CONSTRUCTING NONHOMEOMORPHIC
STOCHASTIC FLOWS

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Abstract

A pure stochastic flow is a process $(X_{st}, 0 \leq s \leq t < \infty)$, taking values in the space of functions from a space M to itself, such that $X_{tu}(X_{st}(z)) = X_{su}(z)$ for all z , and all $s \leq t \leq u$, with no exceptions: interpret $X_{st}(z)$ as the position at time t of a particle which is at z at time s . A very general existence result is obtained: given a consistent system of finite-dimensional Markov processes on M , with a spatial stochastic continuity condition, there exists a pure stochastic flow with these finite-dimensional distributions. Application: suppose $b(\cdot)$ is the covariance function of a homogeneous $\mathbb{R}^d$ -valued random field on $\mathbb{R}^d$, Lipschitz away from 0, with mild technical conditions; then there exists a pure stochastic flow with this covariance. Let $v(x,y)$ be the time when trajectories from x and y coalesce; if $b(\cdot)$ is homogeneous and isotropic, and satisfies an additional condition, then $P(v(x,y) < \infty)$ is calculated and shown to be > 0 ; also for any countable dense subset D of $\mathbb{R}^d$, any $t > 0$, and any y in $\mathbb{R}^d$, $X_{0t}(D)$ contains $X_{0t}(y)$ with probability 1. Such covariances exist when $d = 2$.

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Dedication

Dedicated to T.E. Harris on his sixty-sixth birthday.

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I owe a huge debt of gratitude to T.E. Harris, who inspired me to study this subject, corrected numerous errors, and furnished the material of Section 16. Discussions with Richard Arratia and Richard Durrett have been very helpful.

Footnotes to appear on page 1 of Section 1.

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Part I

Introduction

1. BackgroundA. The theory of stochastic flows

The study of stochastic flows emerged from the idea of regarding strong solutions of Ito stochastic differential equation as functions of the initial point. Suppose that the state space is a Riemannian manifold V (for example $V = \mathbb{R}^d$) and let $(X_t(x), t \geq 0, x \text{ in } V)$ denote the solution process. If the coefficients of the stochastic differential equation ("s.d.e.") are bounded and Lipschitz, a version of this process exists which is almost surely continuous in t and x (see Prioret [22]). If the coefficients have bounded derivatives of all orders, a version of the process exists such that, almost surely, $x \mapsto X_t(x)$ is a (random) diffeomorphism (Elworthy [9], Carverhill and Elworthy [5], Ikeda and Watanabe [14]). Weaker conditions are given by Kunita [17] and Meyer [20] for the flow to be homeomorphic and one-to-one respectively.

The case of smooth (or maybe C^4) coefficients has led to important generalizations, in which stochastic flows are constructed without reference to solutions of stochastic differential equations on V . Elworthy [8] showed how a system of s.d.e. on V could be 'lifted to the diffeomorphism group', and obtained $(X_t(\cdot), t \geq 0)$ as the solution of an s.d.e. on the diffeomorphism group of V with $X_0 = \text{identity}$. An important paper of Baxendale [2] **begins** with the notion of a Brownian motion on the diffeomorphism group, and shows how such a process is characterized by the covariance of the motions of pairs of points in V . The covariance in turn corresponds to a real separable Hilbert space continuously included in the

Frechet space $C^\infty(TV)$ of smooth sections of the tangent bundle, and to an infinite-dimensional Wiener process thereon.

When $V = \mathbb{R}^2$, Harris [12] showed that a C^2 - bounded covariance tensor $b(\cdot)$ of a homogeneous $\mathbb{R}^2$ - valued random field in $\mathbb{R}^2$ can be used to construct a stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ with values in the space of continuous mappings of $\mathbb{R}^2$ to itself, such that $(X_t(x) \equiv X_{0t}(x), t \geq 0)$ is a Brownian motion in $\mathbb{R}^2$ for each x in $\mathbb{R}^2$, and the angle-brackets process of $(X_t(y))$ and $(X_t(x))$ satisfies (2.2) below.

Baxendale and Harris [3] generalized these methods to construct and examine stochastic flows in $\mathbb{R}^d$, $d \geq 2$, with isotropic covariance tensors. In order to express these stochastic flows as solutions of Ito s.d.e's (as mentioned in the first paragraph), an infinite number of driving Brownian motions are needed.

The first attempt to study the case where the covariance $b(\cdot)$ is non-differentiable is the second paper of Harris [13], which was inspired by work of Arratia [1]. Harris treats only the one-dimensional case, and assumes that $b(\cdot)$ is Lipschitz on closed intervals not containing zero and has a spectral distribution which is not of pure jump type; thus $b(\cdot)$ is continuous, but not necessarily differentiable at zero. When either (i) $b''(0)$ is finite, or (ii) $1 - b(x) \geq c|x|^{2-\varepsilon}$ in a neighbourhood of 0 for some $\varepsilon > 0$, then Harris is able to construct what we call a pure stochastic flow (Definition 3.1 below) $(X_{st}, 0 \leq s \leq t < \infty)$ on $\mathbb{R}^1$. The measure-theoretic aspects of his construction depend on the fact that $\mathbb{R}^1$ is totally ordered, and so right-continuous versions of X_{st} are determined by the values of $X_{st}(x)$ for rational x . Harris was able to demonstrate the phenomenon of **coalescence** (which can never occur when the covariance is C^2); i.e. when (ii) holds,

$P(X_{ot}(\mathbf{R}^1) \cap I \text{ is finite}) = 1$ for every $t > 0$, whenever I is a compact interval of $\mathbf{R}^1$.

The present paper has been written to establish the truth of a conjecture of Harris; namely that pure stochastic flows, whose covariances are not C^2 , may be constructed in dimension 2 or more, and that something similar to the coalescence phenomenon mentioned above can occur. The most difficult problems in this project were measure-theoretic ones (sections 4-10 below); these were solved by using and extending Nelson's theory [21] of Radon probability measures on function spaces.

B. Infinite particle systems

In the one-dimensional case, Arratia [1] uses a system of coalescing Brownian motions on the line to study interacting particle systems on $\mathbf{Z}$, such as the voter model and coalescing random walks. There is at least an intuitive connection between the continuous coalescing systems that we study in Sections 14 and 16, and the voter model in $\mathbf{Z}^d$, $d \geq 2$. (See Liggett [19]). It remains to be seen whether this will prove fruitful. Some results about the voter model in $\mathbf{Z}^2$ have recently been obtained by Cox and Griffiths [6].

C. Turbulent diffusion

Kesten and Papanicolaou [16] show that the motion of particles in a C^3 random velocity field can be approximated by the action of a stochastic flow. A more sophisticated result was obtained by Kunita [18]. We conjecture that stochastic flows with finite coalescence times (see Sections 14, 16 and 17) may provide models of turbulent diffusion in which short-range forces are strong enough to make particles coalesce.

2. Outline of the main results

The overall structure of this paper is as follows. An object called a pure stochastic flow is defined in Section 3. Sections 4,5,6 and 10 present an abstract way to construct a pure stochastic flow; Sections 7,8,9 and 10 provide another construction with better measurability properties. Part V considers the concrete problem of constructing a pure stochastic flow on $\mathbb{R}^d$ with a given (usually nondifferentiable) covariance; a criterion (in the isotropic case) for trajectories to coalesce in finite time is also given. Part VI returns to an abstract setting to consider what happens to the space as a whole under a flow in which this kind of coalescent behavior occurs.

The chain of reasoning tends to be rather long. To help the reader, we present now a concatenation of some of the main results, so that the logical sequence of the later sections may be more easily understood.

Suppose (T,d) is an infinite separable metric space, and let $M = T \cup \{\infty\}$ be the one-point compactification of T . Let $\Gamma = M^M$, regarded as the space of functions from M to M with the topology of pointwise convergence.

Suppose that for each finite subset $\{z_1, \dots, z_k\}$ of distinct points of T ,

we are given a Markov process $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$ in T^k , with initial point $(z_1, \dots, z_k)$; assume furthermore that

- (a) the distributions of these processes are consistent, and
- (b) for fixed t , the map $z \rightarrow Z_t(z)$ is uniformly stochastically continuous (i.e. continuous in probability) on (T, d) .

THEOREM A.

