EXPECTATION FUNCTIONALS ASSOCIATED WITH
SOME STOCHASTIC EVOLUTION EQUATIONS

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Expectation Functionals Associated with
Some Stochastic Evolution Equations*

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Abstract. Infinite-dimensional diffusion equations for the expectation functionals associated with some stochastic evolution equations in Hilbert spaces are studied. The asymptotic properties such as the boundedness and the stationary distributions of solutions are also discussed.

1. Introduction

Consider the Ito equation in $\mathbb{R}^n$:

$$(1.1) \quad dx_t = b(x_t)dt + \sigma(x_t)dw_t, \quad x_0 = x,$$

where the notations are standard. It is well-known [1] that, under suitable assumptions on the coefficients, the solution process $\{x_t\}$ is a Markov diffusion in $\mathbb{R}^n$. We denote by $q_t(x, dy)$ the associated transition probability function.

Let ϕ be a smooth function on $\mathbb{R}^n$ and define

$$F_t(x) = \int_{\mathbb{R}^n} \phi(y) q_t(x, dy).$$

Then F_t satisfies the Kolmogorov equation

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$$(1.2) \quad \frac{\partial F_t}{\partial t} = \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} F_t + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} F_t \equiv L F_t, F_0 = \phi,$$

$$\text{where } a_{ij} = \sum_{k=1}^n \sigma_{ik} \sigma_{kj}.$$

If q_t has a smooth density $f(x,t)$, it can be found by solving the Fokker-Planck equation

$$(3.1) \quad \frac{\partial f}{\partial t} = L^* f, f(x,0) = \delta(x),$$

where L^* is the formal adjoint of L and $\delta(x)$ is the Dirac-delta function. Due to these connections, the method of partial differential equations has been applied successfully to studying stochastic differential equations in $\mathbb{R}^n$. However this method has not been explored extensively in the case of stochastic partial differential equations. However, see the papers [2]-[5]. One of the main reasons is the difficulty in dealing with diffusion equations in infinite dimensions.

In the present paper, we shall first review some known results on some stochastic evolution equations in Hilbert spaces. Then we define the infinitesimal generator of a diffusion process as the solution to such a stochastic evolution equation. After proper definitions, the existence of solutions to the associated diffusion equations in infinite dimensions will be studied through a variational formulation. Finally we will present some results on the asymptotic properties, such as the boundedness and the stationary distribution of the solution as $t \rightarrow \infty$.

2. Some Stochastic Evolution Equations

Let H be a real, separable Hilbert space, and V a real, reflexive Banach space such that the inclusions $V \subset H \subset V'$ are densely defined and compact, where V' is the dual of V . We denote by $\|\cdot\|$, $|\cdot|$, $\|\cdot\|_*$ the norms in V , H and V' , respectively, and by $\langle \cdot, \cdot \rangle$, $(\cdot, \cdot)$ the inner product in H and the duality between V and V' .

Suppose that $A: V \rightarrow V'$ and $\Sigma: H \rightarrow H$ are bounded linear operators. We denote

by w_t a V' -valued Brownian motion [6] with

$$E w_t = 0$$

and

$$E \langle w_t, \phi \rangle \langle w_s, \psi \rangle = (t \wedge s) (\phi, \psi), \quad \forall \phi, \psi \in V.$$

Consider the linear stochastic equation

$$(P1) \quad du_t + (Au_t)dt = \Sigma dw_t, \quad 0 < t < T, \quad u_0 = g \in H.$$

Then we have the existence theorem

Theorem 2.1 Let A be coercive such that $(-A)$ generates a strongly continuous semigroup $\{T_t\}$ on H such that $(T_t \Sigma)$ is Hilbert-Schmidt $\forall t > 0$. Then the problem

$$(P1) \text{ has a unique solution } u_t \in L^2((0, T) \times \Omega; V) \cap C([0, T]; L^2(\Omega; H)).$$

Furthermore the solution is given by

$$(2.1) \quad u_t = T_t g + \int_0^t T_{t-s} \Sigma dw_s$$

Remark: $\{u_t\}$ is a Gauss-Markov process in H with the mean and the covariance operator given by

$$(2.2) \quad E u_t = T_t g,$$

$$(2.3) \quad \Gamma_t = \int_0^t (T_s R T_s^*) ds,$$

where

$$(2.4) \quad R = (\Sigma \Sigma^*).$$

The linear stochastic evolution equation has been treated by many authors. For details, see [7] - [9] among others.

