EXISTENCE RESULTS FOR DOUBLY NONLINEAR HIGHER ORDER PARABOLIC EQUATIONS ON UNBOUNDED DOMAINS

by

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ABSTRACT

We prove the existence of weak (or "energy") solutions of the homogeneous Dirichlet initial-boundary value problem for some equations of the form $\partial(Bu)/\partial t + Au = f$, where A and B are nonlinear monotone operators deriving from convex functionals and the spatial domain is an arbitrary open set of R^n. In particular, our existence theorem applies (for any $p,q > 1$ and any $m,n \geq 1$) if A and B are defined by:

$$Au = (-1)^m \sum_{|\alpha|=m} D^\alpha(|D^\alpha u|^{p-1}\operatorname{sgn} D^\alpha u), \quad (Bu)(x) = |u(x)|^{q-1}\operatorname{sgn} u(x)$$

We start from an existence result of Grange and Mignot [20] and follow some methods of Alt and Luckhaus [3]. In addition, we use Nikol'skii spaces (spaces involving Hölder conditions in the L^p metric) to perform a key compactness argument of the proof.

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1. Introduction

J.L. Lions [23, p. 525] points out the interest of studying evolution problems for doubly nonlinear equations

$$\frac{\partial}{\partial t}(Bu) + Au = f$$

where both A and B are nonlinear operators. A typical example is the initial-boundary value problem

$$\frac{\partial}{\partial t}(|u|^{q-1}\text{sgn } u) + A_1 u = f \quad \text{on} \quad \Omega \times (0,T) \equiv Q \tag{1.1}$$

u satisfies null Dirichlet boundary conditions on $\partial\Omega \times (0,T)$ (1.2)

$$u = u_0 \quad \text{on} \quad \Omega \quad \text{for} \quad t = 0 \tag{1.3}$$

$$A_1 u = (-1)^m \sum_{|\alpha|=m} D^\alpha(|D^\alpha u|^{p-1}\text{sgn } D^\alpha u) \tag{1.4}$$

where u is an extended real-valued function $u = u(x,t)$, $(x,t) \in \Omega \times (0,T)$, $\Omega \subset R^n$, $0 < T < \infty$ and the sum extends to all x-derivatives of order m. (Thus the order of operator A_1 and equation (1.1) is 2m). This problem is doubly nonlinear for $p \neq 2$ and $q \neq 2$. Initial condition (1.3) will be replaced by $Bu(0) = Bu_0$ for more general examples. For $p = 2$ $A_1 = (-\Delta)^m$ and for $m = 1$ $-A_1$ is the nonisotropic p-laplacian.

The case $m = 1$, $p = 2$, $f = 0$ of equation (1.1) is related to the well-known porous media equation

$$\frac{\partial v}{\partial t} - \Delta(|v|^M \text{ sgn } v) = 0$$

by the change $v = |u|^{q-1}$ sgn u, $q - 1 = 1/M$. Notice that $q > 1$ is equivalent to $M > 0$.

In this paper we prove an existence result (Theorem 5.1) which includes problem (1.1)-(1.4) for any $p,q > 1$, any $m,n \geq 1$ and any open set $\Omega \subset R^n$.

($p,q \in R$; $m,n \in N$). (1.2) holds in the sense that $u(\cdot,t) \in W(\Omega)$ for almost all $t \in (0,T)$, where the space $W(\Omega)$ is similar to the Sobolev space $W_0^{m,p}(\Omega)$. (The definition of $W(\Omega)$ is found in H5.1, Section 5; see also Appendix I).

A modified existence result (Theorem 8.1) allows the replacement of (1.1) by

$$\frac{\partial}{\partial t}(|u|^{q-1}\operatorname{sgn} u) + A_1 u + |u|^{r-1}\operatorname{sgn} u = f \qquad (1.5)$$

for any $r > 1$. (A_1 is given by (1.4)). Although the solutions given by these theorems are very weak, they have remarkable properties (finite speed of propagation, extinction in finite time, nonexistence of global nonnegative solutions, localization of the support) depending on p,q,m (and r in case (1.5)): see [9,10].

In Section2 we present some notations and results on convex functionals and monotone operators deriving from convex functionals. In Section 3 we prove an abstract formula of integration by parts (Proposition 3.1), following methods used by Mignot [25], Bamberger [6] and Alt and Luckhaus [3] to obtain related concrete results. This formula allows to handle the very weak derivative $\partial(Bu)/\partial t$ and implies an "energy" estimate for the solutions of Theorem 5.1. (This suggests the name "energy solutions" for them). Alt and Luckhaus [3] already exploit this kind of estimate. In the case of problem (1.1)-(1.4) with $f = 0$ the energy estimate reads (where $1/q' = 1 - 1/q$)

$$\frac{1}{q'}\int_\Omega |u(x,t)|^q dx + \sum_{|\alpha|=m}\int_0^t\int_\Omega |D^\alpha u(x,\tau)|^p dx\,d\tau = \frac{1}{q'}\int_\Omega |u_0(x)|^q dx$$

In Section 4 we state an abstract existence theorem of Grange and Mignot [20]. In Section 5 we state Theorem 5.1 and give some examples. Both A and B are assumed to be (singlevalued) derivatives of continuous convex functionals (on two different spaces). Since the operator A_1 of (1.4) is only partially coercive on unbounded domains, we set a coerciveness hypothesis for the operator B.

The proof of Theorem 5.1 (Section 6) begins applying the theorem of Grange and Mignot [20] to some approximating problems. Thus we avoid discretization. (Papers [3] and [20] use time discretization). The estimates for passing to the limit are obtained from Proposition 3.1. The problem of passing to the limit for the operator B is solved by means of a compactness lemma of Lions-Aubin (see Appendix IV) and the use of Nikol'skii spaces (see Appendixes II and III). Proposition 3.1 is again used at the end of the proof in order to perform a monotonicity argument for the operator A. General background for the proof is found in the book of Lions [23].

In section 7 and 8 we sketch extensions of Proposition 3.1 and Theorem 5.1 to sums of spaces and operators, following a device of Lions [23].

Related work

Grange and Mignot [20] prove an abstract existence theorem (see Section 4) which includes problem (1.1)-(1.4) if Ω is bounded and $1/q > 1/p - m/n$. (Paper [20] also considers multivalued operators). For the second order case (i.e. for $m = 1$), the existence results of Raviart [28] and Alt and Luckhaus [3] apply to problem (1.1)-(1.4) if Ω is bounded. Paper [3] includes systems of second order equations with nonhomogeneous boundary conditions and does not require power bounds for the operator B. Existence results for related doubly nonlinear second order equations are found in [21,4,11,2,22,17,19].

We do not consider here the questions of uniqueness and regularity. The above references and [25,6,7] include some results on these questions (only for second order problems).

Finally, we give some references on existence results for higher order problems if $q = 2$ or $p = 2$. The case $q = 2$ of (1.1) is well-known ($\forall m \geq 1$): see e.g. the books of Lions [23] and Brézis [14]. The case $p = 2$ (Ω bounded, $\forall m \geq 1$) can be dealt with Theorem 2 of Brézis [12].

Notations

X, Y: real Banach spaces

p' : conjugate exponent of p: $1/p + 1/p' = 1$

X' : topological dual of X

J^* : conjugate convex functional of J; see (2.3)

Ω : arbitrary open set of R^n

$\bar{\Omega}$: closure of Ω in R^n

$Q = \Omega \times (0,T)$

a.e.: almost everywhere

$x = (x_1,\dots,x_n) \in \Omega$

D^α (α is a multi-index of integers): a derivative with respect to $x_1 \dots x_n$ (t excluded) of order $|\alpha|$

$W(\Omega)$: Sobolev-like space: see H5.1 (Section 5).
$W(\Omega) = W_0^{m,p}(\Omega) \cap L^q(\Omega)$ if $1 < q < p$ or if Ω is bounded (see Appendix I).

