RATE OF GROWTH OF THE COALESCENT SET
IN A COALESCING STOCHASTIC FLOW

By

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IMA Preprint Series # 249

July 1986

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

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Rate of growth of the coalescent set
in a coalescing stochastic flow

by

R.W.R. Darling

Summary

Consider an isotropic stochastic flow in $\mathbb{R}^d$ (i.e. a simultaneous random, correlated motion of all points in space), where $d = 1, 2$ or 3 , such that the joint law of the motion of two particles allows the particles to meet and coalesce in finite time. The coalescent set J_t is a random subset of $\mathbb{R}^d$ consisting of the initial positions of particles which have coalesced by time t with the particle which started at 0 . We show that the expected volume of J_t grows at a rate proportional to $\sqrt{t}$ when $d = 1$, and at rates close to proportional to $t/\log t$ (resp. t) when $d = 2$ (resp. $d = 3$). We give an example of a coalescing stochastic flow when $d = 3$. These results are analogous to growth rates of expected population size of a surviving type in the 'invasion process' described by Clifford and Sudbury.

Abbreviated Title: Growth of the coalescent set

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Title Pg Footnotes: AMS 1980 sub. class. Prim. 60G60; Sec. 60K35, 60J70.

Key words and phrases: stochastic flow, coalescence, isotropic covariance, Bessel process, invasion process, coalescent set.

Research supported by National Science Foundation grant DMS-8502802.

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Appendix A. Passage times for Bessel processes

Appendix B. Construction of an isotropic covariance of coalescing
type in dimension 3.

References

An isotropic stochastic flow in $\mathbb{R}^d$ may be roughly described as follows: at time 0 , every point in $\mathbb{R}^d$ begins to execute a random motion, such that the marginal law of the motion of a single particle is a Brownian motion with the appropriate initial position, but the motions of two or more particles are, in general, correlated. The covariance between the motions of two points at time t depends only on the distance between them, not on their absolute positions in space. In dimension $d = 1, 2$ or 3 , it is possible to specify the correlation structure in such a way that the trajectories of any two particles will eventually meet almost surely ($d = 1$ or 2), or with positive probability ($d = 3$); at the time when they meet, they are defined to coalesce. Examples may be found in Harris [8], for $d = 1$; in Darling [5], for $d = 2$; and in Appendix B below, for $d = 3$.

In order to understand the coalescence phenomenon, it is natural to try to calculate the probability law of the mass of the set of particles which have coalesced with a given particle (say the particle initially at 0) by time t ; this assumes, of course, that this 'coalescent set' is Lebesgue measurable. This turns out to be fairly straightforward when $d = 1$ (see Section 3 below), but difficult in dimensions 2 and 3. This is not altogether surprising, since the qualitative nature of the coalescence phenomenon is much more complex in dimensions 2 and above: for example all the points on the boundary of a certain set might have coalesced with 0 , while a point in the interior might not have; this phenomenon is impossible in dimension 1, where the flow is

order-preserving. However Fubini's theorem and comparison with certain 'Wiener sausages' allows us to estimate the expected mass of the coalescent set, and these estimates form the main results of this paper.

There is a profound analogy with infinite particle systems on $\mathbb{Z}^d$, such as the invasion process considered by Sudbury [13]. Each integral lattice point is initially occupied by a particle of a different type which proceeds to attack the neighbouring sites, and replace the particles formerly there with some of its own type. Sudbury shows that the expected number of particles at time t , which are of the same type as the particle at the origin at time t , has growth rate $\text{const.} \times t/\log t$ when $d = 2$, and $\text{const.} \times t$ when $d = 3$. These rates are within the bounds computed below for the expected mass of the coalescent set. Related problems were considered by Cox and Griffeath [4].

Acknowledgments. The technical assistance of Hans Weinberger (Minnesota) played a crucial role in this project. Richard Arratia and Maury Bramson gave valuable insights into the problem. The financial support of the National Science Foundation is gratefully acknowledged.

1. Isotropic stochastic flows - definitions and construction

The set-up we describe here is quite similar to the one discussed by Baxendale and Harris [3], except that the latter restrict themselves to C^2 covariances.

Fix $d \geq 1$, and let M denote the one-point compactification of $\mathbb{R}^d$. Let $F = M^M$, considered as the space of functions from M to M with the product topology. For any topological space Y , let $\mathcal{B}(Y)$ denote the Borel σ -algebra.

Definition 1.1. A family $(X_{st}, 0 \leq s \leq t < \infty)$ of random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, and taking values in $(F, \mathcal{B}(F))$, will be called a time-homogeneous pure stochastic flow with independent increments (or "pure stochastic flow", for short), if (1.1) to (1.4) hold (notation: $X_{st}(x, \omega)$ means the image of x in M under the mapping $X_{st}(\omega)$, i.e. the location at time t of the particle which was at x at time s).

$$(1.1) \quad X_{tt}(x, \omega) = x, \text{ all } \omega \text{ in } \Omega, t \geq 0, \text{ and } x \text{ in } \mathbb{R}^d$$

$$(1.2) \quad X_{tu}(X_{st}(x, \omega), \omega) = X_{su}(x, \omega) \text{ for all } 0 \leq s \leq t \leq u, \\ \text{all } \omega \text{ in } \Omega \text{ (no exceptions), and } x \text{ in } \mathbb{R}^d.$$

$$(1.3) \quad \text{The processes } (X_{s+h, t+h}, 0 \leq s \leq t < \infty) \\ \text{have the same law for all } h \geq 0.$$

$$(1.4) \quad X_{st}, X_{s't'}, \dots \text{ are independent if } s \leq t \leq s' \leq t' \leq \dots.$$

In order to call the process a pure stochastic flow on $\mathbb{R}^d$, we also insist on the obvious condition:

$$(1.5) \quad P(X_{st}(x) = \infty) = P(X_{st}(\omega) \in \mathbb{R}^d) = 1, \text{ all } x \in \mathbb{R}^d, \text{ all } s \leq t.$$

The class of pure stochastic flows we are concerned with consists

of those in which the motion of any set of k particles, $k = 1, 2, \dots$, is a diffusion process in $(\mathbb{R}^d)^k$ with zero drift. We wish to characterize the stochastic flow by the law of these 'k-point motions', which in fact is completely specified by the generator of the two-point motions.

Definition 1.2. Suppose b is a function from $\mathbb{R}^d$ into the set of $d \times d$ real matrices, with entries $(b^{pq}(x), 1 \leq p, q \leq d, x \in \mathbb{R}^d)$, with the algebraic properties necessary for it to be the covariance function of a homogeneous $\mathbb{R}^d$ -valued Gaussian random field on $\mathbb{R}^d$. A pure stochastic flow on $\mathbb{R}^d$ is said to have (spatially and temporally homogeneous) covariance b , if the generator of any k -point motion $(X_{0t}(x_1), \dots, X_{0t}(x_k), t \geq 0)$, for $x_1, \dots, x_k$ in $\mathbb{R}^d$, is

$$(1.6) \quad Af(z_1, \dots, z_k) = \frac{1}{2} \sum_{1 \leq i, j \leq k} \sum_{1 \leq p, q \leq d} b^{pq}(z_j - z_i) (\partial^2 / \partial z_i^p \partial z_j^q) f(z_1, \dots, z_k)$$

for $f \in C_0^\infty((\mathbb{R}^d)^k)$ (i.e. C^∞ functions with compact support), where $z_i = (z_i^1, \dots, z_i^d)$.

Remark. The essential content of (1.6) is that for all x, y in $\mathbb{R}^d$,

$$(1.6') \quad \lim_{t \downarrow 0} (1/t) \text{Cov}(X_t(y+x) - (y+x), X_t(y) - y) = b(x)$$

Definition 1.3. A pure stochastic flow on $\mathbb{R}^d$ with covariance b is said to be isotropic if b satisfies:

$$(1.7) \quad b(x) = G^* b(Gx) G,$$

for every $d \times d$ orthogonal matrix G (here G^* denotes G transpose).

According to Yaglom [16], such a covariance may be expressed as

$$(1.8) \quad b^{pq}(x) = (B_L(|x|) - B_N(|x|))x^p x^q / |x|^2 + B_N(|x|)\delta^{pq}, \quad x \neq 0,$$

where $B_L(r) = b^{pp}(re_p)$ for every $1 \leq p \leq d$, and

$B_N(r) = b^{pp}(re_q)$ for every $p \neq q$. (Naturally $B_N \equiv 0$ if $d = 1$).

For computational convenience, we define

$$(1.9) \quad v_L(r) = 1 - B_L(r), \quad v_N(r) = 1 - B_N(r), \quad r \geq 0,$$

and normalize so that

$$(1.10) \quad v_L(0) = v_N(0) = 0, \text{ i.e. } b(0) = \text{identity.}$$

(Note: (1.10) and (1.6) imply that for each fixed x ,

$(X_t(x) - x, t \geq 0)$ is a d -dimensional standard Brownian motion.)

For b to be a covariance at all, certain technical conditions must be satisfied. From Yaglom [16], equation (4.37), we see that B_L and B_N must be of the form

$$(1.11) \quad B_L(r) = \int_0^\infty F^{(1)}(r\lambda) M_1(d\lambda) + \int_0^\infty F^{(2)}(r\lambda) M_2(d\lambda), \quad r > 0;$$

$$B_N(r) = \int_0^\infty F^{(3)}(r\lambda) M_1(d\lambda) + \int_0^\infty F^{(4)}(r\lambda) M_2(d\lambda), \quad r > 0;$$

where $F^{(i)}$, $1 \leq i \leq 4$, are certain special functions depending only on the dimension d , and M_1 and M_2 are any pair of positive measures on $(0, \infty)$ such that the integrals in (1.11) and (1.12) are finite, with

$\lim_{r \downarrow 0} B_L(r) = \lim_{r \downarrow 0} B_N(r) = 1$, and such that:

(1.12) at least one of M_1, M_2 has a non-zero absolutely continuous part (with respect to Lebesgue measure on $(0, \infty)$).

The smoothness of the functions $F^{(i)}$ guarantees that B_L and B_N are both Lipschitz continuous on $[c, \infty)$ for all $c > 0$. It follows from (1.8) that $x \rightarrow b^{pq}(x)$ is Lipschitz continuous outside the ball with radius c , centred at 0 , for all $c > 0$. We see that all the technical conditions stated in Darling [5], Section 11, are now satisfied.

We assume henceforward that the measures M_1 and M_2 satisfy all the conditions described above.

Using the functions V_L and V_N defined in (1.8) and (1.9) above, define

$$(1.13) \quad H(r) = \exp\left\{-(d-1) \int_1^r [V_N(s)/sV_L(s)] ds\right\}, \quad r > 0.$$

which is well-defined because V_N and V_L are both non-negative.

THEOREM 1.4

Suppose there exists a C^2 function $S: (0, \infty) \rightarrow (0, \infty)$ such that

$$(1.14) \quad S'(r) = H(r) \quad \text{for } r > 0, \text{ and } S(0+) = 0.$$

Then there exists an isotropic pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ satisfying (1.1)-(1.6) (i.e.. with covariance b).

Proof. See Darling [5], Section 14.

Knowing that a pure stochastic flow exists, it makes sense to talk about the coalescence time of the particles started at space points x and y , namely

$$(1.15) \quad v(x, y, \omega) = \inf\{t \geq 0 : X_{0t}(x, \omega) = X_{0t}(y, \omega)\}.$$

Condition (1.2) ensures that for $t > v(x, y, \omega)$, $X_{0t}(x, \omega)$ remains the same as $X_{0t}(y, \omega)$. Thus we are able to define:

Definition 1.5: The coalescent set process for the point $\emptyset$ (any other point could be used instead) is the increasing set-valued process

$$(1.16) \quad \begin{aligned} J_t(\omega) &= \{x: X_{0t}(x, \omega) = X_{0t}(\emptyset, \omega)\} \\ &= \{x: v(\emptyset, x, \omega) \leq t\} . \end{aligned}$$

Of course, for a homeomorphic stochastic flow, $J_t(\omega)$ equals $\{\emptyset\}$ for all t . However there is a class of isotropic pure stochastic flows for which $J_t(\omega)$ can grow.