As $t \rightarrow \infty$, for T_t being a contraction, we have

$$(2.5) \quad \lim_{t \rightarrow \infty} T_t g = 0, \quad \forall g \in H.$$

If, in addition,

$$\sup_{t > 0} \text{tr } \Gamma_t < \infty,$$

then there exists a nuclear operator Γ on H such that

$$(2.7) \quad \lim_{t \rightarrow \infty} (\Gamma_t h, h) = (\Gamma h, h) \quad \forall h \in H. \quad \text{Since } u_t \text{ is Gaussian, we expect that it}$$

converges weakly to a Gaussian random variable with mean zero and the covariance operator Γ .

Theorem 2.2 Suppose that the condition (2.6) is satisfied. Then the solution u_t of (P1) has the invariant (stationary) distribution $\mu \in N(0, \Gamma)$, a centered Gaussian distribution with the covariance operator Γ . In addition Γ satisfies the following equation

$$(2.8) \quad \langle A\Gamma g, h \rangle + \langle A\Gamma h, g \rangle = (Rg, h), \quad \forall g, h \in V. \quad \#$$

The weak convergence to the Gaussian invariant distribution can be shown by the method of characteristic functionals, as done by Chow [5] and Zabczyk [10], (see also [11] for further discussion).

Remarks: The covariance Γ may be determined from (2.8) explicitly for some special cases. In particular

(i) when A is self-adjoint,

$$\Gamma = \frac{1}{2} A^{-1} R;$$

(ii) when A is a normal operator and A commutes with R , we get

$$\Gamma = (A + A^*)^{-1} R.$$

Next we consider the nonlinear stochastic evolution equation

$$(P2) \quad du_t + B(u_t)dt = \Sigma dw_t, \quad 0 < t < T, \quad u_0 = g \in H,$$

where, in contrast with (P1), B is a nonlinear operator. For monotone nonlinear operators, the existence and uniqueness theorem has been proved by Bensoussan and Temam [12] and Pardoux [13]. Let us assume

(B1) $B: V \rightarrow V'$ satisfies

$$\|B(u)\|_* \leq \alpha \|u\|^{p-1} \quad \text{for some } \alpha > 0, p > 1, \quad \forall u \in V,$$

(B2) B is coercive: $\exists$ constants $\alpha > 0, \beta, \gamma \geq 0$

$$2 \langle B(u), u \rangle + \beta \|u\|^2 + \gamma \geq \alpha \|u\|^p + \text{tr. } R, \quad \forall u \in V,$$

(B4) B is monotone, that is,

$$2 \langle B(u) - B(v), u-v \rangle + \lambda |u-v|^2 > 0, \forall u, v \in V.$$

Then we have

Theorem 2.3 The problem (P2) has a unique solution u_t which is a diffusion process in H . Furthermore we have

$$u_t \in C([0, T]; L^2(\Omega; H)) \cap L^p((0, T) \times \Omega; V)_{\#}$$

Remark: This is a special case of a general existence theorem proved by Pardoux [13].