$D'(Q)$: space of real-valued (Schwartz) distributions on Q

$D'(0,T;X)$: space of X-valued distributions on the real interval $(0,T)$.

$L^p(0,T;X)$: space of (classes of) measurable functions u from $(0,T)$ to X such that $\|u\|_X$ belongs to $L^p(0,T;R)$. It is reflexive if X is reflexive and $1 < p < \infty$.

We refer to Brézis [14, Appendix] for background material about the spaces of vector-valued functions $L^p(0,T;X)$.

2. On convex functionals and monotone operators deriving from convex functionals

In this section we present some notations and results on convexity to be used later. These results are known and we only sketch proofs of some specific statements. We refer to Barbu [8] or Ekeland and Teman [18] for proofs of fundamental results and further information.

Let Y be a real Banach space. We assume

$$\left.\begin{array}{l} J: Y \to R \text{ is a \underline{convex} continuous functional possessing} \\ \text{an everywhere defined Gateaux derivative } B: Y \to Y' \end{array}\right\} \tag{2.1}$$

Since B coincides with the subdifferential ∂J of J, B is a monotone operator and for each $u \in Y$ Bu is characterized by

$$J(w) - J(u) \geq (Bu, w - u) \quad \text{for all} \quad w \in Y \tag{2.2}$$

where $(\cdot,\cdot)$ stands for the duality between Y' and Y.

Remark 2.1. If Y is reflexive, hypothesis (2.1) implies that B is demicontinuous, i.e. continuous from Y strong to Y' weak. This follows as in the Hilbert space case (see Brézis [14, Corollary 2.5]) noting that monotone operators from Y to Y' are locally bounded (see e.g. Barbu [8, p. 44]).

Let $J^*: Y' \to R \cup \{+\infty\}$ be the conjugate convex functional of J, defined by

$$J^*(v) = \sup_{w \in Y} ((v,w) - J(w)) \tag{2.3}$$

Lemma 2.1 Under hypothesis (2.1):

I. J^* is (strongly and) weakly lower semicontinuous (l.s.c.) on Y' and takes at least a finite value.

II. $B^{-1} = \partial J^*$.

III. J^* is bounded below by $-J(0)$.

IV. J^* is coercive, i.e. $J^*(v) \to +\infty$ as $\|v\|_{Y'} \to +\infty$.

I and II are standard results. III follows taking $w = 0$ in (2.3).

Proof of IV. Let $S \subset Y'$; IV is equivalent to

$$J^*(S) \text{ bounded in } R \Rightarrow S \text{ bounded in } Y'.$$

From (2.3) for all $w \in Y$

$$|(v,w)| \leq J^*(v) + \max\{J(w) + J(-w)\}.$$

Thus $J^*(S)$ bounded implies

$$|(v,w)| \leq C_w \quad \forall v \in S, \forall w \in Y$$

where C_w depends only on w. And by the Principle of Uniform Boundedness (Banach-Steinhaus)

$$\|v\|_{Y'} \leq \text{Constant} \quad \forall v \in S.$$ Q.E.D.

Remark 2.2 In this paper J is finite everywhere, B is singlevalued, J^* may have the value $+\infty$ and B^{-1} may be multivalued and not everywhere defined. In Theorem 5.1 J^* is finite and continuous, while B^{-1} is everywhere defined (i.e., B is surjective). In Example 2.1 (see below) B^{-1} is a classical singlevalued operator. The usefulness of J^* in the present paper is due to the fact that $B^{-1} = \partial J^*$, i.e. J^* is a potential functional for B^{-1} (in a generalized sense).

Now we consider the functional $u \to J^*(Bu)$ from Y to R, which turns out to be finite everywhere.

Lemma 2.2 Under hypothesis (2.1):

I. $J^*(Bu) = (Bu,u) - J(u)$ for all $u \in Y$

II. The functional $u \to J^*(Bu)$ is continuous for all $u \in Y$ if Y is reflexive.

III. For all $u_1, u_2 \in Y$

$$J^*(Bu_2) - J^*(Bu_1) \geq (Bu_2 - Bu_1, u_1) \tag{2.4}$$

$$J^*(Bu_2) - J^*(Bu_1) \leq (Bu_2 - Bu_1, u_2) \tag{2.5}$$

Proof of I. From (2.3)

$$J^*(Bu) = \sup_{w \in Y} ((Bu,w) - J(w))$$

and from (2.2) for all $u,w \in Y$

$$(Bu,w) - J(w) \leq (Bu,u) - J(u).$$

Thus the former supremum is attained for $w = u$.

Proof of II. It is implied by I, B demicontinuous (see Remark 2.1) and J continuous.

Proof of III. (2.5) is obtained from (2.4) multiplying by -1 and interchaning u_1 and u_2. (2.4) follows from I and (2.2) with $u = u_2$ and $w = u_1$.

Remark 2.3 In general the functional $u \to J^*(Bu)$ is not convex. Nevertheless, in some important particular cases it is convex (see Example 2.1 below).

Example 2.1 (Operator B of equation (1.1)). Let Ω be an arbitrary open set of R^n, $Y = L^q(\Omega)$, $1 < q < \infty$. Taking

$$J(u) = \frac{1}{q} \int_\Omega |u(x)|^q dx$$

then $(Bu)(x) = |u(x)|^{q-1} \operatorname{sgn} u(x)$ and

$$J^*(v) = \frac{1}{q'} \int_\Omega |v(x)|^{q'} dx \ , \quad J^*(Bu) = \frac{1}{q'} \int_\Omega |u(x)|^q \, dx$$

(In this example B and B^{-1} are bijective and everywhere continuous).

3. Formula of integration by parts

The purpose of this section is to generalize in some directions the classical formula

$$\int_0^t \int_\Omega \frac{\partial \beta(u(x,\tau))}{\partial t} u(x,\tau) dx d\tau = \int_\Omega \int_0^t \frac{\partial v(x,\tau)}{\partial t} \beta^{-1}(v(x,\tau)) d\tau dx =$$

$$= \int_\Omega j^*(\beta(u(x,t))) dx - \int_\Omega j^*(\beta(u(x,0))) dx$$

where $\Omega \subset R^n$, $\beta(s) = \frac{dj(s)}{ds}$, $\beta^{-1}(s) = \frac{dj^*(s)}{ds}$, $v = \beta(u)$.

We shall prove an abstract result following methods used by Mignot [25], Bamberger [6] and Alt and Luckhaus [3] to obtain related concrete results.

Let X, Y be real Banach spaces. We set the following notations and hypotheses.

H3.1 Y is reflexive, $X \subset Y$, X is dense in Y and the imbedding $X \to Y$ is continuous. The compatible dualities (X',X) and (Y',Y) are both denoted $(\cdot,\cdot)$.

H3.2 J and B satisfy (2.1).

H3.3 $u \in L^p(0,T;X)$, $1 < p < \infty$, $0 < T < \infty$.

H3.4 Bu, $\frac{d}{dt}(Bu) \in L^{p'}(0,T;X')$ where the derivative d/dt is in the sense of (vector-valued) distributions of $D'(0,T;X')$, see e.g. Lions [23, p. 7].

Remark 3.1 Hypothesis H3.4 implies (see e.g. [23, p. 7] or [14, Appendix]) that Bu is continuous from $[0,T]$ to X' (eventually after redefining it on a set of zero measure of $[0,T]$). Thus $Bu(t)$ makes sense for all $t \in [0,T]$ and not only almost everywhere.

