INFINITE TIME OPTIMAL CONTROL AND PERIODICITY

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Infinite Time Optimal Control and Periodicity

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Abstract:

For smooth nonlinear systems

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^r u_i(t) X_i(x(t))$$

the infinite time optimal control problems: Maximize

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(x(t), u(t)) dt \quad (\text{average yield criterion})$$

or

$$\lim_{T \rightarrow \infty} \int_0^T e^{-\delta t} g(x(t), u(t)) dt \quad (\text{discounted criterion})$$

are considered, where the initial value $x(0)$ may be free or restricted. We study the existence of optimal periodic solutions for the above problems: If approximately optimal solutions have a limit point in the interior of some control set, then there exist approximately optimal periodic solutions. This result is applied to the growth of linear control semigroups and to a three dimensional predator-prey harvesting model.

Running title: Periodic Optimal Control

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1. Introduction

Consider a control system described by

$$(1.1) \quad \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^r u_i(t) X_i(x(t)), \quad t \geq 0,$$

where $u(t) := (u_i(t)) \in U \subset \mathbb{R}^r$ and $x(t) \in \mathbb{R}^d$. We are interested in optimal control problems where either an average criterion

$$(1.2) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(x(t), u(t)) dt$$

or a discounted criterion

$$(1.3) \quad \lim_{T \rightarrow \infty} \int_0^T e^{-\delta t} g(x(t), u(t)) dt$$

have to be maximized, the initial value $x(0) = x_0$ may be free or restricted.

Results for the problem on $\mathbb{R}_+^0 = [0, \infty)$ are rather scarce (cf. Rockafellar [10] or Aubin and Clarke [4]). Hence a common practice in applications has been to restrict attention to constant controls u yielding a steady state x_u of (1.1) -- then, in a second step, one tries to reach the optimal equilibrium. A more sophisticated approach allows for trajectories and controls of some common period $\tau > 0$. Up to a certain degree, this takes into account the dynamics of the system and often leads to substantial improvements, notably in chemical and aerospace engineering. The relations between the steady state and the periodic optimization problems (for criterion (1.2)) have been studied to some extent, cf. Colonius [7] for a recent presentation.

In this paper we address the problem from the following point of view: under which conditions does the general problem on $\mathbb{R}_+^0$ have a periodic solution? As simple counterexamples show, this reduction is not always

possible. Even for bounded trajectories, and under periodic forcing, the solutions of (1.1) may show a quite complicated behavior. Note that methods of Fourier expansion are not helpful, since (1.1) is nonlinear, the control values are restricted and an approximation on the unbounded time interval $[0, \infty)$ is needed. Instead, finite time controllability properties and control sets (cf. Kliemann [8]) are crucial.

In Section 2, we recall the notion and some properties of control sets, and define limit sets for trajectories of (1.1) corresponding to a control $u(t)$. We show that every limit set has nonvoid intersection with some control set. Section 3 presents the main result of this paper: Periodic solutions are approximately optimal for the infinite time problem, if approximately optimal solutions have a limit point in the interior of some control set. In Sections 4 and 5 this result is applied to the growth of linear control semigroups, motivated by a stability problem from linear, parameterexcited stochastic systems, and to a three dimensional predator-prey harvesting model.

2. Control Sets

In this paper we consider nonlinear control systems of the form

$$(2.1) \quad \dot{x} = f(x, u)$$

on a paracompact, connected finite dimensional C^∞ manifold M (of dimension m), where we assume that for each $u \in U$ the vector fields $f(\cdot, u)$ are globally Lipschitz continuous, and U is a subset of $\mathbb{R}^r$. The set of admissible controls is $\tilde{U} = \{u : \mathbb{R}_+ \rightarrow U \text{ measurable}\}$.

For the system (2.1) there is a control theoretic decomposition of M as follows: Let $\phi(t, x, u(t))$ denote the solution of (2.1) at time t with initial value x under the control action $u \in \tilde{U}$, and define the positive orbit of x at time t as $\tilde{O}^+(x, t) = \{y \in M, \text{ there exists } u \in \tilde{U} \text{ such that } y = \phi(t, x, u)\}$, $\tilde{O}_{\leq T}^+(x) = \bigcup_{0 \leq t \leq T} \tilde{O}^+(x, t)$ and $\tilde{O}^+(x) = \tilde{O}_{< \infty}^+(x)$. For the negative orbit we use the superscript " $-$ ". A set $D \subset M$ is called a controllability set of (2.1), if $\overline{D} \subset \tilde{O}^+(x)$ for all $x \in D$. For each controllability set D there is a unique maximal (with respect to set inclusion) controllability set $\hat{D} \supset D$, and these maximal sets are called control sets for (2.1). (As each set $\{x\} \subset M$ is always a controllability set by the above definition, we include as control sets only those one point sets $\{x\}$ that are rest points for (2.1), i.e. $f(x, u) = 0$ for some $u \in U$.) A control set D is called invariant, if $\overline{D} \supset \tilde{O}^+(x)$ for all $x \in D$, all other control sets are variant. Note that because of maximality control sets are either disjoint or identical (and always path connected).

In this set up M can be uniquely decomposed as

$$(2.2) \quad M = A \cup B \cup C$$

where A is the set of points outside control sets,

B contains the points in some variant control set and

C those points that are elements of some invariant control set,

see Arnold and Kliemann [1] for further details.

From now on we specialize to systems that are linear in the control parameter

$$(2.3) \quad \dot{x} = X_0(x) + \sum_{i=1}^r u_i X_i(x),$$

where we assume that all solutions exist for all times $t \in \mathbb{R}_+$. Note that for these systems the closures of the (positive or negative) orbits at time t coincide for measurable and for piecewise constant controls, also for $U \subset \mathbb{R}^r$ and the convex hull $\text{co } U \subset \mathbb{R}^r$. Hence the (variant and invariant) control sets for the different classes of admissible controls agree. The importance of control sets in this context lies in the fact that they describe the possible limit points of the trajectories of (2.3) in compact sets:

Let $K \subset M$ be compact and C -invariant (i.e. $\phi(t, x, u) \in K$ for all $t \geq 0$, and all $x \in K$, $u \in U$), and let U be compact and convex.