In the situation described above, there exists a Γ -valued process $(X_{st}, 0 \leq s \leq t < \infty)$, defined on a probability space $(\Omega, \mathcal{F}, P)$ with the following properties. (Notation: for y in M and ω in Ω , $X_{st}(y, \omega)$ means $X_{st}(\omega)(y)$.)

- (i) $X_{tt}(\omega) = \text{identity for all } \omega, \text{ and all } t \geq 0.$
- (ii) *If $0 \leq s \leq t \leq u$, then $X_{tu}(X_{st}(z, \omega), \omega) = X_{su}(z, \omega)$ for all z in M and all ω in Ω , with no exceptions.*
- (iii) *For fixed $s \geq 0$, the law of $(X_{su}(z_1), \dots, X_{su}(z_k), u \geq s)$ in M^k is the same as that of $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0).$*

- (iv) *If $0 \leq s \leq t \leq s' \leq t' \dots$, then $X_{st}, X_{s't'}, \dots$ are independent.*
- (v) *The Borel probability measure on Γ induced by X_{st} is Radon, for each fixed $0 \leq s \leq t$.*

Proof.

See sections 4,5,6 and 10. The intermediate step between the processes $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$ and the flow is an object called a "convolution semigroup of probability measures" (Definitions 4.12). It is convenient to use Radon measures here, so as to be able use the projective limit theorem of Bochner [4]. The stochastic continuity condition is imposed to ensure that the operation of composition of functions, i.e., $(f,g) \rightarrow f \circ g$, is measurable in a certain sense.

What is lacking in Theorem A is any conclusion about whether $x \rightarrow X_{st}(x)$ is a measurable map. To remedy this defect, let μ be any Radon measure on M such that $\mu(\{\infty\}) = 0$, and replace (b) by assumptions (c) and (d) as follows:

- (c) for fixed $t \geq 0$ and z in T , the distribution of $Z_t(z)$ is absolutely continuous with respect to μ , and
- (d) for fixed $t \geq 0$ and for $[\mu]$ - almost all z in T , the canonical (i.e.

Radon) version of Z_t is almost surely continuous at z . (Proposition 7.5 gives a criterion for this in terms of the finite-dimensional distributions.)

THEOREM A'

Under conditions (a), (c) and (d), the conclusions of Theorem A hold, and moreover:

(vi) *For each ω in Ω , and each $0 \leq s \leq t$, the discontinuities of the mapping $y \rightarrow X_{st}(y, \omega)$ form a set of μ -measure zero,*

(vii) *For each $0 \leq s \leq t$, the map $M \times \Omega \rightarrow M$ given by $(z, \omega) \rightarrow X_{st}(z, \omega)$ is $\mu \otimes P$ -measurable.*

Proof. See Sections 7 to 10.

In Part V we take M to be the one-point compactification of $\mathbb{R}^d$, and μ to be Lebesgue measure. Suppose that for z in $\mathbb{R}^d$, $b(z) = (b^{pq}(z))$ is a $d \times d$ matrix representing the covariance function of a spatially homogeneous mean-zero $\mathbb{R}^d$ -valued Gaussian random field U on $\mathbb{R}^d$; i.e., $b(z) = \text{Cov}(U(0), U(z))$. Assume furthermore, that

(C.1) $b(0) = I$. (A normalization condition only)

(C.2) A certain mild positive-definiteness condition; see Section 11.

(C.3) For $1 \leq p, q \leq d$, the mapping $z \rightarrow b^{pq}(z)$ is Lipschitz continuous on the complement of every open ball in $\mathbb{R}^d$, centered at the origin, and is also continuous at zero (possibly non-Lipschitz).

(C.4) There exist constants $c > 0$ and $\alpha > 1$ such that $G_L(z) \geq \alpha G_N(z)$

whenever $0 \leq |z| \leq c$, where G_L and G_N are certain functions defined in (13.1) and (13.2). (A more refined condition is available in the isotropic case: see Proposition 14.3.)

THEOREM B.

Under assumptions (C.1) to (C.4), the conditions of Theorem A are satisfied for the Markov processes $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$ with generator

$$(2.1) \quad A^{(k)} f(y_1, \dots, y_k) = \frac{1}{2} \sum_{1 \leq i, j \leq k} \sum_{1 \leq p, q \leq d} b^{pq}(y_j - y_i) D_1^p D_1^q f(y_1, \dots, y_k)$$

for f in $C_0^\infty((\mathbb{R}^d)^k)$ and $(y_1, \dots, y_k)$ in $(\mathbb{R}^d)^k$, where $D_1^p = \partial / \partial y_1^p$.

Let $(X_{st}, 0 \leq s \leq t < \infty)$ be the pure stochastic flow constructed according to Theorem A, and abbreviate $X_{0t}(z)$ to $X_t(z)$. Then

$$(2.2) \quad d \langle X^p(z_1), X^q(z_j) \rangle_t = b^{pq}(X_t(z_j) - X_t(z_1)) dt.$$

Proof. See Sections 11, 12 and 13.

A special case of the situation above is described in detail in Section 14, namely the case where the covariance is isotropic. If $(Z_t(z_t), \dots, Z_t(z_k), t \geq 0)$ are as in Theorem B, define $\rho_t(y) = |Z_t(y) - Z_t(0)|$. In the isotropic case, $(\rho_t(y), t \geq 0)$ is a one-dimensional diffusion, whose scale function has a derivative H as in (14.9) below. Consider the following conditions:

(C.4') There exists a solution S of the differential equation $S'(t) = H(t)$, $t > 0$, such that $S(0+) = 0$.

(C.5) Condition (C.4') holds, and

$$\int_{(0,1)} S(u)/[S'(u)(1-B_L(u))]du < \infty \quad (\text{see (14.16)}).$$

THEOREM C.

Theorem B holds in the isotropic case, under assumptions (C.1) to (C.3) and (C.4'). Let $(X_{st}, 0 \leq s \leq t \leq \infty)$ be the corresponding pure stochastic flow defined on $(\Omega, \mathcal{F}, P)$, and define

$$v(x, y, \omega) = \inf \{t : X_{0t}(x, \omega) = X_{0t}(y, \omega)\}.$$

Then $\{\omega : v(x, y, \omega) \leq t\}$ is P -measurable for all x, y . Moreover

(a) If (C.5) is false, then $P(v(x, y) < \infty) = 0$ for $x \neq y$.

(b) If (C.5) is true, then:

(i) $P(v(x, y) < \infty) = 1 - S(|x - y|)/S(\infty)$.

(ii) Let $q(t, x, y)$ denote $P(v(x, y) \leq t)$; then $(x, y) \mapsto q(t, x, y)$ is Lebesgue measurable on $(\mathbb{R}^d)^2$, and $q(t, x, y) \rightarrow 1$ as $y \rightarrow x$.

(iii) For each $0 \leq s < t$, and each countable dense subset D of $\mathbb{R}^d$, and each y in $\mathbb{R}^d$, $X_{st}(y)$ belongs to $X_{st}(D)$ almost surely.

Proof. See sections 14 and 17.

The text contains three important examples. Section 16 is devoted to an example, lifted from an unpublished manuscript of T.E. Harris, to show that assumptions (C.1) to (C.3), (C.4') and (C.5) can be satisfied simultaneously in dimension two. (Thus the theory presented herein is not vacuous!) The last part of Section 6 describes a method of constructing a convolution semigroup of probability measures, and a pure stochastic flow, by composing suitable random transformations of M at each of the jump

times of a Poisson process; the author thanks K. Parthasarathy for explaining this construction. An application of this method is made at the end of section 17, to illustrate a certain pathological behavior that pure stochastic flows may exhibit; namely that it may happen that $P[\mu(X_{0t}(M) > 0)] > 0$, for all $t \geq 0$, even though assertion (iii) of Theorem C is true.

The methods which are used in this paper are as follows: Sections 4 to 10 are measure-theoretic, with Nelson [12] as the principal reference. The proofs in the remaining sections are elementary martingale theory, stochastic differential equations, and the theory of one-dimensional diffusions. An exception is Section 16, which is mainly an application of Bessel function techniques. Lesser known facts about Radon probability measures are collected in an Appendix.

3. Pure Stochastic Flows.

For every topological space Y studied in the sequel, $\mathbb{B}(Y)$ and $\mathbb{B}_0(Y)$ will denote the Borel sigma-algebra and the Baire sigma-algebra respectively. (See Appendix A). $C(Y)$ (resp. $C_0(Y)$) denotes the continuous functions (resp. with compact support) from Y to $\mathbb{R}$.