Proposition 3.1 Assume H3.1 to H3.4. Then

I. Bu is continuous from $[0,T]$ to Y' weak. In particular, $Bu \in L^\infty(0,T;Y')$.

II. For all $s,t \in [0,T]$, $s < t$,

$$J^*(Bu(t)) - J^*(Bu(s)) = \int_s^t \left(\frac{d}{dt} (Bu(\tau)),\ u(\tau)\right) d\tau \tag{3.1}$$

Remark 3.2 II implies that the function $t \to J^*(Bu(t))$ is absolutely continuous from $[0,T]$ to R and

$$\frac{d}{dt} (J^*(Bu(t))) = \left(\frac{d}{dt} (Bu(t)), u(t)\right) \quad \text{for almost all } t \in (0,T)$$

Remark 3.3 Let $g: R \to R$ be a continuous function. If the functional $v \to g(J^*(v))$ is uniformly convex (see e.g. Vainberg [30, p. 121]) or equal to an uniformly convex norm equivalent to $\|\cdot\|_{Y'}$, then Bu is continuous from $[0,T]$ to Y' (strong). Furthermore, $u \in C([0,T];Y)$ if, in addition, B is injective and B^{-1} is continuous on range (B). Thus, $u \in C([0,T]; L^q(\Omega))$ for Y and J as in Example 2.1, since the $L^{q'}$ norm, $1 < q' < \infty$, is uniformly convex.

Remark 3.4 Let Y be a Hilbert space (identified to its dual) and B the identity operator. Thus $J(u) = J^*(u) = (1/2)\|u\|_Y^2$. Then Proposition 3.1 and relation $u \in C([0,T];Y)$ are well-known and frequently used: see e.g. [23].

Proof of Proposition 3.1. The following proof is strongly influenced by that of Alt and Luckhaus [3, Lemma 1.5].

Step 1. The function $t \to J^*(Bu(t))$ is measurable because the function $u \to J^*(Bu)$ is continuous on Y (Lemma 2.2). Now $J^*(Bu) \in L^1(0,T;R)$ from (2.5), H3.3, H3.4 and J^* bounded below (Lemma 2.1).

Step 2. Let $0 < s < t < T$. From (2.5)

$$J^*(Bu(\tau)) - J^*(Bu(\tau - h)) \leq (Bu(\tau) - Bu(\tau - h), u(\tau))$$

holds for almost all $\tau \in (s,t)$ and all $h > 0$ small enough. Integrating in τ between s and t we obtain

$$\int_{t-h}^{t} J^*(Bu(\tau))d\tau - \int_{s-h}^{s} J^*(Bu(\tau))d\tau \leq \int_{s}^{t} (Bu(\tau) - Bu(\tau - h), u(\tau))d\tau \qquad (3.2)$$

We divide by h and let $h \to 0$. Hypothesis H3.4 implies (see Brézis [14, Appendix]) that the differential quotient converges (strongly) in $L^{p'}(0,T;X')$ to the derivative (notice that $1 < p' < \infty$). Since $J^*(Bu) \in L^1$ (Step 1), we obtain that

$$J^*(Bu(t)) - J^*(Bu(s)) \leq \int_{s}^{t} \left(\frac{d}{dt} (Bu(\tau)), u(\tau)\right)d\tau \qquad (3.3)$$

for almost all $s,t \in (0,T)$, $s < t$.

Step 3. Bu is (strongly) measurable from $[0,T]$ to Y', because B is demicontinuous (recall Remark 2.1). This is true without assuming Y' separable: see Brézis [13, Appendix IV]. From (3.3) and J^* bounded below $J^*(Bu) \in L^\infty(0,T;R)$. Since J^* is coercive (Lemma 2.1), $Bu \in L^\infty(0,T;Y')$. This and $Bu \in C([0,T];X')$, (recall Remark 3.1), imply that

$$Bu \in C([0,T];Y' \text{ \underline{weak}})$$

by Lemma 8.1 of Lions and Magenes [24, p. 297]. (Here we use the reflexivity of Y', but it is not necessary that X' be reflexive.) In particular, $Bu(t) \in Y'$ and $J^*(Bu(t))$ makes sense for all $t \in [0,T]$ (not only for almost all t). Since J^* is weakly l.s.c. (Lemma 2.1), the function $t \to J^*(Bu(t))$ is l.s.c. from $[0,T]$ to R.

Step 4. We are gonig to prove that

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h (Bu(\tau) - Bu(0),u(\tau))d\tau = 0 \tag{3.4}$$

For $p \neq \infty$ (i.e. $p' \neq 1$), Bu is Hölder continuous (of exponent $1/p$) from $[0,T]$ to X', because of H3.4 (see again [14, Appendix]). Thus

$$|(Bu(\tau) - Bu(0),u(\tau))| \le C\ \tau^{1/p}\ \|u(\tau)\|_X$$

holds for almost all $\tau \in (0,T)$, where we also use hypothesis H3.3. By Hölder inequality (notice that $(p'/p + 1)/p' = 1$)

$$\int_0^h |(Bu(\tau) - Bu(0),u(\tau))|d\tau \le C_1\ h\left(\int_0^h \|u(\tau)\|_X^p d\tau\right)^{1/p}$$

This and H3.3 imply (3.4).

For $p = \infty$ it is enough to observe that

$$|(Bu(\tau) - Bu(0),u(\tau))| \le C_2\ g(\tau)$$

where $g(\tau) = \|Bu(\tau) - Bu(0)\|_{X'}$ is a continuous function with $g(0) = 0$.

Step 5. From Step 3 $Bu(0) \in Y'$. We set

$$Bu(t) = Bu(0) \quad \text{for} \quad t < 0.$$

Thus we obtain trivially for $h > 0$

$$\int_{-h}^{0} J^*(Bu(\tau))d\tau = h\, J^*(Bu(0)) \tag{3.5}$$

We remark that extension to negative t will be used for $Bu(t)$ but not for $u(t)$. Repeating Step 2 with $s = 0$ and taking into account (3.4) and (3.5) we obtain for almost all $t \in (0,T)$

$$J^*(Bu(t)) - J^*(Bu(0)) \leq \int_0^t \left(\frac{d}{dt} (Bu(\tau)), u(\tau)\right) d\tau \tag{3.6}$$

Step 6 We set $Bu(t) = Bu(T)$ for $t > T$ and argue as in Step 5, considering $u(\tau + h)$ and $u(\tau)$ instead of $u(\tau)$ and $u(\tau - h)$ and using (2.4) instead of (2.5). We obtain for almost all $t \in (0,T)$

$$J^*(Bu(T)) - J^*(Bu(t)) \geq \int_t^T \left(\frac{d}{dt} (Bu(\tau)), u(\tau)\right) d\tau \tag{3.7}$$

Step 7 Finally, (3.6) and (3.7) hold for all $t \in [0,T]$ because $J^*(Bu)$ is l.s.c. (Step 3). Thus, we have proved (3.1) for $t = T$ and $s = 0$. The same proof applied to the interval $[s,t]$ shows that (3.1) holds for all $s,t \in [0,T]$, $s < t$. Q.E.D.

4. An existence theorem of Grange and Mignot

Let V_1 and V_2 be separable and reflexive real Banach spaces. We set the following hypotheses.

H4.1 V_1 and V_2 satisfy H3.1 with $X = V_1$ and $Y = V_2$. In addition, the imbedding $V_1 \to V_2$ is compact.

H4.2 Φ_A and A satisfy (2.1) with $J = \Phi_A$, $B = A$ and $Y = V_1$. Φ_B and B satisfy (2.1) with $J = \Phi_B$ and $Y = V_2$. A and B are bounded on bounded sets.

H4.3 (Coerciveness of A). There exists γ, $1 < \gamma < \infty$, such that

$$\liminf_{\|u\|_{V_1} \to \infty} \frac{\Phi_A(u)}{\|u\|_{V_1}^{\gamma}} > 0$$

H4.4 $u_0 \in V_1$

H4.5 $f \in L^{\infty}(0,T;V_1')$, $\frac{df}{dt} \in L^{\gamma'}(0,T;V_1')$

Theorem 4.1 (Grange-Mignot [20]). Under hypotheses H4.1 to H4.5, there exists u such that

$$u \in L^{\infty}(0,T;V_1), \quad Bu \in L^{\infty}(0,T;V_2'), \quad Au \in L^{\infty}(0,T;V_1')$$

$$Bu(0) = Bu_0$$

$$\frac{d}{dt}(Bu) + Au = f \quad \text{in} \quad L^{\infty}(0,T;V_1')$$

where the derivative d/dt is in the sense of (vector-valued) distributions of $D'(0,T;V_1')$.