2.1. Definition: For a trajectory $\phi(\cdot, x, u)$ of (2.3) define

$$\hat{\omega}(x, u) = \bigcap_{n \in \mathbb{N}} \text{cl} \{ (\phi(t, x, u), u(t+\cdot)), t \geq n \} \subset M \times L_{2, \text{loc}}(\mathbb{R}_+, \mathbb{R}^r), \text{ where}$$

"cl" denotes the closure taken with respect to the given topology on M and the weak topology on compact intervals in $L_{2, \text{loc}}(\mathbb{R}_+, \mathbb{R}^r)$. The set $\omega(x, u) = \{y \in M, \text{ there exists a sequence } t_k \rightarrow \infty \text{ with } \phi(t_k, x, u) \rightarrow y\}$ is called the limit set of $\phi(\cdot, x, u)$. Observe that $\omega(x, u) = \{y \in M, \text{ there exists } v \in U \text{ such that } (y, v) \in \hat{\omega}(x, u)\}$.

2.2. Definition: A nonempty set $L \subset \omega(x, u)$ is called positively invariant, if $\phi(\cdot, y, v) \subset L$ for all $y \in L$ and all $v \in \underline{U}$ with $(y, v) \in \hat{\omega}(x, u)$.

2.3. Lemma: The set $\omega(x, u)$ is nonempty, compact, connected and positively invariant.

Proof:

(i) The sets $cl\{\phi((t, x, u), u(t+\cdot)), t \geq n\}$ are nonempty and compact for each $n \in \mathbb{N}$. Finitely many of these sets have a nonvoid intersection, hence $\hat{\omega}(x, u) \neq \emptyset$ and it follows that $\omega(x, u)$ is nonempty, compact and connected (compare e.g. Nemytskii and Stepanov [9], Chapter V, Theorems 3.01 and 3.09).

(ii) For positive invariance it suffices to show that for $x_n \rightarrow x_0$, $u_n \rightarrow v$ one has $\phi_n := \phi(t, x_n, u_n) \rightarrow \phi_0 := \phi(t, x_0, v)$ for all $t \geq 0$. This follows (even uniformly on compact intervals) from the estimate

$$\begin{aligned} |\phi_n(t) - \phi_0(t)| &\leq |x_n - x_0| + \left| \int_0^t \{X_0(\phi_n(t)) + \sum_{i=1}^r u_{n,i}(t)X_i(\phi_n(t)) \right. \\ &\quad \left. - X_0(\phi_0(t)) + \sum_{i=1}^r v_i(t)X_i(\phi_0(t))\} dt \right| \\ &\leq |x_n - x_0| + \left| \int_0^t \{X_0(\phi_n(t)) - X_0(\phi_0(t)) \right. \\ &\quad \left. + \sum_{i=1}^r u_{n,i}(t)[X_i(\phi_n(t)) - X_i(\phi_0(t))]\} dt \right| \\ &\quad + \left| \int_0^t \left\{ \sum_{i=1}^r [u_{n,i}(t) - v_i(t)]X_i(\phi_0(t)) \right\} dt \right|. \end{aligned}$$

The first and third summand converge by assumption; the second one is bounded from above by $K \int_0^t |\phi_n(t) - \phi_0(t)| dt$, where K is a Lipschitz constant for the vector fields $X_0, \dots, X_r$. Now Gronwall's inequality implies uniform

convergence on compact time intervals. ■

2.4. Lemma: (i) Each compact, positively invariant set L of $\phi(\cdot, x, u)$ contains a minimal, positively invariant set $\tilde{L}$.

(ii) Each trajectory $\bigcup_{t \geq 0} \{\phi(t, y, v), (y, v) \in \omega(x, u)\}$ in a minimal, positively invariant set $\tilde{L}$ is dense in $\tilde{L}$.

Proof: This can be proved exactly as in the classical case, see e.g.

Nemytskii and Stepanov [9], Chapter V, Theorems 7.02 and 7.06. ■

2.5. Proposition: Let $\phi(\cdot, x, u)$ be a trajectory of (2.3) in a compact, C -invariant set K and $\omega(x, u)$ its limit set. Then there exists a control set D of (2.3) such that $\omega(x, u) \cap D \neq \emptyset$.

Proof: By Lemmas 2.3 and 2.4 the limit set $\omega(x, u)$ contains a point y in a minimal, positively invariant set $\tilde{L}$. $\tilde{L}$ in turn contains a dense trajectory $\phi(\cdot, y, v)$ and thus y lies in some control set D . ■

While proposition 2.5 analyzes the relation between the limit set of a trajectory and control sets, the following results describe the behavior of trajectories which actually hit a control set.

2.6 Proposition: Let $\phi(\cdot, x, u)$ be a trajectory of (2.3) and assume that for some control set D one has $\phi(t_i, x, u) \in D$ ($i = 1, 2$), where $t_1 < t_2$. Then $\phi(t, x, u) \in D$ for all $t \in [t_1, t_2]$.

Proof: For some $t \in [t_1, t_2]$ denote $y = \phi(t, x, u)$. Then y is reachable from the point $x_1 = \phi(t_1, x, u)$ hence approximately reachable from any $z \in D$ by continuous dependence of the solutions of (2.3) on initial values. On

the other hand $x_2 = \phi(t_2, x, u)$ is reachable from y , hence any $z \in D$ is approximately reachable from y . ■

2.7. Corollary: (i) If $\phi(\cdot, x, u)$ has a limit point in $\text{int } D$, then

$\phi(t, x, u) \in D$ for all $t > t_0$, where $t_0 = \inf\{t > 0, \phi(t, x, u) \in D\}$.

(ii) For invariant control sets C one even has: If there exists $t_0 \in \mathbb{R}^+$ with

a) $\phi(t_0, x, u) \in \bar{C}$, then $\phi(t, x, u) \in \bar{C}$ for all $t > t_0$,

b) $\phi(t_0, x, u) \in \text{int } C$, then $\phi(t, x, u) \in \text{int } C$ for all $t > t_0$.

Proof:

(i) follows immediately from proposition 2.6.

