Reinforcement by a Thin Layer with
Oscillating Thickness

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Abstract

We study a singular perturbation problem involving variational functionals that nearly degenerate on a thin layer with rapidly varying thickness. One interpretation is the reinforcement of a rod in torsion by a thin, oscillatory coating of a very strong material.

1. Introduction

Several recent papers have studied "reinforcement problems" [2,4,7,13], so called because they include the torsion of a constant cross-section elastic rod reinforced by a thin layer of very strong material. Mathematically, these are singular perturbation problems involving elliptic operators that nearly degenerate on a thin layer near the boundary. The present paper considers the analogous problem when the reinforcing layer has rapidly varying thickness. Our method is variational, employing the notion of Γ -convergence [1,6,10,11,12].

A full statement of our results is given in section 2; the main conclusion is Theorem 2.4, which identifies the Γ -limit of a certain sequence of functionals. This introduction will focus instead on the simpler linear case, which displays all the essential features of the general one. No familiarity with the theory of Γ -convergence is assumed.

We begin with a simply connected domain $\Omega \subset \mathbb{R}^2$, and perturb it by adjoining an oscillatory layer Σ_ϵ , giving $\Omega_\epsilon = \overline{\Omega} \cup \Sigma_\epsilon$. The thickness of Σ_ϵ varies periodically with period $S_\epsilon L$, where $L = |\partial\Omega|$ and $1/S_\epsilon \in \mathbb{N}$; its mean thickness is R_ϵ . Both R_ϵ and S_ϵ tend to zero, with $R_\epsilon \sim S_\epsilon \sim \epsilon$. (See figure 1.)

We are interested in the solutions of

$$(1.1) \quad L_\epsilon u_\epsilon = f \text{ in } \Omega_\epsilon, \quad u_\epsilon = 0 \text{ on } \partial\Omega_\epsilon,$$

where L_ϵ is defined by

$$(1.2) \quad L_\epsilon u = \nabla \cdot (a_\epsilon(x) \nabla u), \quad a_\epsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ \epsilon & \text{if } x \in \Sigma_\epsilon \end{cases}.$$

Equation (1.1) is to be understood in the weak sense at the interface $\partial\Omega$, where a_ϵ is discontinuous. Thus it asserts that $\Delta u_\epsilon = f$ in Ω and

$\varepsilon \Delta u_\varepsilon = f$ in Σ_ε , with the "transmission boundary condition" that u_ε and $a_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu}$ are continuous across $\partial\Omega$.

For the interpretation in terms of torsional rigidity, we view Ω_ε as the cross section of an elastic rod. The domain Ω is occupied by a material with unit shear modulus, while the material in the reinforcing layer Σ_ε has shear modulus ε^{-1} . The torsional rigidity of the resulting rod is $4\int \Omega_\varepsilon dx$, where u_ε solves $L_\varepsilon u_\varepsilon = -1$.

There is another mechanical interpretation of (1.1), involving antiplane shear; in this context our problem is one of "weak attachment." Once again Ω_ε is viewed as the cross section of an elastic rod, but this time the material in Σ_ε has shear modulus ε . The boundary of the rod is clamped, and a body load f acts in the lengthwise direction. The solution of $L_\varepsilon u_\varepsilon = f$ represents, in this interpretation, the lengthwise displacement of the rod.

Our problem is significant from the mathematical viewpoint because it involves the interaction of small coefficients, a thin domain, and oscillatory geometry. Thus it combines features of other, previously studied problems involving small coefficients on thin domains, e.g. [8,15], or thin domains with rapidly oscillatory geometry, e.g. [9,14]. It may seem esoteric to consider rods reinforced by an oscillatory layer. We believe, however, that this is a step toward more practical applications - for example understanding the boundary behavior of a liquid crystal as it meets a rough surface.

Our goal is to characterize u_ε , the solution of (1.1), in the limit as $\varepsilon \rightarrow 0$. It turns out that the homogeneous Dirichlet condition is replaced by one of Robin type: the limiting equation is

$$(1.3) \quad \Delta u_0 = f \quad \text{in } \Omega, \quad \frac{\partial u_0}{\partial \nu} + E c_k u_0 = 0 \quad \text{at } \partial\Omega.$$

Here $E = \lim \epsilon/R_\epsilon$, $k = \lim R_\epsilon/S_\epsilon$, and c_k is the energy of the solution of a certain elliptic boundary value problem, depending on both k and the form of the layer thickness variation (see (2.3)). We shall establish

(1.4a) Convergence of solutions: for any $f \in L^2(\mathbb{R}^2)$, the solution of (1.1) tends in $L^2(\Omega)$ to that of (1.3) as $\epsilon \rightarrow 0$.

(1.4b) Convergence of the spectrum: the eigenvalues of the operator L_ϵ converge, as $\epsilon \rightarrow 0$, to those of the limiting operator (1.3).