PROPOSITION 1.6

Suppose there exists a C^2 function $S: (0, \infty) \rightarrow (0, \infty)$ such that (1.14) holds, and moreover

$$(1.17) \quad \int_0^1 S(u) / [S'(u) v_L(u)] du < \infty .$$

Then the coalescence times defined in (1.15) satisfy:

$$(1.18) \quad P(v(x, y) = \infty) = S(|x - y|) / S(\infty) .$$

$$(1.19) \quad \lim_{y \rightarrow x} \mathbb{E}[e^{-v(x, y)}] = 1 .$$

Proof. See Darling [5], Section 14

Observe that if $S(\infty) = \infty$ (which is true for a class of examples in dimensions 1 and 2) then $P(v(x, y) < \infty) = 1$, and so each x in $\mathbb{R}^d$ is almost certain to belong to $J_t(\omega)$ for sufficiently large t .

2. The distribution of the right endpoint of the coalescent set when $d=1$.

Coalescing stochastic flows on the line were studied by Harris [8], who was inspired by an example of Arratia [1,2]. Darling [5] slightly extended the class of covariances for which pure stochastic flows may be constructed. The theory of the preceding section simplifies considerably, because the isotropy condition is vacuous, and there is no 'transverse' correlation function B_N .

PROPOSITION 2.1 (Harris [8], Darling [5])

Suppose $b: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, non-negative definite, $b(0) = 1$ (for convenience only), the spectral distribution of b is not of pure jump type, b is Lipschitz outside each interval $(-c, c)$, $c > 0$, and moreover

$$(2.1) \quad \int_0^1 y \, dy / (1-b(y)) < \infty .$$

Then there exists a pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ on $\mathbb{R}^1$ with covariance b (Definitions 1.1, 1.2) such that:

$$(2.2) \quad (x, \omega) \rightarrow X_{st}(x, \omega) \text{ is } m \otimes P\text{-measurable (} m = \text{Lebesgue measure),}$$

$$(2.3) \quad J_t(\omega) \text{ (see (1.16)) is an interval of } \mathbb{R}^1, \text{ hence Lebesgue measurable for every } t \geq 0 \text{ and } \omega \in \Omega; \text{ also } m(J_t(\omega) \cap A) \text{ is } P\text{-measurable for every Lebesgue measurable set } A \text{ in } \mathbb{R}^1.$$

$$(2.4) \quad P(v(x, y) < \infty) = 1, \text{ for all } x, y. \text{ (see (1.15)).}$$

$$(2.5) \quad \text{For fixed } t > 0, P(v(x, y) \leq t) \rightarrow 1 \text{ as } y \rightarrow x.$$

Remark. A consequence of the condition on the spectral distribution is that

$$(2.6) \quad b(y) \rightarrow 0 \text{ as } y \rightarrow \infty.$$

In view of the simple structure of the coalescent set in dimension 1, it is natural to consider the law of the right endpoint, namely

$$(2.7) \quad R_t(\omega) = \sup\{y: y \in J_t(\omega)\} = \sup\{y: v(0, y, \omega) \leq t\}.$$

It follows from (2.3) that R_t is a random variable for each t .

Harris [8], Lemma 3.4, shows that for each $y > 0$, the process $(X_{0t}(y) - X_{0t}(0), t \geq 0)$ has the same law P_y as the diffusion $(\xi_t, t \geq 0)$, started at y , with absorption at 0 , and with generator

$$(2.8) \quad Lf(x) = (1-b(x))f''(x), \quad f \in C_0^\infty(\mathbb{R}^1).$$

(The form of the generator may be calculated from (1.6)).

It follows that for $y > 0$,

$$(2.9) \quad \begin{aligned} P(R_t < y) &= P(X_{0t}(y) - X_{0t}(0) > 0) \\ &= P_y(\xi_s > 0, 0 \leq s \leq t) = P_y(\tau_0 > t), \end{aligned}$$

where

$$(2.10) \quad \tau_0 = \inf\{s: \xi_s = 0\}.$$

PROPOSITION 2.2

The following formula holds for every spatially- and temporally-homogeneous pure stochastic flow on the line, with covariance b as in Proposition 2.1: for R_t as in (2.7), and $y > 0$,

$$(2.11) \quad \lim_{t \rightarrow \infty} P(R_t/\sqrt{t} < y) \text{ exists and } = \alpha(y), \text{ where}$$

$$(2.12) \quad \alpha(y) = \int_1^{\infty} (y/2)(\pi r^3)^{-1/2} e^{-y^2/4r} dr \quad (\text{independent of } b).$$

Remarks 1. Let Q_y denote the law of standard Brownian motion, started at y , and let τ_0 denote the first passage time to 0 . Then,

$$(2.13) \quad \alpha(y) = Q_y(\tau_0 > 2) .$$

(see Ito and McKean (1974) p. 26).

2. Arratia [1,2] has studied a stochastic flow on the line which corresponds to the case $b(0) = 1$, $b(y) = 0$ for $y \neq 0$. In other words, all the points on the real line execute independent Brownian motion, with the rule that when two trajectories collide, they coalesce. For this system,

$$(2.14) \quad P(R_t/\sqrt{t} < y) = \alpha(y) , \text{ all } t \geq 0 , \text{ all } y > 0 .$$

The proposition asserts that, for every stochastic flow with covariance as in Proposition 2.1, the right endpoint of the coalescent set has the same asymptotic behavior, for large time, as for the Arratia system.

Proof. Step 1. For $r > 0$, (2.9) shows that

$$P(R_t/r < y) = P_{yr}(\tau_0 > t) .$$

The function $\alpha: (0, \infty) \rightarrow [0, 1]$ is continuous and increasing.

Therefore it suffices to prove that, given $\varepsilon > 0$.

$$(2.15) \quad \limsup_{r \rightarrow \infty} P_{yr}(\tau_0 > r^2) \leq \alpha(y(1+2\varepsilon)) ,$$

$$(2.16) \quad \liminf_{r \rightarrow \infty} P_{yr}(\tau_0 > r^2) \geq \alpha(y(1-2\varepsilon)) ,$$

replacing t by r^2 in (2.11)

Step II. Fix $y > 0$ and $\varepsilon > 0$, and define for each $r > 0$

$\sigma_r(x) = [2(1-b(rx))]^{1/2}$, and

$$(2.17) \quad \eta_{r,t} = r^{-1} \xi_t(ry)$$

where $(\xi_t(ry), t \geq 0)$ has generator (2.8) with $\xi_0(ry) = ry$. If $(W_t, t \geq 0)$ is Brownian motion, the process (2.17) has a representation:

$$(2.18) \quad \eta_{r,t} = y + r^{-1} \int_0^{t \wedge \tau_0} \sigma_r(\eta_{r,s}) dW_s.$$

Define $\delta > 0$ by the formula

$$(2.19) \quad 1-\delta = [(1+\varepsilon)^2(1-\varepsilon/2)]^{-1}.$$

It follows from (2.6) that there exists $c > 0$ such that

$$(2.20) \quad \sigma_r(x)^2 = 2(1-b(rx)) > 2(1-\delta) \quad \text{for } x \geq c/r.$$

For $r > c/y$, define a stopping-time

$$(2.21) \quad \tau_c = \inf\{t: \xi_t(ry) = c\} = \inf\{t: \eta_{r,t} = c/r\}.$$

For $t \leq \tau_c$, the continuous martingale $(\eta_{r,t}, t \geq 0)$ has a quadratic variation process satisfying

$$(2.22) \quad \langle \eta_r \rangle_t \geq 2(1-\delta)t/r^2, \quad t \leq \tau_c.$$

Hence until time τ_c , the process η_r is running at least as fast as $(2(1-\delta))^{1/2}/r$ times a Brownian motion, and so

$$\begin{aligned} P_{ry}(\tau_c > t) &\leq Q_0(\inf\{s: (2(1-\delta))^{1/2} r^{-1} W_s \geq y-c/r\} > t) \\ &= Q_0(\inf\{u: (2(1-\delta))^{1/2} W_{ut/2} \geq yr-c\} > 2) \\ &= Q_0(\inf\{u: W_u \geq (yr-c)[t(1-\delta)]^{-1/2}\} > 2) \\ (2.23) \quad &\leq \alpha[yr(t(1-\delta))^{-1/2}], \text{ since } \alpha \text{ is increasing.} \end{aligned}$$

Step III. The stopping-time τ_0 is the sum of τ_c and the time taken to reach 0 from c . Thus

$$(2.24) \quad P_{ry}(\tau_0 > t) \leq P_{ry}(\tau_c > t(1-\varepsilon/2)) + P_c(\tau_0 > t\varepsilon/2).$$

The function α is strictly increasing; therefore there exists $T > 0$

such that

$$(2.25) \quad P_c(\tau_0 > t\varepsilon/2) < \alpha(y(1+2\varepsilon)) - \alpha(y(1+\varepsilon)) \quad \text{if } t \geq T,$$

using the fact that $P_c(\tau_0 < \infty) = 1$. Now combine (2.23)-(2.25), taking $t = r^2$: for $r \geq T^2$,

$$\begin{aligned} P_{ry}(\tau_0 > r^2) &\leq \alpha(y(1-\delta))^{-1/2} (1-\varepsilon/2)^{-1/2} + \alpha(y(1+2\varepsilon)) - \alpha(y(1+\varepsilon)) \\ &= \alpha(y(1+2\varepsilon)), \text{ by (2.19).} \end{aligned}$$

This concludes the proof of (2.15).

Step IV: To prove (2.16), redefine $\delta > 0$ so that

$$(2.26) \quad (1+\delta)^{1/2} = (1-\varepsilon)/(1-2\varepsilon).$$

It follows from (2.6) that there exists $c > 0$ such that $|b(x)| < \delta$ for $|x| > c$. In particular,

$$(2.27) \quad \delta_r(x)^2 = 2(1-b(rx)) < 2(1+\delta) \quad \text{for } x \geq c/r.$$

For τ_c as in (2.21), we observe that

$$(2.28) \quad \langle \eta_r \rangle_t < 2(1+\delta)t/r^2, \quad \text{for } t \leq \tau_c.$$

Comparing τ_c with the first passage time for Brownian motion,

$$\begin{aligned} P_{ry}(\tau_c > t) &\geq Q_0(\inf\{s: [2(1+\delta)]^{1/2} r^{-1} W_s = y-c/r\} > t) , \\ &= Q_0(\inf\{u: [2(1+\delta)]^{1/2} W_{ut/2} \geq yr-c\} > 2) \\ &= Q_0(\inf\{u: W_u \geq (yr-c)[t(1+\delta)]^{-1/2}\} > 2) \\ &= \alpha[(yr-c)(t(1+\delta))^{-1/2}] \end{aligned}$$

$$(2.29) \quad \geq \alpha[yr(1-\varepsilon)(t(1+\delta))^{-1/2}] \quad \text{if } r > c/(y\varepsilon)$$

since α is an increasing function. Taking $t = r^2$,

$$P_{ry}(\tau_0 > r^2) \geq P_{ry}(\tau_c > r^2) \geq \alpha(y(1-2\varepsilon))$$

if $r > c/(y\varepsilon)$, by (2.26) and (2.29). Now (2.16) follows, which completes the proof.