(ii) follows from the C -invariance of $\bar{C}$ and $\text{int } C$ for invariant control sets, compare e.g. Arnold and Kliemann [2]. ■

We now introduce an assumption under which we prove our main results:

(2.4) For all control sets D_α and all $x \in D_\alpha$ we assume: There exists a time $T > 0$ (depending only on the control set D_α) such that for all open neighborhoods $U(x) \cap \text{int } \overset{+}{0}_{\leq T}(x) \neq \emptyset$ and $\text{int } \overset{-}{0}_{\leq T}(x) \cap U(x) \neq \emptyset$.

This condition (2.4) is e.g. satisfied in the following set up: Assume that the vector fields $X_0, \dots, X_r$ in (2.3) are C^∞ and that the distribution $\Delta_{\tilde{L}}$ in TM , generated by the Lie algebra $\tilde{L} = \tilde{L}(X_0, \dots, X_r)$ has the maximal integral manifolds property (see Sussmann [12]), then (2.3) "lives" on one of those maximal integral manifolds. Hence we can restrict ourselves without loss of generality to the case $\Delta_{\tilde{L}}(x) = T_x M$ for all $x \in M$. This implies local

accessibility of the control system (2.3) and hence (2.4) for all $x \in M$, all $T > 0$.

2.8. Lemma: Let $C \subset M$ be an invariant control set and assume (2.4). Then

- (i) C is closed in M ,
- (ii) $\text{int } C \neq \emptyset$
- (iii) $\mathcal{Q}^+(x) \supset \text{int } C$ for all $x \in C$.

For a proof see Kliemann [8] Lemma 2.1. Variant control sets $B \subset M$ need not have nonvoid interior:

2.9. Example: Consider in $\mathbb{R}^2$ the control system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{with} \quad U = [0, a] \subset \mathbb{R}.$$

The eigenvalues of the systems matrix are $\lambda_{1,2} = -\frac{u}{2} \pm \sqrt{\frac{1}{4}u^2 - 1}$. Hence for $u \equiv 0$ the system moves on circles centered at 0 and thus each of these circles is contained in a control set. For $u > 0$ one has $\text{Re } \lambda < 0$ and so the circles are the control sets, each of them being variant with void interior. The point $\{0\}$ is the only invariant control set. Of course the system satisfies the above Lie algebra condition in $\mathbb{R}^2 \sim \{0\}$. ■

But if for some (variant) control set B we have $\text{int } B \neq \emptyset$, then exact controllability holds in $\text{int } B$:

2.10. Lemma: Let $D \subset M$ be a control set with $\text{int } D \neq \emptyset$ and assume (2.4). Then $\mathcal{Q}^+(x) \supset \text{int } D$ for all $x \in D$.

Proof: For each $x \in D$ we have by definition that $\tilde{O}^+(x)$ is dense in D . For $y \in \text{int } D$ it follows from (2.4) that $\text{int } \tilde{O}_{\leq T}^-(y) \cap \text{int } D \neq \emptyset$, hence $\text{int } \tilde{O}_{\leq T}^-(y) \cap \tilde{O}^+(x) \neq \emptyset$. Hence there exists a control $u(\cdot)$ such that for some $t \in \mathbb{R}^+$ $\phi(t, x, u) = y$. ■

Note that variant control sets with nonvoid interior are never closed.

The next proposition is concerned with a crucial step for the existence of periodic controls: finite time controllability. We define the following first hitting time map

$$(2.5) \quad h : M \times M \rightarrow \mathbb{R}^+ \cup \{\infty\}, \quad h(x, y) = \inf_{u \in \tilde{U}} \{t > 0, \phi(t, x, u) = y\}.$$

Note that h is not upper semicontinuous in general.

2.11. Proposition: Let $D \subset M$ be a control set and assume (2.4). Let

$K_1 \subset D$, $K_2 \subset \text{int } D$ be compact sets, then there exists a time

$T = T(K_1, K_2) < \infty$ such that $h(x, y) \leq T$ for all $x \in K_1$, $y \in K_2$, where h is defined by (2.5).

Proof:

(i) For $x \in K_1$, $y \in K_2$ we show that there is an open neighborhood $U(x)$ such that $h(z, y) \leq t_x < \infty$ for all $z \in U(x)$.

By (2.4) there exist $T < \infty$ and $y_1 \in \text{int } D \cap \tilde{O}_{\leq T}^-(y)$, let $U(y_1)$ be an open neighborhood of y_1 contained in $\text{int } D \cap \tilde{O}_{\leq T}^-(y)$. For $x \in K_1$ there exists a control $u \in \tilde{U}$ and a time $t_1 < \infty$ such that $\phi(t_1, x, u) = y_1$ by Lemma 2.10. The solutions of (2.3) depend continuously on the initial value, hence there exists an open neighborhood $U(x)$ such that $\phi(t_1, z, u) \in U(y_1)$

for all $z \in U(x)$. Putting this together yields $U(x) \subset \overset{-}{O}_{\sim t_1+T}(y)$, hence $h(z,y) \leq t_1 + T$ for all $z \in U(x)$.

(ii) For $x \in K_1$, $y \in K_2$ we show that $h(x,z) \leq t_y < \infty$ for all z in some open neighborhood of y .

Let $x_1 \in \text{int } D$ and $u_1 \in \mathcal{U}$, $t_1 < \infty$ such that $\phi(t_1, x, u_1) = x_1$ by Lemma 2.10. By (2.4) there exist $T < \infty$ and $y_1 \in \text{int } D \cap \overset{+}{O}_{\sim T}(x_1)$, let $U(y_1)$ be an open neighborhood of y_1 contained in $\text{int } D \cap \overset{+}{O}_{\sim T}(x_1)$. Using again Lemma 2.10, there exist $u_2 \in \mathcal{U}$, $t_2 < \infty$ such that $\phi(t_2, y_1, u_2) = y$. The solution of (2.3) under the control action u_2 defines a (semi-) group of homeomorphisms on M , thus at time t_2 the open set $U(y_1)$ is mapped onto an open neighborhood $U(y)$. I.e. $U(y) \subset \overset{+}{O}_{\sim t_1+T+t_2}(x)$ which means $h(x,z) \leq t_1 + T + t_2$ for all $z \in U(y)$.