Remark 4.1. In [20] A and B may be multivalued. We have stated the theorem for the singlevalued case.

5. Existence theorem (statement and examples)

Function spaces

Let Ω be an arbitrary open set of R^n. We shall use the real space $L^q(\Omega)$ and a Sobolev-like space $W(\Omega)$. We set the following notations and definitions

$$\|u\|_{W(\Omega)} = \|u\|_{L^q(\Omega)} + |u|_{W(\Omega)} \tag{5.1}$$

$$|u|_{W(\Omega)} = \sum_{\overline{m}<|\alpha|<m} \|D^\alpha u\|_{L^p(\Omega)} \tag{5.2}$$

where m and $\overline{m}$ are nonnegative integers and $|\alpha|$ is the order of the derivative D^α.

H5.1 Let $1 < p,q < \infty$. We define the space $W(\Omega) = W(\Omega,p,q,m,\overline{m})$ as the closure of $C_0^\infty(\Omega)$ in the real Banach space

$$\{u \in L^q(\Omega): \; D^\alpha u \in L^p(\Omega), \; \overline{m} < |\alpha| < m\}, \; 0 < \overline{m} < m, \; m > 1$$

with the norm defined by (5.1). (See also (5.2); in general, (5.2) is only a seminorm). We write $W'(\Omega)$ for the topological dual of $W(\Omega)$.

(Some facts on the space $W(\Omega)$ are collected in Appendix I.)

Hypotheses on the operator A

A is the (singlevalued) subdifferential of a convex functional I on $W(\Omega)$. We set the hypotheses on A in semi-abstract form.

H5.2 (Derivative of convex functional). I and A satisfy (2.1) with $J = I$, $B = A$ and $Y = W(\Omega)$.

H5.3 (Coerciveness with respect to the seminorm). For all $u \in W(\Omega)$

$$I(u) > c_1 \; |u|^p_{W(\Omega)} - c_2, \quad c_1 > 0$$

H5.4 (Boundedness). For all $u \in W(\Omega)$

$$\|Au\|_{W'(\Omega)} < c_3 \; \|u\|^{p-1}_{W(\Omega)} + c_4$$

Notice that H5.3, Lemma 2.2-I and Lemma 2.1-III imply

$$(Au,u) \geq c_1 |u|^p_{W(\Omega)} - c_5, \; c_1 > 0, \quad \forall u \in W(\Omega) \tag{5.3}$$

The operator B

Let $\beta\colon \Omega \times R \to R$ be a function and define B by

$$(Bu)(x) = \beta(x,u(x)) \tag{5.4}$$

H5.5 $\beta(x,s)$ is a Caratheodory function: measurable in x for all $s \in R$ and continuous in s for almost all $x \in \Omega$.

H5.6 $\beta(x,s)$ is nondecreasing in s for almost all $x \in \Omega$.

H5.7 For almost all $x \in \Omega$ and for all $s, s_1, s_2 \in R$

$$\beta(x,0) = 0 \tag{5.5}$$

$$|\beta(x,s)| \geq c_6 |s|^{q-1}, \; c_6 > 0, \text{ (coerciveness)} \tag{5.6}$$

$$|\beta(x,s_2) - \beta(x,s_1)| \leq \begin{cases} c_7 |s_2 - s_1|^{q-1} & \text{if } 1 < q \leq 2 \\ c_7(|s_1| + |s_2|)^{q-2} \, |s_2 - s_1| & \text{if } q > 2 \end{cases} \tag{5.7}$$

We note some easy consequences of these hypotheses on B. Setting

$$j(x,s) = \int_0^s \beta(x,\sigma)d\sigma, \quad J(u) = \int_\Omega j(x,u(x))dx,$$

then J and B satisfy (2.1) with $Y = L^q(\Omega)$, B and J are bounded on bounded sets and

$$J^*(Bu) \geq c_8 \, \|u\|^q_{L^q(\Omega)} \, , \; c_8 > 0, \quad \forall u \in L^q(\Omega) \tag{5.8}$$

Let us sketch the proof of (5.8). From (5.5) and (5.7) with $s_1 = 0$

$$|j(x,s)| \leq C|s|^q$$

Setting $w = \mu|v|^{q'-1}\text{sgn } v$ in (2.3) with $\mu > 0$ small enough (μ independent of v), we obtain

$$J^*(v) \geq C \int_\Omega |v|^{q'} dx, \quad c > 0, \quad \forall v \in L^{q'}(\Omega)$$

Now (5.8) follows from (5.6).

Remark 5.1 B is continuous by (5.7) or by a well-known result for Nemytskii operators on L^p spaces. B is also surjective, since it is monotone, continuous and coercive (from a theorem of Minty-Browder, see e.g. [8, p. 40] or [23, p. 171]). Furthermore, J^* is continuous. In fact, from H5.6, (5.5)-(5.6) $j(x,s) \geq C|s|^q$, $C > 0$. Then from (2.3) and $|vw| \leq \varepsilon|w|^q + C_\varepsilon|v|^{q'}$ we obtain

$$J^*(v) \leq C \int_\Omega |v|^{q'} dx, \quad \forall v \in L^{q'}(\Omega)$$

This implies that J^* is continuous, since it is convex, lower semicontinuous and everywhere finite: see e.g. [18,p. 13].

Statement of the existence theorem

Theorem 5.1 Assume H5.1 to H5.7 and

$$u_0 \in L^q(\Omega), \quad f \in L^{p'}(0,T;W'(\Omega)) \tag{5.9}$$

where $0 < T < \infty$. Then there exists u such that

$$u \in L^p(0,T;W(\Omega)) \tag{5.10}$$

$$Bu(0) = Bu_0 \tag{5.11}$$

$$\frac{d}{dt}(Bu) + Au = f \quad \text{in} \quad L^{p'}(0,T;W'(\Omega)) \tag{5.12}$$

where the derivative d/dt is in the sense of (vector-valued) distributions of $D'(0,T;W'(\Omega))$.

((5.12) holds if and only if <u>the differential equation holds in $D'(Q)$</u>: see Remark 5.3 below. In addition, the above relations imply (5.13) to (5.15): see below).

<u>Remark 5.2</u> From H5.4 and (5.10)

$$Au, \frac{d}{dt}(Bu) \in L^{p'}(0,T;W'(\Omega)) \tag{5.13}$$

Thus $Bu(0)$ makes sense (Remark 3.1). Furthermore, from Proposition 3.1

$$Bu \in C([0,T];\ L^{q'}(\Omega)\ \underline{\text{weak}}) \tag{5.14}$$

$$u \in L^{\infty}(0,T;L^{q}(\Omega)) \tag{5.15}$$

where we use also (5.8) to obtain (5.15). <u>For $\beta(x,s) = |s|^{q-1}\,\mathrm{sgn}\, s$ (Example 2.1)</u> we know (Remark 3.3) that <u>$u \in C([0,T];L^{q}(\Omega))$ and the initial condition (5.11) is equivalent to $u(0) = u_0$</u>.

<u>Remark 5.3</u> $L^{p'}(0,T;W'(\Omega))$ is the dual of $L^{p}(0,T;W(\Omega))$, because $W(\Omega)$ is reflexive. Since $C_0^{\infty}(\Omega)$ is dense in $W(\Omega)$, $L^{p'}(0,T;W'(\Omega))$ is identified to a subspace of $D'(Q)$. (More details are found in Appendix I.) Thus (5.12) holds if and only if $\partial(Bu)/\partial t + Au = f$ holds in $D'(Q)$: see Brezis [14, Appendix], specially Propositions A.6 and A.7 and Corollary A.2.