These are both direct consequences of a more general statement, that the associated quadratic functions Γ -converge. Let us write

$$(1.5a) \quad F_\epsilon(u) = \begin{cases} \int_\Omega |\nabla u|^2 dx + \epsilon \int_{\Sigma_\epsilon} |\nabla u|^2 dx & \text{if } u \in H_0^1(\Omega_\epsilon) \\ + \infty & \text{otherwise} \end{cases}$$

for the functional associated with L_ϵ , and

$$(1.5b) \quad F_0(u) = \begin{cases} \int_\Omega |\nabla u|^2 dx + E c_k \int_{\partial\Omega} |u|^2 ds & \text{if } u \in H^1(\Omega) \\ + \infty & \text{otherwise} \end{cases}$$

for that associated with the limiting equation. The assertion of Γ -convergence has two parts:

(1.6a) Upper bound on F_0 : If $v_\epsilon \in H_0^1(\Omega_\epsilon)$ and $v_\epsilon \rightarrow v_0$ in $L^2(\mathbb{R}^2)$ as $\epsilon \rightarrow 0$, then $F_0(v_0) \leq \liminf F_\epsilon(v_\epsilon)$.

(1.6b) Attainability of the bound: If $v_0 \in H^1(\Omega)$, then there exist

$v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ such that $v_\varepsilon \rightarrow v_0$ in $L^2(\mathbb{R}^2)$ and $F_0(v_0) = \lim F_\varepsilon(v_\varepsilon)$.

This is the content of Theorem 2.4, specialized to the case at hand.

It is a general fact that Γ -convergence of a sequence of functionals implies convergence of the solutions of the associated minimization problems (see Propositions 2.2 and 2.3). In the present context the proof is not difficult, so we sketch it for the sake of completeness. One verifies that the functionals F_ε are equicoercive on L^2 : any family $\{v_\varepsilon\}$ with $|F_\varepsilon(v_\varepsilon)| \leq C$ is compact in $L^2(\mathbb{R}^2)$ with the topology of strong convergence (see [2]). If u_ε solves (1.1), then it minimizes $F_\varepsilon(u) + 2 \int_{\Omega_\varepsilon} u f dx$. One verifies that $F_\varepsilon(u_\varepsilon)$ remains bounded as $\varepsilon \rightarrow 0$; therefore there is a sequence $u_{\varepsilon(k)}$ which converges to a limit u_0 in $L^2(\mathbb{R}^2)$. We claim that u_0 minimizes $F_0(u) + 2 \int_{\Omega} u f dx$. Indeed,

$$\begin{aligned} (1.7) \quad \inf_u F_0(u) + 2 \int_{\Omega} u f dx &\leq F_0(u_0) + 2 \int_{\Omega} u_0 f dx \\ &\leq \liminf_{k \rightarrow \infty} F_{\varepsilon(k)}(u_{\varepsilon(k)}) + 2 \int_{\Omega_{\varepsilon(k)}} u_{\varepsilon(k)} f dx , \end{aligned}$$

by (1.6a), while

$$\inf_u F_0(u) + 2 \int_{\Omega} u f dx \geq \liminf_{k \rightarrow \infty} F_{\varepsilon(k)}(u_{\varepsilon(k)}) + 2 \int_{\Omega_{\varepsilon(k)}} u_{\varepsilon(k)} f dx ,$$

by (1.6b); therefore equality must hold in (1.7). Since F_0 is strictly convex, the limit u_0 is unique, and so the entire sequence $\{u_\varepsilon\}$ converges to it.

The convergence of the spectrum is proved by a similar argument, using the minimax characterization of the j^{th} eigenvalue of L_ε :

$$\lambda_{\epsilon}^j = \inf_{D_{\epsilon}^j} \sup_{u \in D_{\epsilon}^j} \frac{F_{\epsilon}(u)}{\int_{\Omega_{\epsilon}} |u|^2 dx},$$

where D_{ϵ}^j ranges over j -dimensional subspaces of $H_0^1(\Omega_{\epsilon})$. Details can be found in [3], or in section 3.9 of [1]. (These references state a hypothesis of uniform ellipticity, but equicoercivity in L^2 is all that the proof requires.)

It remains to say a word about (1.6a,b). If $u \in H^1(\Omega)$, then there is an extension u_{ϵ} which vanishes on $\partial\Omega_{\epsilon}$, with $|\nabla u_{\epsilon}| \sim 1/\epsilon$ in the layer Σ_{ϵ} . It is easy to achieve

$$(1.8) \quad \epsilon \int_{\Sigma_{\epsilon}} |\nabla u_{\epsilon}|^2 dx \leq C \int_{\partial\Omega} |u|^2 ds$$

for some constant C . To prove (1.6a,b) we must show that $C = Ec_k$ is the best constant for (1.8), simultaneously for all $u \in H^1(\Omega)$, asymptotically as $\epsilon \rightarrow 0$. In other words, we must identify energy of the most efficient extension u_{ϵ} .

2. Preliminaries and statement of main result

This section gives the definition and the main properties of Γ -convergence, and states our main result, Theorem 2.4. For simplicity our discussion is restricted to the two-dimensional case and to periodic layers. The extension to n dimensions and to the "locally periodic" case will be discussed in section 4.