(iii) A standard compactness argument now finishes the proof of proposition 2.11. ■

Finally we mention a result for existence and uniqueness of invariant control sets:

2.12. Proposition: Suppose that (2.4) is satisfied and that $K \subset M$ is an invariant compact set for the system (2.3). Then:

- (i) There exists an invariant control set $C \subset K$ (and C is compact, $\text{int } C \neq \emptyset$).
- (ii) Denote by $S = \{x_u \in M, X_0(x_u) + \sum_{i=1}^r u_i X_i(x_u) = 0 \text{ for some } (u_i)_{i=1,\dots,r} \in \mathcal{U}\}$ the steady states of (2.3) for constant controls u

and assume that all $x \in S \cap K$ are asymptotically stable.

(a) For each $x \in S \cap K$ there exists a control set $D \subset K$ with $x \in D$.

(b) If for each $u \in U$ there is exactly one corresponding $x_u \in S \cap K$, and all x_u , $u \in U$ are asymptotically stable, globally in K , then C is unique in K and

$$C = \overline{O^+(x)} \quad \text{for any } x \in S \cap C.$$

Proof: (i) See Arnold and Kliemann [1] and Lemma 2.8.

(ii)(a) Follows from the definition of control sets.

(b) Is an easy consequence of asymptotic stability, globally in K and of Lemma 2.8. The last assertion follows from the definition of control sets. ■

Note that even (ii)(b) does not exclude the existence of additional variant control sets in K .

3. A Periodicity Principle

In this section we prove that a class of infinite time optimal control problems has periodic solutions with performance arbitrarily close to the optimal one. We consider the following situations: Given the system (2.3)

$$\dot{x} = X_0(x) + \sum_{i=1}^r u_i X_i(x) \quad \text{on } M \quad \text{with } U \subset \mathbb{R}^r \text{ convex, compact and the}$$

performance criterion

$$(3.1) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t g(x,u) dt = \kappa(x_0, u) \quad (\text{infinite time average yield criterion})$$

or

$$(3.2) \quad \int_0^{\infty} e^{-\delta t} g(x,u) dt = \eta(x_0, u) \quad (\text{infinite time discounted criterion})$$

where we assume that g is continuous in both variables. Denote

$$\kappa = \sup_{u \in \tilde{U}} \sup_{x_0 \in M} \kappa(x_0, u)$$

and assume that κ is finite (If x_0 is fixed, we write

$\kappa(x_0) := \sup_{u \in \tilde{U}} \kappa(x_0, u)$). Then one says that (3.1) has ϵ -optimal periodic

solutions, if for each $\epsilon > 0$ there exists a periodic solution (x_p, u_p)

(depending on ϵ) such that $\kappa(x_p, u_p) > \kappa - \epsilon$, and similarly for problem

(3.2) and η . Note that we do not assume that there exists an optimal trajectory with performance equal to κ , or η .

We first analyze the infinite time average yield problem (3.1) where we have to assume that κ is the supremum over all bounded trajectories $\phi(\cdot, x, u) \subset M$, see example 3.4 at the end of this section for a counter-example in the unbounded case. For bounded trajectories we know from proposition 2.5

that the limit points $\omega(x,u)$ intersect some control set D of (2.3). If $\omega(x,u)$ is a one point set, we have the following preliminary result:

3.1. Proposition: Suppose that g is affine in u . If for $\varepsilon > 0$ there exists a trajectory (x,u) with $\kappa(x,u) > \kappa - \varepsilon$ and $\omega(x,u) = \{y\} \subset M$, then there exists a stationary solution (x_s, u_s) with $\kappa(x_s, u_s) > \kappa - \varepsilon$.

Proof: If $\omega(x,u) = \{y\}$, then there exists a constant control $\tilde{u}$ such that $\phi(t, y, \tilde{u}) = y$ for all $t \in \mathbb{R}^+$ by Lemma 2.3. Since $\phi(t, x, u) \rightarrow \phi(t, y, \tilde{u})$ and by continuity of g one sees that $\kappa(y, \tilde{u}) > \kappa - \varepsilon$. ■

The main result of this section is

3.2. Theorem: If for $\varepsilon > 0$ there exists a bounded trajectory (x,u) with $\kappa(x,u) > \kappa - \varepsilon$ and $\omega(x,u) \cap \text{int } D \neq \emptyset$ for some control set D , then there exists a periodic solution (x_p, u_p) with $\kappa(x_p, u_p) > \kappa - \varepsilon$.

Proof: Let (x,u) be a bounded trajectory with $\kappa(x,u) > \kappa - \varepsilon$ and let $y = \phi(t_0, x, u) \in \text{int } D$. The trajectory $\phi(\cdot, y, u(t_0 + \cdot))$ has the same limit points as $\phi(\cdot, x, u)$, hence in particular an accumulation point $y_0 \in \text{int } D$. By Corollary 2.7 $\phi(t, y, u) \in D$ for all $t > 0$ and by assumption $\phi(\cdot, y, u) \subset K$ for some compact set $K \subset D$. Hence by Proposition 2.11, there exists a $T > 0$ such that $h(z, y) \leq T$ for all $z \in K$.

By assumption there exists t_1 (arbitrarily large) such that

$$\frac{1}{t_1} \int_0^{t_1} g(\phi(t, y, u(t_0 + t)), u) dt > \kappa - 2\varepsilon \quad \text{and}$$

$$\begin{aligned}
& \frac{1}{t_1+T} \int_0^{t_1+T} g(\phi(t, y, u(t_0+t)), u) dt \\
&= \frac{1}{t_1+T} \int_0^{t_1} g(\phi(t, y, u(t_0+t)), u) dt + \frac{1}{t_1+T} \int_{t_1}^{t_1+T} g(\phi(t, y, u(t_0+t)), u) dt \\
&> \frac{1}{t_1+T} \int_0^{t_1} g(\phi(t, y, u(t_0+t)), u) dt + \frac{1}{t_1+T} T \cdot \min_{(z,u) \in K \times U} g(z, u) \\
&> \kappa - 3\varepsilon - \varepsilon \quad \text{for } t_1 \text{ sufficiently large.}
\end{aligned}$$

Define a periodic control

$$u_p = \begin{cases} u(t_0+t) & \text{for } t \in [0, t_1] \\ u_c(t) & \text{for } t \in (t_1, t_1+t_2] \end{cases}$$

where u_c steers $\phi(t_1, y, u)$ to y in time $t_2 \leq T$, and continue u_p periodically for $t > t_1 + t_2$. Then $\phi(\cdot, y, u_p)$ is periodic and in each

period we have $\frac{1}{t_1+t_2} \int_0^{t_1+t_2} g(\phi(t, y, u(t_0+t)), u) dt > \kappa - 4\varepsilon$. ■

For invariant control sets C we can strengthen this result to:

3.3 Corollary: If for $\varepsilon > 0$ there exists a solution (x, u) with $\kappa(x, u) > \kappa - \varepsilon$ and a time $t_0 > 0$ such that $\phi(t_0, x_0, u) \in \text{int } C$, then there exists a periodic solution (x_p, u_p) with $\kappa(x_p, u_p) > \kappa - \varepsilon$.

