ANALYSIS OF BROWNIAN FUNCTIONALS
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ANALYSIS OF BROWNIAN FUNCTIONALS

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§1. Introduction and Background

We analyze functionals of the form

$$(1) \quad \phi(\dot{B}(t), t \in \mathbb{R}^1), \quad \dot{B}(t) = \frac{d}{dt} B(t),$$

where $\{B(t)\}$ is a Brownian motion and hence $\{\dot{B}(t)\}$ is a white noise. A functional expressed in the form (1) is often called a Brownian functional, although the expression has only formal significance.

It is important to note that $\{\dot{B}(t)\}$ is taken to be a system of variables of ϕ in the expression (1) and that it is a continuous analogue of i.i.d. (independent identically distributed) Gaussian random variables.

1.1. The first thing we should do for the setup is to introduce the probability distribution, denote it by μ , of $\{\dot{B}(t)\}$. We start with the characteristic functional

$$(2) \quad C(\xi) = E(\exp[i\dot{B}(\xi)]) = \exp \left[-\frac{1}{2} \|\xi\|^2 \right], \quad \xi \in E,$$

where E is a test function space (a nuclear space) being a member of a Gel'fand triple

$$E \subset L^2(\mathbb{R}^1) \subset E^*,$$

and where $\|\cdot\|$ is the $L^2(\mathbb{R}^1)$ - norm. Then, the Bochner-Minlos theorem guarantees the existence and the uniqueness of the measure μ on E^* such that

$$(3) \quad C(\xi) = \int_{E^*} \exp[i\langle x, \xi \rangle] d\mu(x),$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form connecting E and E^* . Thus, μ -almost every x in E^* is viewed as a sample function of $\dot{B}$, and a member $\phi(x)$ in the complex Hilbert space $(L^2) = L^2(E^*, \mu)$ is a realization of the formal expression (1). The $\phi(x)$ is also called a Brownian functional.

1.2. Rotation group. We introduce the infinite dimensional rotation group. An orthogonal transformation of $L^2(\mathbb{R}^1)$ is called a rotation of E if it is a homeomorphism of E . The collection of all rotations of E forms a group. We

denote it by $O(E)$. If E is not specified, it is denoted by O_∞ .

Definition. The group $O(E)$ is called a rotation group of E , or often called an infinite dimensional rotation group.

Let g^* be the adjoint of g in $O(E)$, which is defined in a usual manner;

$$\langle g^*x, \xi \rangle = \langle x, g\xi \rangle, \quad x \in E^*, \quad \xi \in E.$$

Then, we have

Theorem 1. For any $g \in O(E)$

$$g^* \mu = \mu$$

holds.

Proof. It is easy to see that the characteristic functional of $g^*\mu$ is equal to $C(\xi)$, which proves the assertion.

1.3. Observation.

The measure μ is supported by the infinite dimensional sphere $S^\infty(\sqrt{\infty})$, which is as thick as (or perhaps as thin as) ℓ^2 . We are able to give an intuitive as well as a mathematical interpretation of this statement.

Theorem 1 says μ is invariant under the rotations, and therefore, roughly, μ is a uniform measure over the sphere $S^\infty(\sqrt{\infty})$. In addition, once a direction is fixed, the measure μ is concentrated on the equator.

This naive observation leads us to consider an analogy with the Hilbert space $L^2(S^2, d\sigma)$, $d\sigma$ being the uniform measure on S^2 , and its direct sum decomposition:

$$(4) \quad L^2(S^2, d\sigma) = \sum_{n=0}^{\infty} \oplus K_n,$$

where K_n is spanned by the spherical harmonics of degree n .

1.4 Fock space.

Take a complete orthonormal system (abbr. c.o.n.s.) $\{\xi_n\}$ in $L^2(\mathbb{R}^1)$ such that $\xi_n \in E$ for every n . Then, each linear functional $\langle x, \xi_n \rangle$ is viewed as a random variable on (E^*, μ) and standard Gaussian in distribution. We see further that $\{\langle x, \xi_n \rangle\}$ is a system of i.i.d. random variables.