Henceforth M will denote an infinite compact Hausdorff space with a countable base. Such a space is metrizable, and so we may consider M as a compact metric space with metric $\rho(\cdot, \cdot)$.

Let Γ denote the product space M^M , considered as the space of functions from M to M with the product topology. Γ is compact by Tychonoff's theorem. A base for the open sets of Γ is given by the class of all sets of the form

$$\{f \in \Gamma : f(x_1) \in G_1, \dots, f(x_n) \in G_n\}$$

where $x_1, \dots, x_n$ are arbitrary points in M , and $G_1, \dots, G_n$ are arbitrary open sets in M . The Borel structure of Γ has been studied by Nelson [21].

Definition 3.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $(X_{st}, 0 \leq s \leq t < \infty)$ of measurable mappings from $(\Omega, \mathcal{F})$ to $(\Gamma, \mathbb{B}(\Gamma))$ will be called a **pure stochastic flow** on M if (a) and (b) hold:

$$(3.1) \quad (a) \quad X_{tt}(\omega) = \text{identity, for all } t \geq 0, \text{ and all } \omega \text{ in } \Omega.$$

(3.2) (b) $X_{tu}(\omega) \circ X_{st}(\omega) = X_{su}(\omega)$ ($\circ$ denotes composition) whenever $0 \leq s \leq t \leq u$, for all ω in Ω , with no exceptions.

If we use the notation $X_{st}(y, \omega)$ instead of $X_{st}(\omega)(y)$, for y in M , then (3.2) becomes:

(3.2') $X_{tu}(X_{st}(y, \omega), \omega) = X_{su}(y, \omega)$, y in M , ω in Ω .

Loosely speaking, each random variable X_{st} is a 'random transformation' of M , and (3.2') is a kind of semigroup property, often called the flow property. For fixed $s \geq 0$ and $y_1, \dots, y_k$ in M , $(X_{st}(y_1), \dots, X_{st}(y_k), t \geq s)$ is an M^k -valued stochastic process, called a k-point motion.

Here are two obvious conditions to place on a pure stochastic flow:

(R.1) **Time homogeneity:** The processes $(X_{s+h, t+h}, 0 \leq s \leq t < \infty)$ have the same law for all $h \geq 0$.

(R.2) **Independent increments:** for $s \leq t$, let E_{st} denote the sigma-algebra generated by $(X_{uv}, s \leq u \leq v \leq t)$. We require that $E_{st}, E_{s't'}, \dots$ are independent with respect to P if $s \leq t \leq s' \leq t' \leq \dots$.

Definition 3.2. A pure stochastic flow on M will be called a **time-homogeneous pure stochastic flow with independent increments** if (R.1) and (R.2) hold.

Part II. Construction of a stochastic flow assuming spatial stochastic continuity

The main result of Part II is Theorem 6.1, which explains how to obtain a pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ given that the desired finite dimensional distributions of each X_{st} satisfy a spatial stochastic continuity condition, which turns out to be easy to verify in cases of interest. Sections 4 and 5 develop the necessary (and somewhat novel) measure-theoretic machinery.

4. Convolution of measures with respect to composition of functions.

Let M and Γ be as in Section 3. Suppose Q and R were a pair of probability measures on $B(\Gamma)$ such that the operation of composition of functions, $(f, g) \rightarrow (f \circ g)$ were $Q \otimes R$ - measurable. Then it would be possible to construct a convolution $Q * R$ by pushing forward the the product measure $Q \otimes R$. Unfortunately it does not seem to be possible to prove the measurability of composition with respect to the completed product σ -algebra, for the class of measures we wish to consider.

However we can avoid this problem by using an even larger σ -algebra on

[illegible]

$\Gamma \times \Gamma$, as we shall see.

Define the canonical Γ -valued random variable Z to be the map from $M \times \Gamma$ to M given by

$$(4.1) \quad Z(x, f) = f(x), \quad x \text{ in } M, \quad f \text{ in } \Gamma.$$

Definition 4.1. A probability measure Q on $\mathcal{B}(\Gamma)$ will be called **stochastically continuous** on a subset E of M if the random variables $(Z(x), x \in M)$ on $(\Gamma, \mathcal{B}(\Gamma), Q)$ are stochastically continuous for x in E ; in other words, for each x in E and every $\varepsilon > 0$,

$$(4.2) \quad Q(\{f: \rho(f(x), f(y)) > \varepsilon\}) \rightarrow 0 \quad \text{as } y \rightarrow x.$$

If this convergence is uniform over x in E , Q will be called **uniformly stochastically continuous** on E .

Remark 4.2. The set $G = \{f: \rho(f(x), f(y)) > \varepsilon\}$ is an open Baire subset of Γ , as noted by Nelson [21, p. 636]. To see that G is open, express G as the union of the open sets $\{f: \rho(f(x), w) < \varepsilon/2, \rho(f(y), z) < \varepsilon/2\}$, over all points z and w in M such that $\rho(z, w) \geq 2\varepsilon$. Since G is open, G^c is closed and hence compact in Γ . Moreover G^c is a G_δ set, since it is the intersection $\bigcap_n \{f: \rho(f(x), f(y)) < \varepsilon + 1/n\}$ of open subsets of Γ .

Definition 4.3. A probability measure Q on $\mathbb{B}_0(\Gamma)$, or its unique Radon extension to $\mathbb{B}(\Gamma)$ (see Appendix A.4), is said to satisfy a **composability condition** if either of the following two conditions holds:

(4.3) Q is stochastically continuous on M (and hence uniformly stochastically continuous, by compactness of M),

or

(4.4) M is the one-point compactification of an infinite separable metric space (T, d) , (write $M = T \cup \{\infty\}$), Q is uniformly stochastically continuous on T with respect to d , and (4.5) holds:

$$(4.5) \quad Q(\{f: f(\infty) = \infty\}) = 1, \quad Q(\{f: f(x) = \infty\}) = 0 \quad \text{for } x \in T.$$

For the remainder of this section we will work mostly with condition (4.4); however all the results below remain true under the simpler condition (4.3).

Recall that a **Dynkin system** in a set Ω is a class $\mathbb{D}$ of subsets of Ω such that Ω is in $\mathbb{D}$, and $\mathbb{D}$ is closed under taking complements and countable disjoint unions. A countably additive non-negative set function on $\mathbb{D}$ is called a **premeasure** on $\mathbb{D}$.

Definition 4.4. Suppose $(Y_i, \mathcal{E}_i, \lambda_i)$, $i=1, 2, \dots, n$, are probability spaces. For E contained in $Y_1 \times \dots \times Y_n$, $\mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n$ denotes

$\{y_1 : (y_1, y_2, \dots, y_n) \in E\}$. The $(\lambda_1, \dots, \lambda_n)$ - **left product Dynkin system**, denoted $\Lambda_{\lambda_1, \dots, \lambda_n}$, is defined as follows: a subset E of $Y_1 \times \dots \times Y_n$ is an element of $\Lambda_{\lambda_1, \dots, \lambda_n}$ if all the following conditions hold.

$$\begin{aligned}
 (4.6) \quad & E \cap \{y_2, \dots, y_n\} \in E_1 \text{ for all } (y_2, \dots, y_n) \in Y_2 \times \dots \times Y_n, \\
 & y_2 \rightarrow \lambda_1(E \cap \{y_2, \dots, y_n\}) \text{ is } \lambda_2\text{-measurable for all } y_3, \dots, y_n, \\
 & y_3 \rightarrow \int \lambda_1(E \cap \{y_2, \dots, y_n\}) \lambda_2(dy_2) \text{ is } \lambda_3\text{-measurable for all} \\
 & \quad y_4, \dots, y_n \\
 & \dots \\
 & y_n \rightarrow \int \dots \int \lambda_1(E \cap \{y_2, \dots, y_n\}) \lambda_2(dy_2) \dots \lambda_{n-1}(dy_{n-1}) \text{ is } \lambda_n\text{-} \\
 & \quad \text{measurable.}
 \end{aligned}$$

It is easy to check that $\Lambda_{\lambda_1, \dots, \lambda_n}$ is closed under taking complements

and countable unions of disjoint sets. The **left product premeasure**

$\lambda_1 \otimes_L \dots \otimes_L \lambda_n$ on $\Lambda_{\lambda_1, \dots, \lambda_n}$ is defined by

$$(4.7) \quad \lambda_1 \otimes_L \dots \otimes_L \lambda_n(E) = \int_{y_n} \dots \left[\int_{y_2} \lambda_1(E \cap \{y_2, \dots, y_n\}) \lambda_2(dy_2) \right] \dots \lambda_n(dy_n)$$

Remarks 1. $E_1 \times \dots \times E_n$ is contained in $\Lambda_{\lambda_1, \dots, \lambda_n}$, and $\lambda_1 \otimes_L \dots \otimes_L \lambda_n$

is an extension to $\Lambda_{\lambda_1, \dots, \lambda_n}$ of the usual product measure $\lambda_1 \otimes \dots \otimes \lambda_n$.