Proof: Same as before with Corollary 2.7(ii). ■

Note that under condition (2.4) any invariant control set has nonvoid interior. Proposition 3.1 and theorem 3.2 guarantee ε -optimal periodic solutions for the infinite time average yield problem, except for the case

where the set of limit points has more than one element and does not intersect the interior of some control set.

If the ϵ -optimal trajectories are unbounded, one cannot expect ϵ -optimal periodic solutions for (3.1'), since the optimal value κ does not depend on the behavior on finite time intervals.

3.4. Example: Let $M = (0, \infty)$ and consider the problem

$$\dot{x} = 3 - 2u, \quad u \in U = [1, 2]$$

$$g(x, u) = 5 - 4u.$$

The unique control set $C = M = (0, \infty)$ is invariant and one may start at any (interior) point $x_0 \in C$. The solutions are $\phi(t, x_0, u) = (3-2u)t + x_0$, $\kappa(x_0, u) = 5 - 4u$ and $\kappa = 1$ is obtained for $u \equiv 1$, where the corresponding solution is $\phi(t, x_0, 1) = t + x_0$. Any periodic solution has to use controls in $[\frac{3}{2}, 2]$, where $g(x, u) < -1$, and hence any periodic solution has performance < 0 . ■

For the discounted problem (3.2) one cannot expect ϵ -optimal periodic solutions, if one starts outside some control set. Our result in this case is:

3.5. Theorem: If for $\epsilon > 0$ there exists a trajectory (x, u) with

$\eta(x, u) > \eta - \epsilon$ and

(i) if $\omega(x, u) \cap \text{int } D \neq \emptyset$ for some control set D ,

or

(ii) if there exists $t_0 > 0$ such that $\phi(t_0, x, u) \in \text{int } C$ for some invariant control set C ,

then there exists a solution (x_p, u_p) with $\eta(x_p, u_p) > \eta - \epsilon$ and (x_p, u_p) is

periodic after the time of first entrance into int D (or int C respectively).

Proof: Use the same ideas as in the proof of Theorem 3.2 and Corollary 3.3. ■

3.6. Remarks:

- (a) If problems (3.1) or (3.2) have a solution that actually realizes κ (or η) then in general there need not be a periodic control having performance κ (or η).
- (b) Consider the discounted problem (3.2) for $\delta \rightarrow 0$: If one starts outside some control set D, then the initial part of the trajectory until the entrance into int D will carry less and less weight, as the period of the solution constructed in theorem 3.5 grows. In this sense the solutions of problem (3.2) converge towards those of (3.1) for $\delta \rightarrow 0$.
- (c) Theorem 3.5 remains valid, if instead of $e^{-\delta t}$ one uses any discount function $\phi(t)$ which decreases monotonically to 0 as $t \rightarrow \infty$ and such that η is still finite.
- (d) Let us mention once again that outside control sets the system (2.3) cannot have any solution which is periodic in the state variable $x \in M$, since any periodic solution lies in some control set.

4. Growth of Linear Control Semigroups

Consider a linear, parameter controlled system

$$(4.1) \quad \dot{x} = ux \quad \text{in } \mathbb{R}^d$$

with $u \in N \subset \text{gl}(d, \mathbb{R})$ compact. According to the remarks after (2.3) we can restrict ourselves to piecewise constant controls and we define the systems group $\tilde{G}$ and the systems semigroup $\tilde{S}$, describing the orbits of (4.1) by

$$\tilde{G} = \{e^{t_r A_r} \dots e^{t_1 A_1}, A_i \in N, t_i \in \mathbb{R}, 1 \leq i \leq r \in \mathbb{N}\}$$

$$\tilde{S} = \{e^{t_r A_r} \dots e^{t_1 A_1}, A_i \in N, t_i \in \mathbb{R}^+, 1 \leq i \leq r \in \mathbb{N}\}.$$

$\tilde{G}$ is a connected Lie subgroup of $\text{Gl}(d, \mathbb{R})$ with Lie algebra $\mathfrak{g} = \underline{\text{LA}}(N)$, the Lie algebra generated by N in $\text{gl}(d, \mathbb{R})$. Let $\tilde{G}_t$ and $\tilde{S}_t$ denote the (semi-) group with $\sum_{i=1}^r t_i \leq t$.

The problem is to describe the growth rate, i.e. the Lyapunov exponent of the system (4.1) (which is the growth rate of $\{\tilde{S}_t, t > 0\}$). This can be done using the spectral radius or the operator norm. We define

$$\beta(t) = \frac{1}{t} \log r(\tilde{S}_t), \quad r(\tilde{S}_t) = \sup_{g \in \tilde{S}_t} r(g),$$

$r(g)$ the spectral radius of $g \in \text{Gl}(d, \mathbb{R})$

$$(4.2) \quad \beta = \lim_{t \rightarrow \infty} \beta(t) \quad (= \sup_{t > 0} \beta(t)),$$

and

$$\delta(t) = \frac{1}{t} \log \|S_t\|, \quad \|S_t\| = \sup_{g \in S_t} \|g\|$$

$\|g\|$ the operator norm of $g \in \text{Gl}(d, \mathbb{R})$,

$$(4.3) \quad \delta = \lim_{t \rightarrow \infty} \delta(t) \quad (= \inf_{t > 0} \delta(t)).$$

Of course $\beta \leq \delta$, and we are looking for conditions that guarantee $\beta = \delta$. The answer to this problem is important for the analysis of the stability of moments of associated stochastic systems and for the description of large deviations in these systems, compare Arnold et al. [3] for this idea and for the following set-up.