Let Ω be a simply connected, bounded, open subset of $\mathbb{R}^2$, with $\partial\Omega$ of class $C^{1,1}$ (i.e. $\partial\Omega$ has a Lipschitz continuous normal vector). We denote by L the length of $\partial\Omega$, and suppose that $d:\mathbb{R} \rightarrow (0,\infty)$ is an L -periodic, continuous function with mean value one. The minimum and maximum values of d will be called m and M respectively. We consider positive numbers $R_\varepsilon, S_\varepsilon$ with $S_\varepsilon^{-1} \in \mathbb{N}$, such that $\lim R_\varepsilon = \lim S_\varepsilon = 0$ as $\varepsilon \rightarrow 0$. For each ε , let

$$\Sigma_\varepsilon = \{y + tv(y) : y \in \partial\Omega, 0 < t < R_\varepsilon d(s(y)/R_\varepsilon)\},$$

$$\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon,$$

where $v(y)$ is the outer unit normal at $y \in \partial\Omega$, and $s(y)$ parametrizes $\partial\Omega$ by arclength. This Ω_ε is an open set with oscillating boundary, converging to Ω as $\varepsilon \rightarrow 0$ (figure 1). Finally, let $G:H^1(\Omega) \rightarrow [0,\infty]$ be an $L^2(\Omega)$ -lower semicontinuous functional such that $G(u) \geq \int_\Omega |\nabla u|^2 dx$. We are interested in the asymptotic behavior as $\varepsilon \rightarrow 0$ of the minimization problems

$$(2.1) \quad \min \left\{ G(u) + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u|^2 dx + 2 \int_{\Omega_\varepsilon} f(x)u dx : u \in H_0^1(\Omega_\varepsilon) \right\}.$$

If $G(u) = \int_\Omega |\nabla u|^2 dx$ then this is the variational formulation of (1.1).

Our analysis uses the notion of Γ -convergence introduced by

DeGiorgi and Franzoni in [12]. We recall the definition and the main properties of interest here (see [1,6,10,11]) for more information).

Definition 2.1: If X is a metric space, let $\{F_h\}$ be a sequence of functions from X into $\mathbb{R}$, and let $x \in X$. Set

$$(2.2) \quad \Gamma^-(X) \liminf_{h \rightarrow \infty} F_h(x) = \inf \{ \liminf_{h \rightarrow \infty} F_h(x_h) : x_h \rightarrow x \}$$

$$\Gamma^-(X) \limsup_{h \rightarrow \infty} F_h(x) = \inf \{ \limsup_{h \rightarrow \infty} F_h(x_h) : x_h \rightarrow x \} .$$

When these coincide and equal $F(x)$, we say that $\{F_h\}$ Γ -converges to F at the point x, and we write $F(x) = \Gamma^-(X) \lim F_h(x)$.

The utility of Γ -convergence in the calculus of variations is due to the fact that, with a mild coercivity hypothesis, the Γ -convergence of functionals implies the convergence of their minimizers:

Proposition 2.2. Let $\{F_h\}$ be a sequence of functions from X to $\mathbb{R}$. Assume that

- (i) for each $x \in X$, $F(x) = \Gamma^-(X) \lim F_h(x)$;
- (ii) the functions F_h are equicoercive on X, i.e. for each $t \in \mathbb{R}$ there exists a compact subset K_t of X such that $\{x \in X : |F_h(x)| \leq t\} \subset K_t$ for every h.

Then $\liminf_h [\inf_X F_h] = \min_X F$. Moreover, if $\{x_h\}$ is a sequence in X with $\liminf_h F_h(x_h) = \min_X F$ and if $x_h \rightarrow x$, then x is a minimum point of F on X.

A second important property is that Γ -convergence is unaffected by the addition of a "lower order term":

Proposition 2.3: Let $\{F_h\}$ be a sequence of functions from X to $\mathbb{R}$ which Γ -converges to F . If $H: X \rightarrow \mathbb{R}$ is finite and continuous at $x \in X$, then $\Gamma(X) \lim_h [F_h + H](x) = F(x) + H(x)$.

These definitions and properties extend in the obvious way to a family of functions F_ϵ , where $\epsilon \rightarrow 0^+$ is a real parameter.

By virtue of these results, our task is reduced to studying the $\Gamma(X)$ convergence of the functionals

$$F_\epsilon(u) = G(u) + \epsilon \int_{\Sigma_\epsilon} |\nabla u|^2 dx + \chi_{H_0^1(\Omega_\epsilon)},$$

where χ_A is the indicator function of a set A (equal to 0 on A and $+\infty$ elsewhere). We must specify a space X in which the family $\{F_\epsilon\}$ is equicoercive, and calculate the associated Γ -limit F_0 . The minimizers of (2.1) will then converge as $\epsilon \rightarrow 0$ to a solution of the limit problem $\min[F_0(u) + 2 \int f u dx]$. The natural choice for X is $L^2(\mathbb{R}^2)$, since the family $\{F_\epsilon\}$ is equicoercive in $L^2(\mathbb{R}^2)$ if $\liminf_{\epsilon \rightarrow 0} \epsilon/R_\epsilon > 0$ (see Theorem III.3 of [2]).

The Γ -limit of F_ϵ is not hard to compute when $R_\epsilon \ll \epsilon$ or $R_\epsilon \gg \epsilon$. Identifying $u \in H^1(\Omega)$ with its extension by zero to all of $\mathbb{R}^2$, the methods in [2] give

$$R_\epsilon \ll \epsilon \Rightarrow \Gamma(L^2(\mathbb{R}^2)) \lim F_\epsilon(u) = G(u) + \chi_{H_0^1(\Omega)}$$

$$R_\epsilon \gg \epsilon \Rightarrow \Gamma(L^2(\mathbb{R}^2)) \lim F_\epsilon(u) = G(u).$$

When $R_\epsilon \sim \epsilon$, however, the situation is different: the Γ -limit of F_ϵ

depends on the behavior of the ratio R_ϵ/S_ϵ . Its form is given by the following theorem, which is our main result.