2. The order of integration in (4.7) is: $\lambda_2, \lambda_3, \dots, \lambda_n$. Interchanging the

order is not allowed.

PROPOSITION 4.5

Let $c: \Gamma \times \Gamma \rightarrow \Gamma$ be the function $c(f,g) = f \circ g$ (composition). Suppose Q and R are probability measures on $\mathbb{B}_0(\Gamma)$ such that either Q satisfies composability condition (4.3), or Q satisfies composability condition (4.4) and R satisfies (4.5). Then c is $(\Lambda, \mathbb{B}_0(\Gamma))$ -measurable, where $\Lambda = \Lambda_{Q,R}$ is the left product Dynkin system

(Definition 4.4).

Proof. Suppose A is an arbitrary element of $\mathbb{B}_0(\Gamma)$; then $c^{-1}(A)$ is in Λ if and only if the following two assertions hold:

$$(4.8) \quad \{f: f \circ g \in A\} \in \mathbb{B}_0(\Gamma);$$

$$(4.9) \quad g \rightarrow Q(\{f: f \circ g \in A\}) \text{ is } R\text{-measurable.}$$

We assume that Q satisfies (4.4) and R satisfies (4.5); (the proof is simpler in the other case). The proof depends on the following Lemma.

LEMMA 4.6.

Assertions (4.8) and (4.9) hold when $A = \{h: h(x) \in G\}$, for any x in M , and any open subset G of M .

Proof. Assertion (4.8) is trivially true, since $\{f: f \circ g \in A\} = \{f: f(g(x)) \in G\}$, which is Baire. As for assertion (4.9), there are two cases:

Case I: $x \in T$. Let φ be an arbitrary bounded continuous function on M , and let (y_n) be a sequence in T converging to y in T . The random variables $(\varphi(Z(y_n)), n \geq 1)$ (see (4.1)), defined on the probability space $(\Gamma, \mathbb{B}_0(\Gamma), Q)$, are uniformly bounded and converge in probability to $\varphi(Z(y))$, by (4.3); hence they converge in mean, and so the map

$$(4.10) \quad y \rightarrow \mathbb{E}^Q [\varphi(Z(y))] = \int_{\Gamma} \varphi(Z(y, f)) Q(df)$$

is continuous on T .

Fix x in T , and an open subset G of M . The map $g \rightarrow g(x)$ from Γ to M is continuous, by the definition of the product topology. Composing this map with the continuous map appearing in (4.10), we see that

$$(4.11) \quad g \rightarrow \int_{\Gamma} \varphi(Z(g(x), f)) Q(df)$$

is continuous from Γ to $\mathbb{R}$, on the set $\{g: g(x) \in T\}$.

Urysohn's Lemma ensures that there exists a sequence of bounded functions (φ_n) in $C(M)$ which converge pointwise to the indicator of the open set G . Thus if A is the set specified in the Lemma, and if $g(x)$ is in T , then

$$Q(\{f: f \circ g \in A\}) = Q(\{f: f(g(x)) \in G\})$$

$$= \int_{\Gamma} 1_G(Z(g(x), f)) Q(df), \text{ (notation as in (4.1)),}$$

$$= \int_{\Gamma} \lim \phi_n(Z(g(x), f)) Q(df)$$

$$= \lim \int_{\Gamma} \phi_n(Z(g(x), f)) Q(df)$$

using Lebesgue's bounded convergence theorem. Thus $Q(\{f: f \circ g \in A\})$ is a limit of continuous functions of g by (4.11), and is therefore a Baire measurable function of g on the set $\{g: g(x) \in T\}$, which is of R -measure one. Hence the mapping $g \rightarrow Q(\{f: f \circ g \in A\})$ is R -measurable.

Case II: $x = \infty$. On the set $\{g: g(\infty) = \infty\}$, it is true that $Q(\{f: f \circ g \in A\}) = Q(\{f: f(\infty) \in G\}) = 1_G(\infty)$ by (4.5). Since $R(\{g: g(\infty) = \infty\}) = 1$, it follows that assertion (4.9) holds in this case also. $\square$

Proof of Proposition 4.5, continued

Let S denote the collection of all subsets A of Γ for which (4.8) and (4.9) hold. Then S is a σ -algebra containing all sets A of the form described in Lemma 4.6. However such sets generate $B_0(\Gamma)$. This proves that $c^{-1}(A)$ belongs to A for all A in $B_0(\Gamma)$, as desired. $\square$

Definition 4.7. Suppose that Q and R are Radon probability measures on $B_0(\Gamma)$, or their unique Radon extensions to $B(\Gamma)$, such that either Q satisfies (4.3); or Q satisfies (4.4) and R satisfies (4.5). By virtue of Proposition 4.5, we may define a probability measure $Q * R$ on $B_0(\Gamma)$,

(whose unique Radon extension to $\mathbb{B}(\Gamma)$ will also be denoted $Q * R$), by the formula

$$(4.12) \quad Q * R(A) = (Q \otimes_L R) \circ c^{-1}(A) = \int Q(\{f : f \circ g \in A\}) R(dg),$$

for A in $\mathbb{B}_0(\Gamma)$; note that countable additivity of $Q * R$ follows from countable additivity of $Q \otimes_L R$.

We shall call $Q * R$ the composition convolution, or simply the convolution, of Q and R .

Remarks 4.8.

1. Suppose V and W are independent Γ -valued random variables on a probability space $(\Omega, \mathcal{E}, P)$, with distributions Q and R respectively.

Then (4.12) could be written as:

$$Q * R(A) = \mathbb{E}[\mathbb{E}[1_A(V \circ W) | W]].$$

According to the reasoning above, the expression above is well-defined, even though $P(V \circ W \in A)$ may not be (i.e. $V \circ W$ may fail to be a random variable). In other words, $\{(f, g) : f \circ g \in A\}$ may fail to be $Q \otimes R$ -measurable.

2. By virtue of the last remark, Fubini's theorem does not apply to (4.12); in particular, reversing the order of integration is not allowed.

3. The Riesz representation theorem provides an alternative way of making the same construction, under the assumptions of Proposition 4.5; $Q \ast R$ is the unique Radon probability measure on $\mathbb{B}(\Gamma)$ representing the positive linear functional Φ on $C(\Gamma)$, where

$$(4.13) \quad \Phi(\psi) = \int \int \psi(f \circ g) Q(df) R(dg) \quad , \quad \psi \in C(\Gamma).$$

(By (4.11), the right side is well-defined at least for ψ of the form $\psi(h) = \phi_1(h(x_1)) \dots \phi_m(h(x_m))$, where $x_1, \dots, x_m$ are points in M and $\phi_1, \dots, \phi_m$ are in $C(M)$; the linear span of such functions ψ is dense in $C(\Gamma)$ by the Stone-Weierstrauss theorem.)

In order to iterate the composition convolution of probability measures on $\mathbb{B}_0(\Gamma)$, we need the next result.

PROPOSITION 4.9.

Suppose Q and R are probability measures on $\mathbb{B}_0(\Gamma)$ (or their unique Radon extensions to $\mathbb{B}(\Gamma)$) both of which satisfy composability condition (4.4) (resp. (4.3)). Then $Q \ast R$ also satisfies (4.4) (resp. (4.3)).

Proof. Suppose Q and R both satisfy (4.4). First we shall check that $Q \ast R$ satisfies (4.5). Let $H = \{h : h(\infty) = \infty\}$. Then

$$Q * R(H) \geq \int_H Q(\{f: f \circ g(\infty) = \infty\}) R(dg) = Q(H)R(H) = 1.$$

On the other hand if x is in T , and if K denotes $\{h: h(x) = \infty\}$, then

$$Q * R(K) = \int_K Q(\{f: f \circ g(x) = \infty\}) R(dg) + \int_{K^c} Q(\{f: f \circ g(x) = \infty\}) R(dg) = 0,$$

since $R(K) = 0$, and $Q(\{f: f \circ g(x) = \infty\}) = 0$ for g in K^c . This verifies

(4.5) for $Q * R$.