3. The expected mass of the coalescent set when $d = 1$.

PROPOSITION 3.1

When $d = 1$, the mass $m(J_t)$ of the coalescent set satisfies:

$$(3.1) \quad \lim_{t \rightarrow \infty} t^{-1/2} \mathbb{E}[m(J_t)] = 4/\sqrt{\pi}.$$

Proof. By symmetry, it is clear in dimension 1 that

$$\mathbb{E}[m(J_t)] = 2\mathbb{E}[R_t] = 2 \int_0^{\infty} P(R_t \geq y) dy.$$

Change variables, using $y = x\sqrt{t}$. Then for $t > 0$,

$$(3.2) \quad t^{-1/2} \mathbb{E}[m(J_t)] = 2 \int_0^{\infty} P(R_t/\sqrt{t} \geq x) dx$$

We wish to let $t \rightarrow \infty$, and interchange limit and integral on the right side, using Lebesgue's dominated convergence theorem. For this, we need to dominate $P(R_t/\sqrt{t} \geq x)$ by a function in $L^1([0, \infty), dx)$, uniformly in $t \geq 0$. As we saw in (2.9),

$$(3.3) \quad P(R_t/\sqrt{t} \geq x) = P_{x\sqrt{t}}(\xi_s = \emptyset \text{ for some } s \in [0, t]) = P_{x\sqrt{t}}(\tau_0 \leq t),$$

where the martingale $(\xi_t, t \geq 0)$ has quadratic variation process

$$\langle \xi \rangle_t = 2 \int_0^t (1 - b(\xi_s)) ds \leq 4t.$$

Comparing $(\xi_t, t \geq 0)$ with Brownian motion run at four times the usual rate, and using the reflection principle, we see that

$$(3.4) \quad \begin{aligned} P_{x\sqrt{t}}(\tau_0 \leq t) &\leq Q_{x\sqrt{t}}(\tau_0 \leq 4t) = Q_0(\tau_{x\sqrt{t}} \leq 4t) \\ &= 2 \int_{x\sqrt{t}}^{\infty} e^{-r^2/4t} (8\pi t)^{-1/2} dr \end{aligned}$$

$$= 2(8\pi)^{-1/2} \int_x^\infty e^{-s^2/4} ds .$$

Call the last expression $g(x)$. Then by Fubini's theorem

$$\int_0^\infty g(x) dx = 2(8\pi)^{-1/2} \int_0^\infty s e^{-s^2/4} ds < \infty .$$

According to (3.3) and (3.4), $P(R_t/\sqrt{t} \geq x) \leq g(x)$, so Lebesgue's dominated convergence theorem may be applied to (3.2), giving:

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-1/2} E[m(J_t)] &= 2 \int_0^\infty \lim_{t \rightarrow \infty} P(R_t/\sqrt{t} \geq x) dx \\ &= 2 \int_0^\infty Q_y(\tau_0 \leq 2) dy \\ &= 4 \int_0^\infty dy \left\{ \int_y^\infty e^{-r^2/4} \pi^{-1/2} (1/2) dr \right\} \text{ (Reflection principle)} \\ &= 2\pi^{-1/2} \int_0^\infty dr \{ r e^{-r^2/4} \} \quad \text{(Fubini)} \\ &= 4\pi^{-1/2} . \end{aligned}$$

4. A formula for the expected mass of the coalescent set when $d \geq 2$

The existence theorem for isotropic pure stochastic flows on $\mathbb{R}^d$ (Theorem 1.4 above) unfortunately says nothing about the spatial measurability of the random mappings $X_{st}(\cdot, \omega): M \rightarrow M$, where $M = \mathbb{R}^d \cup \{\omega\}$. Therefore there is no guarantee that the coalescent set $J_t(\omega)$ is a Lebesgue measurable subset of $\mathbb{R}^d$, almost surely.

However, if m denotes the extension to M of Lebesgue measure on $\mathbb{R}^d$, it is trivially true (since one-point motions are Brownian) that for any measurable subset A of M ,

$$(4.1) \quad m(A) = 0 \Rightarrow \int_M P(X_{st}(x) \in A) m(dx) = 0, \quad 0 \leq s \leq t.$$

(It is proved in Darling [6] that condition (1.14) implies that the integrand in (4.1) is measurable in x .) In this case, given any positive integer n , a discrete time construction found in Section 6 of Darling [6] allows us to construct a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, on which are defined a family of $\Gamma(=M^M)$ -valued random variables $(W_{j/n, k/n}, j \leq k, j, k \in \mathbb{Z}^+)$, with the following properties:

(4.2) Their joint law as $(\Gamma, \mathcal{B}_0(\Gamma))$ -valued random variables, is the same as $(X_{j/n, k/n}, j \leq k, j, k \in \mathbb{Z}^+)$.

(4.3) Whenever $i \leq j \leq k$ in $\mathbb{Z}^+$,

$$W_{j/n, k/n}(W_{i/n, j/n}(x, \omega), \omega) = W_{i/n, k/n}(x, \omega), \quad x \in M, \quad \omega \in \Omega_n.$$

(4.4) $(x, \omega) \rightarrow W_{j/n, k/n}(x, \omega)$ is $m \otimes P_n$ -measurable, for each $j \leq k$.

For $t = j/n$, let $J_{n,t}(\omega) = \{y: W_{0t}(y, \omega) = W_{0t}(\emptyset, \omega)\} \subseteq \mathbb{R}^d$. Then

(4.4) shows that $\mathbb{E}[m(J_{n,t}(\omega))]$ is well-defined when $t = j/n$, and by Fubini's theorem it equals:

$$(4.5) \quad \begin{aligned} &= \int_M P_n(\{\omega: \min\{i: W_{0,i/n}(y, \omega) = W_{0,i/n}(0, \omega)\} \leq j\}) m(dy) \\ &= \int_M P(v(0, y) \leq t) m(dy) \end{aligned}$$

in the terminology of (1.15); these equalities depend on (4.3) and (4.2) respectively. Hence the integral (4.5) represents the expected mass of the coalescent set in the corresponding discrete-time isotropic pure stochastic flow, for $t = j/n$, $j = 0, 1, 2, \dots$.

We shall now explain how the integral (4.5) may be treated as a function of t , for all $t \geq 0$. Observe that $v(0, y)$ is the first passage time to 0 of the 'distance process' $(|X_{0t}(y) - X_{0t}(0)|, t \geq 0)$, started at $|y|$. Formula (3.11) of Baxendale and Harris (1985), or direct computation from (1.6), shows that the distance process has the same law as the one-dimensional diffusion process $(\rho_t, t \geq 0)$, started at $|y|$ and with absorption at 0, whose generator is:

$$(4.6) \quad Rf(x) = V_L(x)f''(x) + ((d-1)/x)V_N(x)f'(x), \quad f \in C_0^\infty((0, \infty)).$$

Let P_r denote the law of this process when $|y| = r$, and introduce the stopping-times

$$(4.7) \quad \tau_a = \inf\{t: \rho_t \leq a\}, \quad \text{for } 0 \leq a \leq |y|.$$

Then

$$(4.8) \quad P(v(0, y) \leq t) = P_{|y|}(\tau_0 \leq t).$$

By the theory of singular diffusions, the last expression is a continuous function of y . Hence the integral (4.5) makes sense for all $t \geq 0$,

and converting into polar coordinates yields the following result:

PROPOSITION 4.1

The expected mass of the coalescent set at time t , at least for the discrete-time stochastic flows discussed above, is given by the formula:

$$(4.9) \quad A(d)\Psi_d(t) \text{ , where } \Psi_d(t) = \int_0^\infty r^{d-1} P_r(\tau_0 \leq t) dr ;$$

here $A(d)$ is the surface area of the unit sphere in $\mathbb{R}^d$, and P_r is the law of the diffusion $(p_t, t \geq 0)$, described above, with $p_0 = r$.

Since the $d = 1$ case was already treated in Section 3, we shall henceforward consider only the cases $d = 2$ and $d = 3$ (no isotropic pure stochastic flows of coalescent type are known in dimension $d \geq 4$). The main results of this paper may be summarized as follows.

THEOREM 4.2

Suppose that the functions $B_L(r)$ and $B_N(r)$ are both $O(r^{-3/2})$ (resp. $O(r^{-1})$) when $d = 2$ (resp. $d = 3$), as $r \rightarrow \infty$. For $\Psi_d(t)$ as in (4.9), there exists a constant $c_2 > 0$ (not depending on the covariance b), and $c_1 > 0$, $c_4 > 0$ (depending on b), and given $\delta > 0$, there exists a constant $c_3 > 0$ (depending on δ and b) such that

$$(4.10) \quad c_3 t (\log t)^{-5/3 - \delta} \leq \Psi_2(t) \leq c_2 t / \log t .$$

$$(4.11) \quad c_4 t (\log t)^{-2} \leq \Psi_3(t) \leq c_1 t (\log t)$$

for all sufficiently large t .

Proof. See sections 7 and 8 below.

Remark 4.3

Up to a dimensional constant, $\psi_d(t)$ may be considered as the expected mass of the coalescent set for a given isotropic stochastic flow, of coalescent type (i.e. (1.17) holds), in dimension d . Loosely speaking, $\psi_2(t)$ is comparable to $\text{const.} \times t/\log t$, and $\psi_3(t)$ is comparable to $\text{const.} \times t$, where the amount of deviation from these rates may depend on the precise form of the covariance. This confirms the remarks made in the last paragraph of the introduction, about the analogy with certain infinite particle systems.

5. Comparison results for hitting probabilities

In order to estimate the integrand in (4.9), we wish to study two classes of diffusion processes on the positive real line. The first type of diffusion process is $\sqrt{2}$ times the Bessel (δ) process, for arbitrary real $\delta > 1$: the generator is:

$$(5.1) \quad G^{(\delta)}f(x) = f''(x) + ((\delta-1)/x)f'(x), \quad f \in C_0^\infty((0, \infty)).$$

Let Q_x^δ denote the probability measure on $C([0, \infty); \mathbb{R})$ induced by the diffusion $(\xi_t, t \geq 0)$ with generator $G^{(\delta)}$, started at $x > 0$. Also define, for $a > 0$,

$$(5.2) \quad \tau_a = \inf\{t: \xi_t \leq a\}$$

$$(5.3) \quad u^{(\delta)}(a, x, t) = Q_x^\delta(\tau_a \leq t) \quad \text{for } x \geq a, \quad t \geq 0.$$

Relevant results concerning $u^{(\delta)}(a, x, t)$ are collected in Appendix A.

The second type of diffusion process is the process (ρ_t) described in Section 4, whose generator R is given in (4.6); this process has the same law as the distance between two particles under the (isotropic) stochastic flow. On the basis of the examples described in Appendix B, and in Darling [5], Section 16, we shall assume that the functions V_L and V_N , which appear in the definition of R , satisfy the following conditions:

$$(5.4) \quad V_L \text{ and } V_N \text{ are continuous on } [0, \infty), \text{ and}$$

Lipschitz continuous on $[c, \infty)$ for each $c > 0$.

$$(5.5) \quad V_L(0) = V_N(0) = 0; \quad 0 < V_L(r) < 2 \quad \text{and} \quad 0 < V_N(r) < 2, \\ \text{if } r > 0.$$

$$(5.6) \quad \text{For } 0 < r < 1, \quad V_L(r) \text{ and } V_N(r) \text{ behave in such a way that}$$

(1.14) and (1.17) hold (so coalescence is possible).

(5.7) $d = 2$: $V_L(r)$ and $V_N(r)$ are both $1 + O(r^{-3/2})$ as $r \rightarrow \infty$

(5.8) $d = 3$: $V_L(r)$ and $V_N(r)$ are both $1 + O(r^{-1})$ as $r \rightarrow \infty$.

From (5.7), (5.8) we see that $G^{(d)}$ and R are similar for large x .

