TWO DIMENSIONAL NAVIER-STOKES FLOW WITH MEASURES AS INITIAL VORTICITY

BY

YOSHIKAZU GIGA

TETSURO MIYAKAWA

HIROFUMI OSADA

IMA Preprint Series # 259

August 1986

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

#	Author(s) Title
120	**D.R.J. Chillingworth**, Three Introductory Lectures on Differential Topology and Its Applications
121	**Giorgio Vergara Caffarelli**, Green's Formulas for Linearized Problems with Live Loads
122	**F. Chiarenza and N. Garofalo**, Unique Continuation for Nonnegative Solutions of Schrodinger Operators
123	**J.L. Ericksen**, Constitutive Theory for some Constrained Elastic Crystals
124	**Minoru Murata**, Positive solutions of Schrodinger Equations
125	**John Maddocks and Gareth P. Parry**, A Model for Twinning
126	**M. Kaneko and M. Wooders**, The Core of a Game with a Continuum of Players and Finite Coalitions: Nonemptiness with Bounded Sizes of Coalitions
127	**William Zame**, Equilibria In Production Economies with an Infinite Dimensional Commodity Space
128	**Myrna Holtz Wooders**, A Tiebout Theorem
129	**Abstracts** for the Workshop on Theory and Applications of Liquid Crystals
130	**Yoshikazu Giga**, A Remark on A Priori Bounds for Global Solutions of Semi-linear Heat Equations
131	**M. Chipot and G. Vergara-Caffarelli**, The N-Membranes Problem
132	**P.L. Lions and P.E. Souganidis**, Differential Games and Directional Derivatives of Viscosity Solutions of Isaacs' Equations II
133	**G. Capriz and P. Giovine**, On Virtual Effects During Diffusion of a Dispersed Medium in a Suspension
134	**Fall Quarter Seminar Abstracts**
135	**Umberto Mosco**, Wiener Criterion and Potential Estimates for the Obstacle Problem
136	**Chi-Sing Man**, Dynamic Admissible States, Negative Absolute Temperature, and the Entropy Maximum Principle
137	**Abstracts** for the Workshop on Oscillation Theory, Computation, and Methods of Compensated Compactness
138	**Arie Leizarowitz**, Tracking Nonperiodic Trajectories with the Overtaking Criterion
139	**Arie Leizarowitz**, Convex Sets in R^n as Affine Images of some Fixed Set in R^n
140	**Arie Leizarowitz**, Stochastic Tracking with the Overtaking Criterion
141	**Abstracts** from the Workshop on Amorphous Polymers and Non-Newtonian Fluids
142	**Winter Quarter Seminar Abstracts**
143	**D.G. Aronson and J.L. Vazquez**, The Porous Medium Equation as a Finite-speed Approximation to a Hamilton-Jacobi Equation
144	**E. Sanchez-Palencia and H. Weinberger**, On the Edge Singularities of a Composite Conducting Medium
145	**Jon C. Luke**, Soliton Solutions in a Class of Fully Discrete Nonlinear Wave Equations
146	**Chi-Sing Man and H. Cohen**, A Coordinate-Free Approach to the Kinematics of Membranes
147	**J.L. Lions**, Asymptotic Problems in Distributed Systems
148	**Reiner Lauterbach**, An Example of Symmetry Breaking with Submaximal Isotropy Subgroup
149	**Abstracts** from the Workshop on Metastability and Incompletely Posed Problems
150	**B. Bozar-Karaklewicz and Jerry Bona**, Wave-dominated Shelves: A Model of Sand-Ridge Formation by Progressive, Infragravity Waves
151	**Abstracts** from the Workshop on Dynamical Problems in Continuum Physics
152	**V.I. Oliker**, The problem of Embedding S^n Into R^{n+1} with Prescribed Gauss
153	**R. Batra**, The force on a Lattice Defect in an Elastic Body
154	**J. Fleckinger and Michael Lapidus**, Eigenvalues of Elliptic Boundary Value Problems with and Indefinite Weight Function
155	**R. Kohn and M. Vogelius**, Thin Plates with Rapidly Varying Thickness, and Their relation to Structural Optimization
156	**M. Gurtin**, Some Results and Conjectures in the Gradient Theory of Phase Transitions
157	**A. Novick-Cohen**, Energy Methods for the Cahn-Hilliard Equation
158	**M. Biroli and U. Mosco**, Wiener Estimates for Parabolic Obstacle Problems
159	**E. Bennett and W. Zame**, Prices and Bargaining in Cooperative Games
160	**W.A. Harris and Y. Sibuya**, The n-th Roots of Solutions of Linear Ordinary Differential Equations
161	**Millard F. Beatty**, Some Dynamical Problems in Continuum Physics
162	**P. Bauman and D. Phillips**, Large-Time Behavior of Solutions to a Scalar Conservation laws in Several Space Dimensions
163	**A. Novick-Cohen**, Interfacial Instabilities in Directional Solidification of Dilute Binary Alloys: The Kuramoto-Sivashinsky Equation.
164	**H.F. Weinberger**, On Metastable Patterns in Parabolic Systems
165	**D. Arnold and R.S. Falk**, Continuous Dependence on the Elastic Coefficients for a Class of Anisotropic Materials
166	**I.J. Bakelman**, The Boundary Value Problems for Non-linear Elliptic Equation and the Maximum Principle for Euler-Lagrange Equations
167	**Ingo Muller**, Gases and Rubbers
168	**Ingo Muller**, Pseudoelasticity in Shape Memory Alloys - an Extreme Case of Thermoelasticity
169	**Luis Magalhaes**, Persistence and Smoothness of Hyperbolic Invariant Manifolds for Functional Differential Equations
170	**A. Damlamian and M. Vogelius**, Homogenization limits of the Equations of Elasticity in Thin Domains
171	**H.C. Simpson and S.J. Spector**, On Hadamard Stability in Finite Elasticity
172	**J.L. Vazquez and C. Yarur**, Isolated Singularities of the Solutions of the Schrodinger Equation with a Radial Potential
173	**G. Dal Maso and U. Mosco**, Wiener's Criterion and Γ-Convergence
174	**John H. Maddocks**, Stability and Folds
175	**R. Hardt and D. Kinderlehrer**, Existence and Partial Regularity of Static Liquid Crystal Configurations
176	**M. Nerukar**, Consruction of Smooth Ergodic Cocyles for Systems with Fast Periodic Approximations
177	**J.L. Ericksen**, Stable Equilibrium Configurations of Elastic Crystals
178	**Patricio Aviles**, Local Behavior of Solutions of Some Elliptic Equations
179	**S.-N. Chow and R. Lauterbach**, A Bifurcation Theorem for Critical Points of Variational Problems
180	**R. Pego**, Phase Transitions: Stability and Admissibility In One Dimensional Nonlinear Viscoelasticity
181	**Mariano Giaquinta**, Quadratic Functions and Partial Regularity
182	**J. Bona**, Fully Discrete Galerkin Methods for the Korteweg De Vries Equation
183	**J. Maddocks and J. Keller**, Mechanics of Robes
184	**F. Bernis**, Qualitative Properties for some nonlinear higher order
185	**F. Bernis**, Finite Speed of Propagation and Asymptotic Rates for some Nonlinear Higher Order Parabolic Equations with Absorption
186	**S. Reichelstein and S. Reiter**, Game Forms with Minimal Strategy Spaces
187	**T. Ding**, An Answer to Littlewood's Problem on Boundedness
188	**J. Rubinstein and R. Mauri**, Dispersion and Convection in Periodic Media
189	**W.H. Fleming and P.E. Souganidis**, Asymptotic Series and the Method of Vanishing Viscosity
190	**H. Beirao Da Veiga**, Existence and Asymptotic Behavior for Strong Solutions of Navier-Stokes Equations in the Whole Space
191	**L.A. Caffarelli, J.L. Vazquez, and N.I. Wolanski**, Lipschitz Continuity of Solutions and Interfaces of the N-Dimensional Porous Medium Equation
192	**R. Johnson**, m-Functions and Floquet Exponents for Linear Differential System:
193	**F.V. Atkinson and L.A. Peletier**, Ground States and Dirichlet Problems for $-\Delta = F(U)$ in R^2
194	**G. Dal Maso, U. Mosco**, The Wiener Modulus of a Radial Measure
195	**H. A. Levine and H.F. Weinberger**, Inequalities between Dirichlet and Neumann Eigenvalues
196	**J. Rubinstein**, On the Macroscopic Description of Slow Viscous Flow Past a Random Array of Spheres
197	**G. Dal Maso and U. Mosco**, Wiener Criteria and Energy Decay for Relaxed Dirichlet Problems

Two Dimensional Navier-Stokes Flow with Measures as Initial Vorticity*

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060, Japan

Tetsuro Miyakawa
Department of Mathematics
Hiroshima University
Hiroshima 730, Japan

Hirofumi Osada
Department of Mathematics
Nara Women's University
Nara 630, Japan

*Research partly supported by the Grand-in-Aid for Scientific Research No. A 60302001, the Japan Ministy of Education,Science and Culture.

Two Dimensional Navier-Stokes Flow with Measures as Initial Vorticity

Yoshikazu GIGA, Tetsuro MIYAKAWA & Hirofumi OSADA

Introduction

This paper studies the nonstationary flow of a viscous incompressible fluid in R^2 when the initial vorticity is very singular. The governing equations of motion are the Navier-Stokes equations

$$(1)\qquad \begin{aligned} &u' - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0,\\ &u \to 0 \quad \text{as} \quad |x| \to \infty, \quad u(x,0) = a(x), \quad \nabla\cdot a = 0, \end{aligned}$$

where u and p represent unknown velocity and pressure, respectively, $\nu > 0$ is the kinematic viscosity, $(u\cdot\nabla) = \sum_i u^i\partial/\partial x_i$, $\nabla\cdot u = \sum_i \partial u^i/\partial x_i$ and $u' = \partial u/\partial t$. The density of the fluid is assumed to be one by a normalization.

We consider problem (1) in two dimensions assuming that the initial vorticity

$$\nabla \times a = \partial a^2/\partial x_1 - \partial a^1/\partial x_2$$

is a finite Radon measure on R^2 and discuss the solvability. The velocity fields of this type include those with vortex sheets and point sources of vorticity, which are both important in the vortex theory for ideal fluids. It is very recent that a rigorous relation between solutions of the Euler equations (system (1) with $\nu = 0$) and the classical theory of the motion of point vortices is studied ; See, e.g., Marchioro and Pulvirenti [19],[20] and Turkington [31]. For the Navier-Stokes system (1) Benfatto, Esposito and Pulvirenti [3] constructed a global smooth solution, assuming that initial vorticity is a finite pure point measure which is small compared with ν, i.e.,

$$\nabla \times a = \sum_{j=1}^{m} \alpha_j \ \delta(x-z_j)$$

and ν is sufficiently large compared with $\sum_j |\alpha_j|$; here, $\delta(x-z_j)$ is the Dirac measure supported at $z_j \in R^2$. In other words, results in [3] say that point source vorticities can diffuse following the Navier-Stokes flow provided ν is large. We note that this result does not follow from classical theories for the Navier-Stokes system as developed by Leray [17], Ladyzhenskaya [16] or Temam [30]. As pointed out in [3], classical existence results for (1) fail to work since the initial velocity a, with $\nabla\times a$ a measure, is not necessarily square-summable, even locally.

Our main goal in this paper is to show that there is a smooth global (in time) solution to (1), assuming only that the initial vorticity $\nabla\times a$ is a finite measure on R^2. Evidently, this improves the result of [3] since no restriction is imposed on ν as well as on the size and the form of $\nabla\times a$.

To show the existence, we adopt a standard procedure. We first regularize the initial velocity a, consider the corresponding regular solutions of (1), and then take a subsequence converging to the desired solution of the original problem. As is well known, to carry out this process one needs good a priori estimates for regular solutions. For this purpose we study the vorticity equation for $v = \nabla\times u$:

$$v' - \nu\Delta v + (u\cdot\nabla)v = 0, \tag{2a}$$

$$u = K*v \tag{2b}$$

for smooth initial data $v(x,0) = \nabla\times a$. where K is the vector function:

$$K(x_1,x_2) = (-x_2,x_1)/2\pi|x|^2, \quad x = (x_1,x_2)$$

and $*$ denotes the convolution on R^2. These equations are formally obtained by taking $\nabla\times$ of (1) and using the condition $\nabla\cdot u = 0$. We note that there is no vorticity stretching term in (2a) since the space dimension equals 2.

We regard (2a) as a linear parabolic equation for v with coefficients u and write the corresponding fundamental solution as $\Gamma_u(x,t;y,s)$, $t \geq s$. A bound for Γ_u due to the third author [25] yields our key a priori estimates:

$$\begin{aligned} C_1(t-s)^{-1}\exp[-C_2|x-y|^2/(t-s)] &\leq \Gamma_u(x,t;y,s) \leq \\ &\leq C_3(t-s)^{-1}\exp[-C_4|x-y|^2/(t-s)] \end{aligned} \tag{3}$$

where the positive constants C_j, $j=1,2,3,4$, depend only on ν and L^1-norm of $\nabla\times a$. Estimate (3) makes it possible to control the behavior of v as $t \to 0$ uniformly in approximation so that solutions with regularized initial data converges to a solution of the original problem, i.e., the problem (1) with $\nabla \times a$ a finite(Radon) measure on R^2. Estimates of the form of (3) with C_j independent of the smoothness of coefficients were first established by Aronson [1], Aronson and Serrin [2] for linear equations of divergence form. Osada [25] extends estimates in Aronson [1] to a class of linear equations of non-divergence form which includes equation (2a) as a typical example.

Existence problem for nonlinear evolution equations with measures as initial data has recently attracted many mathematicians. For example, McKean [22], Osada and Kotani [24] and Sznitman [29] study the existence and uniqueness of solutions for the Burgers equation

$$u' + uu_x = \nu u_{xx}, \quad x \in R^1$$

with $u(x,0) = c\delta(x)$, $c > 0$. For the problem

$$u' - \Delta u + u^p = 0, \quad x \in R^n ; \quad u(x,0) = c\delta(x), \quad c > 0,$$

Brezis and Friedman [5] prove the existence of a solution for $0 < p < 1+2/n$ and nonexistence for $p \geq 1+2/n$; see also [35] for more general initial data. Their existence results are extended to more general equations of the form $u' - \Delta u + f(u) = 0$ by Niwa [23]. For the problem

$$u' + f(u)_x = 0, \quad x \in R^1 ; \quad u(x,0) = \delta(x),$$

Liu and Pierre [18] discuss existence, (non-)uniqueness and asymptotic behavior of entropy solutions under various assumptions on the form of the function f. Our main result may be understood as an example of the existence results for nonlinear parabolic equations involving measures as initial data, since our result yields in particular global solutions to the problem (2a),(2b) when $v(x,0)$ is an arbitrary finite measure on R^2.