Secondly we shall check the uniform stochastic continuity of $Q * R$ on

T . Given $\varepsilon > 0$ and $\delta > 0$, there exist $\eta > 0$ and $\gamma > 0$ such that:

$$(4.14) \quad Q(\{f: d(f(z), f(w)) > \varepsilon\}) \leq \delta/2 \text{ whenever } d(z, w) \leq \eta, \text{ and}$$

$$(4.15) \quad R(\{g: d(g(x), g(y)) > \eta\}) \leq \delta/2 \text{ whenever } d(x, y) \leq \gamma.$$

Let J denote the Baire set (see Remark 4.2) $\{g: d(g(x), g(y)) > \eta\}$ in Γ , for

some fixed x and y in T with $d(x, y) \leq \gamma$; then

$$\begin{aligned} Q * R(\{h: d(h(x), h(y)) > \varepsilon\}) &= \int Q(\{f: d(f \circ g(x), f \circ g(y)) > \varepsilon\}) R(dg) \\ &\leq Q(\Gamma)R(J) + \int_{J^c} Q(\{f: d(f \circ g(x), f \circ g(y)) > \varepsilon\}) R(dg) \leq \delta/2 + R(J^c)\delta/2 \leq \delta. \end{aligned}$$

Since ε and δ are arbitrary, this verifies the uniform stochastic

continuity of $Q * R$ on T . □

COROLLARY 4.10.

Suppose $Q_1, \dots, Q_n$ are probability measures on $\mathbb{B}_0(\Gamma)$ (or their

unique Radon extensions to $\mathbb{B}(\Gamma)$ which satisfy composability condition (4.4) (resp. (4.3)). By virtue of Proposition 4.9, we may define the convolution $Q_1 * \dots * Q_n$, for $n \geq 3$, inductively as follows:

$$(4.16) \quad Q_1 * \dots * Q_n = (Q_1 * \dots * Q_{n-1}) * Q_n.$$

(The operation $*$ is not necessarily associative.) Moreover

(i) Every such composition convolution satisfies the composability condition (4.4) (resp. (4.3)), and

(ii) Let $c_n: \Gamma^n \rightarrow \Gamma$ be the mapping $c_n(f_1, \dots, f_n) = f_1 \circ \dots \circ f_n$. Then c_n is $(\Lambda_{Q_1, \dots, Q_n}, \mathbb{B}_0(\Gamma))$ - measurable (see definition (4.4), and

$$(4.17) \quad Q_1 * \dots * Q_n = (Q_1 \otimes_L \dots \otimes_L Q_n) \circ c_n^{-1}.$$

Proof. Only (ii) requires proof. The pair of assertions hold for $n=2$, by Proposition 4.5 and Definition 4.7. Consider the inductive hypothesis that both assertions hold when $n = m$, for some $m \geq 2$. Then for A in $\mathbb{B}_0(\Gamma)$, the mapping.

$$f_{m+1} \rightarrow (Q_1 * \dots * Q_m) (\{h: h \circ f_{m+1} \in A\})$$

is Q_{m+1} - measurable, by assertion (i) and Proposition (4.5). Referring to

(4.6), we see that this, together with the $(\Lambda_{Q_1, \dots, Q_m}, \mathbb{B}_0(\Gamma))$ -

measurability of c_m (which holds by induction), shows that c_{m+1} is

$(\Lambda_{Q_1, \dots, Q_{m+1}}, \mathbb{B}_0(\Gamma))$ - measurable. Moreover by (4.7),

$$(Q_1 \otimes_L \dots \otimes_L Q_{m+1})(c_{m+1}^{-1}(A)) = \\ \int \dots \int Q_1(\{f_1 : f_1 \circ \dots \circ f_{m+1} \in A\}) Q_2(df_2) \dots Q_{m+1}(df_{m+1})$$

Using (4.7) again, this is equal to

$$\int (Q_1 \otimes_L \dots \otimes_L Q_m)(\{(f_1, \dots, f_m) : f_1 \circ \dots \circ f_{m+1} \in A\}) Q_{m+1}(df_{m+1}), \\ = \int (Q_1 * \dots * Q_m)(\{h : h \circ f_{m+1} \in A\}) Q_{m+1}(df_{m+1})$$

using the inductive hypothesis for (4.17). Using (4.12) and (4.17), this is

$$= (Q_1 * \dots * Q_m) * Q_{m+1}(A) = Q_1 * \dots * Q_{m+1}(A).$$

This completes the induction and the proof. $\square$

Definition 4.11. A map $w : \Gamma^r \rightarrow \Gamma^k$ will be called a **cascade** if there exist integers $1 \leq a(i) \leq b(i) \leq r$, $i = 1, 2, \dots, k$, such that

$$(4.18) \quad w(f_1, \dots, f_r) = (g_1, \dots, g_k), \text{ where } g_i = f_{a(i)} \circ f_{a(i)+1} \circ \dots \circ f_{b(i)}.$$

The map w will be called a **perfect cascade** if $a(1) = 1$, $b(i) + 1 = a(i+1)$ for $i = 1, 2, \dots, k-1$, and $b(k) = r$.

Definition 4.12. A family of probability measures $\{Q_t, t \geq 0\}$ on $\mathbb{B}_0(\Gamma)$

(or their unique Radon extension to $\mathbb{B}(\Gamma)$), all of which satisfy

composability condition (4.4) (resp. (4.3)), will be called a **convolution**

semigroup if $Q_s * Q_t = Q_{s+t}$ for all $s, t \geq 0$, and if $Q_0(\{e\}) = 1$, where $e :$

$M \rightarrow M$ is the identity map.

PROPOSITION 4.13

- (i) Suppose $Q_1, \dots, Q_r$ are probability measures on $\mathbb{B}_0(\Gamma)$ (or their unique Radon extensions to $\mathbb{B}(\Gamma)$) which satisfy composability condition (4.4) (resp. (4.3)). If $w: \Gamma^r \rightarrow \Gamma^k$ is a cascade (Definition 4.11), then w is $(\bigwedge_{Q_1, \dots, Q_r}, \mathbb{B}_0(\Gamma^k))$ -measurable.

- (ii) Consider the following commutative diagram:

$$(4.19) \quad \begin{array}{ccc} u: \Gamma^m & \longrightarrow & \Gamma^k \\ \downarrow w & & \downarrow \text{proj}(j) \\ v: \Gamma^n & \longrightarrow & \Gamma^j \end{array}$$

where u and v are cascades, w is a perfect cascade, j is less than or equal to k , and $\text{proj}(j)$ denotes projection onto the first j factors. Let us represent the mapping w as $w(\underline{f}) = \underline{g}$ where

$$(4.20) \quad \underline{f} = (f_{1,1}, \dots, f_{1,r(1)}, \dots, f_{n,1}, \dots, f_{n,r(n)}), \quad \underline{g} = (g_1, g_2, \dots, g_n), \quad \text{and} \\ g_1 = f_{1,1} \circ \dots \circ f_{1,r(1)}.$$

(here $r(1) + \dots + r(n) = m$). Suppose that $\{Q_t, t \geq 0\}$ is a convolution semigroup (Definition 4.12) and $s(i, q) \geq 0$ for $1 \leq i \leq n$, $1 \leq q \leq r(i)$. Let $t(i)$ denote $s(i, 1) + \dots + s(i, r(i))$. Let P_1 denote the probability measure

$$(Q_{s(1,1)} \otimes_L \dots \otimes_L Q_{s(1,r(1))} \otimes_L \dots \otimes_L Q_{s(n,1)} \otimes_L \dots \otimes_L Q_{s(n,r(n))}) \circ u^{-1}$$

on $\mathbb{B}_0(\Gamma^k)$ (such a measure exists by (i)), and let P_2 denote the probability measure $(Q_{t(1)} \otimes_L \dots \otimes_L Q_{t(n)}) \circ v^{-1}$ on $\mathbb{B}_0(\Gamma^j)$. Then

$$(4.21) \quad P_2 = P_1 \circ \text{proj}(j)^{-1}.$$

Proof. (i) Each co-ordinate function g_i appearing in (4.18) is

$(c_r^{-1}(\mathbb{B}_0(\Gamma)), \mathbb{B}_0(\Gamma))$ - measurable by Corollary 4.10, and hence w is also.