<u>Examples of operators A satisfying H5.2 to H5.4</u>

<u>Example 5.1</u>

$$Au = \sum_{\overline{m} < |\alpha| \le m} (-1)^{|\alpha|} D^{\alpha}(|D^{\alpha}u|^{p-1}\,\mathrm{sgn}\, D^{\alpha}u)$$

In particular, $A = A_1$ of (1.1) if $\overline{m} = m$.

<u>Example 5.2</u> Let $m = \overline{m}$,

$$Au = (-1)^m \sum_{|\alpha|=m} D^\alpha(|D^m u|_2^{p-2} D^\alpha u)$$

where $|D^m u|_2$ is the euclidean norm of the vector of all m-order derivatives. (For $m = 1$, $-A$ is the isotropic p-laplacian). The operators of Examples 5.2 and 5.3 are isotropic (i.e. rotationally invariant).

Example 5.3 Let m be even and $\overline{m} = m > 2$,

$$Au = \Delta^{m/2}(|\Delta^{m/2}u|^{p-1} \operatorname{sgn} \Delta^{m/2}u)$$

This operator satisfies H5.3 (coerciveness) because of Calderon-Zygmund inequality and the density of $C_0^\infty(\Omega)$ in $W(\Omega)$. (This inequality establishes that $\|\Delta^{m/2}u\|_p$ and $\|D^m u\|_p$ are equivalent on $C_0^\infty(\Omega)$ if $1 < p < \infty$).

Example 5.4 We give now a more general example. Let α be a multi-index of integers as those used for derivatives, $\xi = \{\xi_\alpha : \overline{m} \leq |\alpha| \leq m\}$ and R^N the vector space formed by the ξ. Consider the function

$$F: \Omega \times R^N \to R$$

such that $F(x,\xi)$ is measurable in x for each $\xi \in R^N$, convex and continuously derivable in ξ for almost all $x \in \Omega$. Set $A_\alpha = \partial F/\partial \xi_\alpha$ and assume

$$|A_\alpha(x,\xi)| \leq c_1|\xi|^{p-1} + g_1(x), \quad g_1 \in L^{p'}(\Omega), \quad \overline{m} \leq |\alpha| \leq m$$

$$|F(x,\xi)| \geq c_2|\xi|^p - g_2(x), \; c_2 > 0, \; g_2 \in L^1(\Omega).$$

Finally, set

$$Au = \sum_{\overline{m} \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, Du(x)), \quad I(u) = \int_\Omega F(x, Du(x))dx$$

where $Du(x) = \{D^\alpha u(x): \overline{m} \leq |\alpha| \leq m\}$. Then I and A satisfy hypotheses H5.2 to H5.4.

Notice that for $\overline{m} = 0$, A of Example 5.4 is a particular case of operator of the calculus of variations and of quasilinear elliptic operator in generalized divergence form (see e.g. [23] and [16], respectively).

Example for the operator B

Example 5.5 If we take (for any $q > 1$)

$$\beta(x,s) = a(x)|s|^{q-1}\text{sgn } s,\ a > 0,\ a \text{ and } 1/a \in L^\infty(\Omega)$$

the operator B defined by (5.4) satisfies H5.5 to H5.7. (Of course, for $a \equiv 1$ we obtain Example 2.1).

Remark 5.4 We shall use the functional I_G and the operator A_G induced by J and A on an open subset G of Ω. Since $C_0^\infty(\Omega)$ is dense in $W(\Omega)$, the operator L of zero extension from $W(G)$ to $W(\Omega)$ is an isometric isomorphism and commutes with D^α, $\overline{m} < |\alpha| < m$ (proof as for the Sobolev spaces $W_0^{m,p}$: see e.g. [1]). Thus if we define for $U \in W(G)$

$$I_G(U) = I(LU),\quad A_G U = L^* ALU$$

then I_G and A_G satisfy H5.2 to H5.4 with Ω replaced by G. Notice that $A_G U$ is the restriction to G of the distribution ALU. An analogous remark for the operator B is straightforward.

6. Proof of Theorem 5.1

We shall use the capital letter U_k for the solution of the approximating problem (defined on $\Omega_k \times (0,T)$) and the small letter u_k for the zero extension of U_k to $\Omega \times (0,T)$.

Approximating problems

Let $\{G_k\}$ be an increasing sequence of open balls covering R^n. We set

$$\Omega_k = \Omega \cap G_k$$

Let $\{f_k\}$ be a sequence of $C_0^\infty((0,T);W'(\Omega))$ such that

$$f_k \to f \quad \text{in} \quad L^{p'}(0,T;W'(\Omega)) \tag{6.1}$$

(Convergence will be in the strong sense unless otherwise stated). Let $\{u_{0k}\} \subset C_0^\infty(\Omega)$ and G_k be so chosen that

$$\text{support } u_{0k} \subset \Omega_k, \quad u_{0k} \to u_0 \quad \text{in} \quad L^q(\Omega) \tag{6.2}$$

We shall use the same symbols for the restrictions of u_{0k} to Ω_k and of f_k to $\Omega_k \times (0,T)$. Consider the equation

$$\frac{d}{dt}(B_k U_k) + A_k U_k + \frac{1}{k}|U_k|^{\lambda-1} \operatorname{sgn} U_k = f_k \quad \text{on} \quad \Omega_k \times (0,T) \tag{6.3}$$

where A_k and B_k are the operators induced on Ω_k by A and B (see Remark 5.4) and

$$\lambda > \max\{p,q\}$$

We are going to apply the existence theorem of Grange-Mignot (Theorem 4.1) in the following way. We take

$$V_1 = W_0^{m,p}(\Omega_k) \cap L^\lambda(\Omega_k), \qquad V_2 = L^q(\Omega_k)$$

(The norm of the intersection is defined as the sum of the norms). The imbedding $V_1 \to V_2$ is compact, since the weak convergence in L^λ and the a.e. convergence imply the strong convergence in L^q if $q < \lambda$ and the domain is bounded (see e.g. [23, p. 144]). The operator A of Theorem 4.1 is now defined by

$$u \to A_k u + (1/k)|u|^{\lambda-1}\operatorname{sgn} u$$

Hypothesis H4.3 holds with $\gamma = p$. This follows from H5.3, $p < \lambda$ and Poincaré-Friedrichs inequality. (This inequality implies that the seminorm $|u|_W$ is equivalent on bounded domains to the norm of $W_0^{m,p}$). Thus, from Theorem 4.1 equation (6.3) has a solution $U_k \in L^\infty(0,T;V_1)$ such that

$$B_k U_k(0) = B_k u_{0k} \quad \text{on} \quad \Omega_k$$

Estimates of u_k

We take the duality product of (6.3) and U_k, integrate in t between 0 and t and apply Proposition 3.1. In addition, we consider the zero extension u_k of U_k to $\Omega \times (0,T)$. Notice that $(A_k U_k, U_k) = (Au_k, u_k)$ by Remark 5.4. So we obtain for all $t \in [0,T]$

$$\begin{aligned} J^*(Bu_k(t)) + \int_0^t (Au_k(\tau),u_k(\tau))d\tau + \frac{1}{k}\int_0^t\int_\Omega |u_k|^\lambda dx\, d\tau = \\ = \int_0^t (f_k(\tau),u_k(\tau))d\tau + J^*(Bu_k(0)) \end{aligned} \tag{6.4}$$

(The initial condition for u_k reads $Bu_k(0) = Bu_{0k}$ by the first relation of (6.2)). From (6.2) and Lemma 2.2, $J^*(Bu_k(0))$ is bounded (independently of k). Now, the coerciveness properties (5.3) and (5.8) imply in the usual way (see e.g. [23, p. 163]) that

$$u_k \text{ is bounded in } L^\infty(0,T;L^q(\Omega)) \text{ and in } L^p(0,T;W(\Omega)) \tag{6.5}$$