Theorem 2.4: Assume that $\lim \epsilon/R_\epsilon = E \in (0, \infty)$ and that
 $\lim R_\epsilon/S_\epsilon = k \in [0, \infty]$. Then for each $u \in H^1(\Omega)$,

$$\Gamma^-(L^2(\mathbb{R}^2)) \lim F_\epsilon(u) = G(u) + E c_k \int_{\partial\Omega} |u(y)|^2 dy ,$$

where the constant c_k is defined by

$$(2.3) \quad c_k = \min \left\{ \int_0^L ds \int_0^{d(s)} (k^2 \left| \frac{\partial v}{\partial s} \right|^2 + \left| \frac{\partial v}{\partial t} \right|^2) dt : v(s, 0) = 1, v(s, d(s)) = 0, \right. \\ \left. v(0, t) = v(L, t) \right\} .$$

Here $\int_0^L f ds$ denotes the average value of a periodic function f on $[0, L]$. It is easy to see that

$$(2.4) \quad c_0 = \int_0^1 \frac{1}{d(s)} ds \quad \text{and} \quad c_\infty = \frac{1}{m} \quad (m = \min d(s)) ,$$

achieved (if d is differentiable) by

$$(2.5) \quad w_0 = 1 - t/d(s) \quad \text{and} \quad w_\infty = (1 - t/m)_+ ,$$

respectively.

3. Calculation of the Γ -limit

This section presents the proof of Theorem 2.4. Throughout we shall write L^2 and H^1 for $L^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$, respectively, and we identify $u \in H^1(\Omega)$ with $u \cdot 1_\Omega$, its extension by zero to all of $\mathbb{R}^2$. Setting

$$F^+(u) = \Gamma^-(L^2) \limsup F_\varepsilon(u) , \quad F^-(u) = \Gamma^-(L^2) \liminf F_\varepsilon(u) ,$$

our task is to show that

$$(3.1) \quad \forall u \in H^1(\Omega) , \quad F^+(u) \leq G(u) + Ec_k \int_{\partial\Omega} |u(y)|^2 dy$$

$$(3.2) \quad \forall u \in H^1(\Omega) , \quad F^-(u) \geq G(u) + Ec_k \int_{\partial\Omega} |u(y)|^2 dy .$$

Since $\partial\Omega$ is $C^{1,1}$, each $u \in H^1(\Omega)$ has an extension $\tilde{u} \in H^1$. Also, since d can be uniformly approximated by smooth functions, there is no loss of generality in assuming that it is smooth. In the estimates that follow, we shall use the same letter C for all positive constants.

Proof of (3.1): Let $w(s,t)$ be the solution of the minimization problem (2.3). It extends by periodicity to all (s,t) with $0 \leq t \leq d(s)$; and it is Lipschitz continuous,

$$(3.3) \quad \left| \frac{\partial w}{\partial s} \right|^2 + \left| \frac{\partial w}{\partial t} \right|^2 \leq C ,$$

because it solves an elliptic differential equation. (In the extreme cases $k = 0$ and $k = \infty$, w is given by (2.5), which are Lipschitzian by inspection.)

If $(s(x), t(x))$ are the tangential and normal coordinates of a point $x \in \Sigma_\epsilon$, then we define for every $u \in H^1(\Omega)$ and every $\epsilon > 0$

$$u_\epsilon(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ \tilde{u}(x) w\left(\frac{s(x)}{S_\epsilon}, \frac{t(x)}{R_\epsilon}\right) & \text{if } x \in \Omega_\epsilon \setminus \Omega, \end{cases}$$

where $\tilde{u}$ is an H^1 extension of u . This u_ϵ lies in $H_0^1(\Omega_\epsilon)$, and it is easy to see that $u_\epsilon \rightarrow u|_\Omega$ in L^2 . Moreover,

$$\begin{aligned} (3.4) \quad \limsup_{\epsilon} \epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx &= \limsup_{\epsilon} \epsilon \int_{\Sigma_\epsilon} |\tilde{u}|^2 |\nabla [w\left(\frac{s(x)}{S_\epsilon}, \frac{t(x)}{R_\epsilon}\right)]|^2 dx \\ &= \limsup_{\epsilon} \epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} |\tilde{u}(s, t)| [S_\epsilon^{-2} |\frac{\partial w}{\partial s}\left(\frac{s}{S_\epsilon}, \frac{t}{R_\epsilon}\right)|^2 + R_\epsilon^{-2} |\frac{\partial w}{\partial t}\left(\frac{s}{S_\epsilon}, \frac{t}{R_\epsilon}\right)|^2] dt \\ &= \limsup_{\epsilon} E \int_0^L ds \int_0^{d(s/S_\epsilon)} |\tilde{u}(s, t R_\epsilon)|^2 [k^2 |\frac{\partial w}{\partial s}\left(\frac{s}{S_\epsilon}, t\right)|^2 + |\frac{\partial w}{\partial t}\left(\frac{s}{S_\epsilon}, t\right)|^2] dt. \end{aligned}$$