For $x > 0$, let P_x denote the probability measure on $C([0, \infty); \mathbb{R})$ induced by the diffusion $(\xi_t, t \geq 0)$ with generator R (see (4.6)), started at x , and with absorption at 0 . (Condition (5.6) implies that 0 is an accessible boundary point.)

We now state a well-known fact, described for example by Ito and McKean [10], p. 154.

LEMMA 5.1

For $a \geq 0$, $q(a, x, t) = P_x(\tau_a \leq t)$ is the solution of the parabolic partial differential equation:

$$(5.9) \quad \partial q / \partial t = Rq,$$

$$q(a, a, t) = 1 \text{ for } t \geq 0; \quad q(a, x, 0) = 0 \text{ for } x > a.$$

Our main technical tool will be the following result, suggested by Hans Weinberger (Minnesota):

PROPOSITION 5.2

Given $0 < \varepsilon < 1$, choose $a > 0$ such that

$$(5.10) \quad 1 - \varepsilon < V_L(x) < 1 + \varepsilon \text{ and } 1 - \varepsilon < V_N(x) < 1 + \varepsilon, \text{ for } x \geq a$$

(possible by virtue of (5.7), (5.8)). Then, in the notation of (5.3),

$$(5.11) \quad u^{(\beta)}(a, x, t(1-\varepsilon)) \leq P_x(\tau_a \leq t) \leq u^{(\gamma)}(a, x, t(1+\varepsilon))$$

where $\beta-1 = (d-1)(1+\varepsilon)/(1-\varepsilon)$ and $\gamma-1 = (d-1)(1-\varepsilon)/(1+\varepsilon)$.

Proof. Eliminate a from the notation, writing $u^{(\beta)}(x, t)$ for $u^{(\beta)}(a, x, t)$, etc. Let $w(x, t) = q(x, t) - u^{(\beta)}(x, t(1-\varepsilon))$ (see Lemma 5.1). Differentiating with respect to t ,

$$\begin{aligned} w_t(x, t) &= q_t(x, t) - (1-\varepsilon)u_t^{(\beta)}(x, t(1-\varepsilon)) \\ &= V_L(x)q_{xx} - (1-\varepsilon)u_{xx}^{(\beta)} + x^{-1}[(d-1)V_N(x)q_x - (\beta-1)(1-\varepsilon)u_x^{(\beta)}]. \end{aligned}$$

Now $q_{xx} > 0$, $q_x < 0$, $u_{xx} > 0$, $u_x < 0$. Hence $V_L(x)q_{xx} > (1-\varepsilon)q_{xx}$ and $(d-1)V_N(x)q_x > (d-1)(1+\varepsilon)q_x = (\beta-1)(1-\varepsilon)q_x$. Hence

$$w_t < (1-\varepsilon)(q_{xx} - u_{xx}^{(\beta)}) + x^{-1}(\beta-1)(1-\varepsilon)(q_x - u_x^{(\beta)}).$$

Therefore

$$\begin{aligned} (5.12) \quad w_t &> (1-\varepsilon)w_{xx} + (\beta-1)(1-\varepsilon)x^{-1}w_x, \quad x \geq a, t \geq 0 \\ w(a, t) &= 0, \quad t \geq 0; \quad w(x, 0) = 0, \quad x > a. \end{aligned}$$

The solution to (5.12), with equality instead of inequality, is identically zero by the maximum principle. Hence $w(x, t) \geq 0$ for all $x \geq a, t \geq 0$. This proves the first inequality of (5.11).

For the second inequality, take instead

$$w(x, t) = u^{(\gamma)}(x, t(1+\varepsilon)) - q(x, t).$$

Then

$$\begin{aligned} w_t(x, t) &= (1+\varepsilon)u_t^{(\gamma)}(x, t(1+\varepsilon)) - q_t(x, t) \\ &= (1+\varepsilon)u_{xx}^{(\gamma)} - V_L(x)q_{xx} + x^{-1}[(\gamma-1)(1+\varepsilon)u_x^{(\gamma)} - (d-1)V_N(x)q_x] \\ &> (1+\varepsilon)(u_{xx}^{(\gamma)} - q_{xx}) + x^{-1}(\gamma-1)(1+\varepsilon)[u_x^{(\gamma)} - q_x]. \end{aligned}$$

Therefore

$$w_t > (1+\varepsilon)w_{xx} + (\gamma-1)(1+\varepsilon)x^{-1}w_x,$$

with boundary conditions as in (5.12). Hence $w(x,t) \geq 0$ by the same reasoning as before.

Proposition 5.2 is of most use when the 'constants' a, ε, γ and β are allowed to be functions of t . For this, we state

COROLLARY 5.3

Suppose $a = a(t)$ is an increasing function of t , such that $a(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then there exist continuous functions $\beta(t)$ and $\gamma(t)$, such that

$$(5.13) \quad \gamma(t) < d < \beta(t)$$

$$(5.14) \quad \beta(t)-d \text{ and } d-\gamma(t) \text{ are } O(a(t)^{-3/2}) \text{ when } d = 2,$$

$$(5.15) \quad \beta(t)-d \text{ and } d-\gamma(t) \text{ are } O(a(t)^{-1}) \text{ when } d = 3,$$

and for $x \geq a(t)$,

$$(5.16) \quad u^{(\beta(t))}(a(t), x, t/2) \leq P_x(\tau_{a(t)} \leq t) \leq u^{(\gamma(t))}(a(t), x, 2t)$$

Proof. Suppose $d = 2$. According to (5.7), $|1-V_N(x)|$ and $|1-V_L(x)|$ are $O(a(t)^{-3/2})$ for $x \geq a(t)$. From Proposition 5.2, we see that

(5.16) holds, for t sufficiently large, whenever

$$\beta(t)-d > 2(d-1) \sup_{x \geq a(t)} \max(|1-V_L(x)|, |1-V_N(x)|)$$

and similarly for $\gamma(t)$. This gives (5.14). The $d = 3$ case is similar.

6. Some integrals related to the Wiener sausage

Let $u^{(\delta)}(a, x, t)$ be as in (5.3), namely $Q_x^\delta(\tau_a \leq t)$ where Q_x^δ is the law of $\sqrt{2}$ times the Bessel(δ) process, for $\delta \geq 1$, $a > 0$, $x \geq a$, and $t \geq 0$. Define for $\gamma \geq 1$

$$(6.1) \quad I(\gamma, \delta, a, t) = \int_a^\infty x^{\gamma-1} u^{(\delta)}(a, x, t) dx.$$

We need to know the behavior of this integral for large t .

Remark. Recall that the Wiener sausage C_t^a refers to the points in space within distance a of the trajectory of a Brownian particle up to time t . To aid the reader's intuition, we remark that

$A(d)I(d, d, a\sqrt{2}, t)$ is the expected volume of the Wiener sausage C_t^a in dimension d (here $A(d)$ is the same dimensional constant as in (4.9)).

To verify this, observe that

$$E[m(C_t^a)] = E_0[m(\{y: \inf_{s \leq t} |B_s - y| \leq a\})]$$

where $(B_t, t \geq 0)$ denotes Brownian motion in $\mathbb{R}^d$, and ' E_0 ' means that the motion starts at 0 . This is equal to

$$\begin{aligned} & E_y[m(\{y: \inf_{s \leq t} |B_s| \leq a\})] \quad (\text{symmetry}) \\ &= \int 1_{\{(y, w): \inf_{s \leq t} |B_s| \leq a\}} d(m \otimes P_y) \quad (\text{Fubini}) \\ &= \int_{\mathbb{R}^d} \tilde{P}_y(\inf_{s \leq t} |B_s| \leq a) m(dy) \\ &= A(d) \int_0^\infty r^{d-1} Q_r^d(\tau_{a\sqrt{2}} \leq t) dr = A(d)I(d, d, a\sqrt{2}, t) \end{aligned}$$

where $\tilde{P}_y$ denotes the law of Brownian motion started at y .

LEMMA 6.1

The integral (6.1) is finite for all $\gamma \geq 1$ and $\delta \geq 1$.

Proof. We deduce from Lemma A.3(ii) that $I(\gamma, \delta, a, t) \leq I(\gamma, 1, a, t)$.

Moreover Proposition A.2 shows

$$I(\gamma, 1, a, t) = \int_a^\infty x^{\gamma-1} (\pi t)^{-1/2} \left\{ \int_{x-a}^\infty e^{-s^2/4t} ds \right\} dx.$$

By Fubini's theorem, this is

$$= (\pi t)^{-1/2} \int_0^\infty ds e^{-s^2/4t} \int_a^{s+a} x^{\gamma-1} dx < \infty.$$

LEMMA 6.2

For all $\delta \geq 1$, $u^{(\delta)}(a, x, t) = u^{(\delta)}(1, x/a, t/a^2)$, and

$$(6.2) \quad I(\gamma, \delta, a, t) = a^\gamma I(\gamma, \delta, 1, t/a^2)$$

Proof. The first assertion follows from the fact that for all $\lambda > 0$, $u^{(\delta)}(a\lambda, x\lambda, t\lambda^2) = u^{(\delta)}(a, x, t)$, which is an immediate consequence of Lemma A.1. The second assertion follows from the first by integrating as in (6.1).

PROPOSITION 6.3

Take $a = 1$, $\gamma \geq 1$ and $\delta \geq 1$, and consider the Laplace transform.

$$(6.3) \quad \bar{I}(\gamma, \delta, 1, p) = \int_0^\infty e^{-pt} I(\gamma, \delta, 1, t) dt.$$

(i) If $\delta > 2$ and $\delta > \delta - 2$, then

$$(6.4) \quad \bar{I}(\gamma, \delta, 1, p) \sim c(\gamma, \delta) p^{(\delta-\gamma-4)/2}, \text{ as } p \downarrow 0, \text{ when}$$

$$c(\gamma, \delta) = 2^{\gamma-\delta+1} \Gamma((\gamma-\delta+2)/2) \Gamma(\gamma/2) / \Gamma((\delta-2)/2).$$

(ii) On the other hand if $\gamma = \delta = 2$, then

$$(6.5) \quad \bar{I}(2, 2, 1, p) = K_1(\sqrt{p}) / [p^{3/2} K_0(\sqrt{p})]$$

(K_r is a modified Bessel function of the third kind.)

Proof. (i) Combine (6.1) and (6.3) and interchange the order of integration to obtain:

$$\begin{aligned} \bar{I}(\gamma, \delta, 1, p) &= \int_1^{\infty} x^{\gamma-1} \left\{ \int_0^{\infty} e^{-pt} u^{(\delta)}(a, x, t) dt \right\} dx \\ (6.6) \qquad &= \int_1^{\infty} x^{\gamma-r-1} K_r(x\sqrt{p}) dx / [p K_r(\sqrt{p})] \end{aligned}$$

by formula (A.6) (Appendix), where $r = (\delta-2)/2$.

According to a formula for Mellin transforms on p 331 of Erdelyi [7],

$$(6.7) \quad \int_0^{\infty} x^{\gamma-r-1} K_r(x\sqrt{p}) dx = p^{-(\gamma-r)/2} 2^{\gamma-r-2} \Gamma\left(\frac{\gamma-2r}{2}\right) \Gamma(\gamma/2) .$$

On the other hand, for $\delta > 2$ (i.e. $r > 0$), and for each $x > 0$,

$$x^r p^{r/2} K_r(x\sqrt{p}) \rightarrow 2^{r-1} \Gamma(r) \quad \text{as } p \downarrow 0 .$$

Lebesgue's bounded convergence theorem implies that, as $p \downarrow 0$,

$$(6.8) \quad p^{r/2} \int_0^1 x^{\gamma-r-1} K_r(x\sqrt{p}) dx \rightarrow 2^{r-1} \Gamma(r) \int_0^1 x^{\gamma-2t-1} dx = 2^{r-1} \Gamma(r) / (\gamma-2r)$$

since $\gamma-2r-1 = \gamma-(\delta-2)-1 > -1$ by assumption.