In Section 1 we start with local existence results for problem (1) in R^n, $n \geq 2$, with initial velocity a in L^p, $p > n$ and show that the solution is regular for $t > 0$. For later use we discuss higher regularity up to $t = 0$. Since (1) is parabolic, these results are conceptually well known. However, it is difficult to find the appropriate version in the literature since initial velocity a is not necessarily square-summable, i.e., initial energy may be infinite.

From Section 2 we consider only the two dimensional flow. We extend the local solution obtained in Section 1 to global smooth solutions, appealing to the vorticity equation (2a)(2b). An argument of this sort is found in McGrath [21]. Our global existence results in Section 2 improve recent results in [19,20] as well as that of [21], by relaxing assumptions on the initial data.

As a byproduct of our existence results, we prove in section 2 a persistent property of our solutions in Sobolev spaces $W^{m,p}(R^2)$, $p > 2$, $m = 0,1,2,\ldots$ Namely, we shall show that if $a \in W^{m,p}(R^2)$ and $\nabla\times a \in L^q(R^2)$ with $1/q = 1/p+1/2$, then the corresponding solution stays in $W^{m,p}(R^2)$ for all time and bounds for the solution on each finite time interval are independent of the viscosity ν. Such a uniform bound enables us to take a subsequence converging as $\nu \to 0$ to a solution of the Euler equations. In fact we construct a global solution to the Euler solutions under the same assumptions on a.

Persistent property of this sort is systematically studied by Kato [15] and Ponce [27] for the solutions of (1) with finite energy.

Since our solution may have infinite energy, our results are not included in either of [15] and [27]. After we completed this work, we learned that Kato and Ponce [34] extends their results to solutions which may have infinite energy. Their result covers our results for $m \geqq 2$. However, our results for $m = 0,1$ are not contained even in [34]. In particular, our existence result for the Euler equations seems new for initial data $a \in L^p(R^2)$, $\nabla\times a \in L^q(R^2)$, $1/q = 1/p+1/2$.

Section 3 establishes our key a priori estimates for smooth solutions constructed in Section 2. It is crucial that our bound depends only on L^1-norm of initial vorticity $\nabla\times a$ and is independent of regularity of a.

In Section 4 we apply our a priori estimate in Section 3 and prove our main existence result. More precisely, we construct a global solution to (1) as well as (2a)(2b) when the initial vorticity is a finite measure on R^2 and prove the regularity for $t > 0$ as well as some decay estimates as $t \to \infty$. We clarify the meaning of the convergence to initial velocity as $t \to 0$ by using Lorentz spaces. We further show that our solution is unique provided that the pure point part of $\nabla\times a$ is small. We note that there is no restriction on the size of the continuous part of $\nabla\times a$. This result covers the uniqueness result of [3] since they assume that $\nabla\times a$ is a finite pure point measure and that its total variation is small.

We are grateful to Professor Masao Yamazaki for some suggestions on Lorentz spaces. We are also grateful to Professor Noriaki Suzuki for valuable suggestions on the proof of Lemma 4.4.

1. Local Solutions in R^n with Initial Data in L^p

This section establishes some local existence results in L^p for the Navier-Stokes system (1) in R^n, $n \geq 2$, without assuming that the initial energy is finite. Although there are many references on the local existence in R^n, only a few results are available unless we assume that the initial energy is finite (see e.g., [7, 12, 14, 16, 33, 34]) ; so we give here the details of our version for later use. The basic tool for constructing solutions is a standard successive approximation which goes back to Leray [17] and is systematically studied in [10, 11, 14, 32, 33, 34].

We shall also discuss higher regularity up to $t = 0$ which is used in the sequel. Since the equation is semilinear parabolic, regularity for $t > 0$ and up to $t = 0$ is conceptually well known (see, e.g., [7], [10],[34]). However, we state here our version and give a complete proof for later use and for the reader's convenience, since our argument contains new technical aspects and our result does not follow from a simple combination of known results.

In what follows we use the following notation : BC denotes the class of bounded continuous functions. $L^p(R^n)$ represents the space of L^p-vector or tensor functions on R^n, as well as the space of L^p-scalar functions on R^n ; the norm of f in $L^p(R^n)$ is denoted by $\|f\|_p$. We write $BC([0,T) ; L^p(R^n))$ simply as $B_{p,T}$. The norm of $u(x,t)$ in $B_{p,T}$ is defined by

$$|u|_{p,T} = \sup_{0 \leq t < T} \|u\|_p(t).$$

If $f = (f^1, \ldots, f^n)$ is a vector function on R^n, ∇f denotes the tensor $\partial_i f^j$, $1 \leq i, j \leq n$, where $\partial_i = \partial/\partial x_i$. Similarly, for a nonnegative integer k, $\nabla^k f$ denotes the tensor $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f^j$, $\alpha_1 + \ldots + \alpha_n = k$. The expression $\partial_t f$ denotes the time derivative of f.

As is a usual practice ([7,10,11,12,14,32,33,34]), to solve (1) we transform (1) to its integral form:

$$(1.1) \qquad u(t) = e^{\nu t \Delta} a + S[u](t), \quad t \geq 0,$$

where

$$(1.2) \qquad S[u](t) = S[u,u](t) \; ; \quad S[u,w](t) = -\int_0^t e^{\nu(t-s)\Delta} P(u \cdot \nabla) w(s) ds.$$

Here $e^{t\Delta}$ is the solution operator for the heat equation ; P is a singular integral operator of convolution type (see [7]) and is the orthogonal projection on $L^2(R^n)$ onto the subspace of divergence-free vector fields. A solution u of (1.1) is called a mild solution of the initial value problem (1) since (1) and (1.1) is equivalent provided u and a are smooth and decaying at space infinity. It turns out that the solutions treated in this paper are all smooth and satisfy the equations in the classical sense for $t > 0$. However, we should be careful to the behavior of the solutions as $t \to 0$ in order to understand the meaning of initial condition. We first prepare basic estimates in L^p on the bilinear map $S[u,w]$. We observe $\sum_j u^j \partial_j w = \sum_j \partial_j (u^j w)$ ($\equiv \nabla \cdot (u \otimes w)$ in short) provided $\nabla \cdot u = 0$. This gives another expression of S :

$$(1.2') \qquad S[u,w](t) = -\int_0^t e^{\nu t-s)\Delta} P\nabla\cdot(u \otimes w)ds$$

$$= -\int_0^t \nabla\cdot e^{\nu(t-s)\Delta} P(u \otimes w)ds$$

since P and ∇ commute with $e^{\nu t\Delta}$. Since $\nabla\cdot u = 0$, we may replace (1.2) by (1.2').

Lemma 1.1. *Let* $2 \leqq n < p < \infty$, $T > 0$ *and* $\sigma = 1/2-n/2p$. *Then*

(i) $|S[u,w]|_{p,T} \leqq M(\nu T)^{\sigma}|u|_{p,T}\,|w|_{p,T}/\nu$ *Provided that* $\nabla\cdot u = 0$;

(ii) $|(\nu t)^{1/2}\,\nabla S[u,w]|_{p,T} \leqq M(\nu T)^{\sigma}|u|_{p.T}|(\nu t)^{1/2}\,\nabla w|_{p,T}/\nu$;

(iii) $|\nabla S[u,w]|_{q,T} \leqq M(\nu T)^{\sigma}|u|^{2\sigma}_{p,T}\,|(\nu t)^{1/2}\nabla u|^{1-2\sigma}_{p,T}|\nabla w|_{q,T}/\nu$;

with $1/q = 1/p + 1/n$, *where* M *is a positive constant depending only on* n *and* p.

Proof. We use the well-known estimates :

$$(1.3) \qquad \|\nabla e^{\nu t\Delta}\, f\|_r \leqq C(\nu t)^{-1/2-(n/s-n/r)/2}\,\|f\|_s, \quad 1 \leqq s \leqq r \leqq \infty,$$

$$(1.4) \qquad \|Pf\|_r \leqq C\|f\|_r, \qquad 1 < r < \infty, \text{ (see [13, Chap 9]).}$$

(i) Apply (1.4) to (1.2') to get

$\|S[u,w]\|_p(t) \leqq C\int_0^t \|\nabla\cdot e^{\nu(t-s)\Delta}\,(u \otimes w)(s)\|_p\, ds.$

Applying (1.8) and Hölder's inequality to the right hand side yields

$$\|S[u,w]\|_p(t) \leqq C\int_0^t [\nu(t-s)]^{-1+\sigma} \|u \otimes w\|_{p/2}(s)ds$$

$$\leqq C\int_0^t [\nu(t-s)]^{-1+\sigma}\|u\|_p(s)\|w\|_p(s)ds$$

$$\leqq M_1(\nu T)^{\sigma}|u|_{p,T} \, |w|_{p,T}/\nu, \qquad 0 \leqq t < T,$$

where M_1 depends only on n and p. This proves (i) with $M = M_1$.

(ii) A similar argument gives

$$(\nu t)^{1/2}\|\nabla S[u,w]\|_p(t) \leqq C(\nu t)^{1/2} \int_0^t [\nu(t-s)]^{-1+\sigma} \|u\|_p(s)\|\nabla w\|_p(s)ds$$

$$\leqq C(\nu t)^{1/2} \int_0^t [\nu(t-s)]^{-1+\sigma}(\nu s)^{-1/2}ds \times |u|_{p,T}|(\nu t)^{1/2} \nabla w|_{p,T}$$

$$\leqq M_2(\nu T)^{\sigma}|u|_{p,T}|(\nu t)^{1/2} \nabla w|_{p,T}/\nu \;, \qquad 0 \leqq t < T,$$

which shows (ii) with $M = M_2$.

(iii) Similarly to the proof of (ii), we take ∇ of $S[u,w]$ to get

$$\|\nabla S[u,w]\|_q(t) \leqq C\int_0^t [\nu(t-s)]^{-1/2} \|(u\cdot\nabla)w\|_q(s)ds$$

$$\leqq C\int_0^t[\nu(t-s)]^{-1/2} \|u\|_\infty(s)\|\nabla w\|_q(s)ds$$

where C depends only on p and n. Since $p > n$ the Gagliardo-Nirenberg inequality [9. p.24. Theorem 9.3] yields

$$\|u\|_\infty \le C\|u\|_p^{2\sigma}\|\nabla u\|_p^{1-2\sigma}.$$

We thus have

$$\|\nabla S[u,w]\|_q(t)$$
$$\le C\int_0^t [\nu(t-s)]^{-1/2}(\nu s)^{-1/2+\sigma}\|u\|_p^{2\sigma}(s)\|(\nu s)^{1/2}\nabla u\|_p^{1-2\sigma}(s)\|\nabla w\|_q(s)ds$$

$$\le M_3(\nu T)^\sigma |u|_{p,T}^{2\sigma}|(\nu t)^{1/2}\nabla u|_{p,T}^{1-2\sigma}|\nabla w|_{q,T}/\nu$$

with M_3 depending only on n and p. Taking $M = \max(M_1, M_2, M_3)$, we complete the proof.

We begin by constructing a local solution in L^p, $p > n$.

<u>Proposition 1.2</u>. (i) <u>Suppose that the initial velocity</u> a <u>is in</u> $L^p(R^n)$ <u>for some</u> $p > n$ <u>and</u> $\nabla\cdot a = 0$. <u>Then there is a unique local solution</u> u <u>of</u> (1.1) <u>such that</u> $u \in B_{p,T}$ <u>for some</u> $T > 0$ <u>and</u>

(1.5) $$|u|_{p,T} \le 2\|a\|_p .$$

(ii) <u>The time</u> T <u>can be taken so that</u>

(1.6) $$T \ge C\nu^{-1+1/\sigma} / \|a\|_p^{1/\sigma}, \quad \sigma = 1/2 - n/2p ;$$

(1.7a) $$(\nu t)^{1/2}\nabla u \in B_{p,T} \text{ with } |(\nu t)^{1/2}\nabla u|_{p,T} \le C\|a\|_p$$

<u>and</u>

(1.7b) If $\nabla a \in L^q(R^n)$ with $1/q = 1/p+1/n$, then $\nabla u \in B_{q,T}$ and $|\nabla u|_{q,T} \leq 2\|\nabla a\|_q$.

where C *depends only on* n *and* p.

(iii) *Let* m *be a nonnegative integer and suppose that* $\nabla^k a \in L^p(R^n)$ *for* $k = 0,\ldots, m$. *Then in addition to* (i)(ii) *the time* T *can be taken so that*

$$\nabla^k u \in B_{p,T} \quad \text{and} \quad |\nabla^k u|_{p,T} \leq C', \quad k = 0,\ldots, m; \tag{1.8a}$$

$$(\nu t)^{1/2} \nabla^{m+1} u \in B_{p,T} \quad \text{and} \quad |(\nu t)^{1/2} \nabla^{m+1} u|_{p,T} \leq C'; \tag{1.8b}$$

$$\nabla^k \partial_t^h u \in B_{p,T} \quad \text{and} \quad |\nabla^k \partial_t^h u|_{p,T} \leq C' \quad \text{for} \quad k+2h \leq m, \tag{1.8c}$$

where C' *depends only on* n, m, p *and bounds for* ν *and* $\|\nabla^k a\|_p$, $k = 0,\ldots, m$.

Proof. (i),(ii). Consider the successive approximation for (1.1) :

$$u_{j+1} = u_0 + S[u_j], \quad u_0 = e^{\nu t\Delta} a, \quad j = 0,1,\ldots \tag{1.9}$$

Lemma 1.1 (i) and the estimate $\|e^{\nu t\Delta} a\|_p \leq \|a\|_p$ together yield

$$|u_{j+1}|_{p,T} \leq \|a\|_p + M(\nu T)^{\sigma}|u_j|^2_{p,T}/\nu .$$

This implies that, for all $j \geqq 0$

(1.10a) $$|u_j|_{p,T} \leqq K \equiv 2r\theta^{-1}\|a\|_p, \quad r = 1 - (1-\theta)^{1/2} < 1$$

provided

(1.11) $$0 < \theta \equiv 4\|a\|_p M(\nu T)^{\sigma}/\nu < 1.$$

For θ, $0 < \theta < 1$ we take $T > 0$ so that (1.11) holds.