Since

$$\tilde{u}(s, t R_\epsilon) = u(s, 0) + \int_0^{t R_\epsilon} \frac{\partial \tilde{u}}{\partial t}(s, \tau) d\tau,$$

we have

$$\begin{aligned} (3.5) \quad |\tilde{u}(s, t R_\epsilon)|^2 &\leq |u(s, 0)|^2 + 2|u(s, 0)| \int_0^{M R_\epsilon} \left| \frac{\partial \tilde{u}}{\partial t}(s, \tau) \right| d\tau \\ &\quad + M R_\epsilon \int_0^{M R_\epsilon} \left| \frac{\partial \tilde{u}}{\partial t}(s, \tau) \right|^2 d\tau \end{aligned}$$

for every $t \in [0, d(s/S_\epsilon)]$. Combining (3.3) - (3.5) with the facts that

$$\lim \int_0^L \int_0^{MR_\epsilon} |u(s, 0)| \cdot \left| \frac{\partial \tilde{u}}{\partial \tau} (s, \tau) \right| d\tau ds = 0$$

$$\lim MR_\epsilon \int_0^L \int_0^{MR_\epsilon} \left| \frac{\partial \tilde{u}}{\partial t} (s, \tau) \right|^2 d\tau ds = 0$$

we conclude that $\limsup \epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx$ is bounded by

$$\begin{aligned} & \limsup E \int_0^L ds |u(s, 0)|^2 \int_0^{d(s/S_\epsilon)} [k^2 \left| \frac{\partial w}{\partial s} \left(\frac{s}{S_\epsilon}, t \right) \right|^2 + \left| \frac{\partial w}{\partial t} \left(\frac{s}{S_\epsilon}, t \right) \right|^2] dt \\ &= E \int_0^L |u(s, 0)|^2 ds \left\{ \int_0^L d\sigma \int_0^{d(\sigma)} [k^2 \left| \frac{\partial w}{\partial s} (0, t) \right|^2 + \left| \frac{\partial w}{\partial t} (0, t) \right|^2] dt \right\} \\ &= Ec_K \int_{\partial\Omega} |u(y)|^2 dy, \end{aligned}$$

and so

$$F^+(u) \leq \limsup F_\epsilon(u_\epsilon) \leq G(u) + Ec_K \int_{\partial\Omega} |u(y)|^2 dy.$$

Proof of (3.2): We must show that whenever $u_\epsilon \rightarrow u|_\Omega$ in L^2 ,

$$G(u) + Ec_K \int_{\partial\Omega} |u(y)|^2 dy \leq \liminf F_\epsilon(u_\epsilon).$$

By the definition of F_ϵ , we need only consider the case $u_\epsilon \in H_0^1(\Omega)$ and

$$(3.6) \quad \int_\Omega |\nabla u_\epsilon|^2 dx + \epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx \leq C.$$

Since G is L^2 -lower semicontinuous we have $G(u) \leq \liminf G(u_\epsilon)$, and our task is to prove that

$$(3.7) \quad Ec_k \int_{\partial\Omega} |u(y)|^2 dy \leq \liminf \epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx .$$

Notice that the sequence $\{u_\epsilon\}$ is uniformly bounded in $H^1(\Omega)$, by (3.6). Their traces on $\partial\Omega$ are therefore bounded in $H^{1/2}(\partial\Omega)$, and strongly convergent in $L^2(\partial\Omega)$ to the boundary trace of u .

The case $k = 0$ is relatively easy. On the one hand

$$Ec_0 \int_{\partial\Omega} |u(y)|^2 dy = \lim \frac{\epsilon}{R_\epsilon} \int_0^L \frac{1}{d(s/S_\epsilon)} |u_\epsilon(s,0)|^2 ds ;$$

on the other hand

$$|u_\epsilon(s,0)|^2 = \left| \int_0^{R_\epsilon d(s/S_\epsilon)} \frac{\partial u_\epsilon}{\partial t}(s,t) dt \right|^2 \leq R_\epsilon d(s/S_\epsilon) \int_0^{R_\epsilon d(s/S_\epsilon)} |\nabla u_\epsilon(s,t)|^2 dt .$$

These combine to give (3.7) when $k = 0$.

The case $k = \infty$ is only slightly more difficult. We have

$$(3.8) \quad Ec_\infty \int_{\partial\Omega} |u(y)|^2 dy = \lim \frac{\epsilon}{mR_\epsilon} \int_0^L |u_\epsilon(s,0)|^2 ds ,$$

and

$$(3.9) \quad |u_\epsilon(s,0) - u_\epsilon(s,mR_\epsilon)|^2 = \left| \int_0^{mR_\epsilon} \frac{\partial u_\epsilon}{\partial t}(s,t) dt \right|^2 \\ \leq mR_\epsilon \int_0^{mR_\epsilon} |\nabla u_\epsilon(s,t)|^2 dt .$$