Thus from H5.4 and the boundedness of B

$$\begin{aligned} &Au_k \text{ is bounded in } L^{p'}(0,T;W'(\Omega)) \\ &Bu_k \text{ is bounded in } L^\infty(0,T;L^{q'}(\Omega)) \\ &Bu_k(T) \text{ is bounded in } L^{q'}(\Omega) \end{aligned}$$

(We recall that Bu_k is continuous from $[0,T]$ to $L^{q'}(\Omega)$ weak and thus $Bu_k(T)$ makes sense and belongs to $L^{q'}(\Omega)$). Therefore, there exist u,v,ξ,χ and a subsequence of $\{u_k\}$ (which we denote as the original sequence) such that

$$u_k \to u \quad \text{in} \quad L^\infty(0,T;L^q(\Omega)) \quad \text{weak star} \tag{6.6}$$

$$u_k \to u \quad \text{in} \quad L^p(0,T;W(\Omega)) \quad \text{weak} \tag{6.7}$$

$$Bu_k \to v \quad \text{in} \quad L^\infty(0,T;L^{q'}(\Omega)) \quad \text{weak star} \tag{6.8}$$

$$\frac{d}{dt}(Bu_k) \to \frac{dv}{dt} \quad \text{in} \quad D'(0,T;L^{q'}(\Omega)) \quad \text{and in} \quad D'(Q) \tag{6.9}$$

$$Bu_k(T) \to \xi \quad \text{in} \quad L^{q'}(\Omega) \quad \text{weak} \tag{6.10}$$

$$Au_k \to \chi \quad \text{in} \quad L^{p'}(0,T;W'(\Omega)) \quad \text{weak} \tag{6.11}$$

In addition, we are going to prove that

$$w_k \equiv (1/k)|u_k|^{\lambda-1} \operatorname{sgn} u_k \to 0 \quad \text{in} \quad L^{\lambda'}(Q) \tag{6.12}$$

From (6.4) and (6.5)

$$(1/k) \int_0^T \int_\Omega |u_k|^\lambda \, dxdt < C \quad \text{(a constant)}$$

$$\int_0^T \int_\Omega |w_k|^{\lambda'} \, dxdt = (1/k^{\lambda'}) \int_0^T \int_\Omega |u_k|^\lambda \, dxdt < C/k^{\lambda'-1}$$

which proves (6.12).

Passing to the limit in the differential equation

Let be any $\phi \in C_0^\infty(Q)$. We use ϕ as test function for (6.3). Since support $\phi \subset \Omega_k \times (0,T)$ for k large enough, we can replace U_k by its zero extension u_k. Furthermore, taking into account that (6.1), (6.9), (6.11) and (6.12) imply the convergence in $D'(Q)$, we successively obtain:

$$dv/dt + \chi = f \tag{6.13}$$

$$dv/dt \in L^{p'}(0,T;W'(\Omega)) \tag{6.14}$$

$$\frac{d}{dt}(Bu_k) \to \frac{dv}{dt} \text{ weakly in } L^{p'}(0,T;W'(\Omega)) + L^{\lambda'}(Q) \tag{6.15}$$

Proof of $v = Bu$

This is the key point of the existence proof. Let G be an open ball such that $\overline{G} \subset \Omega$. We shall use a compactness lemma of Lions-Aubin (Lemma IV.1, in Appendix IV) and the compact imbedding

$$H^{s,p_0}(G) \to L^{p_0}(G)$$

where $H^{s,p_0}(G)$ is a Nikol'skii space (see Appendix II). In fact

$$Bu_k \text{ is bounded in } L^{p_0}(0,T;H^{s,p_0}(G)) \tag{6.16}$$

This follows from (6.5) and Lemma III.3 (in Appendix III) taking $p_0 < p$ (and $p_0 > 1$).

Now we apply Lemma IV.1 with $v_k = Bu_k$, $p_1 = \lambda'$,

$$E_0 = H^{s,p_0}(G),\ E = L^{p_0}(G),\ E_1 = W'(G) + L^{\lambda'}(G) + L^{p_0}(G)$$

Recall that $\lambda > p$ (thus $\lambda' < p'$) and notice that $L^{\lambda'}(Q) = L^{\lambda'}(0,T;L^{\lambda'}(\Omega))$. Thus from Lemma IV.1, (6.15), (6.16) and (6.8)

$$Bu_k \to v \ \underline{\text{(strongly)}} \text{ in } L^{p_0}(0,T;L^{p_0}(G)) = L^{p_0}(Gx(0,T)) \tag{6.17}$$

and for some subsequence

$$Bu_k \to v \ \underline{\text{a.e.}} \text{ on } Gx(0,T) \tag{6.18}$$

Now we use a monotonicity argument (we follow e.g. [3, p. 323]). For any $R > 0$, we define a truncation operator P_R by

$$(P_R w)(x) = \begin{cases} -R & \text{if } w(x) < -R \\ w(x) & \text{if } |w(x)| \leq R \\ R & \text{if } w(x) > R \end{cases}$$

Take any $\varepsilon > 0$ andy any $\phi \in L^q(Gx(0,T))$. Since $P_R(B(u + \varepsilon\phi) - Bu_k)$ is uniformly bounded (for fixed R), (6.18) implies that as $k \to \infty$

$$P_R(B(u + \varepsilon\phi) - Bu_k) \to P_R(B(u + \varepsilon\phi) - v)$$

a.e. in $Gx(0,T)$ and strongly in $L^\mu(Gx(0,T))$ for any $\mu \geq 1$, $\mu \neq \infty$. In particular, in $L^{q'}$. This, (6.6) and the monotonicity of B imply that as $k \to \infty$

$$0 \leq \int_0^T \int_G P_R(B(u + \varepsilon\phi) - Bu_k) \cdot (u + \varepsilon\phi - u_k)dxdt \to$$

$$\to \varepsilon \int_0^T \int_G P_R(B(u + \varepsilon\phi) - v) \cdot \phi \, dxdt$$

Dividing by ε, letting $\varepsilon \to 0^+$ and using Lebesque's dominated convergence we obtain

$$0 \leq \int_0^T \int_G P_R(Bu - v) \cdot \phi \, dxdt$$

Which implies $v = Bu$ on $Gx(0,T)$, thus $v = Bu$ on $\Omega \times (0,T) \equiv Q$.

<u>Proof of $\xi = Bu(T)$ and $Bu(0) = Bu_0$</u>

From Lemma IV.1 and $v = Bu$ we also obtain

$$Bu_k \to Bu \quad \text{(strongly) in } C([0,T]; E_1) \tag{6.19}$$

(6.19) and (6.10) imply $\xi = Bu(T)$ on G (thus on Ω) and

$$Bu_k(T) \to Bu(T) \quad \text{in } L^{q'}(\Omega) \text{ weak} \tag{6.20}$$

Analogously, (6.19), (6.2), $Bu_k(0) = Bu_{0k}$ and the continuity of B imply $Bu_0 = Bu(0)$ and

$$J^*(Bu_k(0)) = J^*(Bu_{ok}) \to J^*(Bu_0) = J^*(Bu(0)) \tag{6.21}$$

where we have used that the functional $u \to J^*(Bu)$ is continuous (Lemma 2.2). Alternatively, we can use that J^* is also continuous (Remark 5.1).

Proof of $\chi = Au$

From (6.13), (6.14), $v = Bu$, $u \in L^p(0,T;W(\Omega))$ and Proposition 3.1 we obtain

$$\int_0^T (\chi(t),u(t))dt = J^*(Bu(0)) - J^*(Bu(T)) + \int_0^T (f(t),u(t))dt \qquad (6.22)$$

We recall that J^* is weakly lower semicontinuous on $L^{q'}(\Omega)$. Let $k \to \infty$ in (6.4) with $t = T$. From (6.1), (6.7), (6.20) and (6.21) we obtain

$$\limsup_{k \to \infty} \int_0^T (Au_k(t),u_k(t))dt \leq J^*(Bu(0)) - J^*(Bu(T)) + \int_0^T (f(t),u(t))dt$$

This and (6.22) imply $\chi = Au$ by a standard monotonicity argument: see e.g. [23, p. 160].