Form Hermite polynomials in $\langle x, \xi_n \rangle$'s of the form

$$\prod_k H_{n_k}(\langle x, \xi_k \rangle / \sqrt{2}) \quad (\text{finite product}),$$

$\sum n_k$ being the degree of the polynomial. Such polynomials are mutually orthogonal in (L^2) . The subspace spanned by such polynomials of degree n is denoted by H_n . The entire space (L^2) admits a direct sum decomposition:

$$(5) \quad (L^2) = \sum_{n=0}^{\infty} \oplus H_n. \quad (\text{Fock space}).$$

Note that H_n is the collection of the Itô multiple Wiener integral of degree n (cf. the decomposition (4)).

§2. Tools from analysis.

In order to carry out the analysis on (L^2) we provide two basic tools from analysis. One is the integral representation and the other uses particular subgroups of the rotation group $O(E)$.

2.1. Integral representation

We introduce the $\mathcal{T}$ -transform given by

$$(6) \quad (\phi)(\xi) = \int_{E^*} \exp[i\langle x, \xi \rangle] \phi(x) d\mu(x) \equiv \hat{\phi}(\xi), \quad \xi \in E, \phi \in (L^2).$$

The collection $\{\hat{\phi}; \phi \in (L^2)\}$ is made into a reproducing kernel Hilbert space denoted by $\mathcal{H}$ with kernel $C(\xi-\eta)$, $(\xi, \eta) \in E \times E$, so that

$$\mathcal{H} \cong (L^2) \text{ under } \mathcal{T}.$$

Note that the $\mathcal{T}$ -transform looks like the ordinary Fourier transform, but not quite. An important remark is that the functional $\exp[i\langle x, \xi \rangle]$ in the expression (6) can play a role of a test functional and $\phi(x)$ can be taken to be a generalized functional introduced later.

Theorem 2. If $\mathcal{T}$ is restricted to H_n , we have

$$(7) \quad \begin{aligned} (\phi)(\xi) &= i^n C(\xi) U(\xi), & \phi &\in H_n \\ U(\xi) &= \int_{R^n} \int F(u_1, \dots, u_n) \xi(u_1) \dots \xi(u_n) du^n, & F &\in L^2(R^n) \\ \phi &\rightarrow F & \text{one-to-one,} \\ \|\phi\|_{(L^2)} &= \sqrt{n!} \|F\|_{L^2(R^n)}, \end{aligned}$$

where $\widehat{L^2(R^n)} = \{F \in L^2(R^n); F \text{ symmetric}\},$

The proof uses the fact that $\mathcal{T}$ carries the generating function of the Hermite polynomials

$$\exp[\sqrt{2} t \langle x, \eta \rangle - t^2] = \sum_k \frac{t^k}{k!} H_k(\langle x, \eta \rangle / \sqrt{2}). \quad \|\eta\| = 1.$$

to the functional

$$C(\xi) \exp[i\sqrt{2} t(\eta, \xi)] = \sum_k \frac{t^k}{k!} (\sqrt{2}i)^k (\eta, \xi)^k.$$

This theorem establishes the following isomorphism (up to constant $\sqrt{n!}$):

$$(8) \quad H_n \cong L^2(\mathbb{R}^n) \quad \text{under } \mathcal{T}.$$

The functional $U(\xi)$ in (7) is called the U -functional associated with $\phi \in H_n$. For a general $\phi \in (L^2)$, the U -functional is obtained by

$$U(\xi) = \int \phi(x+\xi) d\mu(x), \quad \xi \in E. \quad (\text{see [5]}).$$

2.2. Subgroups of $O(E)$.

- 1) Take a c.o.n.s. $\{\xi_n\}$ as before, and let E_n be the subspace of E spanned by $\{\xi_1, \dots, \xi_n\}$.