Taking ∇ of (1.9) and then applying Lemma 1.1 (ii), together with (1.3) and (1.10a), yields

$$|(\nu t)^{1/2} \nabla u_{j+1}|_p \leqq C\|a\|_p + M(\nu T)^{\sigma}|u_j|_p|(\nu t)^{1/2} \nabla u_j|_p /\nu$$

$$\leqq C\|a\|_p + MK(\nu T)^{\sigma}|(\nu t)^{1/2} \nabla u_j|_p /\nu.$$

Here and hereafter we drop the subscript T to simplify the notation. By definition of K and (1.11), the second term of the right hand side is $\leqq (r/2)|(\nu t)^{1/2} \nabla u_j|_p$. Hence

(1.10b) $$|(\nu t)^{1/2} \nabla u_j|_p \leqq 2C\|a\|_p \equiv N \quad \text{for all} \quad j \geqq 0$$

with C depending only on p and n.

Similarly we take ∇ of (1.9) and apply Lemma 1.1 (iii) (1.10a) and (1.10b) to get

$$|\nabla u_{j+1}|_q \leqq \|\nabla a\|_q + ML(\nu T)^{\sigma}\|a\|_p|\nabla u_j|_q/\nu$$

with L depending only on p and n. If θ is sufficiently small, say $0 < \theta < 2/L$, then the above estimate gives

$$|\nabla u_{j+1}|_q \leq \|\nabla a\|_q + \frac{1}{2}|\nabla u_j|_q$$

which yields the bound

(1.10c) $$|\nabla u_j|_{q,T} \leq 2\|\nabla a\|_q \quad \text{for all} \quad j \geq 0.$$

Here and hereafter we fix T so that (1.11) holds with $\theta L < 2$.

We shall now prove that u_j and $(\nu t)^{1/2}\nabla u_j$ (resp. ∇u_j) are Cauchy sequences in $B_{p,T}$ (resp. $B_{q,T}$). The difference $w_j = u_{j+1} - u_j$ satisfies

(1.12) $$w_j = S[u_{j+1}] - S[u_j]$$

$$= S[u_j, w_{j-1}] + S[w_{j-1}, u_{j-1}], \quad j = 1,2,\ldots$$

Lemma 1.1 (i) implies that

$$|w_j|_p \leq M(\nu T)^{\sigma}(|u_j|_p + |u_{j-1}|_p)|w_{j-1}|_p /\nu .$$

By (1.10a) and (1.11) this yields

$$|w_j|_p \leq 2MK(\nu T)^{\sigma}|w_{j-1}|_p = r|w_{j-1}|_p$$

so that

(1.13a) $$|w_j|_p \leq r^j|w_0|_p \leq 2Kr^j.$$

This shows that u_j is a Cauchy sequence in $B_{p,T}$.

We next take ∇ of (1.12) and apply Lemma 1.1 (ii) to get

$$|(\nu t)^{1/2}\nabla w_j|_p \leq M(\nu T)^\sigma[|u_j|_p|(\nu t)^{1/2}\nabla w_{j-1}|_p + |w_{j-1}|_p|(\nu t)^{1/2}\nabla u_{j-1}|_p]/\nu$$

$$\leq MK(\nu T)^\sigma[|(\nu t)^{1/2}\nabla w_{j-1}|_p + 2Nr^{j-1}]/\nu$$

$$\leq \frac{r}{2}\ |(\nu t)^{1/2}\nabla w_{j-1}|_p + Nr^j$$

by (1.10a), (1.10b), (1.11) and (1.13a). By (1.10b) this yields

(1.13b) $$|(\nu t)^{1/2}\nabla w_j|_p \leq Cr^j\|a\|_p \quad \text{for all} \quad j \geq 1,$$

which shows that $(\nu t)^{1/2}\nabla u_j$ is a Cauchy sequence in $B_{p,T}$.

Similarly, applying Lemma 1.1 (iii) together with (1.13a), (1.13b) and (1.10c) yields

$$|\nabla w_j|_q \leq ML(\nu T)^\sigma\|a\|_p|\nabla w_{j-1}|_q/\nu$$

$$+ M(\nu T)^\sigma|w_{j-1}|_p^{2\sigma}|(\nu t)^{1/2}\nabla w_{j-1}|_p^{1-2\sigma}|\nabla u_{j-1}|_q/\nu$$

$$\leq \frac{1}{2}|\nabla w_{j-1}|_q + C'\|\nabla a\|_q r^j$$

with C' independent of j. Thus

$$|\nabla w_j|_q \leq C''\|\nabla a\|_q s^j, \qquad s = \max(r, 1/2), \quad \text{for all} \quad j \geq 1$$

with C'' independent of j. This shows that ∇u_j is a Cauchy sequence in $B_{q,T}$.

The estimate (1.6) is obvious from our choice of θ and T. Since u_j and $(\nu t)^{1/2}\nabla u_j$ (resp. ∇u_j) are Cauchy sequences in $B_{p,T}$ (resp. $B_{q,T}$), we see the limit $u = \lim u_j$ is a solution of (1.1) in $B_{p,T}$ and satisfies (1.5),(1.7a),(1.7b) by passing to the limit $j \to \infty$ in (1.10a),(1.10b),(1.10c). The proof of uniqueness in $B_{p,T}$ is standard (see [12]) so (i),(ii) are proved.

(iii) First we shall prove that, for $k = 0, \ldots, m$

$$(1.14)_k \qquad |\nabla^k u_j|_p \leq C' \qquad \text{for all} \quad j \geq 0 ;$$

$$(1.15)_k \qquad |\nabla^k w_j|_p \leq C' r^j \qquad \text{for all} \quad j \geq 1,$$

with C' independent of j and T. To do this we appeal to induction on k. The case $k = 0$ is nothing but (1.10a) and (1.13a). So we assume that $(1.14)_i$ and $(1.15)_i$ are valid for $i = 0, \ldots, k-1$. Taking ∇^k of (1.9) and (1.12) gives

$$(1.16) \quad \nabla^k u_{j+1} = \nabla^k u_0 + \nabla^k S[u_j]$$
$$= e^{\nu t\Delta}(\nabla^k a) + S[u_j, \nabla^k u_j] + S[\nabla^k u_j, u_j] + \sum_{i=1}^{k-1} \binom{k}{i} S[\nabla^i u_j, \nabla^{k-i} u_j];$$

$$(1.17) \quad \nabla^k w_j = S[u_j, \nabla^k w_{j-1}] + S[\nabla^k w_{j-1}, u_{j-1}]$$

$$+ S[\nabla^k u_j, w_{j-1}] + S[w_{j-1}, \nabla^k u_{j-1}]$$

$$+ \sum_{i=1}^{k-1} \binom{k}{i} (S[\nabla^i u_j, \nabla^{k-i} w_{j-1}] + S[\nabla^{k-i} w_{j-1}, \nabla^i u_{j-1}]).$$

Applying Lemma 1.1 (i) to (1.16) yields

$$|\nabla^k u_{j+1}|_p \leq C\|\nabla^k a\|_p + M(\nu T)^\sigma[2|u_j|_p|\nabla^k u_j|_p + C\sum_{i=1}^{k-1}|\nabla^i u_j|_p|\nabla^{k-i} u_j|_p]/\nu.$$

Using induction assumption, (1.10a) and (1.11), we have

$$|\nabla^k u_{j+1}|_p \leq C\|\nabla^k a\|_p + r|\nabla^k u_j|_p + C'.$$

Since $r < 1$, this implies $(1.14)_k$ provided $\|\nabla^k a\|_p$ is finite. We next apply Lemma 1.1 (i) to (1.17) and use $(1.14)_k$ with $k \leq m$ to get

$$|\nabla^k w_j|_p \leq M(\nu T)^\sigma[(|u_j|_p + |u_{j-1}|_p)|\nabla^k w_{j-1}|_p + C'\sum_{i=0}^{k-1}|\nabla^i w_{j-1}|_p]/\nu.$$

This, together with (1.10a), (1.11) and induction assumption $(1.15)_i$, $i \leq k-1$, yields

$$|\nabla^k w_j|_p \leq r|\nabla^k w_{j-1}|_p + C'r^j.$$

Since $r < 1$, we obtain $(1.15)_k$. The proofs of $(1.14)_k$ and $(1.15)_k$ are completed. We next show that

(1.18a) $$|(\nu t)^{1/2} \nabla^{m+1} u_j|_p \leq C' \quad \text{for all} \quad j \geq 0 ;$$

(1.18b) $$|(\nu t)^{1/2}\nabla^{m+1}w_j|_p \leq C'r^j \quad \text{for all} \quad j \geq 1,$$

by induction on m. The proofs are parallel to those of (1.10b) and (1.13b). We apply Lemma 1.1 (ii) to (1.16) with $k = m+1$ and use (1.10a), (1.11) and $(1.14)_k$, $k \leq m$, to get

$$|(\nu t)^{1/2}\nabla^{m+1}u_{j+1}|_p \leq C\|\nabla^m a\|_p$$

$$+ M(\nu T)^{\sigma}[|u_j|_p|(\nu t)^{1/2}\nabla^{m+1}u_j|_p + C\sum_{k=1}^{m}|\nabla^k u_j|_p|(\nu t)^{1/2}\nabla^{m+1-k}u_j|_p]/\nu$$

$$\leq C\|\nabla^m a\|_p + \frac{r}{2}|(\nu t)^{1/2}\nabla^{m+1}u_j|_p + C',$$

which implies (1.18a). Similarly, applying Lemma 1.1 (ii) to (1.17) with $k = m+1$ gives

$$|(\nu t)^{1/2}\nabla^{m+1}w_j|_p \leq M(\nu T)^{\sigma}[|u_j|_p|(\nu t)^{1/2}\nabla^{m+1}w_{j-1}|_p$$

$$+ |u_{j-1}|_p|(\nu t)^{1/2}\nabla^{m+1}w_{j-1}|_p]/\nu + C'r^j$$

$$\leq r\,|(\nu t)^{1/2}\nabla^{m+1}w_{j-1}|_p + C'r^j.$$

This implies (1.18b). Estimates $(1.15)_k$ and (1.18b) show that $\nabla^k u_j$ and $(\nu t)^{1/2}\nabla^{m+1}u_j$ are Cauchy sequences in $B_{p,T}$. So (1.8a) and (1.8b) follow from $(1.14)_k$ and (1.18a), respectively, through passage to the limit $j \to \infty$.

It remains to prove (1.9c). We shall prove it by induction on $\ell = k+2h \leq m$. We may assume $m \geq 2$, since otherwise (1.8c) is nothing but (1.8a). If $\ell = 0$, (1.8c) is trivial. Suppose that (1.8c) is true for $k+2h \leq \ell-1$. If $h = 0$, (1.8c) is the same as (1.8a) ; so even if $k+2h = k = \ell$, (1.8c) is true. We again appeal to induction on h. Suppose that (1.8c) is true for $\nabla^{k+2}\partial_t^{h-1}u$ with $k+2h = \ell$. Using the equation :

$$u' = \nu\Delta u - P(u\cdot\nabla)u$$

we calculate

$$\begin{aligned}\nabla^k\partial_t^h u &= \nabla^k\partial_t^{h-1}(\nu\Delta u - P(u\cdot\nabla)u) \\ &= \nabla^k\partial_t^{h-1}(\nu\Delta u) - P(u\cdot\nabla^{k+1}\partial^{h-1})u + \textstyle\sum_\alpha C_\alpha P(U_\alpha W_\alpha) \\ &\equiv I_1 + I_2 + I_3,\end{aligned}$$

where C_α is a constant and U_α, W_α are of the form $\nabla^\gamma\partial_t^\delta u$ with $\gamma+2\delta \leq \ell-2$. Since ∇u and $\nabla^{\gamma+1}\partial_t^\delta u$ are in $B_{p,T}$ by induction assumption on ℓ, the Sobolev inequality implies that u and $\nabla^\gamma\partial_t^\delta u$ are in $B_{p,T} \cap BC([0,T) ; BC(R^n))$. From this it easily follows that I_2 and I_3 are in $B_{p,T}$. (Note that $\nabla^{k+1}\partial_t^{h-1}u$ is in $B_{p,T}$ by induction assumption.) Further, our induction assumption shows that I_1 is in $B_{p,T}$. We thus conclude that $\nabla^k\partial_t^h u$ is in $B_{p,T}$ for $k+2h = \ell$. The bounds in (1.8c) are easily obtained from those of I_1, I_2 and I_3. Thus the proof of (1.8c) is completed by induction.

Remark. The basic idea of the above proof goes back to Leray [17], in which he constructed a global regular solution, when $n = 2$, by a successive approximation, assuming that a is in $H^1 \cap L^\infty$. A proof of (i) is given in Giga [12, Theorem 1 and Sect. 4], except for (1.7a).

The next result establishes a regularizing effect for solutions of (1.1) given in Proposition 1.2.

Proposition 1.3. (i) *Let* $a \in L^p(R^n)$ *for some* $p > n$ *and* $\nabla \cdot a = 0$. *Let* u *be the solution to* (1.1) *given in Proposition* 1.2. *Then,* $\nabla^k \partial_t^h u \in BC([\varepsilon, T); L^p(R^n))$ *for all* $k, h \geq 0$ *and* $0 < \varepsilon < T$. *Moreover, we have the bound*

$$\sup_{[\varepsilon,T)} \|\nabla^k \partial_t^h u\|_p(t) \leq C$$

where C *depends only on* ε, p, n, k, h *and a bound for* $\|a\|_p$. *In particular,* u *is smooth for* $t > 0$ *and solves the Navier-Stokes system in the classical sense for* $t > 0$.

(ii) *Suppose further that* $\nabla^k a \in L^p(R^n)$ *for all* $k \geq 0$. *Then* $\nabla^k \partial_t^h u$ *is bounded and continuous on* $R^n \times [0,T)$ *for all* $k, h \geq 0$. *Moreover, we have*

$$\sup_{[0,T)} \|\nabla^k \partial_t^h u\|_\infty(t) \leq C$$

where C *depends only on* p, n, k, h, ν *and bounds for* $\max_{0 \leq \ell \leq k+2h+1} (\|\nabla^\ell a\|_p)$.

Proof. (i) By (1.7) we have $\|\nabla u\|_p(t_0) \leqq C$ for $0 < t_0 < T$ with C depending only on n, p, t_0 and $\|a\|_p$. We then solve the Navier-Stokes system for $t \geqq t_0$ with initial velocity $u(\cdot,t_0)$ and obtain, due to the uniqueness of solutions, $\|\nabla^2 u\|_p(2t_0) \leqq C$. Repeating this process eventually yields that $\|\nabla^m u\|_p(mt_0)$ is bounded by the same constant C so long as $mt_0 < T$. Since t_0 can be taken arbitrarily small, this shows that $\nabla^m u$ is in $BC([\varepsilon,T) ; L^p(R^n))$ for all $\varepsilon > 0$ with a bound C depending only on p, n, m, ε and $\|a\|_p$. Combining this with (1.8c) yields the estimate in (i). The smoothness is immediate from the Sobolev inequality.

(ii) This follows from (1.8c) via the Sobolev inequality. The proof is completed.

