L^2 -stable if and only if $cfl = O(h^{1/2})$ as $h \downarrow 0$; see [1]. Thus, the scheme is too "slow" in the sense that it is not able to "follow the information at the right speed". This is why it must be helped by an (enormous) increase of the numerical speed $h/\Delta t$ as h goes to zero; i.e., $h/\Delta t = O(h^{-1/2})$. Another way of describing this behaviour is to say that the scheme is very "anti-diffusive".

In the non-linear case the scheme may fail to converge to the entropy solution even under the condition $cfl = O(h^{1/2})$. Numerical evidence shows that this to be the case when f is non-convex (or non-concave).

5.3. Definition of the G-1/2 scheme. The qmfe scheme G-1/2 was designed to overcome these disadvantages [3]. This scheme is an explicit qmfe scheme. To define it we only have to specify the space qm flux $f^{G-1/2}$, and the family of projections $\{\Lambda \Pi_h^n\}$.

i. **The flux $f^{G-1/2}$.** The qm flux $f^{G-1/2}$ is defined by (3.1)-(3.2) -(3.3) Part I using the Godunov flux f^G as the monotone flux depending only on the means, and the flux f^e given by (5.2a).

ii. **The projections $\Lambda \Pi$.** By definition, we have that

$$\tilde{u}_h(t, x) = \tilde{u}^n_i \phi^1((x-x_i)/\Delta x_i) \text{ for } (t, x) \in K^n_i.$$

We define $\Lambda \Pi_h^n$ as follows:

$$\Lambda \Pi_h^n(\tilde{u}_h(t, x)) = \tilde{u}^{*n}_i \phi^1((x-x_i)/\Delta x_i) \text{ for } (t, x) \in K^n_i,$$

where

$$\begin{aligned} \tilde{u}^{*n}_i &= \text{sgn}(\tilde{u}^n_i) \max \{0, \Theta^n_i\}, \\ \Theta^n_i &= \min \{ |\tilde{u}^n_i|, |\bar{u}^n_{i+1} - \bar{u}^n_i|, |\bar{u}^n_i - \bar{u}^n_{i-1}|, \\ &\quad \text{sgn}(\tilde{u}^{n+1}_i (\bar{u}^{n+1}_{i+1} - \bar{u}^{n+1}_i)), \\ &\quad \text{sgn}((\bar{u}^n_{i+1} - \bar{u}^n_i)(\bar{u}^n_i - \bar{u}^n_{i-1})) \}. \end{aligned}$$

It is easy to check that this local projection is of type (3.9). In fact, it is slightly more severe because when $\bar{u}_h$ has a local extremum in $\bar{u}^n_i$ the projected degree of freedom $\tilde{u}^{*n}_i$ is equal to zero; i.e., the approximate solution is constant in K^n_i ; and also because in this case a stronger version of the

maximum principle (3.10) is verified:

$$(5.4a) \quad \bar{u}^n_i + \tilde{u}^{*n}_i \in I(\bar{u}^n_{i+1}, \bar{u}^n_i),$$

$$(5.4b) \quad \bar{u}^n_i - \tilde{u}^{*n}_i \in I(\bar{u}^n_{i-1}, \bar{u}^n_i).$$

Now, for the sake of completeness, we write down the algorithm defining the G-1/2 scheme:

(5.5a) Define the degrees of freedom of $u_{0,h}$ as follows : First, set

$$\begin{aligned} \bar{u}^0_i &= \int_I u_0(s) ds / \Delta x_i, \\ \tilde{u}^0_i &= 6 \int_I (s-x_i) u_0(s) ds / \Delta x_i^2; \end{aligned}$$

then, compute $\tilde{u}^{*0}_i$, and set $\tilde{u}^0_i = \tilde{u}^{*0}_i$;

(5.5b) For $n=1, \dots, N-1$ compute the degrees of freedom of u_h^{n+1} as follows : First, solve for

$$\begin{aligned} (\bar{u}^{n+1}_i - \bar{u}^n_i) / \Delta t^n + (f^{G-1/2, n}_{i+1/2} - f^{G-1/2, n}_{i-1/2}) / \Delta x_i &= 0, \\ (\tilde{u}^{n+1}_i - \tilde{u}^n_i) / \Delta t^n + \{ 3(f^{G-1/2, n}_{i+1/2} + f^{G-1/2, n}_{i-1/2}) \\ &\quad - (f(\bar{u}^n_i - \tilde{u}^n_i) + 4f(\bar{u}^n_i) + f(\bar{u}^n_i + \tilde{u}^n_i)) \} = 0; \end{aligned}$$

then, compute $\tilde{u}^{*n+1}_i$, and set $\tilde{u}^{n+1}_i = \tilde{u}^{*n+1}_i$.