In addition, we claim that

$$(3.10) \quad \limsup \int_0^L |u_\epsilon(s,mR_\epsilon)|^2 ds = 0 ;$$

The desired (3.7) follows immediately from (3.8) - (3.10). To prove (3.10), consider any $\bar{s} \in [0, L]$ with $d(\bar{s}/S_\epsilon) = m$. For

$$s \in [\bar{s}, \bar{s} + LS_\epsilon) ,$$

$$\begin{aligned} |u_\epsilon(s, mR_\epsilon)|^2 &= \left| \int_0^{\bar{s}+LS_\epsilon-s} \langle \nabla u_\epsilon(s+t, mR_\epsilon+t) , (1,1) \rangle dt \right|^2 \\ &\leq 2LS_\epsilon \int_0^{\bar{s}+LS_\epsilon-s} |\nabla u_\epsilon(s+t, mR_\epsilon+t)|^2 dt . \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\bar{s}}^{\bar{s}+LS_\epsilon} |u_\epsilon(s, mR_\epsilon)|^2 ds &\leq 2LS_\epsilon \int_{\bar{s}}^{\bar{s}+LS_\epsilon} ds \int_0^{\bar{s}+LS_\epsilon-s} |\nabla u_\epsilon(s+t, mR_\epsilon+t)|^2 dt \\ &\leq 2LS_\epsilon \int_{\bar{s}}^{\bar{s}+LS_\epsilon} ds \int_{mR_\epsilon}^{mR_\epsilon+LS_\epsilon} |\nabla u_\epsilon(s, t)|^2 dt . \end{aligned}$$

It follows that

$$\begin{aligned} \limsup \int_0^L |u_\epsilon(s, mR_\epsilon)|^2 ds &\leq 2L \limsup S_\epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx \\ &\leq C \lim S_\epsilon/\epsilon = 0 , \end{aligned}$$

establishing (3.10).

Now suppose $k \in (0, \infty)$, and define

$$\bar{u}_\epsilon(s) = \int_s^{s+S_\epsilon} u_\epsilon(\sigma, 0) d\sigma , \quad v_\epsilon = w\left(\frac{s}{S_\epsilon} , \frac{t}{R_\epsilon}\right) ,$$

where w solves the minimization problem (2.3). Note that w is smooth,

since we are assuming that d is regular. Also, since $\{u_\epsilon\}$ are uniformly bounded in $H^{1/2}(\partial\Omega)$, $\bar{u}_\epsilon(s) \rightarrow u(s,0)$ in $L^2(0,L)$. The convexity of the function $|z|^2$ gives

$$\epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx \geq \epsilon \int_{\Sigma_\epsilon} \{\bar{u}_\epsilon^2 |\nabla v_\epsilon|^2 + 2\langle \nabla u_\epsilon - \bar{u}_\epsilon \nabla v_\epsilon, \bar{u}_\epsilon \nabla v_\epsilon \rangle\} dx ,$$

so that $\liminf \epsilon \int_{\Sigma_\epsilon} |\nabla u_\epsilon|^2 dx$ is bounded below by

$$(3.11) \quad \liminf \epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} [\bar{u}_\epsilon |\nabla v_\epsilon|^2 + 2\langle \nabla u_\epsilon - \bar{u}_\epsilon \nabla v_\epsilon, \bar{u}_\epsilon \nabla v_\epsilon \rangle] dt .$$

We evaluate the first term:

$$\begin{aligned} (3.12) \quad & \lim \epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} \bar{u}_\epsilon^2 |\nabla v_\epsilon|^2 dt \\ &= \lim E \int_0^L ds |\bar{u}_\epsilon(s)|^2 \int_0^{d(s/S_\epsilon)} [k^2 |\frac{\partial w}{\partial s}(\frac{s}{S_\epsilon}, t)|^2 + |\frac{\partial w}{\partial t}(\frac{s}{S_\epsilon}, t)|^2] dt \\ &= E \left\{ \int_0^L ds \int_0^{d(s)} [k^2 |\frac{\partial w}{\partial s}(s, t)|^2 + |\frac{\partial w}{\partial t}(s, t)|^2] dt \right\} \int_0^L |u(s, 0)|^2 ds \\ &= E c_k \int_{\partial\Omega} |u(y)|^2 dy . \end{aligned}$$

It remains to show that the other terms in (3.11) tend to zero.

Integration by parts gives

$$\begin{aligned}
 (3.13) \quad & 2\epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} \langle \nabla u_\epsilon - \bar{u}_\epsilon \nabla v_\epsilon, \bar{u}_\epsilon \nabla v_\epsilon \rangle dt \\
 & = 2\epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} [v_\epsilon \operatorname{div}(\bar{u}_\epsilon^2 \nabla v_\epsilon) - u_\epsilon \operatorname{div}(\bar{u}_\epsilon \nabla v_\epsilon)] dt \\
 & \quad + 2\epsilon \int_{\Gamma_\epsilon} \bar{u}_\epsilon (u_\epsilon - \bar{u}_\epsilon v_\epsilon) \frac{\partial v_\epsilon}{\partial \nu} dy,
 \end{aligned}$$

where $\Gamma_\epsilon = \partial\Omega \cup \partial\Omega_\epsilon$ is the boundary of the layer. We evaluate the boundary term first: since $u_\epsilon = v_\epsilon = 0$ on $\partial\Omega_\epsilon$, its limiting absolute value is

$$\begin{aligned}
 (3.14) \quad & \lim \left| \frac{2\epsilon}{R_\epsilon} \int_0^L \bar{u}_\epsilon(s) (\bar{u}_\epsilon(s) - u_\epsilon(s,0)) \frac{\partial w}{\partial t} \left(\frac{s}{S_\epsilon}, 0 \right) ds \right| \\
 & \leq C \lim \int_0^L |\bar{u}_\epsilon(s)| \cdot |\bar{u}_\epsilon(s) - u_\epsilon(s,0)| ds = 0.
 \end{aligned}$$