Remark. We note that Proposition 1.3 (ii) also follows from [7, Theorem 3.4] or [34]. However, apparently no estimate of the form (1.6) is given in [7] or [34] for the time T. Moreover, it seems that Proposition 1.3 (i) does not directly follow from the results of [7] or [34].

2. Global Existence and Persistency via the Vorticity Equation

The goal of this section is to show some global existence results for the Navier-Stokes system (1) in R^2 without assuming that the initial energy is finite. As a byproduct we obtain a persistency of our solutions in Sobolev spaces $W^{m,p}(R^2)$, $p > 2$, $m = 0,1,2,...$ This leads to global existence results for the Euler equations by letting $\nu \to 0$. It should be noted that our version of persistency deals with solutions without finite energy, so it is not included in either of [15] and [27]. Since the standard energy method fails to work in our case, we are forced to appeal to the vorticity equation in order to get the desired results. Such a type of argument is found in McGrath [21] under more stringent assumptions on initial vorticity. Here we give our global existence results for both Navier-Stokes and the Euler equations, based on the results in Section 1, which relaxes the assumptions and simplifies the proofs of [21]. In what follows we always assume that the space dimension n equals 2 unless otherwise specified.

Suppose that the initial velocity a is in $L^p(R^2)$ for some $p > 2$ together with all its derivatives. Proposition 1.3 (ii) then says that there is a unique local solution of (1) which is smooth and bounded on $R^2\times[0,T)$. We here take $\nabla\times$ of (1) and get the vorticity equation for $v = \nabla\times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$:

(V-1) $$L_u v \equiv v' - \nu\Delta v + (u\cdot\nabla)v = 0, \quad t \in (0,T),$$
$$v(x,0) = \nabla\times a$$

Since u is bounded on $R^2 \times [0,T)$ together with all its derivatives, the linear parabolic operator L_u has a unique fundamental solution

$$\Gamma_u(x,t\ ;\ y,s), \qquad 0 \leq s < t < T, \qquad x,\ y \in R^2$$

such that $L_u \Gamma_u = 0$ as a function of (x,t) and

$$\lim_{t \downarrow s} \int_{R^2} \Gamma_u(x,t\ ;\ y,s) f(y) dy = f(x)$$

for every $f \in BC(R^2)$; see [8,Chap. 1].

Let us quickly review some properties of Γ_u which are needed later. It is well known that $\Gamma_u > 0$ and that the function

$$w(x,t) = \int_{R^2} \Gamma_u(x,t\ ;\ y,s) f(y) dy \tag{2.1}$$

is a unique bounded classical solution to $L_u w = 0$ $(t > s)$, $w(x,s) = f \in BC(R^2)$; see[8, Chap. 1,2]. Since L_u has no zero-th order term, $w \equiv 1$ is a unique bounded solution to $L_u w = 0$ $(t > s)$, $w(x,s) = 1$. By (2.1) this yields

$$\int_{R^2} \Gamma_u(x,t\ ;\ y,s) dy = 1, \qquad 0 \leq s < t < T. \tag{2.2}$$

The function $\Gamma_u^*(x,t\ ;\ y,s) = \Gamma_u(y,s\ ;\ x,t)$, $0 \leq t < s < T$, is the fundamental solution to the adjoint problem

$$w' + \nu \Delta w - \nabla \cdot (uw) = 0, \qquad 0 \leq t < T,$$

which is the same as

$$w' + \nu\Delta w - (u\cdot\nabla)w = 0$$

since $\nabla\cdot u = 0$. Similarly to (2.2) we have

$$\int_{R^2}\Gamma_u(y,s\ ;\ x,t)dy = 1, \qquad 0 \leq t < s < T. \tag{2.3}$$

The following result is immediately obtained from Propositions 1.2, 1.3 and identities (2.2) and (2.3).

Proposition 2.1. (i) *Suppose that* $\nabla^k a \in L^p(R^2)$, $k = 0,1,2,\dots,$ *for some* $p > 2$ *and that* $\nabla\cdot a = 0$. *Let* u *be the local solution of* (1) *given in Proposition* 1.2. *Then* $v = \nabla\times u$ *is expressed as*

$$v(x,t) = \int_{R^2}\Gamma_u(x,t\ ;\ y,0)(\nabla\times a)(y)dy, \qquad 0 < t < T. \tag{2.4}$$

(ii) *Suppose further that* $\nabla\times a \in L^q(R^2)$ *for some* q *with* $1 \leq q \leq \infty$. *Then*

$$\|v\|_q(t) \leq \|\nabla\times a\|_q, \qquad 0 \leq t < T. \tag{2.5}$$

We next consider how to recover the velocity field u from the solution v of the equation (V-1). Since $\nabla\cdot u = 0$, it is easily seen that

$$\Delta u = \nabla^{\perp}v, \qquad \text{where} \quad \nabla^{\perp}v = (-\partial v/\partial x_2,\ \partial v/\partial x_1).$$

It is thus expected that if u decays as $|x| \to \infty$,

$$u = E * \nabla^{\perp} v = (\nabla^{\perp} E) * v$$

where $E = (2\pi)^{-1} \log|x|$ is a fundamental solution of Δ in R^2 and $*$ denotes the convolution in R^2. We shall now show that this observation is true in our setting. To this purpose we introduce some function spaces. By $\mathcal{M}$ we denote the space of all finite Radon measures on R^2 with norm defined by the total variation. A measurable function f on R^2 is said to be in $L^{p,\infty}(R^2)$, $1 < p < \infty$, if

$$\|f\|_{p,\infty} = \sup_{\lambda > 0} \lambda [\mathrm{mea}\{x \; ; \; |f(x)| > \lambda\}]^{1/p} < \infty$$

where mea is Lebesgue measure in R^2. Although $\|f\|_{p,\infty}$ does not satisfy the usual triangle inequality, it is a pseudo-norm on the linear space $L^{p,\infty}$ and $L^{p,\infty}$ is a Banach space with a norm equivalent to $\|f\|_{p,\infty}$ (see [4]). Such $L^{p,\infty}$ is often called a Lorentz space.

In what follows we let

$$K(x) = \nabla^{\perp} E(x) = (-x_2, x_1)/2\pi|x|^2 \qquad \text{for } x = (x_1, x_2) \in R^2$$

and consider the convolution operator $U = K * V = \int_{R^2} K(x-y)V(y)dy$. Note that $K \in L^{2,\infty}(R^2)$ and that K is not contained in any $L^p(R^2)$, $1 \leq p \leq \infty$.

<u>Lemma 2.2</u>. (i) <u>For</u> $U = K * V$ <u>we have the estimates</u> :

(2.6a) $\|U\|_p \leq C\|K\|_{2,\infty}\|V\|_q$, *if* $1 < q < 2$, $V \in L^q(R^2)$ *and* $1/p = 1/q - 1/2$;

(2.6b) $\|U\|_{2,\infty} \leq C\|K\|_{2,\infty}\|V\|_{\mathcal{M}}$ *for* $V \in \mathcal{M}$;

(2.6c) $\|\nabla U\|_r \leq C\|V\|_r$ *for* $V \in L^r(R^2)$, $1 < r < \infty$,

with C *independent of* V, *where* $\|V\|_{\mathcal{M}}$ *denotes the total variation of the Radon measure* V.

(ii) *Suppose that* $U \in L^p(R^2)$, $2 < p < \infty$, *with* $\nabla \cdot U = 0$ *and that* $\nabla\times U \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then*

$$U = K*(\nabla \times U).$$

(iii) *Suppose that* $U \in L^{2,\infty}(R^2)$ *with* $\nabla \cdot U = 0$ *and that* $\nabla\times U \in \mathcal{M}$. *Then*

$$U = K*(\nabla \times U).$$

Proof. (i). (2.6a) is nothing but the generalized Young's inequality (see [28, p.32]). Since ∇K is a Calderon-Zygmund kernel, (2.6c) follows from the standard theory of singular integral operators ; see [13, Chap. 9]. To show (2.6b) consider the linear operator $Af = f*V$ for any fixed $V \in \mathcal{M}$. It is easily checked that A defines a bounded linear operator on each $L^p(R^2)$, $1 \leq p \leq \infty$, with norm $\leq \|V\|_{\mathcal{M}}$. An interpolation theorem for Lorentz spaces ([4, Theorem 5.3.4]) now implies that A is bounded on $L^{2,\infty}(R^2)$ with norm $\leq C\|V\|_{\mathcal{M}}$. This shows (2.6b).

(ii),(iii). The function $W = K*(\nabla\times U)$ is in $L^p(R^2)$(resp. $L^{2,\infty}(R^2)$)

by (2.6a) and (2.6b), and satisfies $\nabla \cdot W = 0$, $\nabla \times W = \nabla \times U$. Therefore $Z = U - W$ is harmonic on R^2 and belongs to $L^p(R^2)$ (resp. $L^{2,\infty}(R^2)$). The mean-value theorem for harmonic functions yields, for every $x \in R^2$

$$|Z(x)| \leq \text{mea}(B)^{-1} \int_B |Z(y)|dy \leq C\|Z\|_p \quad (\text{resp.} \leq C\|Z\|_{2,\infty})$$

where B is the unit disc in R^2 with center x and C is independent of x. Liouville's theorem for harmonic functions now implies that Z = constant, which must be equal to 0 since $Z \in L^p(R^2)$ (resp. $L^{2,\infty}(R^2)$). This proves (ii) and (iii).

Proposition 2.3. *Let* $\nabla^k a \in L^p(R^2)$, $k = 0,1,\ldots,$ *for some* $p > 2$. *Suppose further that* $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^p(R^2)$ *with* $1/q = 1/p + 1/2$. *Then the local solution* u *given in Section* 1 *is expressed as*

$$\text{(V-2)} \qquad u(x,t) = K*(\nabla \times u) = \int_{R^2} K(x-y)(\nabla \times u)(y,t)dy, \quad 0 \leq t < T.$$

Moreover, the estimate

$$(2.7) \qquad \|u\|_p(t) \leq C\|\nabla \times u\|_q(t) \leq C\|\nabla \times a\|_q, \quad 0 \leq t < T$$

holds with C *depending only on* p.

Proof. By Proposition 1.2 (i), $u(\cdot,t)$ is in $L^p(R^2)$. So (V-2) follows from Lemma 2.2 (ii). (2.7) is then immediately obtained from (V-2), (2.6a) and (2.5). The proof is completed.

We can now prove our global extension results, using estimate (2.7).

Theorem 2.4. *Suppose that* $\nabla^k a \in L^p(R^2)$, $k = 0,1,\ldots,$ *for some* $p > 2$, *and that* $\nabla \cdot a = 0$. *Suppose further that* $\nabla \times a \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then the local solution to* (1) *given in Proposition* 1.2 *extends uniquely to a global (in time) solution* u *such that* $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ *and*

$$|u|_{p,\infty} \leq C\|\nabla\times a\|_q, \qquad |\nabla u|_{q,\infty} \leq C\|\nabla\times a\|_q$$

where C *depends only on* p. *Moreover, the derivatives* $\nabla^k \partial_t^h u$ *belong to* $B_{p,T}$ *for every finite* $T > 0$ *and satisfy*

$$|\nabla^k \partial_t^h u|_{p,T} \leq C$$

with C depending only on p, k, h, T, ν, and bounds for $\max_{0\leq\ell\leq k+2h}(\|\nabla^\ell a\|_p)$ and $\|\nabla\times a\|_q$.

Proof. Take T as in the proof of Proposition 1.2 with $\|a\|_p$ replaced by $C\|\nabla\times a\|_q$, where C is the constant in (2.7). For any $t_0 \in (0,T)$, (2.7) shows that $\|u\|_p(t_0)$ has a bound depending only on p and $\|\nabla\times a\|_q$. Therefore, the argument in the proof of Proposition 1.2 ensures the existence of a unique solution on $[t_0, t_0+T)$ with initial value $u(\cdot,t_0)$. Suppoose now that u extends uniquely to some finite interval $[0,T_1)$. Then (2.7) holds on $[0,T_1)$ as seen from Propositions 2.2 and 2.3. Thus u extends uniquely to the interval $[0,T_1+T)$. Since T is independent of T_1, we conclude that u extends uniquely to the whole interval $[0,\infty)$. By (2.7) and (2.6c), we easily see that $u \in B_{p,\infty}$ and

$\nabla u \in B_{q,\infty}$ with desired bounds. Bounds for $\nabla^k \partial_t^h u$ are obtained from Proposition 1.2 (iii). The proof is completed.

The assumption $\nabla^k a \in L^p(R^2)$, $k = 0,1,\ldots,$ is assumed so that the local solution $u(x,t)$ is enough regular up to $t = 0$. Since the equation (1) is parabolic, it is natural to expect global existence even if we drop regularity assumptions on a.

Theorem 2.5. *Suppose that* $a \in L^p(R^2)$ *for some* $p > 2$ *with* $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^q(R^2)$, $1/q = 1/p + 1/2$. *Then there is a unique global solution* u *of* (1) *such that* $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ *and*

$$|u|_{p,\infty} \leq C\|\nabla \times a\|_q, \quad |\nabla u|_{q,\infty} \leq C\|\nabla \times a\|_q$$

with C *depending only on* p. *Moreover, all derivatives* $\nabla^k \partial_t^h u$ *exist on* $R^2 \times [\varepsilon,\infty)$ *for any* $\varepsilon > 0$ *and satisfy*

$$\sup_{[\varepsilon,T]} \|\nabla^k \partial_t^h u\|_\infty(t) \leq C$$

where C *depends only on* p, T, k, h, ε, ν, *and a bound for* $\|\nabla \times a\|_q$.

Proof. Let u be a local solution in Proposition 1.3 (i). Since we have $\nabla a \in L^q(R^2)$ by Lemma 2.2 (i)(ii), Proposition 1.2 (1.7b) now implies that ∇u is in $B_{q,T}$ for some T. For every t_0, $0 < t_0 < T$ we have

$$\|\nabla^k u(t_0)\|_p \leq C \quad k = 0,1,2,\ldots \tag{2.8}$$

by Proposition 1.3 (i) where $C = C(p, k, t_0, \nu, \|a\|_p)$. Applying Theorem 2.4 with initial data $u(t_0)$, we find that our solution can be extended globally in time due to the uniqueness. In particular we obtain $u \in B_{p,\infty}$ and $\nabla u \in B_{q,\infty}$ and

$$\|u\|_p(t), \quad \|\nabla u\|_q(t) \leq A\|\nabla\times u\|_q(t_0) \qquad t \geq t_0$$

with A depending only on q.

Letting $t_0 \to 0$ show that

$$(2.9) \qquad \|u\|_p(t), \ \|\nabla u\|_q(t) \leq A\|\nabla\times a\|_q, \qquad t \geq 0,$$

which proves the first part of Theorem 2.5.