Since $c_r^{-1}(\mathbb{B}_0(\Gamma))$ is contained in $\Lambda_{Q_1, \dots, Q_r}$, assertion (i) follows.

(ii) As suggested by (4.13), we represent the Radon extensions of P_1 and P_2

by positive linear functionals Φ_1 and Φ_2 (specified below) on $C(\Gamma^k)$ and

$C(\Gamma^j)$ respectively; then (4.19) and (4.20) imply that $v_i(\underline{g}) = u_i(\underline{f})$ for

$i = 1, 2, \dots, j$. Hence for ψ in $C(\Gamma^j)$,

$$\begin{aligned} \Phi_2(\psi) &= \int \dots \int \psi(v_1(\underline{g}), \dots, v_j(\underline{g})) Q_{t(1)}(dg_1) \dots Q_{t(n)}(dg_n), \\ &= \int \dots \int (\psi \circ \text{proj}(j))(u_1(\underline{f}), \dots, u_k(\underline{f})) Q_{s(1,1)}(df_{1,1}) \dots Q_{s(n,r(n))}(df_{n,r(n)}), \end{aligned}$$

since $Q_{t(n)} = Q_{s(1,1)} * \dots * Q_{s(n,r(n))}$ by the semigroup property. So

$$\Phi_2(\psi) = \Phi_1(\psi \circ \text{proj}(j)), \text{ which verifies (4.21). } \square$$

5. A projective system for building a pure stochastic flow.

In Appendix B we reproduce a definition, due to Bochner[4], of a projective system of Radon topological probability spaces. Our programme is to use Bochner's theorem on projective limits to construct a pure stochastic flow. The purpose of this section is to define a suitable projective system, starting from a convolution semigroup of Radon probability measures $\{Q_t, t \geq 0\}$ on $\mathbb{B}(\Gamma)$ (see Definition 4.12).

The index set I will consist of all finite subsets of $[0, \infty)$ which contain 0, and whose cardinality ≥ 3 , partially ordered by inclusion. A typical α in I will be written as $(t_1, t_2, \dots, t_n)$, where $0 = t_1 \leq t_2 \leq \dots \leq t_n$, and $n = |\alpha| \geq 3$. For each α in I , we shall define a Hausdorff space S_α and a Radon probability measure P_α on $\mathbb{B}(S_\alpha)$; since the general definition is rather subtle, we shall first describe S_α explicitly when $|\alpha| = 3$ and when $|\alpha| = 4$.

When $|\alpha| = 3$, $S_\alpha = \{(f, g, f \circ g) : f, g \in \Gamma\} \subset \Gamma^3$.

When $|\alpha| = 4$,

$$S_\alpha = \{((f_2, f_1, f_2 \circ f_1), (f_3, f_2, f_3 \circ f_2), (f_3 \circ f_2, f_1, f_3 \circ f_2 \circ f_1), (f_3, f_2 \circ f_1, f_3 \circ f_2 \circ f_1)) : f_1, f_2, f_3 \in \Gamma\}.$$

In general, when $|\alpha| = n$, define

$$(5.1) \quad J(\alpha) = \{\sigma: \{1,2,3\} \rightarrow \{1,2,\dots,n\} \text{ such that } \sigma(1) < \sigma(2) < \sigma(3)\}.$$

The elements of $J(\alpha)$ may be listed in arbitrary order as $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ where $m = n!/(n-3)!3!$. Let σ be an arbitrary element of $J(\alpha)$, and suppose $\sigma(1) = q$, $\sigma(2) = r$, and $\sigma(3) = s$; thus $1 \leq q < r < s \leq n$. For Γ as in Section 3, define $\psi(\sigma): \Gamma^{n-1} \rightarrow \Gamma^3$ as follows:

$$(5.2) \quad \psi(\sigma)(h_{n,n-1}, \dots, h_{21}) = (h_{s,s-1} \circ \dots \circ h_{r+1,r}, h_{r,r-1} \circ \dots \circ h_{q+1,q}, h_{s,s-1} \circ \dots \circ h_{q+1,q}).$$

(The reason for labelling elements of Γ^{n-1} this way will be apparent in the definition of P_α .) Next define $\psi_\alpha: \Gamma^{n-1} \rightarrow \Gamma^{3m}$, and S_α , by

$$(5.3) \quad \psi_\alpha(\underline{h}) = (\psi(\sigma_1)(\underline{h}), \dots, \psi(\sigma_n)(\underline{h})), \quad \underline{h} = (h_{21}, \dots, h_{n,n-1}).$$

$$(5.4) \quad S_\alpha = \psi_\alpha(\Gamma^{n-1}) \subset \Gamma^{3m}.$$

We give S_α the subspace topology induced from Γ^{3m} , which makes it a Hausdorff space. Thus $\mathbb{B}_0(S_\alpha) = \{C \cap S_\alpha : C \in \mathbb{B}_0(\Gamma^{3m})\}$ and $\mathbb{B}(S_\alpha) = \{C \cap S_\alpha : C \in \mathbb{B}(S_\alpha)\}$. The map ψ_α is a cascade (Definition 4.11), and given a

convolution semigroup $\{Q_t, t \geq 0\}$ of Radon probability measures on $\mathbb{B}(\Gamma)$,

Proposition 4.13 shows that there is a probability measure P_α on $\mathbb{B}_0(S_\alpha)$

given by:

$$(5.5) \quad P_\alpha(C) = (Q_{t_n - t_{n-1}} \otimes_L \dots \otimes_L Q_{t_2 - t_1})(\psi_\alpha^{-1}(C)), \text{ for } C \in \mathbb{B}_0(S_\alpha).$$

We then take the unique Radon extension to $\mathbb{B}(S_\alpha)$, also denoted P_α . We

have now constructed a Radon probability space $(S_\alpha, \mathbb{B}(S_\alpha), P_\alpha)$ for each α

in I , such that S_α is Hausdorff.

The maps $g_{\alpha\beta} : S_\beta \rightarrow S_\alpha$, $\alpha \leq \beta$, are constructed in a natural way as follows. Suppose $\alpha = (t_1, \dots, t_n)$ and $\beta = (u_1, \dots, u_q)$ with $n \leq q$, and suppose χ is the one-to-one increasing map from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, q\}$ such that $t_i = u_{\chi(i)}$. Then χ induces a map $\text{comp}(\chi) : \Gamma^{q-1} \rightarrow \Gamma^{n-1}$ as follows;

$\text{comp}(\chi)(g_{q,q-1}, \dots, g_{21}) = (h_{n,n-1}, \dots, h_{21})$, where

$$(5.6) \quad h_{i+1,i} = g_{\chi(i+1), \chi(i+1)-1} \circ \dots \circ g_{\chi(i)+1, \chi(i)}, \quad i = 1, 2, \dots, n-1.$$

If $J(\alpha) = \{\sigma_1, \dots, \sigma_m\}$, then $J(\beta)$ may be expressed as $\{\tau_1, \dots, \tau_r\}$ where $(\tau_j(1),$

$\tau_j(2), \tau_j(3)) = (\chi(\sigma_j(1)), \chi(\sigma_j(2)), \chi(\sigma_j(3)))$, for $j = 1, 2, \dots, m$. Then the

following diagram commutes:

$$\begin{array}{ccc}
 (5.7) \quad \psi_\beta : \Gamma^{q-1} & \xrightarrow{\hspace{2cm}} & S_\beta \\
 \downarrow \text{comp}(\chi) & & \downarrow \text{proj}(3m) \\
 \psi_\alpha : \Gamma^{n-1} & \xrightarrow{\hspace{2cm}} & S_\alpha
 \end{array}$$

where $\text{proj}(3m)$ is the projection map of Γ^{q-1} onto the first $3m$ factors.

When $J(\alpha)$ and $J(\beta)$ are listed in the manner above, define $g_{\alpha\beta}$ to be the mapping $\text{proj}(3m): S_\beta \rightarrow S_\alpha$. We may now deduce the following facts:

LEMMA 5.1.