When we compare (6.7) and (6.8), as $p \downarrow 0$, we see that (6.8) is insignificant by comparison with (6.7), and therefore

$$(6.9) \quad \int_1^{\infty} x^{\gamma-r-1} K_r(x\sqrt{p}) dx \sim c_1(\gamma, \delta) p^{-(\gamma-r)/2} \quad \text{as } p \downarrow 0 ,$$

where $c_1(\gamma, \delta) = 2^{\gamma-r-2} \Gamma((\gamma-2r)/2) \Gamma(\gamma/2)$. Now for $r > 0$,

$$K_r(x) \sim 2^{r-1} \Gamma(r) x^{-r} \quad \text{as } x \downarrow 0 ,$$

and so (6.6) and (6.9) together show that

$$I(\gamma, \delta, 1, p) \sim p^{r-1-(\gamma/2)} 2^{\gamma-2r-1} \Gamma((\gamma-2r)/2) \Gamma(\gamma/2) / \Gamma(r) .$$

Since $r = (\delta-2)/2$, this gives (6.4).

(ii) When $\gamma = \delta = 2$, (6.6) becomes:

$$\int_1^{\infty} x K_0(x\sqrt{p}) dx / [p K_0(\sqrt{p})]$$

Change variables, using $x = (1+y^2)^{1/2}$. According to Erdelyi [7], p. 331,

$$\begin{aligned} \int_1^{\infty} x K_0(x\sqrt{p}) dx &= \int_0^{\infty} (1+y^2)^{1/2} K_0((1+y^2)^{1/2} \sqrt{p}) y (1+y^2)^{-1/2} dy \\ &= p^{-1/2} K_{-1}(p^{1/2}) = p^{-1/2} K_1(p^{1/2}) . \end{aligned}$$

Formula (6.5) follows.

COROLLARY 6.4

If $\delta > 2$ and $\gamma+2-\delta > 0$, then

$$(6.10) \quad c_2(\gamma, \delta) a^{\delta-2} t^{(\gamma+2-\delta)/2} \leq I(\gamma, \delta, a, t) \leq c_3(\gamma, \delta) a^{\delta-2} t^{(\gamma+2-\delta)/2}$$

for all sufficiently large t , where

$$\begin{aligned} (6.11) \quad c_2(\gamma, \delta) &= 2^{\gamma-\delta} \Gamma((\gamma+2-\delta) \Gamma(\gamma/2) / [\Gamma((\delta-2)/2) \Gamma((\gamma+6-\delta)/2)] \\ c_3(\gamma, \delta) &= 2^{(\gamma+6-\delta)/2} c_2(\gamma, \delta) . \end{aligned}$$

Proof. Proposition 6.3(i) and Karamata's theorem (see Widder [15], p. 192) show that, as $t \rightarrow \infty$,

$$(6.12) \quad \int_0^t [\gamma, \delta, 1, s] ds \sim c(\gamma, \delta) t^{(\gamma+4-\delta)/2} / \Gamma((\gamma+6-\delta)/2) .$$

Denote the expression on the right side by $A(\gamma, \delta, t)$. Since the integrand on the left is increasing in s , we see that

$$t I(\gamma, \delta, 1, t) \geq 0.5 A(\gamma, \delta, t)$$

for all sufficiently large t . Similarly

$$t I(\gamma, \delta, 1, t) \leq \int_t^{2t} I(\gamma, \delta, 1, s) ds \leq 2A(\gamma, \delta, 2t)$$

for all sufficiently large t . Combining the last two inequalities,

$$0.5c_1(\gamma, \delta)t^{(\gamma+2-\delta)/2} \leq I(\gamma, \delta, 1, t) \leq 2c_1(\gamma, \delta)(2t)^{(\gamma+2-\delta)/2}$$

for all sufficiently large t , where $c_1(\gamma, \delta) = c(\gamma, \delta)/\Gamma((\gamma+6-\delta)/2)$.

Now apply Lemma 6.2 to obtain:

$$0.5c_1(\gamma, \delta)a^{\delta-2}t^{(\gamma+2-\delta)/2} \leq I(\gamma, \delta, a, t) \leq 2c_1(\gamma, \delta)a^{\delta-2}(2t)^{(\gamma+2-\delta)/2}$$

from which the result follows.

COROLLARY 6.5

Taking $\gamma = \delta = 2$, there exists a constant $c_0 > 0$, independent of a , such that

$$(6.13) \quad I(2, 2, a, t) \sim c_0 t / \log t \quad \text{as } t \rightarrow \infty.$$

Proof. The $t/\log t$ behavior is well-known. To check that c_0 is independent of a , observe that, by Lemma 6.2,

$$\begin{aligned} I(2, 2, a, t)(\log t)/t &= a^2(\log t)I(2, 2, 1, t/a^2)/t \\ &= I(2, 2, 1, s)(\log s)/s + I(2, 2, 1, s)(\log a^2)/s \end{aligned}$$

where $s = t/a^2$. Let $s \rightarrow \infty$ for fixed a ; the first term tends to a constant independent of a , while the second tends to zero.

PROPOSITION 6.6

Taking $\delta = \gamma$, we have

- (i) $\partial I(\gamma, \gamma, a, t)/\partial t = a^{\gamma-1}(-\partial u^{(\gamma)}/\partial x|_{x=a})$ for all $\gamma \geq 1$;
- (ii) $a^{1-\gamma}I(\gamma, \gamma, a, t)$ is an increasing function of γ .

Proof (i) Differentiating (6.1) through the integral sign, and appealing to Lemma A.1, we obtain:

$$(6.14) \quad \partial I(\gamma, \gamma, a, t) / \partial t = \int_0^{\infty} x^{\gamma-1} [u_{xx} + (\gamma-1)u_x/x] dx$$

where $u = u^{(\gamma)}(a, x, t)$. Integrating by parts, this is simply

$$(6.15) \quad = [x^{\gamma-1} u_x]_a^{\infty}$$

since the other two integrals cancel. Lemma A.4 shows that $x^{\gamma-1} u_x$ goes to zero as $x \rightarrow \infty$, so (i) follows.

(ii) To prove assertion (ii), it suffices to prove that

$$(6.16) \quad \partial / \partial \gamma (a^{1-\gamma} I(\gamma, \gamma, a, t)) \geq 0.$$

Now

$$(6.17) \quad -u_x^{(\gamma)}(a, a, s) = [1 - Q_{a+h}^{(\gamma)}(\tau_a \leq s)]/h + o(h).$$

Using the result of part (i), the left side of (6.16) is

$$\begin{aligned} & \partial / \partial \gamma \int_0^t (-u_x^{(\gamma)}(a, a, s)) ds \\ &= \int_0^t \{ [u_x^{(\gamma)}(a, a, s) - u_x^{(\gamma+\varepsilon)}(a, a, s)] / \varepsilon + o(\varepsilon) \} ds \\ (6.18) &= \int_0^t \{ [Q_{a+h}^{(\gamma)}(\tau_a \leq s) - Q_{a+h}^{(\gamma+\varepsilon)}(\tau_a \leq s)] / (\varepsilon h) + \varepsilon^{-1} o(h) + o(\varepsilon) \} ds. \end{aligned}$$

However Lemma A.3 (ii) shows that for $\varepsilon > 0$,

$$Q_{a+h}^{(\gamma)}(\tau_a \leq s) = u^{(\gamma)}(a, a+h, s) > u^{(\gamma+\varepsilon)}(a, a+h, s) = Q_{a+h}^{(\gamma+\varepsilon)}(\tau_a \leq s).$$

Hence the expression (6.18) is greater than or equal to

$t(\varepsilon^{-1} o(h) + o(\varepsilon))$. Let $h \downarrow 0$, and then let $\varepsilon \downarrow 0$, to obtain (6.16).

PROPOSITION 6.7

Suppose $\gamma = 2$ and $\delta \geq 1$. Then

$$(6.19) \quad I(2, \delta, a, t) = a^{2-\delta} I(\delta, \delta, a, t) - (\delta-2)t$$

Proof. Differentiating through the integral sign in (6.1), and proceeding as in (6.14), we obtain

$$(6.20) \quad \partial I(2, \delta, a, t) / \partial t = \int_a^\infty x [u_{xx}^{(\delta)} + (\delta-1)u_x^{(\delta)} / x] dx$$

where $u^{(\delta)} = u^{(\delta)}(a, x, t)$. Integrating by parts, we obtain:

$$= [xu_x^{(\delta)}]_a^\infty + (\delta-2) \int_a^\infty u_x^{(\delta)} dx.$$

If $1 \leq \delta < 2$, then Proposition A.5 shows $xu_x^{(\delta)} \rightarrow 0$ as $x \rightarrow \infty$. If $\delta \geq 2$, the same conclusion follows from Lemma A.4. Hence (6.20) equals:

$$= a(-u_x^{(\delta)}(a, a, t)) - (\delta-2)$$

since $u^{(\delta)}(a, a, t) = 1$, and $u^{(\delta)}(a, x, t) \downarrow 0$ as $x \rightarrow \infty$. Combining this with Proposition 6.6(i), we see that

$$\partial I(2, \delta, a, t) / \partial t = a^{2-\delta} \partial I(\delta, \delta, a, t) / \partial t - (\delta-2).$$

Integration yields the result.

COROLLARY 6.8

If $1 \leq \delta < 2 < \beta$, then

$$(6.21) \quad I(2, 2, a, t) + (2-\delta)t \geq I(2, \delta, a, t) \geq I(2, \beta, a, t) \\ \geq I(2, 2, a, t) - (\beta-2)t.$$

Proof. Proposition (6.6) (ii) implies

$$a^{2-\delta} I(\delta, \delta, a, t) \leq I(2, 2, a, t) \leq a^{2-\beta} I(\beta, \beta, a, t).$$

The first and third inequalities in (6.21) follow from this and Proposition 6.7. The middle inequality follows from Lemma A.3(ii).

7. Upper bounds for the expected mass of the coalescent set

Our goal in this section is to obtain upper bounds for

$$(7.1) \quad \Psi_d(t) = \int_0^\infty x^{d-1} P_x(T_0 \leq t) dx$$

for $d = 2$ and 3 (This quantity also depends on the covariance b of the stochastic flow.)

PROPOSITION 7.1

Suppose $a = a(t)$ is a positive, increasing function of t such that $a(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then in each dimension d ($d=2$ or 3), there exists a function $v(t)$, taking values in $[1, d]$, such that

$$(7.2) \quad d-v(t) \text{ is } O(a(t)^{-3/2}) \text{ if } d = 2 \text{ and } O(a(t)^{-1}) \text{ if } d = 3$$

and

$$(7.3) \quad \Psi_2(t) \leq a(t)^2/2 + I(2, 2, a(t), 2t) + (2-v(t))2t$$

$$(7.4) \quad \Psi_3(t) \leq a(t)^3/3 + ca(t)^{v(t)-2}(2t)^{(5-v(t))/2}, \text{ for some } c > 0.$$

Proof. For both $d = 2$ and $d = 3$,

$$\begin{aligned} & \int_0^\infty x^{d-1} P_x(\tau_0 \leq t) dx \\ & \leq \int_0^{a(t)} x^{d-1} P_x(\tau_0 \leq t) dx + \int_{a(t)}^\infty x^{d-1} P_x(\tau_{a(t)} \leq t) dx \\ & \leq a(t)^d/d + \int_{a(t)}^\infty x^{d-1} u^{(v(t))}(a(t), x, 2t) dx \end{aligned}$$

by Corollary 5.3, where $v(t)$ is a function as described in

(5.13)-(5.15). By (6.1), this says

$$(7.5) \quad \Psi_d(t) \leq a(t)^d/d + I(d, v(t), a(t), 2t)$$

The case $d = 2$. Corollary 6.8 gives

$$\psi_2(t) \leq a(t)^2/2 + I(2,2,a(t),2t) + (2-\gamma(t))2t ,$$

as desired.