3. Infinite-Dimensional Diffusion Equations

Let $\{u_t\}$ be a diffusion process in H . For any Borel set $\Lambda \subset H$, define the transition probability

$$(3.1) \quad p_t(v, \Lambda) = \text{Prob. } \{u_t \in \Lambda \mid u_0 = v\}, v \in H,$$

and the transition operator

$$(3.2) \quad (P_t \phi)(v) = \int_H \phi(u) p_t(v, du),$$

where ϕ is a bounded-measurable function on H . For convenience, we set

$$(3.3) \quad F_t(v) = (P_t \phi)(v).$$

Consider the linear stochastic equation (P1), where A is assumed to be self-adjoint and strictly positive. By the Remark (i) following Theorem 2.2, (P1) has the invariant distribution $\mu \in N(0, \Gamma)$, where $\Gamma = \frac{1}{2} A^{-1}R$. Let $H = L^2(H, \mu)$ be the space of real functionals on H that are μ -square integrable. For a smooth functional $\phi \in H$, denote by $D^k \phi(h)$ the k -th Frechet derivative of ϕ at $h \in H$. Let $\{e_k\}$ be a complete orthonormal system (C.O.N.S.) in H with e_k 's $\in V$. If $f(x_1, x_2, \dots, x_k)$ is a real function in k variables, a smooth simple functional ϕ on H is of the form $\phi(v) = f(v_1, v_2, \dots, v_k)$, where f is C^∞ and $v_i = (v, e_{n_i})$, $i=1, 2, \dots, k$. We let S be the set of all smooth simple functionals on H , which grow at most like a polynomial.

For $\phi \in S$, as in $\mathbb{R}^n$, we expect the average functional F_t given by (3.3) would satisfy the diffusion equation

$$(3.4) \quad \partial_t F_t(v) = L_0 F_t(v), \quad t > 0, \quad F_0(v) = \phi(v), \quad v \in H,$$

where

$$(3.5) \quad L_0 \phi(v) = \frac{1}{2} \operatorname{tr.} [RD^2 \phi(v)] - \langle Av, D\phi(v) \rangle.$$

Note that, since ϕ is simple, $L_0 \phi(v)$ is well-defined, but, in general, it is just a formal expression. In (3.5) the term $\langle Av, D\phi(v) \rangle$ has no meaning for $\phi \in C_H^1$ and $v \in H$. To overcome this difficulty, one may want to restrict $V \in V$ and $D\phi \in V$ as a V -derivative. Then the term makes sense in V which is a μ -null subset of H . As it turns out, it is possible to define L_0 μ -a.e. by extending the domain of L_0 , say, from S to a much larger set. To this end we need a couple of lemmas. The first lemma is concerned with two integral identities.

Lemma 3.3 Suppose that $A = A^*$ and $AR = RA$ on S . Then $\forall \phi, \psi \in S$, we have

$$(3.6) \quad \int_H (L_0 \phi) \psi \, d\mu = \int_H \phi (L_0 \psi) \, d\mu = \int_H (RD\phi, D\psi) \, d\mu,$$

and
$$\int_H [L_0 \phi(v)]^2 \, d\mu(v)$$

$$(3.7) \quad = \int_H \left\{ \frac{1}{2} |\operatorname{tr.} RD^2 \phi(v)|^2 + |\langle ARD\phi(v), D\phi(v) \rangle|^2 \right\} \, d\mu(v). \quad \#$$

The first identity yields a Dirichlet integral, similar to that of Laplacian in $\mathbb{R}^n$. In a way one may regard L_0 as a Laplace operator on S with respect to μ . Since the results are essentially finite-dimensional, they can be verified directly by integrations by parts. Before giving the next lemma, we recall the Hermite polynomials in $\mathbb{R}^1$. The normalized Hermite polynomial $h_n(x)$ is defined by [14]

$$(3.8) \quad h_n(x) = (-1)^n (n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}); \quad n=0,1,2,\dots$$

Let $m = (m_1, m_2, \dots, m_k)$, $n = (n_1, n_2, \dots, n_k)$ be multiple indices and define the Hermite functionals on H

$$(3.9) \quad H_n^m(v) = H_{n_1, \dots, n_k}^{m_1, \dots, m_k}(v) = \prod_{j=1}^k h_{n_j} \left\langle \frac{\Gamma^{1/2} v, e_{m_j}}{2} \right\rangle,$$

where m_j and n_j are non-negative integers. It is clear that $H_n^m \in S \forall m, n$.