- (i) $g_{\alpha\beta}$ is a continuous, onto map, and $g_{\alpha\alpha}$ is the identity.
- (ii) Assuming that $\{Q_t, t \geq 0\}$ is a convolution semigroup of Radon probability measures on $B(\Gamma)$, it is true that $P_\beta(g_{\alpha\beta}^{-1}(C)) = P_\alpha(C)$ for all C in $B(S_\alpha)$.

Proof. Continuity is immediate because $g_{\alpha\beta}$ is a projection. Let $\psi_\alpha(\underline{h})$ be an arbitrary element of S_α , where $\underline{h}$ is in Γ^{n-1} . Define $\underline{f}$ in Γ^{q-1} by taking

$$f_{x(i), x(i)-1} = h_{i, i-1} \text{ for } i = 2, 3, \dots, n, \text{ and } f_{j, j-1} = e = \text{identity, for all other } j.$$

Then $\text{comp}(\chi)(\underline{f}) = \underline{h}$, and $\psi_\alpha(\underline{h}) = g_{\alpha\beta}(\psi_\beta(\underline{f}))$. This proves that $g_{\alpha\beta}$ is onto. To

prove part (ii), notice $\text{comp}(\chi)$ is a perfect cascade (Definition 4.11), and

ψ_α and ψ_β are cascades. The commutative diagram (5.7) is therefore of the form (4.19), and Proposition 4.13 (ii) shows that P_α and $P_\beta \circ g_{\alpha\beta}^{-1}$ agree

on $B(S_\alpha)$. $\square$

PROPOSITION 5.2.

Suppose the space M (see Section 3) is infinite. Then $\{(S_\alpha, B(S_\alpha), P_\alpha, g_{\alpha\beta})_{\alpha \leq \beta: \alpha, \beta \in I}\}$, as defined above, is a projective system of Radon topological probability spaces (see Appendix B) with the sequential maximality property. Hence there exists a unique projective limit $(\Omega, \underline{E}, P)$ in the sense of Appendix B.

Remark. The central idea of this proof is due to Richard Arratia (U.S.C.).

Proof. The only condition which remains to be checked is the sequential maximality property; this requires a transfinite induction argument.

Step 1. For any (possibly infinite) subset λ of $[0, \infty)$ such that $0 \in \lambda$, let $p(\lambda)$ denote the set of pairs (s, t) with $s, t \in \lambda$ and $s < t$, and let T_λ denote the subset of $\Gamma^{p(\lambda)}$ each of whose elements is a collection of functions $\{f_{rt} : r, t \in \lambda, r < t\}$ for which $f_{rt} = f_{tu} \circ f_{rt}$ whenever $r < t < u$ in λ . For subsets λ and μ of $[0, \infty)$ with $\lambda \subset \mu$, there is a natural projection map

$\pi_{\lambda\mu} : T_\lambda \rightarrow T_\mu$, namely

$$(5.7) \quad \pi_{\lambda\mu}(\{f_{rt} : r, t \in \mu, r < t\}) = \{f_{rt} : r, t \in \lambda, r < t\}.$$

When λ is a finite set, the set S_λ is in one-to-one correspondence with T_λ : indeed, given an element of S_λ , we obtain the corresponding element of T_λ merely by deleting functions in the collection which are repeated.

Moreover for $\alpha < \beta$ in I , the following diagram commutes:

$$\begin{array}{ccc} T_\beta & \longleftrightarrow & S_\beta \\ \downarrow \pi_{\alpha\beta} & & \downarrow g_{\alpha\beta} \\ T_\alpha & \longleftrightarrow & S_\alpha \end{array}$$

Therefore to establish sequential maximality, it suffices to establish it for the equivalent system $\{(T_\alpha, \pi_{\alpha\beta})_{\alpha \leq \beta: \alpha, \beta \in I}\}$.

Step II. The heart of the proof is the following assertion:

(5.8) Given any proper subset λ of $[0, \infty)$ with $0 \in \lambda$, and given any $s > 0$ which is not an element of λ , and given any element $\underline{g}$ of T_λ , there exists an element $\underline{h}$ of T_μ , where $\mu = \lambda \cup \{s\}$, such that $\pi_{\lambda\mu}(\underline{h}) = \underline{g}$.

The construction of the element $\underline{h}$ goes as follows. Since 0 is an element of λ , the following subset of $M \times [0, \infty)$ is nonempty:

$$(5.9) \quad U_0 = \{(x, r) : x \in M, r \in \lambda, r < s\}.$$

Define an equivalence relation $\approx$ on U_0 by: $(x,r) \approx (y,t)$ if there exists w in λ , $\max(r,t) \leq w < s$, such that $g_{rw}(x) = g_{tw}(y)$, with the convention that g_{uu} is the identity map. Let U_1 denote the set of equivalence classes of U_0 under $\approx$.

We shall now show that $\text{card}(U_1) \leq \text{card}(M)$. Let $t(0) = -\infty$, and let $t(1) \leq t(2) \leq \dots$ be a sequence in λ with the following properties:

- (i) $t(n) < s$ for all $n \geq 1$.
- (ii) Let $t_1 = \sup\{t : t \in \lambda, t < s\}$; if $t_1 \in \lambda$, then let $t(n) = t_1$ for all $n \geq 1$.
- (iii) If t_1 is not in λ , let $\lim_n t(n) = t_1$.

Define M_n to be the set $\{g_{r,t(n)}(x) : x \in M, r \leq t(n)\}$, which is a subset of M , and let W be the disjoint union of the sets $\{M_n, n \geq 1\}$. Since M is an infinite set, $\text{card}(W) \leq \text{card}(M)$. Define a mapping $\zeta : U_1 \rightarrow W$ as follows: given an equivalence class c in U_1 , select an arbitrary representative (x,r) ; there is a unique $n \geq 1$ such that $t(n-1) < r \leq t(n)$, so take $\zeta(c)$ to be the element $g_{r,t(n)}(x)$ in M_n . Then ζ is one-to-one; for if $\zeta(c) = \zeta(d)$ in M_n , then c has a representative (x,r) , and d has a representative (y,u) , such that $r,u \leq t(n)$ and $g_{r,t(n)}(x) = g_{u,t(n)}(y)$, which implies $(x,r) \approx (y,u)$, and so $c = d$ in U_1 . The existence of the map ζ shows that $\text{card}(U_1) \leq \text{card}(W)$, and hence $\text{card}(U_1) \leq \text{card}(M)$, as desired. Thus there exists a one-to-one mapping $\theta : U_1 \rightarrow M$.

Now define h as follows:

Case 1. $r < s$, $r \in \lambda$. Let $h_{rs}(x) = \theta(x, r)$ for $x \in M$.

Case 2. $t > s$, $t \in \lambda$. If $x \in U_{r < s}$ Range(h_{rs}), let $h_{st}(x) = g_{rt}(y)$ for any (y, r) in U_0 such that $h_{rs}(y) = x$; if not let $h_{st}(x) = g_{vt}(x_0)$, where (x_0, v) is some fixed element of U_0 .

We must check immediately that the first part of this definition is unambiguous. Suppose (y, r) and (z, u) are elements of U_0 such that $x = h_{rs}(y) = \theta(y, r) = h_{us}(z) = \theta(z, u)$. Since θ is one-to-one, it follows that $(y, r) \approx (z, u)$. Hence there exists w in λ such that $\max(r, u) \leq w < s$ and $g_{rw}(y) = g_{uw}(z)$; therefore $g_{rt}(y) = g_{wt} \circ g_{rw}(y) = g_{wt} \circ g_{uw}(z) = g_{ut}(z)$, which proves that the definition of $h_{st}(x)$ is unambiguous.

Case 3. $r, t \in \lambda$, $r < t$. Let $h_{rt} = g_{rt}$.

To establish (5.8), it suffices to check the following three assertions:

$$(5.10) \quad h_{us} \circ h_{ru} = h_{rs}, \text{ if } r < u < s \text{ and } r, u \in \lambda.$$

$$(5.11) \quad h_{st} \circ h_{rs} = h_{rt}, \text{ if } r < s < t \text{ and } r, t \in \lambda.$$

$$(5.12) \quad h_{tu} \circ h_{st} = h_{su}, \text{ if } s < t < u \text{ and } t, u \in \lambda.$$

To check (5.10), note that for $r < u < s$, $r, u \in \lambda$, $(x, r) \approx (g_{ru}(x), u)$, and so

$\theta(g_{ru}(x), u) = \theta(x, r)$. Hence

$$h_{us} \circ h_{ru}(x) = h_{us}(g_{ru}(x)) = \theta(g_{ru}(x), u) = \theta(x, r) = h_{rs}(x).$$

To check (5.11), note that if $r < s < t$, $r, t \in \lambda$, then $h_{st}(h_{rs}(y)) = g_{rt}(y) = h_{rt}(x)$, by definition of hst.