Set $G_n = \{g|_{E_n} = \text{rotation of } E_n \text{ and } g|_{E_n^\perp} = \text{identity}\}.$

Obviously

$$G_n \cong SO(n).$$

The limit $G_\infty = \varprojlim G_n$ is therefore the subgroup of $O(E)$ involving finite dimensional rotations.

The number operator (or the Ornstein-Uhlenbeck operator)

$$\Delta_\infty = \sum_n \left(\frac{\partial^2}{\partial \xi_n^2} - \langle x, \xi_n \rangle \frac{\partial}{\partial \xi_n} \right)$$

is characterized by G_∞ . The operator Δ_∞ acts on H_n in such a way that

$$\Delta_\infty \phi = -n\phi, \quad \phi \in H_n.$$

Remark i) On each H_n , the class one representation of G_∞ is given (K. Okamoto et. al. 1981).

- 2) Again, take a c.o.n.s. $\{\xi_n\}$. Let π be a permutation of positive integers. Define a transformation g_π on E by

$$g_{\pi} \xi = \sum_n a_n \xi_{\pi(n)} \quad \text{for} \quad \xi = \sum_n a_n \xi_n.$$

The collection

$$\mathcal{H} = \{g \in O(E); g = g_{\pi} \quad \forall_{\epsilon > 0} \quad \exists N(\epsilon). \quad \forall n > N(\epsilon)\}$$

$$\# \{i \leq n; \pi(i) > n\} < \epsilon n \}$$

forms a subgroup of $O(E)$, and is called the Lévy subgroup.

As we shall see later, $\mathcal{H}$ has an intimate connection with the Lévy Laplacian.

3) Whiskers. A one-parameter subgroup $\{g_t\}$ of $O(E)$ is called a whisker if it comes from a diffeomorphism of the one-point compactification of $\mathbb{R}^1$. More precisely, let ψ_t , $-\infty < t < \infty$, be a family of such diffeomorphisms satisfying

$$\psi_t \circ \psi_s = \psi_{t+s}.$$

If g_t is defined by

$$(9) \quad (g_t \xi)(u) = \xi(\psi_t(u)) \sqrt{|\psi_t'(u)|}$$

and if $g_t \in O(E)$ and continuous in t , then $\{g_t\}$ is a one-parameter subgroup of $O(E)$ and is called a whisker.

A most important example of a whisker is the shift $\{s_t\}$:

$$(S_t \xi)(u) = \xi(u-t).$$

It actually describes the time shift. The adjoints

$$T_t \equiv S_t^*, \quad t \in \mathbb{R}^1.$$

form a flow $\{T_t\}$ on (E^*, μ) by Theorem 1. It is called the flow of Brownian motion.

Other examples of whiskers are the dilation $\{\tau_t\}$: $(\tau_t \xi)(u) = \xi(e^t u) e^{t/2}$, and the special conformal transformation $\{\kappa_t\}$ given by

$$(\kappa_t \xi)(u) = \xi\left(\frac{u}{1-tu}\right) \frac{1}{|1-tu|}.$$

If the basic nuclear space is taken to be D_0 :

$$D_0 = \left\{ \xi \in C^\infty; \xi\left(\frac{1}{u}\right) \cdot \frac{1}{|u|} \in C^\infty \right\},$$

then the above transformation are whiskers.

Here is an interesting remark. The above three whiskers form a subgroup isomorphic to $PGL(2, \mathbb{R})$ and the subgroup describes the projective invariance of Brownian motion discovered by P. Lévy.

Also, note that whiskers can not be approximated properly by finite dimensional rotations.

§3. Generalized Brownian functionals.

Let us recall that $\{\dot{B}(t)\}$ is taken to be the system of variables of Brownian functionals. We therefore want to introduce a class of (generalized) Brownian functionals involving polynomials in $\dot{B}(t)$'s.