Finally, let us point out that as the Godunov scheme is monotone for $cfl \in [0, 1]$, the G-1/2 scheme verifies Corollary 3.2 for $cfl \in [0, 0.5]$.

5.4. Some remarks on the viscosity and the flux $f^{G-1/2}$.
The viscosity of the qm flux $f^{G-1/2}$ is given by

$$v^{G-1/2}_{i+1/2} = v^G_{i+1/2} - 2 a_{i+1/2} / (\bar{u}_{i+1} - \bar{u}_i),$$

where $v^G_{i+1/2} = v^G(\bar{u}_{i+1}, \bar{u}_i)$ is the Godunov viscosity, see (2.10) Part I, and the function a is defined as (3.3) Part I, as indicated in the definition of the flux $f^{G-1/2}$. By definition, the sign of $a_{i+1/2}$ is equal to the sign of $(\bar{u}_{i+1} - \bar{u}_i)$, and so, the viscosity $v^{G-1/2}$ is smaller than the Godunov viscosity. In this sense we say that $G-1/2$ is a (first order accurate) scheme that produces less viscosity than the Godunov one.

By other hand, using the viscosities of the fluxes f^G and f^{fe} (see their definition in (2.9) Part I) we can write

$$f^{fe}_{i+1/2} - f^G_{i+1/2} = [v^G_{i+1/2} - v^{fe}_{i+1/2}] (\bar{u}_{i+1} - \bar{u}_i) / 2,$$

where, of course,

$$v^{fe}_{i+1/2} = (f(\bar{u}_{i+1}) - 2f(\xi_{i+1/2}) + f(\bar{u}_i)) / (\bar{u}_{i+1} - \bar{u}_i).$$

By (2.10) Part I the expression in brackets is non-negative if $\xi_{i+1/2}$ belongs to the interval $I(\bar{u}_{i+1}, \bar{u}_i)$, but this is true because of the maximum principle (5.4). Thus, we have

$$(5.5) \quad \text{sgn}(f^{fe}_{i+1/2} - f^G_{i+1/2}) = \text{sgn}(\bar{u}_{i+1} - \bar{u}_i).$$

Finally, we can apply the formal local error analysis used in the fd theory to obtain

$$|f^{fe}_{i+1/2} - f^G_{i+1/2}| = 0.5 |\partial_x f| h + O(h^2),$$

$$\begin{aligned} |f^G_{i+3/2} - f^G_{i+1/2}| &= |\partial_x f| h + O(h^2), \\ |f^G_{i+1/2} - f^G_{i-1/2}| &= |\partial_x f| h + O(h^2). \end{aligned}$$

These expressions, together with (5.5) and the form of the function a (3.3) Part I, show that in monotone smooth regions of the entropy solution we have $a = f^{fe} - f^G$; i.e., $f^{G-1/2} = f^{fe}$.

6. The LRG-qm implicit schemes. In [7] an implicit non-conforming fe-method to solve numerically the neutron transport equation

$$q_x \partial_x u + q_y \partial_y u + \sigma u = f, \text{ where } q_x \text{ and } q_y \in \mathbb{R},$$

was introduced and analysed. This scheme is a Galerkin method of type (2.1), and can be generalized to the nonlinear case using two-point Godunov flux to define a discrete trace of u_h , as in the G-1/2 scheme. The resulting scheme will be called the LRG scheme, and we shall use it to define implicit qmfe schemes. Let us first introduce the LRG scheme.

6.1. The LRG scheme . This is a Galerkin method of type (2.1) defined as follows:

(6.1a) **The restriction of the approximate solution to each element K_i^n belongs to the space $Q^k(K_i^n)$ of polynomials of degree not greater than k in each of their variables.**

(6.1b) **The trace of u_h in ∂K is chosen as follows :**

$$\begin{aligned} u_h(t^n, x) &= \lim_{\epsilon \downarrow 0} u_h(t^n - \epsilon, x), \quad x \in I_i, \\ u_h(t, x_{i+1/2}) &= \xi_{i+1/2}(t), \quad t \in J^n, \end{aligned}$$

where $\xi_{i+1/2}(t)$ is defined using the Godunov flux f^G in the following manner :

$$f(\xi_{i+1/2}(t)) = f^G(u_h(t, x_{i+1/2}^+), u_h(t, x_{i+1/2}^-)),$$

where the Godunov flux f^G is defined in (2.11) Part 1.