By (2.8) and (2.9), Theorem 2.4 now yields

$$(2.10) \qquad \sup_{[t_0,T]}\|\nabla^k\partial_t^h u\|_p(t) \leq C$$

with $C = C(p, k, h, \nu, t_0, T, \|\nabla\times a\|_q)$. The proof is now completed by applying the Sobolev inequality to (2.10).

<u>Remark</u>. The existence results in Marchioro and Pulvirenti [19] and Osada [26] assume that $\nabla\times a \in L^1\cap L^\infty$. This assumption implies that $\nabla\times a \in L^q$, so one can apply Theorem 2.5 to get global existence.

We finally prove a persistent property, in the sense of Kato [15] and Ponce [27], of our solutions to (1). Although our argument involves nothing new, we state our version since it deals with solutions with infinite energy and therefore is not contained in either of [15] and

[27]. In what follows $W^{m,p}(R^2)$, $m = 0,1,\ldots,$ denotes the usual L^p sobolev space. The norm of $W^{m,p}(R^2)$ is written as $\|\cdot\|_{W^{m,p}}$.

Theorem 2.6. Let $a \in W^{m,p}(R^2)$ *for some* $p > 2$ *with* $\nabla\cdot a = 0$ *and* $\nabla\times a \in L^q(R^2)$, *where* $1/q = 1/p + 1/2$. *Then the solution* u *of* (1) *given in Theorem* 2.5 *is in* $BC([0,T]\,;\,W^{m,p}(R^2))$ *for all* $T > 0$ *and satisfies*

$$(2.11) \qquad \sup_{[0,T]}\|u\|_{W^{m,p}}(t) \leq C \quad \textit{uniformly for} \quad \nu > 0.$$

Proof. It suffices to prove (2.11) since $u \in BC([0,T], W^{m,p}(R^2))$ directly follows from (1.8a) and (2.10). Since (2.11) with $m = 0$ is nothing but Theorem 2.5, we may assume $m \geqq 1$. First assume that $m = 1$ and consider the vorticity equation

$$(2.12) \qquad \begin{aligned} &v' - \nu\Delta v + (u\cdot\nabla)v = 0 \quad (t > t_0), \\ &v = \nabla\times u\,, \quad v(x,t_0) = \nabla\times u(x,\, t_0), \end{aligned}$$

where $t_0 > 0$. By (2.9) and (2.10), applying (2.5) to (2.12) yields $\|v\|_p(t) \leqq \|\nabla\times u\|_p(t_0)$ for all $t \geqq t_0$ and therefore by (2.6c), $\|\nabla u\|_p(t) \leqq C\|v\|_p(t) \leqq C\|\nabla\times u\|_p(t_0)$ for all $t \geqq t_0$ with C depending only on p. Since $\nabla u \in B_{p,T}$ by (1.8a) and $\nabla a \in L^p(R^2)$, letting $t_0 \to 0$ yields $\|\nabla u\|_p(t) \leqq C\|\nabla\times a\|_p$ for all $t \geqq 0$. Combining this with (2.11) for $m = 0$ gives (2.11) for $m = 1$.

We next assume that $m = 2$. We apply ∇ to (2.12), multiply the resulting equality by $|\nabla v|^{p-2}\nabla v$ and integrate by parts using $\nabla\cdot u = 0$, to get

(2.13) $$\frac{d}{dt}\|\nabla v\|_p^p \leq C\|\nabla u\|_\infty \|\nabla v\|_p^p, \qquad t \geq t_0$$

with C depending only on p. To estimate $\|\nabla u\|_\infty$ we appeal to the following result of Kato [15, Lemma A3] :

(2.14) $$\|\nabla u\|_\infty \leq C(\|v\|_\infty + \|v\|_2 + \|v\|_\infty \log[1 + (\|\nabla v\|_p / \|v\|_\infty)])$$

where C depends only on p. Using (2.5) and the Sobolev inequality, we have $\|v\|_\infty \leq \|\nabla\times a\|_\infty \leq C\|a\|_{W^{2,p}}$ and $\|v\|_2 \leq \|v\|_q^{1-2/p}\|v\|_p^{2/p} \leq \|\nabla\times a\|_q^{1-2/p}\|\nabla\times a\|_p^{2/p}$. Thus (2.14) gives

$$\|\nabla u\|_\infty \leq C(1 + \log^+\|\nabla v\|_p)$$

with C depending only on p, $\|\nabla\times a\|_q$ and $\|a\|_{W^{2,p}}$. Combining this with (2.13) and integrating with respect to t now yields

(2.15) $$\|\nabla v\|_p(t) \leq C, \qquad \text{for } t \in [t_0, T]$$

where C depends also on T. Since $\nabla^2 u = \nabla K*(\nabla v)$ and since ∇K is a Calderon-Zygmund kernel, (2.15) implies that $\|\nabla^2 u\|_p(t) \leq C$ on $[0,T]$. This implies (2.11) for $m = 2$.

Suppose finally that $m \geq 3$. We apply ∇^k to (2.12), multiply the resulting equality by $|\nabla^k v|^{p-2}\nabla^k v$ and integrate over R^2. Integration by parts using the condition $\nabla\cdot u = 0$ and the Sobolev inquality together imply that, after summation over $k = 0,1,\ldots, m-1$

$$\frac{d}{dt}\|v\|^p_{W^{m-1,p}} \leq C\|u\|_{W^{m-1,p}} \|v\|^p_{W^{m-1,p}}$$

where C depends only on m and p. Integrating this and then using the estimate $\|u\|_{W^{m,p}} \leq C(\|v\|_{W^{m-1,p}} + \|\nabla\times a\|_q)$, which follows from (2.7) and the relation $\nabla^k u = \nabla K*(\nabla^{k-1}v)$, we get (2.11) by induction on m. The proof is completed.

Theorem 2.6 suggests that we can obtain a solution of the Euler equations (system (1) with $\nu = 0$) by passing to the limit $\nu \to 0$. For $m \geq 2$, this is carried out by Kato and Ponce [33] with no assumption on the vorticity $\nabla\times a$. For the cases $m = 0,1$, which are excluded in [33], our Theorem 2.6 gives the following result.

<u>Corollary 2.7</u>. (i) <u>Let</u> $a \in L^p(R^2)$, $p > 2$, $\nabla\cdot a = 0$ <u>and</u> $\nabla\times a \in L^q(R^2)$ <u>with</u> $1/q = 1/p + 1/2$. <u>Then there is a function</u> u <u>such that</u> :

(a) $u : [0,\infty) \to L^p(R^2)$ <u>is bounded and continuours under the weak topology and</u> $u(\cdot,0) = a$.

(b) $P\nabla\cdot(u \otimes u)$ <u>makes sense as an element of</u> $L^\infty(0,\infty ; W^{-1,p/2}(R^2))$.

(c) $u' + P\nabla\cdot(u \otimes u) = 0$ <u>for</u> $t > 0$.

(ii) <u>Let</u> $a \in W^{1,p}(R^2)$, $p > 2$, $\nabla\cdot a = 0$ <u>and</u> $\nabla\times a \in L^q(R^2)$ <u>with</u> $1/q = 1/p + 1/2$. <u>Then there is a function</u> u <u>such that</u> :

(d) $u : [0,\infty) \to W^{1,p}(R^2)$ <u>is bounded and continuous under the weak and</u> $u(\cdot,0) = a$.

(e) $P(u\cdot\nabla)u$ <u>makes sense as an element of</u> $L^\infty(0,\infty ; L^p(R^2))$.

(f) $u' + P(u\cdot\nabla)u = 0$ <u>for</u> $t > 0$.

$\underline{\text{Proof}}$. We fix a and denote by u_ν, $\nu > 0$, the corresponding solution of (1).

(i) From (2.6c) and (2.7) we see that $\|\nabla u_\nu\|_q$ and $\|u_\nu\|_p$ are bounded in $L^\infty(0,\infty)$. Since $q < p$, this implies that u_ν are bounded in $L^\infty(0,\infty\,;\,W^{1,q}(D))$ for any fixed open disc D. Also, Δu_ν and $P(u_\nu\cdot\nabla)u_\nu = P\nabla\cdot(u_\nu \otimes u_\nu)$ are bounded in $L^\infty(0,\infty\,;\,W^{-1,q}(R^2))$ and $L^\infty(0,\infty\,;\,W^{-1,p/2}(R^2))$, respectively. Since $q < p/2$, $W^{-1,p/2}(D) \subset W^{-1,q}(D)$ with continuous injection ; Thus the equation

$$u_\nu' - \nu\Delta u_\nu + P\nabla(u_\nu \otimes u_\nu) = 0, \quad t > 0,$$

implies that u_ν' are bounded in $L^\infty(0,\infty\,;\,W^{-1,q}(D))$. Since D is arbitrary,Lemma 2.1 in [30, Chap.III] ensures the existence of a subsequence of u_ν (which we denote also by u_ν) so that $u_\nu \to u$ a.e. in $R^2\times(0,\infty)$ as $\nu \to 0$. The foregoing observation shows that we may assume $u \in L^\infty(0,\infty\,;\,L^p(R^2))$ and $\nabla u \in L^\infty(0,\infty\,;\,L^q(R^2))$. Since $\nu\Delta u_\nu \to 0$ as $\nu \to 0$ in $L^\infty(0,\infty\,;\,W^{-1,q}(R^2))$, a simple limiting argument gives

$$\frac{d}{dt}(u,\phi) - (u \otimes u, \nabla\phi) = 0 \quad \text{in} \quad t > 0$$

for every smooth and divergence-free vector field ϕ with compact support. We can thus apply de Rham's theorem [30, Chap.I] to conclude that

$$u' + \nabla\cdot(u \otimes u) + \nabla\Pi = 0, \quad t > 0, \tag{2.16}$$

for some distribution Π on $R^2\times(0,\infty)$. Taking the divergence of (2.16) gives

$$\Delta\Pi = -\sum_{j,k}\partial_j\partial_k(u^j u^k),$$

which shows that we may take $\Pi = \sum_{j,k}R_jR_k(u^ju^k)$, where R_j are the Riesz transforms. By the boundedness of the operators R_j in $L^r(R^2)$, $1 < r < \infty$, the function $\nabla\Pi$ is in $L^\infty(0,\infty\,;\,W^{-1,p/2}(R^2))$. Thus (2.16) implies $u' \in L^\infty(0,\infty\,;\,W^{-1,p/2}(R^2))$, so that (c) follows by applying P to (2.16). (b) follows the boundedness of P in $W^{-1,p/2}(R^2)$. From (b) and (c) it follows that u is continuous from $[0,\infty)$ to $W^{-1,p/2}(R^2)$, and so from $[0,\infty)$ to $W^{-1,q}(D)$ for any D. Since $L^p(D) \subset L^q(D) \subset W^{-1,q}(D)$ with continuous injections, Lemma 1.4 in [30,Chap.III] implies that u is continuous from $[0,\infty)$ to $L^p(D)$ under the weak topology. Since D is arbitrary and $\|u\|_p(t)$ is bounded, the Banach-Steinhaus theorem implies (a).

(ii) Theorem 2.6 shows that u_ν are bounded in $L^\infty(0,\infty\,;\,W^{1,p}(R^2))$. Since $p > 2$, the Gagliardo-Nirenberg inequality : $\|f\|_\infty \leq C\|f\|_p^{1-2/p}\|\nabla f\|_p^{2/p}$ yields the boundedness of $P(u_\nu\cdot\nabla)u_\nu$ in $L^\infty(0,\infty\,;\,L^p(R^2))$. This, together with the boundedness of Δu_ν in $L^\infty(0,\infty\,;\,W^{-1,p}(R^2))$, implies that u_ν' are bounded in $L^\infty(0,\infty\,;\,W^{-1,p}(R^2))$. We can thus apply Lemma 2.1 in [30, Chap.III] to conclude that $u_\nu \to u$ as $\nu \to 0$, a.e. in $R^2\times(0,\infty)$. Similarly to the proof of (i), one can show that

$$(2.17) \qquad u' + P\nabla\cdot(u \otimes u) = 0, \quad t > 0.$$

Since $u(\cdot,t) \in W^{1,p}(R^2)$ for a.e. $t > 0$ and $\nabla\cdot u = 0$, we see that

$\nabla \cdot (u \otimes u) = (u \cdot \nabla)u$; Thus (2.17) is rewritten in the form of (f). (e) is easily seen by applying the Gagliardo-Nirenberg inequality. (e) and (f) together imply the continuity of u from $[0,\infty)$ to $L^p(R^2)$. Since u lies in $L^\infty(0,\infty\ ; W^{1,p}(R^2))$, Lemma 1.4 in [30, Chap.III] ensures the continuity of u as asserted in (d). This completes the proof.

Recently, Kato and Ponce [34] extend their results in [15] and [27] to L^p spaces. They prove the persistency of solutions of (1) with $\nu \geq 0$ in $H^{s,p}$, $s > 1 + 2/p$. However, our Theorem 2.6 and Corollary 2.7 are not covered by their results when $m = 0$ or 1.

3. New a Priori Estimates

This section establishes some new a priori estimates for solutions of (1) in R^2, which depend only on the norm of the measure $\nabla\times a$. These estimates allow us to take a subsequence of solutions for the regularized initial data which converges to the desired solution of the original problem. Our argument is based on a comparison theorem of the third author [25] for the fundamental solution of the heat operator $\partial_t - \nu\Delta$ and that of the operator $L_b = \partial_t - \nu\Delta + (b\cdot\nabla)$ with $\nabla\cdot b = 0$. We note that results in [25] extend those in [1],[2] to operators of non-divergence form.

To be more precise, we consider a parabolic operator in R^n $(n \geqq 2)$ of the form :

$$L_b = \partial_t - \nu\Delta + (b\cdot\nabla),$$

under the following assumptions (3.1) and (3.2).

(3.1) The vector function $b = b(x,t)$ is bounded and continuous on $R^n\times[0,T)$ together with all its derivatives, and satisfies $\nabla\cdot b = 0$.

(3.2) There are functions $c^{ij}(x,t)$, $i,j = 1,\ldots, n$, such that

$$\sup |c^{ij}(x,t)| \leqq \alpha, \qquad i,j = 1,\ldots, n,$$

for some $\alpha > 0$ and

$$b^i = \sum_j \partial_j c^{ij}, \qquad i = 1,\ldots, n, \qquad \partial_j = \partial/\partial x_j$$

where b^i is the i-th component of b.

Since b is assumed to be smooth and bounded, L_b has a unique fundamental solution (see [8, Chap. 1,2]), which we denote by $\Gamma_b(x,t\ ;\ y,s)$, $x,y \in R^n$, $0 \leq s < t < T$.