To evaluate the interior term, we recall that $\Delta w = 0$. Therefore

$$\begin{aligned}
 (3.15) \quad & \lim \left| 2\epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} [v_\epsilon \operatorname{div}(\bar{u}_\epsilon^2 \nabla v_\epsilon) - u_\epsilon \operatorname{div}(\bar{u}_\epsilon \nabla v_\epsilon)] dt \right| \\
 & = \lim \left| 2\epsilon \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} (2 v_\epsilon \bar{u}_\epsilon - u_\epsilon) \frac{\partial \bar{u}_\epsilon}{\partial s} \frac{\partial v_\epsilon}{\partial s} dt \right| \\
 & = \lim \left| \frac{2\epsilon}{S_\epsilon^2} \int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} (2 v_\epsilon \bar{u}_\epsilon - u_\epsilon) [u_\epsilon(s+S_\epsilon, 0) - u_\epsilon(s, 0)] \frac{\partial w}{\partial s} \left(\frac{s}{S_\epsilon}, \frac{t}{R_\epsilon} \right) dt \right| \\
 & \leq \lim \frac{C}{\sqrt{\epsilon}} \left(\int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} |\bar{u}_\epsilon|^2 + |u_\epsilon|^2 dt \right)^{1/2} \left(\int_0^L |u_\epsilon(s+S_\epsilon, 0) - u_\epsilon(s, 0)|^2 ds \right)^{1/2}.
 \end{aligned}$$

Since

$$|u_\epsilon(s, t)|^2 \leq \left(\int_t^{R_\epsilon d(s/S_\epsilon)} |\nabla u_\epsilon(s, \tau)| d\tau \right)^2 \leq C R_\epsilon \int_0^{R_\epsilon d(s/S_\epsilon)} |\nabla u_\epsilon|^2 d\tau,$$

we have

$$\int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} |u_\epsilon(s,t)|^2 dt \leq CR_\epsilon^2 \int_0^L \int_0^{R_\epsilon d(s/S_\epsilon)} |\nabla u_\epsilon|^2 dt \leq C\epsilon ,$$

and so

$$\limsup \frac{1}{\sqrt{\epsilon}} \left(\int_0^L ds \int_0^{R_\epsilon d(s/S_\epsilon)} (|\bar{u}_\epsilon|^2 + |u_\epsilon|^2) dt \right)^{1/2} \leq C .$$

It follows, using the *boundedness* of $\{u_\epsilon(s,0)\}$ in $H^{1/2}(0,L)$, that the limit in (3.15) is zero. In combination with (3.11) - (3.14) this gives the desired assertion (3.7), completing the proof of (3.2).

4. Remarks and generalizations

Our limit problem depends on the constant c_k , defined variationally by (2.3). Though there is in general no explicit formula for this constant, we can nevertheless describe its dependence on k :

Proposition 4.1. The function $k \rightarrow c_k$ is increasing and continuous on $(0, \infty]$. It is continuous at $k = 0$, too, under the extra hypothesis that $d'(s)$ is square integrable on $[0, L]$.

Proof. It is obvious that c_k increases with k . Denoting by w_k the minimizer of (2.3), we have for any $0 < h < k < \infty$

$$\begin{aligned} c_k &\leq \int_0^L ds \int_0^{d(s)} (k^2 |\frac{\partial w_h}{\partial s}|^2 + |\frac{\partial w_h}{\partial t}|^2) dt = c_h + (k^2 - h^2) \int_0^L ds \int_0^{d(s)} |\frac{\partial w_h}{\partial s}|^2 dt \\ &\leq c_h + \frac{k^2 - h^2}{h^2} \int_0^L ds \int_0^{d(s)} (h^2 |\frac{\partial w_h}{\partial s}|^2 + |\frac{\partial w_h}{\partial t}|^2) dt = \frac{k^2}{h^2} c_h . \end{aligned}$$

Therefore

$$0 < h < k < \infty \Rightarrow c_h \leq c_k \leq \frac{k^2}{h^2} c_h ,$$

establishing the continuous dependence on $k \in (0, \infty)$.