For further investigation we project the system (4.1) onto the unit sphere S^{d-1} in $\mathbb{R}^d$, and we obtain for $s(t) = \frac{x(t)}{|x(t)|}$ the equation

$$(4.4) \quad \dot{s}(t) = h(u, s) = (u - s^* u s \text{ Id})s, \quad s_0 = x_0 / |x_0|,$$

where $*$ denotes the transpose and Id the $d \times d$ identity matrix. Then

$$(4.5) \quad |x(t, x_0, u)| = |x_0| \int_0^t s(\tau, x_0)^* u(\tau) s(\tau, x_0) d\tau =: |x_0| \int_0^t q(u, s) d\tau.$$

The system (4.1) is linear in x , so we may consider (4.4) on the projective space $\mathbb{P}^{d-1}$ instead of S^{d-1} , which we will do from now on. The Lie algebra generated by the vector fields in (4.4) is $\tilde{L} = \tilde{L}A(h(u, \cdot), u \in N)$, and our assumption, analogously to the condition mentioned after (2.4), is

$$(4.6) \quad \Delta_{\tilde{L}}(s) = T_S P \quad \text{for all } s \in P.$$

The following theorem states that the system (4.1) has one Lyapunov exponent:

4.1. Theorem: Under assumption (4.6) we have $\beta = \delta$ for the system (4.1).

In order to prove this theorem we transform (4.2) and (4.3) into optimal control problems involving ε -optimal trajectories. Define

$$\kappa = \sup_{u \in \tilde{U}} \sup_{s_0 \in P} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t q(u, s) d\tau, \quad \begin{array}{l} \tilde{U} \text{ the measurable controls} \\ \text{with values in } co N \end{array}$$

$$\kappa_p = \sup_{u \in \tilde{U}_p} \sup_{s_0 \in P} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t q(u, s) d\tau, \quad \begin{array}{l} \tilde{U}_p \text{ the periodic measur-} \\ \text{able controls with values} \\ \text{in } co N. \end{array}$$

$$\text{Note that by (4.5) we have } \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t q(u, s) d\tau = \limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t, x_0, u)|,$$

the Lyapunov exponent of the solution of (4.1) starting in x_0 with control u .

From Arnold et al. [3], Theorem 3.1, we cite the following result.

4.2. Lemma: Assume (4.6), then the system (4.1) has exactly one invariant control set C on P , C is compact and $\text{int } C \neq \emptyset$.

4.3. Lemma: $\kappa_p = \beta$.

Proof:

(i) $\beta < \kappa_p$: For a given $\varepsilon > 0$ we find a $g = e^{t_r A_r} \dots e^{t_1 A_1} \in \tilde{S}_T$ with $\sum_{i=1}^r t_i = T$ and $\frac{1}{T} \log r(g) > \beta - \varepsilon$. Define a piecewise constant control u_0 on $[0, T]$ by $u_0(t) = A_i$ for $t \in [t_1 + \dots + t_{i-1}, t_1 + \dots + t_i)$ and continue

T-periodically. Then the periodic system $\dot{x} = u_0 x$ has $g = \Phi(T)$ as the fundamental matrix at time T , its largest characteristic (or Floquet) exponent is therefore $\frac{1}{T} \log r(g)$. Hence there exists $s_0 \in P$ such that for all $t > 0$ $|x(t, s_0, u_0)| > \exp t \cdot (\beta - \epsilon)$. Since ϵ was arbitrary, we see that $\beta < \kappa_p$.

(ii) $\kappa_p < \beta$: The argument above also proves that

$\beta < \alpha_p = \sup_{u \in \tilde{U}_p} \lim_{t \rightarrow \infty} \frac{1}{t} \log r(\Phi_u(t))$, where $\Phi_u(t)$ is the fundamental matrix of $\dot{x} = ux$ at time t . $\beta > \alpha_p$ is clear from the definition, so it remains to

show that $\kappa_p < \alpha_p$. To this end pick $u \in \tilde{U}_p$ with period T , then

$x(t, s_0, u) = \Phi_u(t)s_0 = (P(t)e^{t\Lambda})s_0$, the Floquet decomposition, where $P(t)$ is

T-periodic and Λ a constant matrix with the Floquet exponents. Then

$$\frac{1}{nT} \log |x(nT, s_0, u)| \leq \frac{1}{nT} (\log |P(T)| + \log |e^{nT\Lambda}| + \log |s_0|).$$

Thus for $n \rightarrow \infty$ we have for all $u \in \tilde{U}_p$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |x(t, s_0, u)| \leq \lim_{t \rightarrow \infty} \frac{1}{t} \log r(\Phi_u(t)), \text{ yielding } \kappa_p \leq \alpha_p. \quad \blacksquare$$

4.4. Lemma: $\kappa = \delta$.

Proof: From (4.5) we see that $\delta = \lim_{t \rightarrow \infty} \sup_{u \in \tilde{U}} \sup_{x_0 \in P} \frac{1}{t} \int_0^t q(u, s) d\tau$.

$\kappa \leq \delta$ is obvious from this formulation, hence it remains to show that

$\delta \leq \kappa$: For a sequence $t_k \rightarrow \infty$ there exist (u^k, s_0^k) with

$$\frac{1}{t_k} \int_0^{t_k} q(u^k, s(s_0^k, u^k)) d\tau > \delta(t_k) - \frac{1}{k}. \text{ Hence for any compact time interval}$$

$[0, T] \subset \mathbb{R}_+$ there exists $u^0 : \mathbb{R}_+ \rightarrow \text{co } N$, measurable such that $u^k \rightarrow u^0$ weakly in $L_2([0, T])$. Take a weak accumulation point u_T^0 of u^k on $[0, T]$ and a diagonal sequence in T , then $u^0(t) \in \text{co } N$ via the Cesaro Limit. Let s_0^0 be an accumulation point of s_k^0 , then $s^k := s(s_0^k, u^k) \rightarrow s(s_0^0, u^0) =: s^0$ uniformly on compact time intervals by the argument from the proof of lemma 2.3.(ii).