Theorem 3.1 ([25]). *Suppose that* b *satisfies* (3.1) *and* (3.2). *Then the following estimates hold for the fundamental solution* Γ_b *of* L_b.

(i) *There are positive constants* C_j, $j = 1,2,3,4$, *depending only on* n, α *and* ν *such that*

$$(3.3) \qquad C_1(t-s)^{-n/2}\exp[-C_2|x-y|^2/(t-s)] \leq \Gamma_b(x,t\ ;\ y,s) \leq \\ \leq C_3(t-s)^{-n/2}\exp[-C_4|x-y|^2/(t-s)]$$

for all $x,y \in R^n$ *and* $0 \leq s < t < T$.

(ii) *There is a* β, $0 < \beta < 1$, *depending only on* α *and* ν *such that*

$$(3.4) \qquad |\Gamma_b(x,t\ ;\ y,s) - \Gamma_b(x't'\ ;\ y',s')| \leq \\ \leq C_5(|s-s'|^{\beta/2} + |y-y'|^{\beta} + |t-t'|^{\beta/2} + |x-x'|^{\beta})$$

for all $\tau < t-s,\ t'-s' < \infty$ *and* $x, x', y, y' \in R^n$, *where* C_5 *depends only on* n, ν, α *and* $\tau > 0$.

The smoothness assumption on b is in fact not necessary. It is assumed here only to avoid the lack of uniqueness of the fundamental solution. For the full version of Theorem 3.1 and its proof, we refer the reader to [25].

Let us now consider the vorticity equation in R^2 for $v = \nabla\times u$:

(V-1) $\quad L_u v \quad v' - \nu\Delta v + (u\cdot\nabla)v = 0, \quad t > 0, \quad v(x,0) = \nabla\times a$;

(V-2) $\quad u = K*v, \quad K(x) = (-x_2,\ x_1)/2\pi|x|^2 , \quad x = (x_1,\ x_2)$.

The next two results show that Theorem 3.1 is applicable to L_u provided that the solution u of (1) is smooth on $R^2\times[0,T)$ and $\nabla\times a$ is a finite measure on R^2.

Lemma 3.2 ([25]). *The function* $K = (K^1,\ K^2)$ *given in* (V-2) *is expressed as*

$$K^1 = \partial_1 A^3 + \partial_2 A^1 \ ; \quad K^2 = -\partial_1 A^1 - \partial_2 A^2 ,$$

where

$$A^1 = -x_1^2 x_2^2/\pi|x|^4, \quad A^2 = -3x_1x_2/2\pi|x|^2 + x_1^3x_2/\pi|x|^4,$$
$$A^3 = -3x_1x_2/2\pi|x|^2 + x_1x_2^3/\pi|x|^4.$$

Proof. The lemma is verified by direct calculation.

Lemma 3.3. Let $U = K*V$ with $V \in \mathcal{M}$. Then U is expressed as

$$U^i = \sum_{j=1}^{2} \partial_j C^{ij}, \quad i = 1,2,$$
$$|C^{ij}(x)| \leqq M \quad \text{on} \quad R^2$$

with M depending only on m such that $\|V\|_{\mathcal{M}} \leqq m$.

Proof. We define

$$C^{11} = A^3*V, \quad C^{12} = A^1*V, \quad C^{21} = -A^1*V, \quad C^{22} = -A^2*V,$$

where A^k, $k = 1,2,3$, are the functions introduced in Lemma 3.2. Since each A^k is in $L^\infty(R^2)$, we have $c^{ij} \in L^\infty(R^2)$ with $\|c^{ij}\|_\infty \leq N\|V\|_{\mathcal{M}}$ where N depends only on $\|A^k\|_\infty$, $k = 1,2,3$. The expression for U follows immediately from Lemma 3.2. This proves Lemma 3.3.

Using Theorem 3.1, Lemma 3.2 and 3.3, we now prove our main results in this section.

Proposition 3.4. Let u be the unique global solution of (1) given in Theorem 2.4. Suppose further that $v_0 \equiv \nabla\times a$ is in $L^1(R^2)$ with $\|v_0\|_1 \leq m$, and let Γ_u be the fundamental solution of the operator L_u. Then the following hold.

(3.5) $\|v\|_1(t) \leq \|v_0\|_1$, $v = \nabla\times u$; $\|u\|_{2,\infty}(t) \leq C\|v_0\|_1$, for $t \geq 0$,

where $\|\cdot\|_{2,\infty}$ is the norm of $L^{2,\infty}(R^2)$ and C depends only on $\|K\|_{2,\infty}$.

(3.6)
$$C_1(t-s)^{-1}\exp[-C_2|x-y|^2/(t-s)] \leq \Gamma_u(x,t;y,s) \leq$$
$$\leq C_3(t-s)^{-1}\exp[-C_4|x-y|^2/(t-s)], \quad t > s \geq 0,$$

with C_j, $j=1,2,3,4$, depending only on ν and m.

(3.7a) $\|v\|_r(t) \leq Ct^{-1+1/r}\|v_0\|_1$ for $t > 0$ and $1 < r \leq \infty$,

(3.7b) $\|\nabla u\|_r(t) \leq Ct^{-1+1/r}\|v_0\|_1$ for $t > 0$ and $1 < r < \infty$,

(3.7c) $\|u\|_r(t) \leq Ct^{1/r-1/2}\|v_0\|_1$ for $t > 0$ and $2 < r \leq \infty$,

with C depending only on r, m and ν.

(3.8) $$\sup_{[\varepsilon,T]}\|\nabla^k\partial_t^h u\|_\infty(t) \leq C , \qquad \varepsilon > 0$$

with C depending only on ε, h, k, ν, T and m.

Proof. By the assumption u is smooth and bounded together with its derivatives on each slab $R^2\times[0,T]$. So the fundamental solution Γ_u exists uniquely. Estimates in (3.5) are already shown in (2.5) and (2.6b). The estimate (3.6) is obtained from Theorem 3.1, since Lemma 3.3 applies to $u = K*v$ due to the estimate (3.5) for v.

The estimate (3.7a) follows from (3.6). Lemma 2.2 together with (3.7a) yields (3.7b)(3.7c) except

$$\|u\|_\infty \leq Ct^{-1/2}\|v_0\|_1 .$$

This is deduced by applying the Gagliardo-Nirenberg inequality: $\|u\|_\infty \leq C\|u\|_r^{1-2/r}\|\nabla u\|_r^{2/r}$, $r > 2$ (see [9, p.24 Theorem 9.3]) to (3.7b) and (3.7c) for finite r.

It remains to prove (3.8). Taking $t_0 = \varepsilon/2$, we see by (3.7c)

$$\|u\|_r(t_0) \leq C, \qquad r > 2$$

with C depending only on t_0, r, ν and m, where $t_0 = \varepsilon/2$. Applying Proposition 1.3 (i) with initial data $u(t_0)$ and $p = r$ yields (3.8) by the uniqueness. We thus complete the proof of Proposition 3.4.

Our next result concerns the continuity of the function $v(\cdot,t) = (\nabla\times u)(\cdot,t)$ when $\nabla\times a$ is a measure, and enables us to give a precise meaning to the initial condition $u(\cdot,0) = a$.

Proposition 3.5. *Let* u *and* a *be as in Proposition* 3.4, *and let* $v = \nabla\times u$, $v_0 = \nabla\times a$. *Then for each* $m > 0$ *and* $T > 0$ *the functions* $v(\cdot,t)$, $\|v_0\|_1 \leqq m$, *are equicontinuous from* $[0,T]$ *to* $\mathcal{M}$ *under the topology of weak convergence of measures. In other words, the pairing* $(v(\cdot,t),\phi)$ of $\phi \in BC(R^2)$ *with the measure* $v(\cdot,t)$ *satisfies*

$$(v(\cdot,t),\phi) \to (v(\cdot,s)\phi) \quad \text{as} \quad t \to s$$

for all $s \in [0,T]$, *and the convergence is uniform in* v *with* $\|v_0\|_1 \leqq m$.

Proof. On $\mathcal{M}^+_m = \{\mu \in \mathcal{M} ;\ \mu \geqq 0,\ \ \|\mu\|_{\mathcal{M}} \leqq m\}$ consider the function

$$R(\mu_1,\ \mu_2) = \inf \int_{R^2\times R^2} (|x-y| \wedge 1)\ d\lambda(x,y), \qquad \mu_1,\ \mu_2 \in \mathcal{M}^+_m$$

where the infimum is taken over all measures $\lambda \geqq 0$ on $R^2\times R^2$ such that $\Pi_1\lambda = \mu_1$ and $\Pi_2\lambda = \mu_2$; here Π_1 (resp. Π_2) is the projection from $R^2\times R^2$ onto the first (resp. second) factor, and $\Pi_i\lambda$, $i = 1,2$, is the image of the measure λ by Π_i. For arbitrary measures μ_1 and μ_2 on R^2 with $\|\mu_i\|_{\mathcal{M}} \leqq m$, $i = 1,2$, we define

$$R(\mu_1,\ \mu_2) = R(\mu_1^+,\ \mu_2^+) + R(\mu_1^-,\ \mu_2^-)$$

where μ_i^+ and μ_2^- denote the positive and negative part of μ_i, respectively. It is known (see [6]) that the function R is a distance funciton on $\{\mu \in \mathcal{M};\ \|\mu\|_{\mathcal{M}} \leq m\}$ which defines a topology equivalent to that of weak convergence. We shall use the function R in showing equicontinuity. Without loss of generality we may assume that $v_0 \geqq 0$ and therefore $v(\cdot,t) \geqq 0$ for all $t \geqq 0$. Consider the measures $\mu(t) = v(x,t)dx$ on R^2 and $\lambda(t,s) = \Gamma_u(x,t;y,s)v(y,s)dxdy$ on $R^2_x \times R^2_y$. Then we have $\mu(t) \geqq 0$, $\lambda(t,s) \geqq 0$ and

$$\Pi_1\lambda(t,s) = [\int_{R^2} \Gamma_u(x,t;y,s)v(y,s)dy]dx = v(x,t)dx = \mu(t) \ ;$$

$$\Pi_2\lambda(t,s) = [\int_{R^2} \Gamma_u(x,t;y,s)dx]v(y,s)dy = v(y,s)dy = \mu(s).$$

Note that here we have used the positivity of Γ_u, identity (2.3), integral representation (2.4) for v and the Chapman-Kolmogorov equality :

$$(3.9) \quad \Gamma_u(x,t\ ;\ y,s) = \int_{R^2} \Gamma_u(x,t;z,t')\Gamma_u(z,t';y,s)dz, \quad 0 \leqq s < t' < t.$$

By (3.6) and the definition of R we see that

$$R(\mu(t),\ \mu(s)) \leq \iint_{R^2\times R^2} |x-y|\Gamma_u(x,t;y,s)v(y,s)dxdy$$

$$\leq C_1(t-s)^{-1}\iint_{R^2\times R^2} |x-y| \ \exp[-C_2|x-y|^2/(t-s)]v(y,s)dxdy$$

$$= C(t-s)^{1/2}\|v\|_1(s) \leq C\|v_0\|_1(t-s)^{1/2} \leq mC(t-s)^{1/2}$$

for $0 \leq s < t \leq T$, where C depends only on m and ν. This shows the desired equicontinuity and the proof is completed.

Remark. Proposition 3.5 can be proved directly without introducing R. In fact, since $v(x,t) = \int_{R^2} \Gamma_u(x,t;y,s)v(y,s)dy$, using (3.5) and the upper estimate for Γ_u in (3.6), one can prove, by a standard calculus argument, that $(v(\cdot,t)\ \phi)$ converges to $(v(\cdot,s),\phi)$ uniformly in $s \geqq 0$ and $\|v_0\|_1 \leqq m$ as $t \downarrow s$. Clearly, this implies the equicontinuity of $(v(\cdot,t),\phi)$ on $[0,T]$.

The proof using R seems conceptually simpler since it conceals the calclus argument for the convergence behind. The function R is used in [3],[19] and [20] in a similar context.

The results obtained in this section are applied in Section 4 to construct a global solution of the problem (1) when $\nabla\times a$ is a measure. In particular, Proposition 3.5 is important in giving a precise meaning to the initial condition : $u(\cdot,0) = a$ when $\nabla\times a$ is a measure.

4. Main Theorems

In this section, we apply a priori estimates in Section 3 to construct a global solution to (1) as well as (2a)(2b) when the initial vorticity $\nabla\times a$ is a general finite Radon measure on R^2. It turns out that our solution is smooth for $t > 0$ and has some decay estimates as $t \to \infty$. We also study the convergence to initial velocity a as $t \to 0$. We further show that our solution is unique provided that the pure point part of the measure $\nabla\times a$ is sufficiently small.

We begin by finding a reasonable function space for a when $\nabla\times a$ is a finite Radon measure on R^2 and $\nabla\cdot a = 0$. By (2.6b) and Lemma 2.2(iii), a is expressed as the sum of $K*(\nabla\times a) \in L^{2,\infty}(R^2)$ and a harmonic vector field. Since our initial velocity a is supposed to decay as $|x| \to \infty$, it is natural to assume that a is in $L^{2,\infty}(R^2)$ with $\nabla\cdot a = 0$ and $\nabla\times a \in \mathcal{M}$ so that $a = K*(\nabla\times a)$.

To study the convergence to initial velocity we give a sufficient condition for the continuity under weak* topology of $L^{2,\infty}(R^2)$. Since $L^{2,\infty}(R^2)$ is the dual space of the Lorentz space $L^{2,1}(R^2)$ (see [4]), weak* topology is well defined.

<u>Lemma.4.1</u>. <u>Suppose that</u> $u \in L^\infty(0,T\,;\,L^{2,\infty}(R^2))$ <u>with</u> $\nabla\cdot u = 0$ <u>and that</u> $v = \nabla\times u$ <u>is continuous from</u> $[0,T]$ <u>to</u> $\mathcal{M}$ <u>under the topology of weak convergence of</u> $\mathcal{M}$. <u>Then</u> u <u>is modified on a set of Lebesgue measure zero in</u> $[0,T]$ <u>so that it is continuous from</u> $[0,T]$ <u>to</u> $L^{2,\infty}(R^2)$ <u>under the weak* topology</u>.