Lemma 3.2 In addition to the hypotheses of Lemma 3.2, suppose A has a complete set $\{e_n\}$ of orthonormal eigenfunctions. then the set $\{H_n^m\}$ of all Hermite functionals defined by (3.9) forms a C.O.N.S. in $H_{\#}$

Let λ_n be the eigenvalue corresponding to the eigenfunction e_n of A , and set

$$(3.10) \quad \Lambda_n^m = \sum_{j=1}^k n_j \lambda_j.$$

On S , we introduce a norm $\|\cdot\|_2$ defined by

$$(3.11) \quad \|\phi\|_2 = \left\langle \int_H \left[\frac{1}{2} |\text{tr. } R D^2 \phi|^2 + |\langle A R D \phi, D \phi \rangle|^2 + |\phi|^2 \right] d\mu \right\rangle^{1/2},$$

and let H_2 be the completion of S with respect to this norm.

By invoking the above lemmas, we are able to prove the following theorem.

Theorem 3.1 Under the hypotheses of Lemmas 3.1 and 3.2, the transition operators $\{P_t\}$ defined by (3.2) form a strongly continuous semigroup on H . The infinitesimal generator L_0 of the semigroup defined on S can be extended to a self-adjoint operator on H_2 . Furthermore we have

$$(3.12) \quad L_0 H_n^m = -\Lambda_n^m H_n^m, \quad \forall n, m. \quad \#$$

Remarks. The above results generalize that of Piech [15] who studied the Ornstein-Uhlenbeck semigroup on an abstract Wiener space. The proof relies heavily on the properties of Hermite functionals. Also, we note that, $\forall \phi \in H$,

$$\phi(v) = \sum_{m,n} \phi_n^m H_n^m(v), \text{ so that by (3.2)}$$

$$(3.13) \quad (-L_0)H_n^m = \sum_{m,n} \Lambda_n^m \phi_n^m H_n^m(v).$$

In view of (3.7), the norm (3.11) can be defined equivalently as follows

$$(3.14) \quad \|\phi\|_2^2 = \sum_{m,n} (1 + \Lambda_n^m)^2 (\phi_n^m)^2.$$

Finally we wish to point out that the assumptions on A and R are rather strong. They might be weakened so that it is still possible to construct the associated Hermite functionals.

For the nonlinear evolution equation (P2), the associated diffusion equation reads

$$(3.15) \quad \partial_t F_t = L F_t, \quad F_0 = \phi,$$

where

$$(3.16) \quad L\phi(v) = \frac{1}{2} \text{tr.} [RD^2\phi(v)] - \langle B(v), DF(v) \rangle, \text{ for } \phi \in S.$$

In this case the method of Hermite functionals is not of great value. Suppose that the nonlinear operator takes the special form

$$(3.17) \quad B(v) = Av + Q(v),$$

where A is as before and $Q: V \rightarrow H$ is bounded nonlinear operator. Then an alternative approach is provided by some weak formulation. In what follows, we shall sketch some basic ideas underlying such approach.

Introduce a new norm $\|\cdot\|_R$ on S defined by

$$(3.18) \quad \|\phi\|_R^2 = \int_H \{ (RD\phi, D\phi) + |\phi|^2 \} d\mu.$$

Let H_R be the closure of S in H with respect to the norm $\|\cdot\|_R$, and let H'_R be its dual space. By means of the Dirichlet integral (3.6), define the bilinear form $b(\phi, \psi)$ on $H_R \times H_R$ by

$$(3.19) \quad b(\phi, \psi) = (RD\phi, D\psi)_H + \int_H (RD\phi, D\psi) d\mu.$$