(6.1c) The initial data u_h^0 is taken equal to the L^2 -projection of u_0 on the space of functions piecewise in $P^k(I_i)$, the space of polynomials on I_i of degree no greater than k

(6.1d) For $n=0, \dots, N-1$ the approximate solution u_h^{n+1} is computed as the solution of the following equations:

$$\begin{aligned} & - \iint_K u_h \partial_t \varphi \, dt \, dx + \int_{\partial K} u_h \varphi n_t \, d\gamma - \\ & - \iint_K f(u_h) \operatorname{grad} \varphi \, dt \, dx + \int_{\partial K} f(u_h) \varphi n_{\partial Q} \, d\gamma = 0, \quad \varphi \in Q^k(K), \end{aligned}$$

where the integrals are evaluated using appropriate quadrature formulae. Note that in order to evaluate the integrals on the boundaries $J^{n+1} \times \{x_{i+1/2}\}$ we have to use the definition of the trace of u_h given by (6.1b), and so, at each point of integration we must evaluate a two-point Godunov flux.

In the linear case, assuming that the solution is smooth enough, it was proven that this scheme has an optimal $L^2(0, T; L^2(\mathbb{R}^d))$ -rate of convergence; i.e., $O(\delta^{k+1})$. See [7].

It is also very important to note that when $k = 0$, i.e., when the approximate solution is piecewise constant, the scheme reduces to the implicit Godunov scheme :

$$(\bar{u}_i^n - \bar{u}_i^{n-1})/\Delta t^n + (f^{G,n}_{i+1/2} - f^{G,n}_{i-1/2}) = 0,$$

$$\bar{u}_i^0 = \int_I u_0(x) dx / \Delta x_i, \quad I = I_i.$$

This suggests that this scheme should be taken as the monotone scheme when building qmfe schemes with the LRG one.

6.3. The LRG-qm schemes . The LRG-qm scheme is a qmfe of second type constructed with the LGR scheme and the fully implicit Godunov scheme. To define it we only have to specify the definition of the "means" $\bar{u}_h$, the time qm flux $\bar{u}_h + b$, the space qm flux $f^{\text{LRG-qm}}$ and the local projections $\Lambda \Pi_h^n$.

i. **The means $\bar{u}_h$ and the function b .** The piecewise constant function $\bar{u}_h$ is defined by

$$\bar{u}_h(t, x) = \bar{u}_i^n = (\Delta x_i)^{-1} \int_I u_h(t^n + 0.5 \Delta t^n, x) dx, \text{ for } (t, x) \in K_i^n;$$

the function b is then defined as in (4.7) Part I with $c_0 \in (0, 1/2)$.

ii. **The flux $f^{\text{LRG-qm}}$.** This flux is defined by (3.1)-(3.2)-(3.3) Part I using the Godunov flux f^G as the monotone flux depending only on the means $\bar{u}_h$, and the flux

$$f^{fe,n}_{i+1/2} = (\Delta t^n)^{-1} \int_J f(u_h(t, x_{i+1/2})) dt, \quad J = J^n.$$

iii. **The projections $\Lambda \Pi$.** Note that the mean of $\tilde{u}_h(t^{n+1/2})$ is equal to zero, because of the definition of the means. Thus, we shall take the function $\Lambda \Pi_h^n(\tilde{u}_h|_K)$ to be the L^∞ -projection of $\tilde{u}_h|_K$ on $\tilde{V}'_C(K)$, where

$$\tilde{V}'_C(K) = \{ v_h \in Q^k(K) : \\ |v_h| \leq c \max \{ |\bar{u}^{n_{i+1}} - \bar{u}^{n_i}|, |\bar{u}^{n_i} - \bar{u}^{n_{i-1}}| \}.$$

For this scheme Theorem 4.1 and Corollary 4.2 hold with $s = 1$. Moreover, using the standard formal local error analysis we have been using in these papers, it can be verified that in smooth monotone regions (in time and in space) we recover the high order accuracy of the initial scheme LRG.

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