The Case $d = 3$. Corollary 6.4 and (7.5) give (7.4), since the constant $c_3(d,\gamma(t))$ in (6.10) is uniformly bounded as $t \rightarrow \infty$, when $d = 3$.

COROLLARY 7.2

There exists a constant $c_1 > 0$ (depending on the covariance b) such that

$$(7.6) \quad \psi_3(t) \leq c_1 t \log t .$$

Proof. Choose $a(t) = \log t$ for large t , in Proposition 7.1. Then (7.4) gives:

$$(7.7) \quad \psi_3(t) \leq (\log t)^3/3 + c(\log t)^{1-(3-\gamma(t))}(2t)^{1+(3-\gamma(t))/2} .$$

Now $3-\gamma(t)$ is $O((\log t)^{-1})$ by (7.2), and for all $\alpha > 0$,

$$(7.8) \quad t^{\alpha(\log t)^{-1}} = \exp[\alpha(\log t)^{-1} \log t] = e^{\alpha} .$$

It follows that for some $c_0 > 0$, depending on the covariance b ,

$$(7.9) \quad (2t)^{1+(3-\gamma(t))/2} \leq c_0 t .$$

Since $3-\gamma(t) \geq 0$, (7.7) and (7.9) now give (7.6).

COROLLARY 7.3

There exists a constant $c_2 > 0$ (not depending on the covariance b) such that

$$(7.10) \quad \psi_2(t) \leq c_2 t / \log t .$$

Proof. Choose $\delta > 0$, and let $a(t) = (\log t)^\delta$, for large t , in (7.3). According to Lemma 6.2,

$$I(2,2,(\log t)^\delta, 2t) = (\log t)^{2\delta} I(2,2,1, 2t/(\log t)^{2\delta}) .$$

By Corollary 6.5, there exists $c_0 > 0$ such that

$$I(2,2,1,u) \leq 2c_0 u / \log u, \text{ all sufficiently large } u .$$

Taking $u = 2t(\log t)^{-2\delta}$, we obtain, for t sufficiently large,

$$\begin{aligned} I(2,2,(\log t)^\delta, 2t) &\leq (\log t)^{2\delta} 4c_0 t (\log t)^{-2\delta} / \log(2t(\log t)^{-2\delta}) \\ (7.11) \qquad \qquad \qquad &\leq 5c_0 t / \log(t) . \end{aligned}$$

Moreover as $t \rightarrow \infty$, (7.2) gives

$$(2-\nu(t))2t = 2t O((\log t)^{-3\delta/2}) .$$

Therefore if we choose $\delta > 2/3$, we see from (7.3) that

$$\psi_2(t) \leq 6c_0 t / \log t$$

for all sufficiently large t .

8. Lower bounds for the expected mass

To obtain a lower bound for $\Psi_d(t)$, the key idea is to obtain a lower bound for $P_x(\tau_0 \leq t)$, using the results of Section 5.

LEMMA 8.1

Given $0 < \varepsilon \leq 1/3$, choose $c > 0$ such that

$\max(|1-V_L(x)|, |1-V_N(x)|) \leq \varepsilon$ for $x \geq c$. Suppose $a(t)$ is an increasing function of t , such that $a(t) \geq c$ and $a(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Then there exists a function $\beta(t)$ taking values in $[d, \infty)$ such that

$$(8.1) \quad \beta(t) - d \text{ is } O(a(t)^{-3/2}) \text{ when } d=2, \text{ or } O(a(t)^{-1}) \text{ when } d=3,$$

and

$$(8.2) \quad P_x(\tau_0 \leq 3t) \geq u^{(\beta(t))}(a(t), x, t/2) u^{(\lambda)}(c, a(t), 2t/3) P_c(\tau_0 \leq t)$$

for all $x \geq a(t)$, where $\lambda = (d-1)(1+\varepsilon)/(1-\varepsilon)$.

Proof. For arbitrary $x > y$, let $f_{x,y}$ denote the probability density function of the passage time to y , for the process $(p_t, t \geq 0)$ started at x ; thus

$$(8.3) \quad P_x(\tau_y \leq t) = \int_0^t f_{x,y}(s) ds.$$

By the strong Markov property, the first passage time to 0 , starting from x , may be viewed as the sum of two independent random variables, namely the first passage time to y from x , plus the first passage time to 0 from y , where y is an arbitrary point in $(0, x)$.

The usual convolution formula gives:

$$(8.4) \quad P_x(\tau_0 \leq 3t) = \int_0^{3t} P_x(\tau_y \leq s) f_{y,0}(3t-s) ds.$$

Since $P_x(\tau_y \leq s)$ is increasing in s , we obtain from (8.4) the inequality

$$(8.5) \quad P_x(\tau_0 \leq 3t) \geq \int_t^{3t} P_x(\tau_y \leq s) f_{y,0}(3t-s) ds \geq P_x(\tau_y \leq t) P_y(\tau_0 \leq 2t).$$

Repeat the same construction, replacing (x, y) by (y, c) , and $3t$ by $2t$, to obtain

$$(8.6) \quad P_x(\tau_0 \leq 3t) \geq P_x(\tau_y \leq t) P_y(\tau_c \leq t) P_c(\tau_0 \leq t).$$

Hence when $y = a(t)$ and $x \geq a(t)$, (5.11) and (5.16) give

$$P_x(\tau_0 \leq 3t) \geq u^{(\beta(t))}(u(t), x, t/2) u^{(\lambda)}(c, a(t), t(1-\varepsilon)) P_c(\tau_0 \leq t).$$

The assertion about $\beta(t)$ follows from Corollary 5.3.

PROPOSITION 8.2

Let $\varepsilon = 1/3$, and take c and $a(t)$ as in Lemma 8.1. Then

$$(8.7) \quad \Psi_2(3t) \geq P_c(\tau_0 \leq t) u^{(3)}(c, a(t), 2t/3) [I(2, 2, a(t), t/2) - (\beta(t) - 2)t/2]$$

$$(8.8) \quad \Psi_3(3t) \geq P_c(\tau_0 \leq t) u^{(5)}(c, a(t), 2t/3) I(3, \beta(t), a(t), t/2)$$

where $\beta(t)$ satisfies (8.1).

Proof. By (7.1) and (8.2)

$$\begin{aligned} \psi_d(3t) &\geq \int_{a(t)}^{\infty} x^{d-1} P_x(\tau_0 \leq 3t) dx \\ &\geq P_c(\tau_0 \leq t) u^{(\lambda)}(c, a(t), 2t/3) \int_{a(t)}^{\infty} x^{d-1} u^{(\beta(t))}(a(t), x, t/2) dx \\ (8.9) \quad &= P_c(\tau_0 \leq t) u^{(\lambda)}(c, a(t), 2t/3) I(d, \beta(t), a(t), t/2) \end{aligned}$$

in the notation of (6.1), where $\beta(t)$ satisfies (8.1), and

$\lambda = 1 + (d-1)(1+\varepsilon)/(1-\varepsilon) = 2d-1$. In the case $d = 2$, Corollary 6.8

and (8.9) give (8.7).

In the case $d = 3$, (8.8) follows immediately from (8.9).

COROLLARY 8.3

Given $h > 0$, there exist $c_3 > 0$ (depending on h and on the covariance b), and $c_4 > 0$ (depending on b), such that for t sufficiently large,

$$(8.10) \quad \Psi_2(t) \geq c_3 t (\log t)^{-5/3 - h},$$

$$(8.11) \quad \Psi_3(t) \geq c_4 t (\log t)^{-2-h}.$$

Proof. (i) The Case $d = 2$. According to (A.8) and (A.9),

$$\begin{aligned} u^{(3)}(c, a, s) &= (c/a)(ns)^{-1/2} \int_{a-c}^{\infty} e^{-u^2/4s} du \\ &= (c/a)(2/n)^{1/2} \int_{(a-c)(2s)^{-1/2}}^{\infty} e^{-v^2/2} dv. \end{aligned}$$

Assuming that $a(t)/\sqrt{t} \rightarrow 0$ as $t \rightarrow \infty$, it follows that there exists $c_5 > 0$ such that

$$(8.12) \quad u^{(3)}(c, a(t), 2t/3) \geq c_5/a(t).$$

Choose $\delta > 0$, and let $a(t) = (\log t)^\delta$ for large t . For $t > 0$, $P_c(\tau_0 \leq t)$ is strictly positive, and increasing in t . Thus (8.7), (8.12), (8.1) and (6.2) show that for sufficiently large t ,

$$\begin{aligned} \Psi_2(3t) &\geq c_6 a(t)^{-1} [a(t)^2 I(2, 2, 1, t/2a(t)^2) - O(t a(t)^{-3/2})] \\ &\geq c_7 a(t)^{-1} t / \log t - O(t a(t)^{-5/2}) \quad \text{by (6.13)} \\ &= c_7 t (\log t)^{-1-\delta} - O(t (\log t)^{-5\delta/2}). \end{aligned}$$

For $\delta > 2/3$, the second term becomes insignificant compared to the first as $t \rightarrow \infty$. Now (8.10) follows.

(ii) The Case $d = 3$.

According to (A.8) and (A.10);

$$u^{(5)}(c, a, s) \geq (c/a)^3 u^{(1)}(c, a, s).$$

By the same reasoning as before, if we assume that $a(t)/\sqrt{t} \rightarrow 0$ as $t \rightarrow \infty$, then there exists $c_8 > 0$ such that

$$(8.13) \quad u^{(5)}(c, a(t), 2t/3) \geq c_8 a(t)^{-3}.$$

Since $P_c(\tau_0 \leq t)$ is non-zero and increasing in t , (8.8), (8.13) and Corollary 6.4 give:

$$(8.14) \quad \psi_3(3t) \geq c_9 a(t)^{-3} a(t)^{\beta(t)-2} t^{1-(\beta(t)-3)/2}$$

using the fact that the expression $c_2(3, \beta(t))$ appearing in (6.10) is bounded below by a positive constant as $t \rightarrow \infty$. Take $a(t) = \log t$ for large t , so that $\beta(t)-3$ is $O((\log t)^{-1})$ by (8.1). Hence for some $\alpha > 0$,

$$t^{-(\beta(t)-3)/2} \geq t^{-\alpha(\log t)^{-1}} = e^{-\alpha} > 0.$$

Hence (8.14) gives

$$\psi_3(3t) \geq c_{10} (\log t)^{\beta(t)-5} t \geq c_{11} (\log t)^{-2} t$$

since $\beta(t) \geq 3$. The result follows.

Appendix A

Passage times for Bessel Processes

Consider, for any real $\delta \geq 1$, the diffusion process on $[0, \infty)$, with absorption at 0, with the generator

$$(A.1) \quad G^{(\delta)}g(x) = g''(x) + [(\delta-1)/x]g'(x), \quad g \in C_0^\infty((0, \infty)).$$

If δ is an integer, this process has the same law as the distance between two independent Brownian motions in dimension δ ; it is also $\sqrt{2}$ times the usual Bessel (δ) process, as discussed in Pitman and Yor [12].

If $0 < a \leq x < \infty$, and if the process $(\xi_t, t \geq 0)$ above is started at x , define a stopping-time τ_a as follows:

$$(A.2) \quad \tau_a = \inf\{t: \xi_t = a\}.$$

If Q_x^δ denotes the law of the process, started at x , define

$$(A.3) \quad u^{(\delta)}(a, x, t) = u(x, t) = P_x(\tau_a \leq t), \quad x \leq a, \quad t \geq 0.$$

The following Lemma is well-known (see Ito and McKean [10], p. 154).