3.1. The following diagram, which is a development of the isomorphism (8), shows how to define the class $H_n^{(-n)}$ of generalized Brownian functionals of degree n.

$$\begin{array}{ccc} H_n^{(-n)} & \cong & \begin{array}{c} \text{---} \frac{n+1}{2} \text{---} \\ \text{---} H^{\frac{n+1}{2}}(R^n) \end{array} \\ \uparrow \text{ } \curvearrowright & & \uparrow \text{ } \curvearrowright \\ H_n & \cong & L^2(R^n) \\ \uparrow \text{ } \curvearrowright & & \uparrow \text{ } \curvearrowright \\ H_n^{(n)} & \cong & \begin{array}{c} \text{---} \frac{n+1}{2} \text{---} \\ \text{---} H^{\frac{n+1}{2}}(R^n), \end{array} \end{array}$$

where $H^m(R^n)$ is the subspace of the Sobolev space of order m over R^n involving symmetric functions.

We have freedom on how to sum up the $H_n^{(-n)}$'s. First we introduce the space of test functionals denoted by $(L^2)^+$. Choose a positive increasing sequence c_n .

$$(L^2)^+ = \{ \phi = \sum_n \phi_n ; \phi_n \in H_n^{(n)}, \sum_n c_n \|\phi_n\|_n^2 < \infty \} ,$$

where $\|\cdot\|_n$ is the $H_n^{(n)}$ - norm.

Example $c_n = (1+n)^\alpha$, $\alpha > 0$.

In this case $\phi = \sum \phi_n$ belongs to $(L^2)^+$ if

$$\sum \|(1-\Delta_\infty)^\alpha \phi_n\|^2 < \infty .$$

Set

$$(L^2)^- = \text{the dual space of } (L^2)^+.$$

and call it the space of generalized Brownian functionals.

Remark. There is another way to generalize the notion of Brownian functionals.

Namely, the kernel function F associated with an H_n -functional by Theorem 2 may have a singularity of polynomial order at a fixed point, say at the origin. In such a case the regularization technique may be applied to the kernel as is done by Gel'fand and Shilov.

3.2. Examples of an $(L^2)^-$ - functional.

$$i) \quad \phi = \int f(u) : x(u)^2 : du \text{ or written as } \int f(u) : \dot{B}(u)^2 : du.$$

It is given as an $H^{(-2)}$ -functional whose U -functional is of the form

$U(\xi) = U_\phi(\xi) = \int f(u) \xi(u)^2 du$. $: \dot{B}(u)^2 :$ means the additive renormalization of $\dot{B}(t)^2$ expressible, formally speaking, as $: \dot{B}(t)^2 : = \dot{B}(t)^2 - \frac{1}{dt}$. Similarly we apply the additive renormalization for $\dot{B}(t)^n$.

$$\text{ii) } \phi = \exp[c \int_T f(u) x(u)^2 du] : \text{ or } : \exp[c \int_T f(u) \dot{B}(u)^2 du] :, c < \frac{1}{2},$$

where T is a finite interval. For a suitable choice of the sequence $\{c_n\}$ ϕ is a member of $(L^2)^-$. The associated U-functional is

$$U(\xi) = \exp [c' \int_T f(u) \xi(u)^2 du]. \quad c' = \frac{c}{1-2c}.$$

Since ϕ is an exponential function, we need the multiplicative renormalization which is consistent with the additive one applied to polynomials in $\dot{B}(t)$'s.

iii) The Donsker delta function $\phi = \delta_a(\langle x, x_{[0,t]} \rangle)$ is also a member of $(L^2)^-$. The associated U-functional is given by

$$U(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2t}} \exp \left[\frac{a}{t} \int_0^t \xi(u) du - \frac{1}{2t} \left(\int_0^t \xi(u) du \right)^2 \right].$$

§4. Calculus on $(L^2)^-$.