Fix $\varepsilon > 0$, for k large enough we have

$$\frac{1}{t_k} \int_0^{t_k} q(u^k, s(s_0^k, u^k)) d\tau > \delta(t_k) - \varepsilon,$$

$$\left| \frac{1}{t_k} \int_0^{t_k} (s^k u^k s^k - s^{0*} u^0 s^0) d\tau \right| < \varepsilon \quad \text{and} \quad |\delta(t_k) - \delta| < \varepsilon.$$

Thus

$$\begin{aligned} \frac{1}{t_k} \int_0^{t_k} s^{0*} u^0 s^0 d\tau &> \frac{1}{t_k} \int_0^{t_k} s^k u^k s^k d\tau - \left| \frac{1}{t_k} \int_0^{t_k} (s^k u^k s^k - s^{0*} u^0 s^0) d\tau \right| \\ &> \delta(t_k) - 2\varepsilon > \delta - 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\kappa > \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t s^{0*} u^0 s^0 d\tau > \delta$. ■

Proof of Theorem 4.1: The result follows from $\kappa = \kappa_p$ by using Corollary 3.3, if we can show that initial values for ε -optimal trajectories can be chosen in $\text{int } C$, where C is the unique compact invariant control set from Lemma 4.2. To this end let $A \subset \text{int } C$ be open such that $\bar{A} \subset \text{int } C$. Then A contains a basis of $\mathbb{R}^d$ and for any $u \in \mathcal{U}$ the maximal Lyapunov exponent among the $\limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t, \cdot, u)|$ can be obtained by starting in A , see Cesari [5], section 3.12. Hence for κ (and κ_p) it suffices to consider initial values in $\bar{A} \subset \text{int } C$. ■

5. Management of Interacting Populations

We consider the following three dimensional model of harvested populations

$$\begin{aligned}
 \dot{x}_1(t) &= q_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 x_3 - u x_1 =: g_1(x_1, x_2, x_3, u) \\
 (5.1) \quad \dot{x}_2(t) &= q_2 \left(1 - \frac{x_2}{K_2}\right) + \alpha_2 x_1 x_2 =: g_2(x_1, x_2) \\
 \dot{x}_3(t) &= q_3 \left(1 - \frac{x_3}{K_3}\right) + \alpha_3 x_1 x_3 =: g_3(x_1, x_3)
 \end{aligned}$$

with

$$u(t) \in [0, U_{\max}] \subset \mathbb{R}$$

where the constants q_i , α_i , K_i , $U_{\max}$ are positive and $U_{\max} < q_1$. These equations describe the dynamics of a system, where the predators x_2 and x_3 feed on the prey x_1 , the prey is subject to harvesting, and the harvesting intensity is considered as the control variable. There is no direct interaction between the predators x_2 and x_3 .

Reasonable performance criteria are the average yield

$$(5.2) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T x_1(t) u(t) dt$$

or, taking into account economic considerations,

$$(5.3) \quad \int_0^\infty e^{-\delta t} (p(u, x_1) - c(x_1)) u x_1 dt,$$

where the discount factor δ is positive and p , c are the price and the cost, respectively, of the harvested prey (cf. Clark [6]).

A simple analysis shows the existence of a compact invariant set of the form $K = \{(x_1, x_2, x_3), 0 \leq x_i \leq c_i, i = 1, 2, 3\}$. The function g_1 vanishes for

$$x_1 = 0 \quad \text{and} \quad x_1 = K_1 \left(1 - \frac{\alpha_1}{q_1} x_2 x_3 - \frac{u}{q_1}\right), \quad u \in [0, u_{\max}].$$

For $i = 2, 3$ the functions g_i vanish for

$$x_i = 0 \quad \text{and} \quad x_i = K_i \left(1 + \frac{\alpha_i}{q_i} x_1\right).$$

Thus we obtain the following eight sets of steady states

$$S_1 = \{(0, 0, 0)\}$$

$$S_2 = \{(0, K_2, 0)\}$$

$$S_3 = \{(0, 0, K_3)\}$$

$$S_4 = \{(0, K_2, K_3)\}$$

$$S_5 = \{(x_1, 0, 0), \quad x_1 = K_1 \left(1 - \frac{u}{q_1}\right), \quad u \in [0, u_{\max}]\}$$

$$S_6 = \{(x_1, 0, x_3), \quad x_1 = K_1 \left(1 - \frac{u}{q_1}\right), \quad x_3 = K_3 \left(1 + K_1 \frac{\alpha_3}{q_3} \left(1 - \frac{u}{q_1}\right)\right),$$

$$u \in [0, u_{\max}]\}$$

$$S_7 = \{(x_1, x_2, 0), \quad x_1 = K_1 \left(1 - \frac{u}{q_1}\right), \quad x_2 = K_2 \left(1 + K_2 \frac{\alpha_2}{q_2} \left(1 - \frac{u}{q_1}\right)\right),$$

$$u \in [0, u_{\max}]\}$$

$$S_8 = \{(x_1, x_2, x_3), \quad x_1 = K_1 \left(1 - \frac{\alpha_1}{q_1} x_2 x_3 - \frac{u}{q_1}\right), \quad x_2 = K_2 \left(1 + \frac{\alpha_2}{q_2} x_1\right),$$

$$x_3 = K_3 \left(1 + \frac{\alpha_3}{q_3} x_1\right), \quad u \in [0, u_{\max}]\}.$$

The planes P_i given by $x_i = 0$ are invariant for the control system (5.1) and, having empty interior in R^3 , they cannot be reached from R_+^3 . We therefore concentrate on the three dimensional situation and assume

$$(5.4) \quad S_8 \neq \emptyset \quad \text{and} \quad S_8 \subset R_+^3 =: M, \quad \text{the state space for (5.1).}$$

It is easy to show that if $S_8 = \emptyset$, then all solutions of the control system (5.1) tend towards one of the planes P_i . Furthermore $S_8 \neq \emptyset$ can be expressed explicitly in terms of the parameters of (5.1).

5.1. Existence of ϵ -optimal periodic solutions

If $\alpha_2 \neq \alpha_3$ or $q_2 \neq q_3$, then condition (2.4) is satisfied in the manifold M (see section 5.3). Furthermore there exists an invariant control set $C \subset M$, C is closed with nonvoid interior. The steady states in S_8 are globally asymptotically stable in M (see section 5.2), thus C is unique and $S_8 \subset C$ by Proposition 2.12. For the criteria (5.2) and (5.3) with initial values in $\text{int } C$ we thus infer the existence of ϵ -optimal periodic controls.