LEMMA A.1

For fixed $a > 0$ and $\delta \geq 1$, the function $u(x, t)$ is the unique solution to the parabolic equation $\partial u / \partial t = G^{(\delta)}u$ on the domain $\{(x, t): x \geq a, t \geq 0\}$, with boundary conditions

$$(A.4) \quad u(a, t) = 1 \quad \text{for } t \geq 0; \quad u(x, 0) = 0 \quad \text{for } x > a.$$

Proof. Fix $t > 0$, and evaluate $u(\xi_t, T-t)$ by Ito's formula. On taking expectations, we observe that if u solves the parabolic equation, then $u(x, t) = P_x(\tau_a \leq t)$.

We use the functions $u^{(\delta)}(a, x, t)$, for various δ , for comparison purposes in Section 5. When δ is an odd integer, it is possible to express the function in a relatively simple form. The following formulas are well-known.

PROPOSITION A.2

Let $\bar{u}(x, p)$ denote the Laplace transform of $u(x, t) = u^{(\delta)}(a, x, t)$, i.e.

$$(A.5) \quad u(x, p) = \int_0^{\infty} e^{-pt} u(x, t) dt .$$

Then

$$(A.6) \quad \bar{u}(x, p) = (a/x)^r K_r(xp^{1/2}) [pK_r(ap^{1/2})]^{-1}, \quad r = (\delta-2)/2$$

where K_r is a modified Bessel function of the third kind (See Lebedev (1972), p. 109). For $\delta = 1$, $\delta = 3$, and $\delta = 5$ we may invert the Laplace transform to obtain explicit formulas:

$$(A.7) \quad u^{(1)}(a, x, t) = \int_0^t (1/2)(x-a)(\pi s^3)^{-1/2} \exp(-(x-a)^2/4s) ds$$

$$(A.8) \quad = (\pi t)^{-1/2} \int_{x-a}^{\infty} e^{-s^2/4t} ds . \quad (\text{"Reflection principle"})$$

$$(A.9) \quad u^{(3)}(a, x, t) = (a/x) u^{(1)}(a, x, t)$$

$$(A.10) \quad u^{(5)}(a, x, t) = (a/x)^3 u^{(1)}(a, x, t) + [a^2(x-a)/x^3] \int_{x-a}^{\infty} \exp\{-s^2/4t - (s-x+a)/a\} ds$$

Note: The $(a/x)^3$ term in (A.10) is not a misprint for $(a/x)^2$.

Proof. The derivation of (A.6) is an elementary exercise in Laplace transform techniques, using the boundary conditions (A.4). The relevant inversion formulas are found in Erdelyi [7], pp 245-247. The relevant

Bessel function formulas are

$$K_{1/2}(y) = [\pi/2y]^{1/2} e^{-y}, \quad K_{3/2}(y) = (1+y^{-1})K_{1/2}(y).$$

LEMMA A.3

Let $u = u^{(\delta)}(a, x, t)$; then for each $x \geq a$ and $t > 0$,

- (i) $\partial u / \partial t$, $(-\partial u / \partial x)$, and $\partial^2 u / \partial x^2$ are all positive.
- (ii) $u = u^{(\delta)}$ is a decreasing function of δ .

Proof. (i) the first two functions are positive because $Q_x^\delta(\tau_a \leq t)$ is increasing in t and decreasing in x . By Lemma A.1,

$u_{xx} = u_t + ((\delta-1)/x)(-u_x)$, which is positive by the result of the previous sentence.

(ii) Solve the stochastic differential equations for the processes with generators $G^{(\delta)}$, for all $\delta > 1$, using the same driving one-dimensional Brownian motion. Now the pathwise comparison theorem of Ikeda and Watanabe [9] shows that for $1 \leq \gamma < \beta$,

$$Q_x^\gamma(\tau_a \leq t) \geq Q_x^\beta(\tau_a \leq t)$$

which gives the result.

LEMMA A.4

For all $\delta \geq 1$, and for fixed $a > 0$ and $t > 0$,

$$\lim_{x \rightarrow \infty} x^{\delta-1} (-u_x^{(\delta)}(a, x, t)) = 0.$$

Proof. Step I. Take a Laplace transform:

$$(A.11) \int_0^\infty e^{-pt} \left(\int_a^\infty x^{\delta-1} (-u_x^{(\delta)}) dx \right) dt = - \int_a^\infty x^{\delta-1} \partial / \partial x \left(\int_0^\infty e^{-pt} u^{(\delta)}(a, x, t) dt \right) dx$$

$$= - \int_a^{\infty} x^{\delta-1} \partial/\partial x \{ (a/x)^r K_r(xp^{1/2}) [pK_r(ap^{1/2})]^{-1} \} dx$$

by (A.6), where $r = (\delta-2)/2$. For all r , $K_r(y)$ behaves like $[n/2y]^{1/2} e^{-y}$ as $y \rightarrow \infty$, and therefore the above integral is finite. It follows from (A.11) that

$$(A.12) \quad \int_a^{\infty} x^{\delta-1} (-u_x^{(\delta)}) dx < \infty$$

where, of course, the integrand is positive.

We shall now show that $x^{\delta-1} (-u_x^{(\delta)}(a, x, t))$ is a decreasing function of x , for $x \geq a$.

Step II

$$(A.13) \quad \partial/\partial x \{ x^{\delta-1} (-u_x^{(\delta)}) \} = -x^{\delta-1} [u_{xx}^{(\delta)} + ((\delta-1)/x) u_x^{(\delta)}] = -x^{\delta-1} u_t^{(\delta)}$$

which is negative, since $u^{(\delta)}(u, x, t)$ is an increasing function of t . Since $u_x^{(\delta)} < 0$, it follows that $x^{\delta-1} (-u_x^{(\delta)})$ is a monotone decreasing, non-negative function of x ; therefore $\lim_{x \rightarrow \infty} x^{\delta-1} (-u_x^{(\delta)}(a, x, t))$ exists, and equals $\ell(\delta, a, t) \geq 0$, say. Now (A.12) proves that $\ell(\delta, a, t) = 0$, which completes the proof.

PROPOSITION A.5

For $1 \leq \delta \leq 3$, $0 < -u_x^{(\delta)}(a, x, t) \leq e^{-(x-a)^2/4t} \left[\frac{8t}{a(x-a)} + t^{-1/2} \right] / \sqrt{\pi}$.

Proof. Fix δ and a , and let $v(x, t) = -u_x^{(\delta)}(a, x, t) > 0$. By differentiating the equation $u_t = u_{xx} + [(\delta-1)/x] u_x$ with respect to x , we see that

$$v_t = -u_{xt} = -u_{tx} = -u_{xxx} - [(\delta-1)/x]u_{xx} + (\delta-1)x^{-2}u_x ,$$

and therefore

$$(A.14) \quad v_t < v_{xx}$$

since $u_x < 0$ and $u_{xx} > 0$. As regards boundary conditions,

$$(A.15) \quad v(x, 0) = -u_x^{(\delta)}(a, x, 0) = 0, \text{ for } x \geq a, \text{ and} \\ v(a, t) = -u_x^{(\delta)}(a, a, t) = \lim_{h \downarrow 0} [1 - u^{(\delta)}(a, a+h, t)]/h, \text{ for } t > 0.$$

Since $1 \leq \delta \leq 3$, Lemma A.3(ii) and equation (A.9) imply

$$(A.16) \quad v(a, t) \leq \lim_{h \downarrow 0} [1 - u^{(3)}(a, a+h, t)]/h = -u_x^{(3)}(a, a, t) \\ = [u^{(3)}(a, x, t)/x + (a/x)(\pi t)^{-1/2} e^{-(x-a)^2/4t}]|_{x=a} \\ = (1/a) + (\pi t)^{-1/2}.$$

The maximum principle, together with (A.14), (A.15) and (A.16), implies that $v(x, t) \leq w(x, t)$, where $w(x, t)$ is the solution, for $x > a$ and $t \geq 0$, to the parabolic equation

$$(A.17) \quad w_t = w_{xx}, \\ w(x, 0) = 0 \text{ for } x \geq a; w(a, t) = (1/a) + (\pi t)^{-1/2}, t > 0.$$

Take the Laplace transform:

$$\bar{w}(x, p) = \int_0^\infty e^{-pt} w(x, t) dt,$$

which satisfies

$$\bar{w}(a, p) = (ap)^{-1} + p^{-1/2}.$$

Then $\bar{w}_{xx} + [(\delta-1)/x]\bar{w}_x - p\bar{w} = 0$, $x > a$, and one may easily verify that

$$\bar{w}(x, p) = \left(\frac{x}{a}\right)^{1/2} \left(\frac{1}{ap} + \frac{1}{\sqrt{p}}\right) K_{1/2}(xp^{1/2}) / K_{1/2}(ap^{1/2}) \\ = \left(\frac{1}{ap} + \frac{1}{\sqrt{p}}\right) e^{-(x-a)p^{1/2}}.$$

Inverting the Laplace transform using Erdelyi [7], p. 245,

$$w(x, t) = \frac{2}{a\sqrt{\pi}} \int_{x-a}^{\infty} e^{-u^2/4t} du + (\pi t)^{-1/2} \exp(-(x-a)^2/4t) .$$

Now

$$\int_{x-a}^{\infty} e^{-u^2/4t} du \leq \int_{x-a}^{\infty} e^{-u(x-a)/4t} du = \left(\frac{4t}{x-a} \right) e^{-(x-a)^2/4t} ,$$

and this gives the result.

Appendix B

Construction of an isotropic covariance of coalescing type in dimension 3

The purpose of this section is to show that for suitable choices of the measures M_1 and M_2 in (1.11), it is possible for all of the conditions (1.12), (1.14) and (1.17) to be satisfied, when the dimension $d = 3$. A similar class of examples was presented for the case $d = 2$, in Section 16 of Darling [5], following unpublished work of Harris. In the $d = 2$ case, the measure M_2 was chosen to be zero, while $M_1(d\lambda)$ was taken to be $2(\delta-1)\lambda^{-\delta}1_{[1,\infty)}(\lambda)d\lambda$ for some constant δ , $1 < \delta < 3$; it was shown that (1.14), (1.17), and (5.4) to (5.7) all hold in this case, and moreover the function S appearing in (1.14) satisfies $S(\infty) = \infty$, which implies $P(v(x,y) < \infty) = 1$ in (1.18).