4.1. Since $\{\dot{B}(t)\}$ is taken to be a system of variables of Brownian functionals as in the expression (1), it is quite reasonable to introduce the partial differential operator $\frac{\partial}{\partial B(t)} \equiv \partial_t$. Further note that the parameter t stands for the time, so that ∂_t is appropriate for what we may call the "causal calculus" where the time development is always taken into account explicitly.

Let $U(\xi)$ be the U-functional associated with $\phi \in (L^2)^-$. If U is Fréchet differentiable and if the derivative $\frac{\delta U}{\delta \xi(t)}$ is a U-functional associated with some ϕ'_t in $(L^2)^-$, then ϕ is ∂_t -differentiable and we define

$$\phi'_t = \partial_t \phi.$$

Through the bilinear form $\langle \cdot, \cdot \rangle_\mu$ connecting $(L^2)^+$ and $(L^2)^-$, we can define the adjoint operator ∂_t^* for ∂_t :

$$\langle \partial_t \phi, \psi \rangle_\mu = \langle \phi, \partial_t^* \psi \rangle_\mu.$$

The domains $D(a_t)$ and $D(a_t^*)$ are large enough in $(L^2)^-$.

Theorem 3: i) a_t is an annihilation operator such that

$$a_t : H_n^{(n)} \rightarrow H_n^{(n-1)}$$

and its action is expressed in terms of the kernel:

$$F(u_1, \dots, u_n) \rightarrow n(\delta_t * F_1)(u_1, \dots, u_{n-1}).$$

ii) a_t^* is a creation operator such that

$$a_t^* : H_n^{(-n)} \rightarrow H_{n+1}^{(-n-1)}$$

and that

$$F(u, \dots, u_n) \rightarrow (\delta_t \hat{\otimes} F_1)(u_1, \dots, u_{n+1}).$$

where $\hat{\otimes}$ means the symmetric tensor product.

Example

$$a_t \int f(u) : x(u)^2 : du = 2f(t) x(t).$$

4.2. Laplacians.

1). The number operator Δ_∞ can be generalized to an operator acting on $(L^2)^-$ and is expressible as

$$(10) \quad \Delta_\infty = \int a_t^* a_t dt.$$

2). The Lévy Laplacian Δ_L is given, according to H.-H. Kuo, by

$$(11) \quad \Delta_L = \int a_t^2 (dt)^2.$$

Lévy's original idea was to introduce an operator

$$\tilde{\Delta}_L = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \frac{\partial^2}{\partial \xi_n^2}$$

acting on a space of functionals on $L^2([0,1])$, where (ξ_n) is a c.o.n.s. If $\{\xi_n\}$ is favorably chosen, namely, if it satisfies the property "également dense"

$$\frac{1}{N} \sum_{n=1}^N \xi_n(t)^2 \rightarrow 1 \text{ in } L^1 \text{ (or weakly),}$$

then $\tilde{\Delta}_L U$ is in agreement with

$$(12) \quad \int_0^1 \frac{\delta^2 U}{\delta \xi(t)^2} dt$$

where $\frac{\delta^2 U}{\delta \xi(t)^2}$ is one of the Fréchet derivatives of the second order (see [1]).

Remark. The variation of the Fréchet derivative $U'_\xi(t) = \frac{\delta U}{\delta \xi(t)}$ is expressed in the form

$$\delta U'_\xi(t) = \delta \xi(t) \frac{\delta^2 U}{\delta \xi(t)^2} + \int \frac{\delta^2 U}{\delta \xi(t) \delta \xi(s)} \delta \xi(s) ds.$$

We use the first term to define $\tilde{\Delta}_L$.

This idea of defining the Laplacian can be extended to the case where functionals are defined on $E^*(\supset L^2(R^1))$ instead of $L^2([0,1])$. An également dense c.o.n.s. $\{\xi_n\}$ in $L^2(R^1)$ may be taken in such a way that

$$(13) \quad \xi_n(u) = \tilde{\xi}(f(u)) \sqrt{|f'(u)|},$$

where $\{\tilde{\xi}_n\}$ is an également dense c.o.n.s. in $L^2([0,1])$ and where f is a diffeomorphism from R^1 to $(0,1)$. Then we come to the same expression as (12) and finally the Lévy Laplacian acting on $(L^2)^-$ is given by (11).