Proof. By Lemma 2.2 (iii), $K*v \in L^\infty(0,T\ ;\ L^{2,\infty}(R^2))$ and $u - K*v = 0$ a.e. in $[0,T]$ as an element of $L^{2,\infty}(R^2)$. The assertion is thus obtained if we show the continuity of $U = K*v$. Take an arbitrary sequence t_ℓ in $[0,T]$ with $t_\ell \to t$ as $\ell \to \infty$. By the Banach-Alaoglu theorem we can extract a subsequence, which is again denoted by t_ℓ, such that $U(t_\ell) \to U_\infty$ weakly* in $L^{2,\infty}(R^2)$. By assumption, $\nabla\times U(t_\ell) = (\nabla\times u)(t_\ell) \to (\nabla\times u)(t)$ in the weak topology of measures. On the other hand, weak* convergence in $L^{2,\infty}(R^2)$ implies the convergence in the destribution topology ; so $\nabla\times U(t_\ell) \to \nabla\times U_\infty$ as $\ell \to \infty$. Hence $\nabla\times U_\infty = (\nabla\times u)(t) = v(t)$ and therefore $U_\infty = K*v(t) = U(t)$ does not depend on the choice of t_ℓ. This proves Lemma 4.1.

Theorem 4.2. (*Existence for the Navier-Stokes system*). *Suppose that* $a \in L^{2,\infty}(R^2)$ $\nabla\cdot a = 0$ *and that* $\nabla\times a$ *is a finite measure. Then problem* (1) *has a global solution* u *which is smooth for* $t > 0$ *such that*

(i) $u : [0,\infty) \to L^{2,\infty}(R^2)$ *is bounded and continuous under the weak* topology and* $u(\cdot,0) = a$.

(ii) $c = \nabla\times u : [0,\infty) \to \mathcal{M}$ *is bounded and continuous under the weak topology and* $v(\cdot,0) = \nabla\times a$.

(iii) *The estimates*

$$\|u\|_r(t) \leqq Ct^{1/r-1/2} \quad \text{for} \quad t > 0,\quad 2 < r \leqq \infty\ ; \tag{4.1}$$

$$\|\nabla u\|_r(t) \leqq Ct^{-1+1/r} \quad \text{for} \quad t > 0,\quad 1 < r < \infty \tag{4.2}$$

hold with C *depending only on* r, ν *and* $\|\nabla\times a\|$.

(iv) *For* $0 < \varepsilon < T$ *and nonnegative integers* k, h, *there is a constant* C *such that*

$$\sup_{[\varepsilon,T]} \|\nabla^k \partial_t^h u\|_\infty(t) \leq C.$$

with C *depending only on* ε, k, h, T, ν *and a bound for* $\|\nabla\times a\|_{\mathcal{M}}$.

(v) *The function* $u(t) = u(\cdot,t)$ *solves the integral equation* (1.1) *in* $L^{2,\infty}(R^2)$.

Proof. Define $a_\eta = e^{\nu\eta\Delta}a$ for $\eta > 0$. By generalized Young's inequality and properties of the heat kernel we observe that $\nabla^k a_\eta \in L^p(R^2)$, $k = 0,1,\ldots,$ for all $p > 2$, and that $\nabla\times a_\eta \in L^q(R^2)$ for all $q \geq 1$. So, by Theorem 2.4 there exists uniquely a global smooth solution u_η of (4) with $u_\eta(\cdot,0) = a_\eta$. Since $\|\nabla\times a_\eta\|_1 \leq \|\nabla\times a\|_{\mathcal{M}}$, the estimate (3.8) guarantees that there is a subsequence $u_{\eta'}$ converging to a function $u(x,t)$ uniformly on every compact subset in $(0,\infty)\times R^2$ together with its all derivatives as $\eta' \to 0$. Estimates for u in (iii),(iv) now follows from (3.7c)(3.7b) and (3.8) by the lower semi-continuity of integrals. Since each u_η solves (1) for $t > 0$, evidently the limit $u(x,t)$ solves (1) for $t > 0$.

We next prove (i) and (ii). Since Proposition 3.5 is applicable we see $\nabla\times u_{\eta'}(\cdot,t)$ converges to $\nabla\times u(\cdot,t)$ uniformly on $[0,T]$ as $\eta' \to 0$ under the weak topology of $\mathcal{M}$ by taking a subsequence if necessary. Since $\nabla\times a_\eta$ converges to $\nabla\times a$ under the weak topology of $\mathcal{M}$ as $\eta \to 0$, we conclude that $v = \nabla\times u$ is continuous from $[0,\infty)$ to $\mathcal{M}$ under the weak topology of $\mathcal{M}$ and $v(x,0) = \nabla\times a(x)$. By (3.5) we see $\|v\|_{\mathcal{M}}(t)$ is bounded on $[0,\infty)$. This completes the proof of (ii). Since $\{u_\eta\}$ is bounded in $L^\infty(0,\infty\ ;\ L^{2,\infty}(R^2))$ by (3.5), a subsequence $\{u_{\eta'}\}$ converges to u weakly* in $L^\infty(0,\infty\,;\,L^{2,\infty}(R^n))$. Since $v = \nabla\times u$ satisfies (ii), applying Lemma 4.1 now yields (i).

It remains to prove (v), i.e.

$$u(t) = e^{\nu t\Delta}a - \int_0^t \nabla\cdot e^{\nu(t-s)\Delta}P(u\otimes u)(s)ds \quad \text{in} \quad L^{2,\infty}(R^2).$$

For $\varepsilon > 0$ our solution $u(t)$ solves

$$u(t) = e^{\nu(t-\varepsilon)\Delta}u(\varepsilon) - \int_\varepsilon^t \nabla\cdot e^{\nu(t-s)\Delta}P(u\otimes u)(s)ds, \quad t \geqq \varepsilon$$

in all $L^p(R^2)$, $p > 2$. By (4.1) with $r = 4$ and (1.3) with $r = s = 2$, we have

$$\int_\varepsilon^t \nabla\cdot e^{\nu(t-s)\Delta}P(u\otimes u)(s)ds \to \int_0^t \nabla\cdot e^{\nu(t-s)\Delta}P(u\otimes u)(s)ds \quad \text{as} \quad \varepsilon\to 0$$

in $L^2(R^2)$ and therefore in $L^{2,\infty}(R^2)$ since $L^2(R^2)$ is continuously embedded to $L^{2,\infty}(R^2)$. So we need only show that, for each fixed $t > 0$

$$(*) \qquad e^{\nu(t-\varepsilon)\Delta}u(\varepsilon) \to e^{\nu t\Delta}a \quad \text{weakly* in} \quad L^{2,\infty}(R^2) \quad \text{as} \quad \varepsilon\to 0\,.$$

Assertion (i) and the boundedness of the operators $e^{t\Delta}$ in $L^{2,\infty}(R^2)$ together imply that $e^{\nu(t-\varepsilon)\Delta}u(\varepsilon)$ is bounded in $L^{2,\infty}(R^2)$ for each fixed $t > 0$. On the other hand, it is easily checked that

$$(e^{\nu(t-\varepsilon)\Delta}u(\varepsilon),\,\phi) \to (e^{\nu t\Delta}a,\,\phi) \quad \text{as} \quad \varepsilon\to 0$$

for any smooth vector function ϕ with compact support in R^2. Since such functions ϕ are dense in the Lorentz space $L^{2,1}(R^2)$ (see e.g. [4]), and since $L^{2,\infty}(R^2)$ is the dual of the space $L^{2,1}(R^2)$, (*) follows from the Banach-Steinhaus theorem. This completes the proof of Theorem 4.2.

The next result discusses properties of the vorticity $v = \nabla\times u$ of the solution u obtained in Theorem 4.2. The main assertion is that v has an integral representation in terms of a well-behaved function $\Gamma(x,t;y,s)$, $t > s \geqq 0$, which is obtained as a limit of the fundamental solutions of parabolic operators L_{u_n} with smooth u_n. This representation plays an important role in discussing the uniqueness for solutions constructed in Theorem 4.2.

<u>Theorem 4.3</u>. (<u>Integral representation for</u> $\nabla\times u$). <u>Under the assumption of Theorem</u> 4.2, <u>the vorticity</u> $v = \nabla\times u$ <u>is expressed as</u>

$$v(x,t) = \int_{R^2}\Gamma(x,t;y,0)(\nabla\times a)(dy), \quad t > 0, \tag{4.3}$$

<u>in terms of a continuous function</u> $\Gamma(x,t;y,s)$, $x,y \in R^2$, $t > s \geqq 0$, <u>with the following properties</u> (4.4)-(4.6) :

$$\int_{R^2}\Gamma(x,t;y,s)dy = \int_{R^2}\Gamma(x,t;y,s)dx = 1, \quad t > s \geqq 0 \ ; \tag{4.4}$$

$$\Gamma(x,t;y,s) = \int_{R^2}\Gamma(x,t;z,t')\Gamma(z,t';y,s)dz, \quad t > t' > s \geqq 0 \ ; \tag{4.5}$$

$$C_1(t-s)^{-1}\exp[-C_2|x-y|^2/(t-s)] \leqq \Gamma(x,t;y,s) \leqq \leqq C_3(t-s)^{-1}\exp[-C_4|x-y|^2/(t-s)], \quad t > s \geqq 0, \tag{4.6}$$

with C_j, $j = 1,2,3,4$, depending only on ν and a bound for $\|\nabla\times a\|_{\mathcal{M}}$.

Moreover, the estimate

$$\|v\|_r(t) \leqq Ct^{-1+1/r}, \quad t > 0, \quad 1 \leqq r \leqq \infty \tag{4.7}$$

holds with C depending only on r, ν and a bound for $\|\nabla\times a\|_{\mathcal{M}}$.

Proof. As in the proof of Theorem 4.2, we consider the functions u_η and $v_\eta = \nabla\times u_\eta$. By (2.4)

$$v_\eta(x,t) = \int_{R^2} \Gamma_{u_\eta}(x,t;y,0)(\nabla\times a_\eta)(y)dy, \quad t > 0, \tag{4.8}$$

where Γ_{u_η} is the fundamental solution of $L_{u_\eta} = \partial_t - \nu_\Delta + (u_\eta\cdot\nabla)$. Since $u_\eta = K*v_\eta$ and $\|v_\eta\|_1(t) \leqq \|\nabla\times a\|_{\mathcal{M}}$, Lemma 3.3 implies that the estimates (3.3) and (3.4) with $b = u_\eta$ are uniform in η. We can thus apply Ascoli's theorem to conclude that, by passing to a subsequence of $\{u_{\eta'}\}$,

$$\Gamma_{u_{\eta''}}(x,t;y,s) \to \Gamma(x,t;y,s) \qquad \text{as} \quad \eta'' \to 0 \tag{4.9}$$

uniformly on compact subsets of points $(x,t;y,s)$ with $t > s \geqq 0$, and that the limit function Γ satisfies (4.6). Futher, since $\nabla\times a_\eta \to \nabla\times a$ as $\eta \to 0$ weakly in $\mathcal{M}$, (4.8),(4.9) and the uniform (in η) bound (3.3) with $b = u_\eta$ together yield (4.3). Identities (4.4) and (4.5) are similarly obtained, since they hold for the fundamental solutions Γ_{u_η} (see (2.2),(2.3) and (3.9)). Finally, (4.7) follows from (4.3) and (4.6). The proof is completed.

We next consider the uniqueness problem for our solution obtained in this section. Let us recall the Lebesgue decomposition of a finite Radon measure μ (see [28, vol I, p.22, Theorem 1.13]); μ is written uniquely as

$$\mu = \mu_{pp} + \mu_c$$

where μ_c is the continuous part, i.e., $\mu_c(\{x\}) = 0$ for all $x \in R^2$ and μ_{pp} is the pure point part, i.e., $\mu_{pp} = \sum_{j=1}^{\infty} \alpha_j \delta(x-z_j)$, $\alpha_j \in R$, $z_j \in R^2$. This is easily verified by defining $\mu_{pp} = E \lrcorner \mu$ with $E = \{x \in R^2;\ \mu(\{x\}) \neq 0\}$ and proving that E is a countable set. Here $E \lrcorner \mu$ denotes the Borel measure defined by $E \lrcorner \mu(A) = \mu(A \cap E)$

<u>Lemma 4.4. For any finite Radon measure</u> μ <u>on</u> R^2 <u>we have</u>

$$\limsup_{t \downarrow 0} t^{1-1/r} \| e^{t\Delta} \mu \|_r \leq C_r \| \mu_{pp} \|_{\mathcal{M}} \quad \underline{\text{for all}} \quad r > 1,$$

<u>where</u> C_r <u>depends only on</u> r.

<u>Proof</u>. We first recall a well-known estimate

$$\| e^{t\Delta} \mu \|_r \leq C_r t^{-1+1/r} \| \mu \|_{\mathcal{M}}.$$

Indeed, since the linear operator $Af = f * \mu$ is bounded in both L^1 and L^∞ with operator-norm $\leq \|\mu\|_{\mathcal{M}}$, applying the Riesz-Thorin theorem ([4], [28, Vol.II]) to A yields the estimate if we take f as the heat kernel.

This estimate shows that we need only prove that

$$\lim_{t \downarrow 0} t^{1-1/r} \| e^{t\Delta} \mu \|_r = 0 \quad \text{for all} \quad r > 1 \tag{4.10}$$

provided μ is continuous i.e., $\mu(\{x\}) = 0$ for any $x \in R^2$. Without loss of generality we may assume that $\mu \geqq 0$. For any fixed $\varepsilon > 0$ we take $N > 0$ so that, denoting $B(0,N) = \{x ; |x| \leqq N\}$, $\mu[R^2 \setminus B(0,N)] < \varepsilon$ and hence $\mu_2 = (R^2 \setminus B(0,N)) \lrcorner\, \mu$ satisfies

$$t^{1-1/r}\|e^{t\Delta}\mu_2\|_r \leqq C_r\varepsilon \quad \text{for all} \quad r > 1. \tag{4.11}$$

The support of the measure $\mu_1 = \mu - \mu_2$ is contained in $B(0,N)$ and direct calculation gives

$$\begin{aligned}(t^{1-1/r}\|e^{t\Delta}\mu_1\|_r)^r &= C_r' t^{-1}\int_{R^2}\Big(\int_{|y|\leqq N} \exp[-|x-y|^2/4t]\mu_1(dy)\Big)^r dx \\ &= C_r' t^{-1}\Big(\int_{|x|>2N} + \int_{|x|\leqq 2N}\Big)\Big(\int_{|y|\leqq N} \exp[-|x-y|^2/4t]\mu_1(dy)\Big)^r dx \\ &\equiv I_1(t) + I_2(t).\end{aligned} \tag{4.12}$$

Since $|x-y| > |x|/2$ if $|x| > 2N$ and $|y| \leqq N$, we get

$$I_1(t) \leqq C_r'\|\mu_1\|_{\mathcal{M}}^r\, t^{-1}\int_{|x|>2N} \exp[-r|x|^2/16t]dx \to 0, \text{ as } t \to 0 \tag{4.13}$$

For $I_2(t)$ applying Minkowski's inequality yields

$$\begin{aligned}I_2(t) &\leqq C_r' t^{-1}\int_{|x|\leqq 2N}\Big(\int_{|x-y|>\sigma} \exp[-|x-y|^2/4t]\mu_1(dy)\Big)^r dx \\ &\quad + C_r' t^{-1}\int_{|x|\leqq 2N}\Big(\int_{|x-y|\leqq \delta} \exp[-|x-y|^2/4t]\mu_1(dy)\Big)^r dx \\ &\equiv I_{21}(t) + I_{22}(t),\end{aligned} \tag{4.14}$$

where $\delta > 0$ is to be chosen later. Obviously, for any fixed $\delta > 0$,

(4.15) $$I_{21}(t) \leqq C_r' \text{mea}[B(0,2N)] \|\mu_1\|_{\mathcal{M}}^r t^{-1} \exp(-r\delta^2/4t) \to 0, \quad \text{as} \quad t \to 0$$

where mea is the Lebesgue measure on R^2. On the other hand, Hölder's inequality yields

(4.16) $$I_{22}(t) \leqq C_r' \int_{|x| \leqq 2N} [\mu_1(B(x,\delta)]^{r-1} \{\int_{|x-y| \leqq \delta} \exp[-r|x-y|^2/4t] \mu_1(dy))dx/t$$

$$\leqq C_r'' \sup_{|x| \leqq 2N} [\mu_1(B(x,\delta)]^{r-1} \| e^{r^{-1} t\Delta} \mu_1 \|_1$$

$$\leqq C_r'' \|\mu\|_{\mathcal{M}} \times \sup_{|x| \leqq 2N} [\mu_1(B(x,\delta)]^{r-1}.$$

Where $B(x,\delta) = \{y ; |y-x| \leqq \delta\}$. We shall now show that

(4.17) $$\mu_1[B(x,\delta)] \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{uniformly in} \quad |x| \leqq 2N.$$

The desired result (4.10) then follows (4.19)-(4.15) by taking δ so that $I_{22}(t) < \varepsilon^r$ and recalling that ε is arbitrary.