The construction when $d = 3$ is similar. Take M_2 to be zero and $M_1(d\lambda)$ to be $3(\delta-1)\lambda^{-\delta}1_{[1,\infty)}(\lambda)d\lambda$ where δ is a constant, and $2 < \delta < 3$. (Note the difference to the $d = 2$ case). The precise form of the functions $F^{(i)}$ appearing in (1.11) is given in Yaglom [16], equation (4.37), from which we obtain:

$$(B.1) \quad B_L(r) = (\pi/2)^{1/2} 3(\delta-1) \int_1^{\infty} \{ (r\lambda)^{-3/2} J_{3/2}(r\lambda) - (r\lambda)^{-1/2} J_{5/2}(r\lambda) \} \lambda^{-\delta} d\lambda,$$

$$(B.2) \quad B_N(r) = (\pi/2)^{1/2} 3(\delta-1) \int_1^{\infty} (r\lambda)^{-3/2} J_{3/2}(r\lambda) \lambda^{-\delta} d\lambda,$$

where J_p denotes the Bessel function of order p . The integrals in (B.1) and (B.2) are finite, because $J_p(x) = O(x^{-1/2})$ for real x , as $x \rightarrow \infty$. As $r \downarrow 0$, $J_p(r\lambda)$ behaves like

$$(B.3) \quad (r\lambda)^p / (2^p r(1+p))$$

(Lebedev [11], p. 134), and one may thereby verify that

$$B_L(r) \rightarrow 1 \quad \text{and} \quad B_N(r) \rightarrow 1 \quad \text{as} \quad r \downarrow 0.$$

Now change variables to $u = r\lambda$. According to (1.9),

$$(B.4) \quad V_L(r) = 1 - B_L(r) \\ = (\delta-1)r^{\delta-1} \int_r^\infty u^{-\delta} \{1 - 3(\pi/2)^{1/2} [u^{-3/2} J_{3/2}(u) - u^{-1/2} J_{5/2}(u)]\} du$$

$$(B.5) \quad V_N(r) = 1 - B_N(r) = (\delta-1)r^{\delta-1} \int_r^\infty u^{-\delta} \{1 - 3(\pi/2)^{1/2} u^{-3/2} J_{3/2}(u)\} du.$$

Using the formulas on p. 111 of Lebedev [11], we find that

$$(B.6) \quad (\pi/2)^{1/2} u^{-3/2} J_{3/2}(u) = u^{-3} \sin u - u^{-2} \cos u,$$

$$(B.7) \quad (\pi/2)^{1/2} [u^{-3/2} J_{3/2}(u) - u^{-1/2} J_{5/2}(u)] \\ = u^{-1} \sin u + 2u^{-2} \cos u - 2u^{-3} \sin u.$$

Therefore it follows that

$$(B.8) \quad V_L(r) = C_L r^{\delta-1} - (\delta-1)r^{\delta-1} \int_0^r u^{-\delta} \{1 - 3u^{-1} \sin u - 6u^{-2} \cos u \\ + 6u^{-3} \sin u\} du,$$

$$(B.9) \quad V_N(r) = C_N r^{\delta-1} - (\delta-1)r^{\delta-1} \int_0^r u^{-\delta} \{1 + 3u^{-2} \cos u - 3u^{-3} \sin u\} du,$$

where

$$(B.10) \quad C_L = (\delta-1) \int_0^\infty u^{-\delta} \{1 - 3(\pi/2)^{1/2} [u^{-3/2} J_{3/2}(u) - u^{-1/2} J_{5/2}(u)]\} du,$$

$$(B.11) \quad C_N = (\delta-1) \int_0^\infty u^{-\delta} \{1 - 3(\pi/2)^{1/2} u^{-3/2} J_{3/2}(u)\} du.$$

Using the series expansions of sine and cosine near 0, one may

verify that $\{1 - 3u^{-1}\sin u - 6u^{-2}\cos u + 6u^{-3}\sin u\} = O(3u^2 + O(u^4))$,
and $\{1 + 3u^{-2}\cos u - 3u^{-3}\sin u\} = O(1u^2 + O(u^4))$. Since $2 < \delta < 3$, we
deduce from (B.8) and (B.9) that

$$(B.12) \quad v_L(r) = C_L r^{\delta-1} + O(r^2), \quad v_N(r) = C_N r^{\delta-1} + O(r^2), \quad \text{as } r \downarrow 0.$$

Notice that the series expansions also show that C_L and C_N are
finite. Moreover integration by parts in (B.11) yields:

$$(B.13) \quad C_N = \left[-u^{1-\delta} \{1 - 3(\pi/2)^{1/2} u^{-3/2} J_{3/2}(u)\} \right]_0^\infty + \\ 3(\pi/2)^{1/2} \int_0^\infty u^{1-\delta} d/du [-u^{-3/2} J_{3/2}(u)] du.$$

By previous remarks,

$$u^{1-\delta} \{1 - 3(\pi/2)^{1/2} u^{-3/2} J_{3/2}(u)\} = O(1u^{3-\delta} + O(u^{5-\delta})) \quad \text{as } u \downarrow 0,$$

and is $O(u^{1-\delta})$ as $u \rightarrow \infty$. Hence the first term on the right in (B.13)
vanishes. Using the fact that

$$(B.14) \quad d/du [-u^{-p} J_p(u)] = u^{-p} J_{p+1}(u),$$

we obtain

$$(B.15) \quad C_N = 3(\pi/2)^{1/2} \int_0^\infty u^{-\delta-1/2} J_{5/2}(u) du.$$

According to Watson [14], p. 391,

$$(B.16) \quad C_N = 3(\pi/2)^{1/2} 2^{-\delta-0.5} \Gamma((3-\delta)/2) / \Gamma(2+(\delta/2)) > 0.$$

From (B.10) and (B.11), we see that

$$C_L - C_N = (\delta-1) 3(\pi/2)^{1/2} \int_0^\infty u^{-\delta-1/2} J_{5/2}(u) du = (\delta-1) C_N.$$

Since $2 < \delta < 3$, we see that

$$(B.17) \quad 2/3 < 2C_N/C_L = (2/\delta) < 1.$$

After these preliminary calculations, we may now state:

PROPOSITION B.1

In dimension $d = 3$, if the longitudinal and transverse correlation functions, B_L and B_N , are given by (B.1) and (B.2), then the resulting isotropic covariance satisfies both (1.14) and the coalescence criterion (1.17). Moreover $B_L(r)$ and $B_N(r)$ are both $O(r^{-1})$ as $r \rightarrow \infty$, and the function S in (1.14) is bounded on $(0, \infty)$ (i.e. $S(\infty) < \infty$).

Proof. For S as in (1.14), equations (1.13) and (B.12) show that, as $t \downarrow 0$,

$$\begin{aligned} S'(t) &= \exp\left\{-2 \int_1^t [C_N r^{\delta-1} + O(r^2)] / [C_L r^{\delta} + O(r^3)] dr\right\} \\ (B.18) \quad &= \exp\left\{-(2C_N/C_L) \log t + O(t^{3-\delta})\right\}. \end{aligned}$$

It follows from (B.17) that

$$(B.19) \quad S'(t) = \varphi(t) t^{-2/\delta}, \quad 0 < t < 1$$

where $\varphi(t)$ is $O(1)$. There exists a unique solution to this differential equation on $(0, \infty)$ such that $S(0+) = 0$ and

$$(B.20) \quad a_1 t^{1-2/\delta} \leq S(t) \leq a_2 t^{1-2/\delta}, \quad 0 \leq t \leq 1$$

for some constants $0 < a_1 < a_2$; this is possible because $2/\delta < 1$.

Extending the definition of $S(t)$ to the domain $(0, \infty)$ is straightforward, using (1.13). Thus (1.14) is verified.

As for (1.17), we see from (B.12), and (B.18)-(B.20) that

$$\int_0^1 S(u) / [S'(u) v_L(u)] du \leq a_3 \int_0^1 [u / v_L(u)] du$$

$$\leq a_4 \int_0^1 u^{2-\delta} du < \infty,$$

for some $a_3 > 0$ and $a_4 > 0$. This verifies (1.17).

To study the rate of decay of $B_N(r)$ as $r \rightarrow \infty$, we integrate by parts in (B.5):

$$\begin{aligned} v_N(r) &= r^{\delta-1} [-u^{1-\delta} \{1-3(\pi/2)^{1/2} u^{-3/2} J_{3/2}(u)\}]_r^\infty \\ &+ r^{\delta-1} \int_r^\infty u^{1-\delta} 3(\pi/2)^{1/2} d/du [-u^{-3/2} J_{3/2}(u)] du. \end{aligned}$$

Now apply (B.14), and the fact that $J_p(x)$ is $O(x^{-1/2})$ as $x \rightarrow \infty$, for real x . Thus

$$\begin{aligned} v_N(r) &= 1 - 3(\pi/2)^{1/2} r^{-3/2} J_{3/2}(r) + 3(\pi/2)^{1/2} r^{\delta-1} \int_r^\infty u^{-\delta-1/2} J_{5/2}(u) du \\ (B.21) \quad &= 1 - O(r^{-2}) + O(r^{-1}) = 1 + O(r^{-1}). \end{aligned}$$

Likewise (B.4) and (B.5) show that

$$(B.22) \quad v_N(r) - v_L(r) = 3(\pi/2)^{1/2} (\delta-1) r^{\delta-1} \int_r^\infty u^{-\delta-1/2} J_{5/2}(u) du = O(r^{-1}).$$

Equations (B.21) and (B.22) verify that $B_L(r)$ and $B_N(r)$ are both $O(r^{-1})$ as $r \rightarrow \infty$.

Finally, as $t \rightarrow \infty$, the previous result shows that

$$\begin{aligned} S'(t) &= \exp\left\{-2 \int_1^t [1+O(r^{-1})]/[r+O(1)] dr\right\} \\ &= \exp\{-2 \log t + O(t^{-1})\} \\ &= \beta(t) t^{-2} \end{aligned}$$

where $\beta(t)$ is bounded above by a constant $A > 0$, for $1 \leq t < \infty$.

Hence

$$S(r) - S(1) = \int_1^r S'(t) dt \leq A[-t^{-1}]_1^r = A(1 - 1/r) .$$

Thus $S(\infty) \leq A < \infty$, as desired.

Remark. It is interesting to note that, in this example, the longitudinal and transverse correlation functions, B_L and B_N , when extended to become even functions on $(-\infty, \infty)$, are globally C^1 , but fail to be C^2 at 0 . Loosely speaking, this example (for δ nearly equal to 3) is 'near' the borderline between coalescing and homeomorphic behavior. Baxendale and Harris [3] obtained diffeomorphic behavior when B_L and B_N were C^3 at 0 .

REFERENCES

- [1] Arratia, R. "Coalescing Brownian motions on the line". Ph.D. Thesis, University of Wisconsin, Madison, 1979.
- [2] Arratia, R. "Coalescing Brownian motions, and the voter model on $\mathbb{Z}$." U.S.C. Preprint, 1985.
- [3] Baxendale, P. & Harris, T.E. "Isotropic stochastic flows" (submitted to Annals of Probability) 1985.
- [4] Cox, J.T., & Griffeath, D. "Diffusive clustering in the two dimensional voter model". Annals of Probability, 14, 347-370 (1986).
- [5] Darling, R.W.R. "Constructing non-homeomorphic stochastic flows", I.M.A. Preprint No. 205, University of Minnesota, 1985.
- [6] Darling, R.W.R. "Ergodicity of a measure-valued Markov chain induced by random transformations." I.M.A. Preprint No. 212, University of Minnesota 1986.
- [7] Erdelyi A. (editor). Tables of integral transforms^S, Volume 1. McGraw-Hill, New York, 1954.
- [8] Harris, T.E. "Coalescing and non-coalescing stochastic flows in $\mathbb{R}^1$ ". Stochastic Processes and their Applications. 18, 187-210 (1984).
- [9] Ikeda, N. & Watanabe, S. "A comparison theorem for solutions of stochastic differential equations and its applications", Osaka J. Math. 14, 619-633, (1977).
- [10] Ito, K. & McKean, H.P. "Diffusion processes and their sample paths". Springer-Verlag, Berlin, 2nd printing, 1974.
- [11] Lebedev, N.N. "Special functions and their applications", Dover, New York, 1972.
- [12] Pitman J. & Yor, M. "Bessel processes and infinitely divisible laws. In 'Stochastic Integrals' ed. D. Williams, Lecture Notes in Math 851, pp. 285-370, (1981).
- [13] Sudbury, A. "The size of the region occupied by one type

in an invasion process". J. Appl. Probability 13, 355-356 (1976).

[14] Watson, G.N. "A treatise on the theory of Bessel functions", Cambridge University Press, 1944.

[15] Widder, D.V. "The Laplace transform". Princeton University Press, 1941.

[16] Yaglom, A.M. "Some classes of random fields in n-dimensional space, related to stationary random processes". Theor. Probability Appl. 2, 273-320 (1957).