The proof of Theorem 5.1 is complete.

7. Formula of integration by parts for sums of spaces

Let X_i, $i = 1,\ldots,N$, and Y be real Banach spaces. We set

$$X = \bigcap_{i=1}^N X_i, \quad \|u\|_X = \sum_{i=1}^N \|u\|_{X_i}$$

H7.1 (a) For $i = 1,\ldots,N$, X is dense in X_i and X' is identified to $\sum_{i=1}^N X_i'$.

(b) X and Y satisfy H3.1.

H7.2 J and B satisfy (2.1).

H7.3 $u \in \bigcap_{i=1}^N L^{p_i}(0,T;X_i)$, $1 < p_i < \infty$, $0 < T < \infty$.

H7.4 $Bu, \frac{d}{dt}(Bu) \in \sum_{i=1}^{N} L^{p_i'}(0,T;X_i')$

where the derivative d/dt is in the sense of distributions of $D'(0,T;X')$.

Proposition 7.1 Under hypotheses H7.1 to H7.4, conclusions I and II of Proposition 3.1 hold.

The proof is as that of Proposition 3.1. The convergence of the differential quotient to the derivative in

$$\sum_{i=1}^{N} L^{p_i'}(0,T;X_i')$$

follows splitting Bu and $d(Bu)/dt$ in the form

$$Bu = \sum_{i=1}^{N} v_i; \ \frac{d}{dt}(Bu) = \sum_{i=1}^{N} \frac{dv_i}{dt}; \ v_i \ \text{and} \ \frac{dv_i}{dt} \in L^{p_i'}(0,T;X_i')$$

8. Existence theorem for sums of operators

We consider the spaces $W_i(\Omega)$, $i = 1,\dots,N$:

$$W_i(\Omega) = W(\Omega,p_i,q,m_i,\overline{m}_i), \quad 0 < \overline{m}_i < m_i,$$

defined in H5.1, excepting that now we only require $m_i > 1$ for at least one $i \in \{1,\dots,N\}$ (rather than for each i).

Theorem 8.1 Assume the hypotheses of Theorem 5.1 with the following two modifications:

1) $A = \sum_{i=1}^{N} A_i$ and each operator A_i, $i = 1,\dots,N$, satisfies hypotheses H5.2 to H5.4 with respect to the space $W_i(\Omega)$.

2) $f \in \sum_{i=1}^{N} L^{p_i'}(0,T;W_i'(\Omega))$

Then there exists u such that (5.14), (5.15), $Bu(0) = Bu_0$ and the following relations hold:

$$u \in \bigcap_{i=1}^{N} L^{p_i}(0,T;W_i(\Omega))$$

$$\frac{\partial}{\partial t}(Bu) + Au = f \quad \text{in} \quad D'(Q)$$

The proof is as that of Theorem 5.1, using now the formula of integration by parts of Proposition 7.1. This device on sums of spaces and operators can be found also in the book of Lions [23].

Example 8.1 Theorem 8.1 applies to problem (1.2)-(1.5) ($\forall p,q,r > 1$; $\forall m,n \geq 1$) taking $N = 2$, $W_1(\Omega) = W(\Omega,p,q,m,m)$ and $W_2(\Omega) = W(\Omega,r,q,0,0) = L^r(\Omega) \cap L^q(\Omega)$. (We recall that Ω is an arbitrary open set of R^n). As noted in Remark 5.2, for this example we know, in addition, that $u \in C([0,T];L^q(\Omega))$ and the initial condition can be written $u(0) = u_0$.

Remark 8.1 Set $E = W_1(\Omega)$ of Example 8.1 and

$$F = W_0^{m,p}(\Omega) \cap L^q(\Omega)$$

Assume that $1 < q \leq p$ or $1 < r \leq p$ or Ω bounded. Then Theorem 8.1 also applies to problem (1.2)-(1.5) with E replaced by F. (This replacement is used in [10].) This follows from the following four points. (1) $F \subset E$ in any case. (2) $F = E$ if $1 < q \leq p$ or Ω is bounded (see Appendix I). (3) If $1 < r \leq p < q$ then $L^r(Q) \cap L^q(Q) \subset L^p(Q)$, thus

$$L^p(0,T;E) \cap L^r(Q) \cap L^\infty(0,T;L^q(\Omega)) = L^p(0,T;F) \cap L^r(Q) \cap L^\infty(0,T;L^q(\Omega))$$

and (4) a coerciveness argument as that connecting (6.4) and (6.5) is still valid.

Appendix I. The Sobolev-like space $W(\Omega)$ and the space $L^{p'}(0,T;W'(\Omega))$

In this appendix we allow $1 \leq p,q < \infty$ and refer to Adams [1] for background material. The space $W(\Omega)$ (defined in Section 5, see H5.1) is reflexive if $1 < p,q$ and separable if $1 \leq p,q$.

(1) The equality

$$W(\Omega) = W_0^{m,p}(\Omega) \cap L^q(\Omega)$$

holds (with equivalent norms) in each of the following three cases. I) For all $q \geq 1$ if Ω is bounded is some direction, because of Poincaré-Friedrichs inequality. II) For any open set Ω if $1 \leq q \leq p$, because of Gagliardo-Nirenberg inequalities (see e.g. [26]). III) If $\overline{m} = 0$ (trivally).

(2) Combining point I above with Sobolev imbedding and Hölder inequality, we obtain $W(\Omega) = W_0^{m,p}(\Omega)$ if Ω is bounded and $1/q \geq 1/p - m/n$.

(3) In any case

$$W(\Omega) \supset W_0^{m,p}(\Omega) \cap L^q(\Omega)$$

Furthermore, this inclusion is <u>strict</u> for $\Omega = R^n$ if $q > p$. This follows from the counterexample of [26, Comment 3, p. 125].

On the other hand, we use repeatedly the space $L^{p'}(0,T;W'(\Omega)) \subset D'(Q)$. A distribution $f \in D'(Q)$ belongs to $L^{p'}(0,T;W'(\Omega))$ if and only if f has the form

$$f = f_0 + \sum_{\overline{m} < |\alpha| \leq m} D^\alpha f_\alpha \,, \quad f_0 \in L^{p'}(0,T;L^{q'}(\Omega)), \quad f_\alpha \in L^{p'}(Q)$$

where the sum extends to x-derivatives (t-derivatives excluded). This is proved as for Sobolev spaces: see e.g. [1, Theorems 3.8 and 3.10].

Appendix II Nikol'skii spaces $H^{s,p}(\Omega)$

For simplicity we assume $0 < s < 1$ and $1 < p < \infty$. A function $u \in L^p(\Omega)$ belongs to $H^{s,p}(\Omega)$ if and only if the norm

$$\|u\|_{H^{s,p}(\Omega)} = \left[\|u\|^p_{L^p(\Omega)} + \sup \frac{1}{|h|^s} \|\Delta_h u\|^p_{L^p(\Omega_\varepsilon)}\right]^{1/p}$$

is finite, where the supremum extends to $h \in R^n$, $\varepsilon > 0$, $0 < |h| < \varepsilon$. Δ_h and Ω_ε are defined by

$$\begin{aligned}(\Delta_h u)(x) &= u(x + h) - u(x) \\ \Omega_\varepsilon &= \{x \in \Omega: \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}\end{aligned} \tag{II.1}$$

If Ω satisfies the cone condition, $H^{s,p}(\Omega)$ is continuously imbedded in $L^\lambda(\Omega)$ for some $\lambda > p$. The ranges of the parameters s,p,λ for the imbeddings of Nikol'skii spaces are the same as for the spaces $W^{s,p}$, excepting some limiting cases. See Adams [1, p. 225] for precise references about these results. An alternative reference is Triebel [29, pp. 327-328]. In [29] the Nikol'skii space $H^{s,p}(\Omega)$ is named $B^s_{p,\infty}(\Omega)$, p. 170.