This suggests a variational formulation of the problem (3.4) as in the case of parabolic equations in finite dimensions [16]. In fact, since H is a separable

Hilbert and H_R is dense in H , the weak formulation for the evolution equation in a Hilbert space is applicable here. First, under some boundedness and smoothness assumptions on $Q(v)$, in addition to the hypotheses of Theorem 3.1, the bilinear form b defined by (3.19) is continuous on $H_R \times H_R$. Hence b defines a linear map $L: H_R \rightarrow H_R'$. This may be regarded as an extension of L on S given in (3.16). Next one can show that the form b is coercive, i.e. there exist constants $\alpha > 0$ and β such that

$$(3.21) \quad b(\phi, \phi) + \beta \|\phi\|_H^2 \geq \alpha \|\phi\|_{H_R}^2 \quad \forall \phi \in H_R.$$

Let us denote by Z the space of functions ψ_t such that $\psi_t \in L^2((0, T); H_R)$ and $\frac{d}{dt} \psi_t \in L^2((0, T); H_R')$, equipped with the norm

$$(3.22) \quad \|\psi\|_Z = \left(\int_0^T (\|\psi_t\|_{H_R}^2 + \|\frac{d}{dt} \psi_t\|_{H_R'}^2) dt \right)^{1/2}.$$

Now we regard the equation (3.15) as an evolution equation in H_R' , which is associated with the "energy" equation

$$(3.23) \quad \frac{1}{2} \|F_T\|_H^2 + \int_0^T b(F_t, F_t) dt = \frac{1}{2} \|F_0\|_H^2, \quad F \in Z.$$

If we multiply the equation (3.15) by a smooth function η_t and integrate by parts, we get

$$(3.24) \quad V(F, \eta) = (F_0, \eta_0)_H - (F_T, \eta_T)_H.$$

where

$$(3.25) \quad V(F, \eta) = \int_0^T \langle b(F_t, \eta_t) - \langle \langle \eta_t, F_t \rangle \rangle \rangle dt,$$

and $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the duality between H_R and H_R' . Introduce a Hilbert space X of functions defined by

$$X = \{ \xi \in L^2((0, T); H_R) : \xi_0 \in H \}$$

and its subspace

$$Y = \{\eta. \in Z: \xi_0 \in H, \xi_T = 0\},$$

where ξ_0, ξ_T stand for the right and left limits of ξ_t in H at $t = 0, T$, respectively. Then in view of (3.24), a variational formulation of the initial value problem (3.15) for the parabolic equation can be stated as follow:

Given $\phi \in H$, find $F. \in X$ such that

$$(3.26) \quad V(F., \eta.) = (\phi, \eta_0), \quad \forall \eta. \in T.$$

As mentioned before, under appropriate assumptions, the bilinear form b is coercive and continuous on $H_R \times H_R$. Consequently the bilinear form V on $X \times Y$ has similar properties. Thus one may appeal to a generalized version of Lax-Milgram Theorem [16] to assert that, for a given $\phi \in H$, the variational equation has a unique solution $F.$ in X , which is called the strong (v.s. classical) solution for the problem (3.15). A detailed account for the results described in this section will be given elsewhere [17].

Remarks Other analytic properties, such as regularity and bounds for the solution can also be discussed in the variational formulation. We wish to point out that, beside the references [2]-[5] and [15], Kuo [18] DaPrato [19] have done interesting work on infinite-dimensional parabolic equations.

4. Asymptotic Properties

We are interested in the asymptotic behavior of the nonlinear stochastic system (P2) as $t \rightarrow \infty$. In particular the existence of a stationary disribution for such a system will be discussed. For this to happen, the solution u_t must remain bounded for all time. Let ϕ be a strictly positive definite functional on H such that $\phi(h) \rightarrow \infty$ as $|h| \rightarrow \infty$. It is called a Liapunov functional if $\phi \in H_R$ and

$L \phi(h) \leq 0$, μ a.s. in H . Then the boundedness of solution is assured by a theorem proved by the author [20].