For the behavior as $k \rightarrow \infty$, consider the family $\{w_k\}$. It is bounded in $H^1(\Sigma)$, where $\Sigma = \{(s, t): 0 < s < L, 0 < t < d(s)\}$, since $c_k \leq c_\infty$. Passing to a subsequence of necessary, there is a weak limit $\bar{w} \in H^1(\Sigma)$. It satisfies

$$\begin{aligned} \int_{\Sigma} \left| \frac{\partial \bar{w}}{\partial s} \right|^2 ds dt &\leq \liminf \int_{\Sigma} \left| \frac{\partial w_k}{\partial s} \right|^2 ds dt \leq L \liminf \frac{c_k}{k^2} \\ &\leq L \liminf c_{\infty}/k^2 = 0, \end{aligned}$$

so $\partial \bar{w}/\partial s = 0$. Also, $\bar{w}(s, 0) \equiv 1$ and $\bar{w}(s, d(s)) \equiv 0$. So

$$\begin{aligned} c_{\infty} &\leq \frac{1}{L} \int_{\Sigma} \left| \frac{\partial \bar{w}}{\partial t} \right|^2 ds dt \leq \frac{1}{L} \liminf \int_{\Sigma} \left| \frac{\partial w_k}{\partial t} \right|^2 ds dt \\ &\leq \liminf c_k, \end{aligned}$$

and it follows that $c_k \rightarrow c_{\infty}$ as $k \rightarrow \infty$.

For the continuity at 0 we suppose that d' is square integrable.

Then $w_0 = 1 - t/d(s)$ minimizes (2.3), and for any $k > 0$

$$c_k \leq \int_0^L ds \int_0^{d(s)} [k^2 \left| \frac{\partial w_0}{\partial s} \right|^2 + \left| \frac{\partial w_0}{\partial t} \right|^2] dt = c_0 + k^2 \int_0^L ds \int_0^{d(s)} \left| \frac{\partial w_0}{\partial s} \right|^2 dt.$$

It follows that $c_k \rightarrow c_0$ as $k \rightarrow 0$.

We explained in section one how our problem may be viewed as one of reinforcing a rod to increase its torsional rigidity. According to Proposition 4.1 the rigidity of the reinforced rod increases as k decreases; the most efficient reinforcing layer oscillates slowly ($k = 0$) or not at all.

Our discussion of Theorem 2.4 has been restricted for simplicity to simply connected domains in two dimensions, and to layers with perfectly periodic thickness variation. However, considerable generalization is possible. If $\Omega \subset \mathbb{R}^2$ is not simply connected, then

the theorem still holds, by repeating the proof for each connected component of $\partial\Omega$. Also, by a localization argument, it still holds when the layer is "locally periodic." This means that for almost all $\bar{y} \in \partial\Omega$ there exists a neighborhood U of $\bar{y}$ in $\mathbb{R}^2$ such that

$$U \cap \Sigma_\varepsilon = U \cap \{y + tv(y) : y \in \partial\Omega, 0 < t < R_\varepsilon d(\frac{s(y)}{S_\varepsilon})\},$$

for a suitable periodic function d and suitable positive numbers R_ε , S_ε . In this case the limit functional will be

$$G(u) + \int_{\partial\Omega} E(y) c_K(y) |u(y)|^2 dy,$$

where E and c_K are piecewise constant functions of y . Finally, the analysis extends - with some additional effort - to domains in $n > 2$ dimensions. In that case the layer thickness d depends on all $n - 1$ tangential coordinates of $\partial\Omega$. The proof of (3.1) remains unchanged; that of (3.2) is similar, but $\bar{u}_\varepsilon$ becomes the average over a cube of side S_ε , and the verification of (3.10) requires a slightly different argument. We leave the details of this extension to the industrious reader.

We present one last generalization, concerning the replacement of $|\nabla u|^2$ with $|\nabla u|^p$, $p > 1$. Using the methods of [2] one may study the $\Gamma^-(L^p)$ - limits of functionals of the form

$$G(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, \nabla u) dx + \chi_{W_0^{1,p}(\Omega_\varepsilon)}(u)$$

provided that G is L^p -lower semicontinuous, $G(u) \geq \int_{\Omega} |\nabla u|^p dx$, and f satisfies the following conditions:

- (i) for all x the function $f(x, \cdot)$ is convex;
- (ii) there exists a function $\omega: [0, \infty) \rightarrow [0, \infty)$ which is continuous and increasing, with $\omega(0) = 0$, such that

$$|f(x_1, z) - f(x_2, z)| \leq \omega(|x_1 - x_2|)(1 + |z|^p);$$
- (iii) for some $0 < \lambda < \Lambda$, $\lambda|z|^p \leq f(x, z) \leq \Lambda(1 + |z|^p)$;
- (iv) there is a nonnegative, continuous function $\gamma(x, z)$, convex and p -homogeneous in z , such that $|f(x, z) - \gamma(x, z)| \leq \rho(|z|)(1 + |z|^p)$, for some $\rho: [0, \infty] \rightarrow [0, \infty)$ that is continuous and decreasing, with $\rho(\infty) = 0$.

In this case the limit functional is

$$G(u) + E \int_{\partial\Omega} c_k(y) |u(y)|^p dy ,$$

with $E = \lim \varepsilon / (R_\varepsilon^{p-1})$ and

$$c_k(y) = \min \left\{ \int_0^L \int_0^{d(s)} \gamma(y, k\tau(y) \frac{\partial v}{\partial s}(s, t) + v(y) \frac{\partial v}{\partial t}(s, t)) dt : v(s, 0) = 1 , \right. \\ \left. v(s, d(s)) = 0 , v(0, t) = v(L, t) \right\} ,$$

in which $\tau(y)$ and $v(y)$ are the unit tangent and normal vectors to $\partial\Omega$ at y .

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Figure 1