If, in addition, Ω is bounded, the former continuous imbedding and the definition of the $H^{s,p}$ norm imply that the imbedding

$$H^{s,p}(\Omega) \to L^p(\Omega)$$

is compact by the standard characterization of L^p strong compactness, applied as e.g. in Brézis [15, p. 170].

Remark II.1 $H^{s,p}(R^n)$ is neither reflexive [29, p. 199] nor separable [29, p. 237] and $C_0^\infty(R^n)$ is not dense in $H^{s,p}(R^n)$ [29, p. 172]. Nevertheless, for $0 < \varepsilon < s$ $H^{s,p}(\Omega)$ is continuously imbedded in the reflexive space $W^{s-\varepsilon,p}(\Omega)$ and this is compactly imbedded in $L^p(\Omega)$ if Ω is bounded and smooth. Thus, the proof of $v = Bu$ (Section 6) can be rewritten taking $E_0 = W^{s-\varepsilon,p_0}(G)$, $0 < \varepsilon < s$, instead of $E_0 = H^{s,p_0}(G)$.

Appendix III An estimate of the Nikol'skii norm of Bu

We consider the operator B from $L^q(\Omega)$ to $L^{q'}(\Omega)$ defined by (5.4), under hypotheses H5.5 and H5.7, except relation (5.6). (Hypothesis H5.6 is not required). We set the notation

$$\|D^m u\|_{L^p(\Omega)} = \left[\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p dx\right]^{1/p} \qquad \text{(III.0)}$$

Lemma III.1 Let $2 \leq q < \infty$ and $u \in L^q(\Omega) \cap W^{1,p}(\Omega)$, $1 < p < \infty$. (Ω is an arbitrary open set of R^n). Assume that ω is an open set, $\overline{\omega} \subset \Omega$, $h \in R^n$ and $|h| < \text{dist}(\omega, \partial\Omega)$. Then there exists s such that $0 < s < 1$, $r > 1$ (r defined by (III.3) below) and

$$\|\Delta_h Bu\|_{L^r(\omega)} \leq C \|u\|^{q-1-s}_{L^q(\Omega)} \|D^1 u\|^s_{L^p(\Omega)} |h|^s$$

where notations (II.1) and (III.0) are used and the constant C depends only on p,q,s and the constant of (5.7). (Thus, C is independent of h,ω and Ω).

Proof. From (5.7), $0 < s < 1$ (to be chosen below) and

$$|t_2 - t_1| \leq |t_2 - t_1|^s(|t_2| + |t_1|)^{1-s}$$

we obtain

$$\|\Delta_h Bu\|^r_{L^r(\omega)} \leq C \int_\omega (|u(x+h)| + |u(x)|)^{r(q-1-s)} |\Delta_h u(x)|^{rs} dx \qquad \text{(III.1)}$$

We are going to apply Hölder inequality with exponents $p/(p - rs)$ and $p/(rs)$. This needs some explanations. We choose r,s so that

$$r(q-1-s)p/(p - rs) = q \qquad \text{(III.2)}$$

which is equivalent to

$$1/r = 1/q' + s/p - s/q \qquad \text{(III.3)}$$

Since $r \to q'$ as $s \to 0$, it is actually possible to choose $r > 1$ and $0 < s < 1$. In addition, $p/(rs) > 1$ follows from $q \geq 2$, $0 < s < 1$ and the identity

$$p/(rs) - 1 = p(q - 1 - s)/(qs)$$

Now the lemma follows from (III.1), (III.2), Hölder inequality (applied as explained above) and the well-known relation (see e.g. [15, Proposition IX.3])

$$\|\Delta_h u\|_{L^p(\omega)} \leq C \|D^1 u\|_{L^p(\Omega)} |h| \qquad \text{(III.4)}$$

Remark III.1 Given p, for q and n large enough we cannot assure that $Bu \in W^{1,\lambda}(\Omega)$, $\lambda \geq 1$. This and the study of the case $1 < q < 2$ suggest the use of fractional order spaces. (For $1 < q < 2$ we shall take $s = q - 1$ and $r = p/(q - 1)$).

Lemma III.2 Assume the hypotheses of Lemma III.1, excepting that now $1 < q \leq 2$. Then

$$\|\Delta_h Bu\|_{L^{p/(q-1)}(\omega)} \leq C \|D^1 u\|^{q-1}_{L^p(\Omega)} |h|^{q-1}$$

the constant C depending only on p,q and the constant of (5.7).

Proof. Notice that $p/(q - 1) \geq p > 1$. We set $r = p/(q - 1)$. From (5.7)

$$\|\Delta_h Bu\|^r_{L^r(\omega)} \leq C \int_\omega |\Delta_h u(x)|^p dx$$

The lemma follows from (III.4).

Lemma III.3 Assume $u \in L^q(G)$, $D^m u \in L^p(G)$, $m > 1$, $1 < p,q < \infty$, where G is a bounded open set satisfying the cone condition (see e.g. [1]). Then there exist p_0 and s such that $p_0 > 1$, $0 < s < 1$, $Bu \in H^{s,p_0}(G)$ and

$$\|Bu\|_{H^{s,p_0}(G)} < C_1 + C_2 \|D^m u\|^s_{L^p(G)}$$

where C_1 and C_2 depend only on p,p_0,q,s, the constant of (5.7), G and $\|u\|_{L^q(G)}$ and remain bounded if $\|u\|_{L^q(G)}$ remains bounded. Notation (III.0) is used.

Proof. We are going to use the inequality

$$\|D^1 u\|_{L^p(G)} < C(G)\left(\|D^m u\|_{L^p(G)} + \|u\|_{L^q(G)}\right) \tag{III.5}$$

which holds for all $p,q > 1$ if G is bounded and satisfies the cone condition. (This form of Gagliardo-Nirenberg inequality is found e.g. in Nirenberg [27]).

Consider the $H^{s,p}$ norm defined in Appendix II. The desired bound for the second term of this norm is obtained from (III.5), Lemma III.1 for $q > 2$ and Lemma III.2 for $1 < q < 2$. (We take $s = q - 1$ if $1 < q < 2$). For the first term of the $H^{s,p}$ norm we take into account that

$$\|Bu\|_{L^{q'}(G)} < C\|u\|^{q-1}_{L^q(G)}$$

(this follows from (5.5) and (5.7) with $s_1 = 0$) and choose $p_0 > 1$ in the following way (recall that G is bounded):

$$p_0 < \min\{r,q'\} \qquad \text{if } q > 2$$

$$p_0 < \min\{p/(q-1),q'\} \quad \text{if } 1 < q < 2$$

Appendix IV A compactness lemma of Lions-Aubin

Let E_0, E and E_1 be Banach spaces such that $E_0 \subset E \subset E_1$, the imbedding $E_0 \to E$ is compact and the imbedding $E \to E_1$ is continuous. Assume $1 < p_0, p_1 < \infty$.

Lemma IV.1 Under the above hypotheses, assume that the sequence $\{v_k\}$ is bounded in $L^{p_0}(0,T;E_0)$ and $\{dv_k/dt\}$ is bounded in $L^{p_1}(0,T;E_1)$. Then there exists a subsequence of $\{v_k\}$ which converges strongly both in $L^{p_0}(0,T;E)$ and in $C([0,T];E_1)$.

For E_0 and E_1 reflexive the lemma was proved by Lions, see e.g. [23, p. 58 and p. 142]. Aubin [5] proves the lemma without reflexivity hypotheses.

(In Section 6 we apply the lemma with E_0 nonreflexive. In Remark II.1 we explain that this can be avoided through an additional argument.)

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