Before we come to the domain $D(\Delta_L)$ we introduce a notion of a normal functional due to P. Lévy. An $(L^2)^-$ - functional is called a normal functional of degree n if it is expressed as a finite sum of functional of the form

$$\int \dots \int F(u_1, \dots, u_k) : x(u_1)^{n_1} \dots x(u_k)^{n_k} : du_1 \dots du_k, \quad \sum n_j = n,$$

which is of course living in $\mathcal{H}_n^{(-n)}$. Now we define

$$D(\Delta_L) = \left\{ \phi = \sum \phi_n \in (L^2)^-; \begin{array}{l} \phi_n \text{ normal, degree } n \\ \sum \Delta_L \phi_n \text{ converges in } (L^2)^- \end{array} \right\}$$

Examples i) $\Delta_L \int f(u) : x(u)^2 : du = 2 \int f(u) du$

ii) $\Delta_L \iint F(u, v) : x(u)^2 x(v)^2 : du dv = 4 \iint F(t, v) : x(v)^2 : dv dt.$

iii) The Laplacian Δ_L can not be applied to the generalized functional $\int_0^1 : x(u)x(1-u) : du.$

We can even speak of the usual harmonic property for Brownian functionals.

ϕ in $D(\Delta_L)$ is said to be harmonic if

$$\Delta_L \phi = 0.$$

Example. If $\int f(u) du = 0$, then $\int f(u) : x(u)^2 : du$ is harmonic.

The eigenvalues and eigenfunctionals of Δ_L are also defined in the usual manner.

Example. Let $\phi = : \exp [c \int_0^1 x(u)^2 du] :$, then

$$U(\xi) = \exp \left[\frac{c}{1-2c} \int_0^1 \xi(u)^2 du \right]. \text{ Hence we have}$$

$$\Delta_L \phi = \frac{2c}{1-2c} \phi.$$

Finally, we should like to mention that Δ_L does commute with the Lévy group $\mathcal{L}$ provided that the c.o.n.s. $\{\xi_n\}$ defining $\mathcal{L}$ comes from an également dense c.o.n.s. $\{\xi_n\}$ by the transformation given by (13). This is a significant property that clarifies the relation between the group $\mathcal{L}$ and the Laplacian Δ_L . The fact that

Δ_L commutes with G_∞ is obvious.

3) As a related operator we introduce

$$(13) \quad H = \int (\partial_t^* \partial_t + \partial_t \partial_t^*) dt.$$

Example. For $\phi = \int f(u): x(u)^n: du$, we have

$$H\phi = (2n+1)\phi.$$

§5. Applications.

5.1. Stochastic integrals.

The integral $\int f(u) \partial_u^* \phi du$ can be defined in the following manner.

i) For $\phi \in H_n^{(-n)}$ with kernel $F \in \widehat{H^{\frac{n+1}{2}}}(R^n)$, $\partial_u^* \phi$ is a functional with kernel $\delta_u \hat{\otimes} F$ by Theorem 3. For f in $L^2(R')$, $\int f(u) \delta_u \hat{\otimes} F du = f \hat{\otimes} F$ is a member in $\widehat{H^{\frac{n+2}{2}}}(R^{n+1})$. We can therefore define $\int f(u) \partial_u^* \phi du$ as an $(L^2)^-$ -functional whose kernel is $f \hat{\otimes} F$.

ii) If $\phi = \sum \phi_n$, $\phi_n \in H_n^{(-n)}$ and if $c_{n+1}^2 \leq c_n^2 / n+1$, then

$$\int f(u) \partial_u^* \phi du = \sum_n \int f(u) \partial_u^* \phi_n du.$$

Remark 1°. The integrand ϕ is not necessarily non-anticipating in the measure theoretic sense.