The uniform continuity (4.17) follows from the continuity of μ_1. Indeed suppose that (4.17) were false. Then there would exist $\eta > 0$, $\delta_\ell \downarrow 0$ and x_ℓ with $|x_\ell| \leqq 2N$ so that

(4.18) $$\mu_1[B(x_\ell, \delta_\ell)] \geqq \eta \qquad \text{for all } \ell.$$

By passing to a subsequence we may assume that $x_\ell \to x$ as $\ell \to \infty$. For any $\delta > 0$, $B(x_\ell, \delta_\ell) \subset B(x,\delta)$ provided ℓ is sufficiently large.

Since $\mu_1(\{x\}) = 0$, we have $\lim_{\delta\downarrow 0} \mu_1[B(x,\delta)] = 0$, so $\lim_{\ell\to\infty} \mu_1[B(x_\ell,\delta_\ell)] = 0$ which contradicts (4.18). We thus obtain (4.17). The proof of Lemma 4.4 is now completed.

Theorem 4.5. (Uniqueness). *Suppose that* $a \in L^{2,\infty}(R^2)$, $\nabla\times a \in \mathcal{M}$, *and* $\nabla\cdot a = 0$. *Take* $m > 0$ *so that* $\|\nabla\times a\|_{\mathcal{M}} \leq m$ *and let* u *be the solution of* (1) *given in Theorem* 4.2. *Then we have the following.*

(i) *For all* $p > 2$, *we have*

$$\limsup_{t\downarrow 0} t^{1/2-1/p}\|u\|_p(t) \leq C\|(\nabla\times a)_{pp}\|_{\mathcal{M}} \tag{4.19}$$

with C *depending only on* p, m *and* ν.

(ii) *For each* $p > 2$ *there is a positive constant* $\varepsilon = \varepsilon(p,\nu,m)$ *such that if* $\|(\nabla\times a)_{pp}\|_{\mathcal{M}} < \varepsilon$, *the solution* u *is unique in the class of functions* w *with the following properties* :

(a) $w : [0,\infty) \to L^{2,\infty}(R^2)$ *is weakly* continuous and* $w(\cdot,0) = a$;

(b) $w : (0,\infty) \to L^p(R^2)$ *is continuous and satisfies* (4.19) *for* $p > 2$.

(c) w *solves* (1.1) *in* $L^{2,\infty}(R^2)$.

In particular the solution u *is unique provided that* $\nabla\times a$ *is a continuous measure.*

Proof. (i) Since $p > 2$ and $u = K*v$ with $v = \nabla\times u$, we get by (2.6a)

$$\|u\|_p(t) \leq C\|K\|_{2,\infty}\|v\|_q(t), \quad 1/q = 1/p+1/2$$

with C depending only on p. By (4.3),(4.6) and Lemma 4.4 we see that

$$\limsup_{t\downarrow 0} t^{1-1/q} \|v\|_q(t) \leq C'\|(\nabla\times a)_{pp}\|_{\mathcal{M}}$$

where C' depends only on q, m and ν. Combining these two estimates gives (4.19).

(ii) Let $\tilde{u}$ be another solution of (1) with the same initial data a satisfying properties (a) and (b) above. By (c) the difference $w = u - \tilde{u}$ satisfies

$$w(t) = -\int_0^t \nabla\cdot e^{\nu(t-s)\Delta} P[w \otimes u(s) + \tilde{u} \otimes w(s)]ds,$$

so that, as in the proof of Lemma 1.1 (i),

$$\|w\|_p(t) \leq M\int_0^t (t-s)^{-1/p-1/2}[\|u\|_p + \|\tilde{u}\|_p](s)\|w\|_p(s)ds.$$

Thus, $\|w\|_{p,T} \equiv \sup_{0<t\leq T} t^{1/2-1/p}\|w\|_p(t)$ satisfies

(4.20) $$\|w\|_{p,T} \leq MB(1/2-1/p,\ 2/p)[\|u\|_{p,t} + \|\tilde{u}\|_{p,T}]\|w\|_{p,T}$$

where B is the beta function. We here assume that $(\nabla\times a)_{pp}$ satisfies

(4.21) $$2CMB(1/2-1/p,\ 2/p)\|(\nabla\times a)_{pp}\|_{\mathcal{M}} < 1$$

where C is the constant in (4.19). Estimates (4.19)-(4.21) together imply that if we take $T > 0$ sufficiently small, then $\|w\|_{p,T} \leq c\|w\|_{p,T}$

for some $c < 1$, which yields $w = 0$ on $[0,T]$ since $\|w\|_{p,T}$ is finite. On the interval $[T,\infty)$, both u and $\tilde{u}$ are classical solutions belonging to L^p, so we get $w = 0$ on $[T,\infty)$ by the uniqueness result of Proposition 1.2. The proof is completed.

Thorem 4.5 shows in particular that the solution is unique whenever $\nabla\times a$ is a continuous measure. When the measure $\nabla\times a$ has a density, i.e., when $\nabla\times a$ is in $L^1(R^2)$, we can also prove more regularity at $t = 0$, as shown in the following theorem.

Theorem 4.6. *If* $a \in L^{2,\infty}(R^2)$, $\nabla\cdot a = 0$ *and if* $\nabla\times a \in L^1(R^2)$, *then the (unique) solution* u *of* (1) *belongs to* $BC([0,\infty) ; L^{2,\infty}(R^2))$.

Proof. By assumption, $e^{\nu t\Delta}(\nabla\times a)$ is in $BC([0,\infty) ; L^1(R^2))$. So, by (2.6b), the function $e^{\nu t\Delta} a = e^{\nu t\Delta} K*(\nabla\times a) = K*[e^{\nu t\Delta}(\nabla\times a)]$ belongs to $BC([0,\infty) ; L^{2,\infty}(R^2))$. By (v) of Theorem 4.2 it suffices therefore to show that the function

$$S[u](t) = -\int_0^t \nabla\cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)ds$$

is in $BC([0,\infty) ; L^{2,\infty}(R^2))$. By (1.3) with $r = s = 2$, and (4.1) with $r = 4$,

$$\|S[u]\|_{2,\infty}(t) \leq \|S[u]\|_2(t)$$

$$\leq C\int_0^t (t-s)^{-1/2} s^{-1/2} ds = CB(1/2,\ 1/2) \quad \text{for} \quad t > 0,$$

which implies the boundedness as well as the continuity for $t > 0$. On the other hand, since $\nabla\times a$ contains no pure point part, Theorem 4.5 (i) yields

$$\lim_{t\downarrow 0} t^{1/2-1/p} \|u\|_p(t) = 0 \quad \text{for all} \quad p > 2. \tag{4.22}$$

Hence, using again (1.3) with $r = s = 2$ and (4.1) with $r = 4$ we get, as $t \to 0$,

$$\|S[u]\|_{2,\infty}(t) \leq \|S[u]\|_2(t)$$

$$\leq CB(1/2,\ 1/2)\|u\|_{4,t} \to 0$$

since $\|u\|_{4,t} \equiv \sup_{0<s\leq t} s^{1/4}\|u\|_4(s) \to 0$ as $t \to 0$ by (4.22). This shows the continuity at $t = 0$, and the proof is completed.

Remark. Benfatto, Esposito and Pulvirenti [3] prove existence and uniqueness of solutions to (1) with initial data a such that

$$\nabla\times a = \sum_{j=1}^{m} \alpha_j\ \delta(x-z_j), \quad \alpha_j \in R^1, \quad z_j \in R^2$$

and $\sum_j |\alpha_j|$ is suficiently small. Here $\delta(x-z_j)$ is the Dirac measure supported by z_j. Our uniqueness result covers that of [3] ; moreover, our existence result improves that of [3] since no restriction is imposed on either of size and the form of the measure $\nabla\times a$.

References

1. Aronson, D.G., Bounds for the fundamental solution of a parabolic equation. Bull. Amer. Math. Soc. 73, 890-896 (1968).

2. Aronson, D.G., & J. Serrin, Local behavior of solutions of quasilinear parabolic equations. Arch. Rational Mech. Anal. 25. 81-122 (1967).

3. Benfatto, G., Esposito, R., & M. Pulvirenti, Planar Navier-Stokes flow for singular initial data. Nonlinear Anal. 9, 533-545 (1985).

4. Bergh, J., & J. Löfström, Interpolation Spaces, An Introduction. Berlin Heidelberg New York : Springer-Verlag 1976.

5. Brezis, H., & A. Friedman, Nonlinear parabolic equations involving measures as initial data. J. Math. Pures et appl. 62, 73-97 (1983).

6. Dobrushin, R.L., Prescribing a system of random variables by conditional distributions. Theory Prob. Appl. 15, 458-486 (1970).

7. Fabes, E.B., Jones, B.F., & N.M. Riviere, The initial value problem for the Navier-Stokes equations with data in L^p. Arch. Rational Mech. Anal. 45, 222-240 (1972).

8. Friedman, A., Partial Differential Equations of Parabolic Type. New Jersey : Prentice-Hall 1964.

9. Friedman, A., Partial Differential Equations. New York : Holt, Rinehart & Winston 1969.

10. Fujita, H., & T. Kato, On the Navier-Stokes initial value problem I. Arch. Rational Mech. Anal. 16, 269-315 (1964).

11. Giga, Y., & T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem. Arch. Rational Mech. Anal. 89, 267-281 (1985).

12. Giga, Y., Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. J. Differential Equations 62, 186-212 (1986).

13. Gilbarg, D., & N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed. Berlin Heidelberg New York : Springer-Verlag 1983.

14. Kato, T., Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions. Math.Z. 187, 471-480 (1984).

15. Kato, T., Remarks on the Euler and Navier-Stokes equations in R^2. Nonlinear Functional Analysis and its Applications, F. E. Browder ed., Proc. of Symposia in Pure Math. 45, part 2, 1-8. Providence, RI: Amer. Math. Soc. 1986.

16. Ladyzhenskaya, O.A., The Mathematical Theory of Viscous Incompressible Flow. New York : Gordon & Breach 1969.

17. Leray, J., Etude de diverses équations intégrales non linéaires et de quelques probléms que pose l'hydrodynamique. J. Math. pures et appl., Serie 9, 12, 1-82 (1933).

18. Liu, T.-P., & M. Pierre, Source-solutions and asymptotic behavior in conservation laws. J. Differential Equations 51, 419-441 (1984).

19. Marchioro, C., & M. Pulvirenti, Hydrodynamics in two dimensions and vortex theory. Commun. Math. Phys. 84, 483-503 (1982).

20. Marchioro, C., & M. Pulvirenti, Euler evolution for singular initial data and vortex theory. Commun. Math. Phys. 91, 563-572 (1983).

21. McGrath, F.J., Nonstationary planar flow of viscous and ideal fluids. Arch. Rational Mech. Anal. 27, 329-348 (1968).

22. McKean Jr., H.P., Propagation of chaos for a class of nonlinear

parabolic equations. Lecture series in diff. eq., Session 7 : Catholic Univ. 1967.

23. Niwa, Y., Semilinear heat equations with measures as initial data. preprint.

24. Osada, H., & S. Kotani, Propagation of chaos for the Burgers equation. J. Math. Soc. Japan 37, 275-294 (1985).

25. Osada, H., Diffusion processes with generators of generalized divergence form. J. Math. Kyoto Univ. to appear.

26. Osada, H., Propagation of chaos for the two dimensional Navier-Stokes equations. preprint ; Announcement : Proc. Japan Acad. 62, 8-11 (1986).

27. Ponce, G., On two dimensional incompressible fluids. Commun.Partial Differ. Equations 11, 483-511 (1986).

28. Reed, M., & B. Simon, Methods of Modern Mathematical Physics Vol. I, II ; New York : Academic Press 1972, 1975.

29. Sznitman, A.S., Propagation of chaos result for the Burgers equation. Probab. Th. Rel. Fields 71, 581-613 (1986).

30. Temam, R., Navier-Stokes Equations. Amsterdam : North-Holland 1977.

31. Turkington, B., On the evolution of a concentrated vortex in an ideal fluid. preprint.

32. Wahl, W. von, The Equations of Navier-Stokes and Abstract Parabolic Equations. Braunschweig : Vieweg Verlag 1985.

33. Weissler, F.B., The Navier-Stokes initial value problem in L^p. Arch. Rational Mech. Anal. 74, 219-230 (1980).

34. Kato, T., & G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L^p_s(R^2)$. preprint.

35. Baras, P., & M. Pierre, Problems paraboliques semi-linéaires avec données mesures. Applicable Analysis 18, 111-149 (1984).

Y. GIGA
Department of Mathematics
Hokkaido University
Sapporo, Japan

T. MIYAKAWA
Department of Mathematics
Hiroshima University
Hiroshima, Japan

H. OSADA
Department of Mathematics
Nara Women's University
Nara, Japan