5.2. Global asymptotic stability of S_8 in R_+^3 .

Let $x^* = x^*(u) \in S_8$, i.e. $g_i(x^*, u) = 0$ for $i = 1, 2, 3$. Using arguments from Sieveking ([11], Theorem 4) we construct a Lyapunov function for this system. First note that, for u fixed, we have a predator-prey system,

since for $i \neq j$ either

$$\frac{\partial g_i}{\partial x_j}(x, u) = 0 \quad \text{and} \quad \frac{\partial g_j}{\partial x_i}(x, u) = 0 \quad \text{on } \mathbb{R}_+^3$$

or

$$\frac{\partial g_i}{\partial x_j}(x, u) \cdot \frac{\partial g_j}{\partial x_i}(x, u) < 0 \quad \text{on } \mathbb{R}_+^3.$$

To (5.1) we associate an undirected graph with three knots and an edge $i-j$ if $\frac{\partial g_i}{\partial x_j} \neq 0$ and $\frac{\partial g_j}{\partial x_i} \neq 0$. This graph obviously is a tree. Hence (cf. Sieveking [11], Lemma 5) there are smooth positive functions $k_i(x)$,

$x \in \mathbb{R}_+^3$ with

$$k_i(x) \frac{\partial g_i}{\partial x_j} = -k_j(x) \frac{\partial g_j}{\partial x_i}, \quad \text{for } 1 \leq i, j \leq 3.$$

Define

$$H(x, u) = \sum_{i=1}^3 k_i(x) (x_i - x_i^* \log x_i).$$

Then

$$\frac{d}{dt} H(x(t), u) < 0$$

along solutions of (5.1) in $\mathbb{R}_+^3$ and

$$H(x^*, u) = \min_{x \in \mathbb{R}_+^3} H(x, u).$$

Thus x^* is locally asymptotically stable. Furthermore

$$\lim_{x \rightarrow \partial \mathbb{R}_+^3} H(x, u) = +\infty$$

and thus x^* is globally asymptotically stable in $\mathbb{R}_+^3$.

5.3. Condition (2.4)

The system (5.1) is of the form

$$\dot{x} = X_0(x) + uX_1(x) \quad \text{on } M = \mathbb{R}_+^3 \quad \text{with}$$

$$X_0 = \begin{pmatrix} q_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 x_3 \\ q_2 \left(1 - \frac{x_2}{K_2}\right) + \alpha_2 x_1 x_2 \\ q_3 \left(1 - \frac{x_3}{K_3}\right) + \alpha_3 x_1 x_3 \end{pmatrix}, \quad X_1 = \begin{pmatrix} -x_1 \\ 0 \\ 0 \end{pmatrix}$$

In a straightforward way one computes the Lie brackets

$$X_2 := [X_0, X_1] = \begin{pmatrix} -\frac{q_1}{K_1} x_1^2 \\ \alpha_2 x_1 x_2 \\ \alpha_3 x_1 x_3 \end{pmatrix}$$

$$X_3 := [X_0, X_2] = \begin{pmatrix} -\frac{q_1^2}{K_1} x_1^2 + \frac{q_1 \alpha_1}{K_1} x_1^2 x_2 x_3 + (\alpha_1 \alpha_2 + \alpha_1 \alpha_3) x_1^2 x_2 x_3 \\ \alpha_2 q_1 x_1 x_2 - \alpha_1 \alpha_2 x_1 x_2^2 x_3 + \frac{\alpha_2 q_2}{K_2} x_1 x_2^2 \\ \alpha_3 q_1 x_1 x_3 - \alpha_1 \alpha_3 x_1 x_2^2 x_3 + \frac{\alpha_2 q_3}{K_3} x_1 x_3^2 \end{pmatrix}$$

The vector fields X_0, X_1, X_2 are linearly dependent for all points

(x_1, x_2, x_3) with

$$(5.5) \quad \alpha_3 q_2 - \frac{\alpha_2 q_2}{K_2} x_2 = \alpha_2 q_3 - \frac{\alpha_2 q_3}{K_3} x_3 .$$

The vector fields X_1, X_2, X_3 are linearly dependent for all points (x_1, x_2, x_3) with

$$(5.6) \quad K_3 q_2 x_2 = K_2 q_3 x_3 .$$

For $\alpha_2 = \alpha_3$ and $q_2 = q_3$, these equations define the same line in the x_2 - x_3 plane, i.e. the same plane $P = \{(c \cdot x_1, x_2 = \frac{K_2}{K_3} x_3, d \cdot x_3), c, d \in \mathbb{R}\}$ in $\mathbb{R}^3$. The vector fields X_0 and X_1 are tangent to P in this case and hence all orbits of (5.1) with initial values in P are contained in P , thus (2.4) cannot be met in this situation.

If however $\alpha_2 = \alpha_3$ and $q_2 \neq q_3$, then (5.5) and (5.6) define parallel lines in the x_2 - x_3 plane, hence the vector fields X_i , $i = 0, \dots, 3$ span the whole tangent space at any point in $\mathbb{R}_+^3$.

If $\alpha_2 \neq \alpha_3$, denote the constant $\beta = (\alpha_3 q_2 - \alpha_2 q_3) / (q_3 \alpha_3 - q_2 \alpha_2)$. If $\beta < 0$, then the lines defined by (5.5) and (5.6) do not intersect in the x_2 - x_3 plane for $x_2 > 0, x_3 > 0$, thus again X_i , $i = 0, \dots, 3$ span the tangent space at any point in $\mathbb{R}_+^3$.

If $\beta > 0$, then (5.5) and (5.6) define a line

$$\{(c \cdot x_1, x_2^0, x_3^0), c \in \mathbb{R}, x_2^0 = \frac{q_3}{q_2} \beta K_2, x_3^0 = \beta K_3\} \text{ in } \mathbb{R}_+^3.$$

The vector field X_0 is never tangent to this line in $\mathbb{R}_+^3$, hence condition (2.4) is satisfied in this case.

Summing up, we see that, except for the case $\alpha_2 = \alpha_3$ and $q_2 = q_3$, the assumption (2.4) is always fulfilled.

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INFINITE TIME OPTIMAL CONTROL AND PERIODICITY

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