2°. This method can be extended to the multidimensional time parameter case without essential modification.

5.2. Feynman integrals.

Following the Feynman's idea, we wish to obtain the quantum mechanical propagator by taking the average of $\exp \left[\frac{i}{\hbar} \cdot \text{action} \right]$ over possible trajectories. We propose to take

$$y(\tau) = y_0(\tau) + \sqrt{\frac{\hbar}{m}} B(\tau)$$

as a possible trajectory, where y_0 is the classical path.

Set

$$I = N: \exp \left[\frac{i}{\hbar} \int_0^t \frac{1}{2} m \dot{y}^2 d\tau + \frac{1}{2} \int_0^t \dot{B}(\tau)^2 d\tau \right] : \delta(y(t) - y_2).$$

If V is chosen so that $J = \exp \left[-\frac{i}{\hbar} \int_0^t V d\tau \right]$ is a test functional, then

$$\langle I, J \rangle_\mu = G(y_1, y_2; t)$$

gives us the propagator. Examples of the potentials may be seen in [4]

5.3. Stochastic functional differential equations

An equation involving operators ∂_t , ∂_t^* , $\frac{d}{dt}$ as well as the multiplication $\dot{B}(t) \cdot = \partial_t + \partial_t^*$ is called a stochastic functional differential equation. Such equations arise from actual evolutionary random phenomena and they are often useful in characterizing generalized Brownian functionals.

Examples

$$(1) \quad \phi = : \exp \left[c \int_0^1 x(u)^2 du \right] :$$

is characterized by

$$\begin{cases} \partial_t \phi = 2c' \partial_t^* \phi, \\ \langle \phi, 1 \rangle_\mu = 1. \end{cases} \quad c' = \frac{c}{1-2c}$$

In particular, for $c = -1/2$

$$\partial_t \phi = -\dot{B}(t) \cdot \phi.$$

(2) Donsker's delta function $\phi_t = \delta_a(B(t))$, $B(t) = \langle x, x_{[0,t]} \rangle$. It satisfies

$$\partial_s \phi_t = \begin{cases} \frac{1}{2} (a - \int_0^t \partial_u^* du) \phi_t, & s < t, \\ 0, & s > t. \end{cases}$$

and

$$\langle \phi_t, 1 \rangle_\mu = \frac{1}{\sqrt{2\pi t}} \exp \left[-\frac{a^2}{2t} \right].$$

(3) Oosawa's equation

For the Ornstein-Uhlenbeck process $\int_0^t \exp[-\lambda(t-u)] \dot{B}(u) du$, we define an operator $E(t) = \int_0^t \exp[-\lambda(t-u)] \partial_u^* du$. Given a stochastic functional differential equation

$$\frac{d}{dt} X(t) = -k_+ e^{-\alpha E(t)} X(t) + k_- e^{\alpha E(t)} (1-X(t)), \quad \alpha, k_{\pm} > 0,$$

we can rephrase the above in terms of U-functional U_t associated with $X(t)$:

$$\frac{d}{dt} U_t(\xi) = -k_+ e^{-\alpha L(t, \xi)} U_t(\xi) + k_- e^{\alpha L(t, \xi)} (1-U_t(\xi)),$$

where $L(t, \xi) = \exp \left[\int_0^t e^{-\lambda(t-u)} \xi(u) du \right]$. The solution is

$$U_t(\xi) = k_- \exp \left[-\int_0^t \{k_+ L(u, -\xi) + k_- L(u, \xi)\} du \right] \times \left[c + \int_0^t L(u, \xi) e^{\int_0^u \{k_+ L(v, -\xi) + k_- L(v, \xi)\} dv} du \right].$$

With the help of this expression we can discuss asymptotic properties, asymmetry in time and other properties.

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