Theorem 4.1 If there exists a Liapunov functional ϕ on H and a uniformly positive-definite, locally bounded continuous functional ρ on H such that

$$(4.1) \quad L\phi(h) \leq \rho(h) \text{ a.s. in } h,$$

then, given $g \in H$, the solution u_t of (P2) is bounded a.s. $\forall t > 0$, i.e. $P\{\sup_{t>0} |u_t| < \infty\} = 1$. #

For a proof, one is referred to our paper [20] in which other results on stability of solutions were also given.

Let $\{e_n\}$ be a C.O.N.S. for H as given in §3 and $\pi_n: H \rightarrow H_n$, the projection onto the linear span H_n of $\{e_1, e_2, \dots, e_n\}$. We denote by $u_t^{(n)}$ the solution of a truncated system (P2):

$$(4.2) \quad du_t^{(n)} + \pi_n B(u_t^{(n)}) dt = \Sigma(\pi_n dw_t), \quad u_0^{(n)} = g^{(n)},$$

where $g^{(n)} = \pi_n g$. Let $x(t) = (x_1, \dots, x_n)(t)$ be the coordinate-vector of the solution $u_t^{(n)}$. Then $x(t)$ is a diffusion process in $\mathbb{R}^n$ with the infinitesimal generator

$$(4.3) \quad L_n \phi(x) = \frac{1}{2} \sum_{j,k=1}^n r_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \phi - \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} \phi,$$

where

$$(4.4) \quad r_{jk} = (Re_j, e_k), \quad b_j(x) = \langle B(\pi_n u), e_j \rangle, \quad j, k=1, 2, \dots, n.$$

Suppose $b_j \in C_b^1(\mathbb{R}^n)$. Then the stationary transition probability density function p_n is determined by the adjoint equation

$$(4.5) \quad L_n^* p_n = 0, \quad p_n \geq 0 \text{ and } \int_{\mathbb{R}^n} p_n(x) dx = 1.$$

This yields the stationary distribution ν_n given by [21]

$$(4.6) \quad \nu_n(D) = \int_D p_n(x) dx, \text{ for any Borel set } D \subset \mathbb{R}^n.$$

For example, if b_j is derivable from a potential V_n such that

$$(4.7) \quad b_j = \frac{1}{2} [R_n (\nabla V_n)]_j, \quad j = 1, 2, \dots, n,$$

where $R_n = [r_{ij}]$ is an $n \times n$ matrix. It is not hard to check that, in this case,

the stationary distribution is given by

$$(4.8) \quad \nu_n(D) = \frac{1}{N_n} \int_D e^{-V_n(x)} d\mu_n(x),$$

where

$$(4.9) \quad \mu_n = \mu \circ \pi_n^{-1}, \quad N_n = \int_{\mathbb{R}^n} e^{-V_n(x)} d\mu_n(x).$$

One expects that, under favorable conditions, the sequence $\{\nu_n\}$ of probability measures converges weakly to a stationary distribution for the problem (P2). This is indeed the case if there exists a C^1 -functional V on H of a sublinear growth such that the gradient

$$(4.10) \quad DV = 2R^{-1}Q \text{ in } H.$$

Then we can show that the problem (P2) has a stationary distribution ν given by

$$(4.11) \quad \nu(\Lambda) = \frac{1}{N} \int_{\Lambda} e^{-V(h)} d\mu(h), \quad \forall \Lambda \subset H, \quad N = \int_H e^{-V(h)} d\mu(h).$$

That is, the Radon-Nikodym derivative of ν with respect to μ is

$$(4.12) \quad \frac{d\nu}{d\mu} = \frac{1}{N} e^{-V}.$$

The above result generalizes the work of Marcus [22], who treated a nonlinear parabolic equation perturbed by a white noise. It may be regarded as our special case where A is the Laplacian, Q is a nonlinear function and $R = I$ an identity operator.

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