To check (5.12) note that if $s < t < u$, $t, u \in \lambda$, then

$$\begin{aligned} h_{tu}(h_{st}(x)) &= h_{tu}(g_{rt}(y)) \text{ if } x = h_{rs}(y), \text{ some } r < s, \text{ some } y, \\ &= g_{tu} \circ g_{rt}(y) = g_{ru}(y) = h_{ru}(y), \\ &= h_{su} \circ h_{rs}(y) \text{ by (5.11),} \\ &= h_{su}(x). \end{aligned}$$

On the other hand if x is not in $\cup_{r < s} \text{Range}(h_{rs})$, then

$$h_{tu}(h_{st}(x)) = g_{tu} \circ g_{vt}(x_0) = g_{vu}(x_0) = h_{su}(x),$$

which verifies (5.12). Thus (5.8) is established.

Step III. We shall now prove the following sequential maximality result:

given a sequence $\alpha(1) \leq \alpha(2) \leq \dots$ in I , and given an element ${}^i \underline{f}$ of $T_{\alpha(i)}$ for

each i , such that $\pi_{\alpha(i), \alpha(i+1)}({}^{i+1} \underline{f}) = {}^i \underline{f}$ for each i , there exists a

collection of functions $\{f_{rt} \in \Gamma: 0 \leq r < t < \infty\}$ such that $f_{rt} = i f_{rt}$ whenever $r, t \in \alpha(i)$. Let $\mathcal{X}$ denote the union of the finite subsets $\alpha(1), \alpha(2), \dots$. For $t, u \in \mathcal{X}$, we may unambiguously define $f_{tu}^{\mathcal{X}} = i f_{tu}$ if $t, u \in \alpha(i)$, to obtain an element $f_{\mathcal{X}}$ of $T_{\mathcal{X}}$; for if $r, t, u \in \mathcal{X}$ with $r < t < u$, then there exists j such that r, t, u are all contained in $\alpha(j)$, so $j f_{tu} \circ j f_{rt} = j f_{ru}$, and the corresponding relation holds for $f_{\mathcal{X}}$.

Let D denote the set of all subsets of $[0, \infty)$ which contain $\mathcal{X}$, and introduce a partial order on D as follows: $\lambda \ll \mu$ if λ is contained in μ , and if for each h^{λ} in T_{λ} , there exists h^{μ} in T_{μ} such that $\pi_{\lambda\mu}(h^{\mu}) = h^{\lambda}$ (see (5.7)). D is nonempty, and the Hausdorff maximality theorem implies that D contains a maximal totally ordered subset, say F . Let η denote the union of all the subsets λ belonging to F .

We now show that $\lambda \ll \eta$ for every λ in F . For given λ in F , λ is contained in η , and given h^{λ} in T_{λ} , we may construct h^{η} in T_{η} such that $\pi_{\lambda\eta}(h^{\eta}) = h^{\lambda}$ as follows: given r, t in η , we unambiguously define h_{rt}^{η} to be h_{rt}^{μ} for any $\mu \gg \lambda$ in F such that $r, t \in \mu$. To check that h^{η} is in T_{η} , use the same argument as in the first paragraph of Step III.

We claim that $\eta = [0, \infty)$. For if there exists $s > 0$ not contained in η , then (5.8) shows that $\eta \ll \eta \cup \{s\}$, and hence $\lambda \ll \eta \cup \{s\}$ for every λ in F ,

which contradicts the maximality of F . Thus we have proved that $\gamma \ll \eta = [0, \infty)$. Hence there exists $\underline{f}$ in T_η such that $\pi_{\gamma\eta}(\underline{f}) = \underline{f}^\gamma$, and so $\pi_{\alpha(i), \eta}(\underline{f}) = \underline{f}$ for $i = 1, 2, \dots$, as desired. $\square$

6. Existence theorem for pure stochastic flows assuming spatial stochastic continuity.

THEOREM 6.1.

Let M be an infinite compact Hausdorff space with a countable base, and let Γ be M^M with the product topology. Let $\{Q_t, t \geq 0\}$ be a convolution semigroup of Radon probability measures on $\mathbb{B}(\Gamma)$ (Definition 4.12). Then there exists a probability space $(\Omega, \underline{E}, P)$ on which is defined a time-homogeneous pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ on M , with independent increments (Definition 3.2), such that the distribution of X_{st} is Q_{t-s} , for all $0 \leq s \leq t$.

Proof. Construct a projective system of Radon topological probability spaces as described in Section 5, and let $(\Omega, \underline{E}, P)$ be the projective limit (Proposition 5.2). Given $0 < s < t < \infty$ and ω in Ω , let $\alpha = (0, s, t)$; then define $X_{st}(\omega)$, $X_{os}(\omega)$ and $X_{ot}(\omega)$ in Γ simultaneously by:

$$(6.1) \quad (X_{st}(\omega), X_{os}(\omega), X_{ot}(\omega)) = \omega_\alpha.$$

We need to check two things:

(6.2) $\omega \rightarrow X_{st}(\omega)$ and $\omega \rightarrow X_{os}(\omega)$ are measurable maps from $(\Omega, \underline{E})$ to $(\Gamma, \mathbb{B}(\Gamma))$.

(6.3) The definition of $X_{os}(\omega)$ is not dependent on the choice of t , and the value of $X_{ot}(\omega)$ is the same as would result from taking $\alpha = (0, t, v)$ for some $v > t$.

To check (6.2), let E be an arbitrary element of $\mathbb{B}(\Gamma)$, and let H denote $\{(f, g, f \circ g) \in S_\alpha : f \in \Gamma, g \in E\}$. Then $H = (\Gamma \times E \times \Gamma) \cap S_\alpha \in \mathbb{B}(S_\alpha)$ (see discussion after (5.4)). If $g_\alpha : \Omega \rightarrow S_\alpha$ is the canonical projection (see Appendix B), then $g_\alpha^{-1}(H)$ is in $\underline{E}$. We have thus proved that $\{\omega : X_{st}(\omega) \in E\} = g_\alpha^{-1}(H) \in \underline{E}$; the other part of (6.2) is proved similarly.

To check (6.3), suppose $\gamma = (0, s, v)$ where v is different from t . Without loss of generality, suppose $v > t$, and let $\beta = (0, s, t, v)$. Then

$$\begin{aligned} g_{\alpha\beta}(\omega_\beta) &= \omega_\alpha = (X_{st}(\omega), X_{os}(\omega), X_{ot}(\omega)), \\ (6.4) \quad g_{\gamma\beta}(\omega_\beta) &= \omega_\gamma = (X_{sv}(\omega), X_{os}(\omega), X_{ov}(\omega)). \end{aligned}$$

We see that $X_{os}(\omega)$ is extracted from ω_β in both cases, and so (6.4) would give the same definition as (6.1). By a similar argument, the definition of $X_{ot}(\omega)$ in (6.1) is the same as if it were extracted from ω_β ,

where, $\delta = (0, t, v)$. This verifies (6.3).

Definition (5.4) for S_α , and equation (6.1), ensure that

$$(6.5) \quad X_{tu}(\omega) \circ X_{st}(\omega) = X_{su}(\omega), \quad 0 \leq s < t < u, \quad \omega \in \Omega.$$

We now define $X_{ss}(\omega)$ to be the identity map from M to M , for every ω , so that (6.5) holds also when $0 \leq s \leq t \leq u$. We have now proved that $(X_{st}, 0 \leq s \leq t < \infty)$ is a pure stochastic flow on M .

It follows from (5.5) and the remarks after Definition 4.4 that if $s \leq t \leq u \leq v \leq \dots$ then the joint distribution of $X_{st}, X_{uv}, \dots$ is $Q_{t-s} \otimes Q_{v-u} \otimes \dots$. This verifies that $(X_{st}, 0 \leq s \leq t < \infty)$ is a time-homogeneous pure stochastic flow with independent increments, such that X_{st} has the desired distribution for each $s \leq t$. $\square$

EXAMPLE 6.2,

The author is indebted to K. Parthasarathy (New Delhi) for explaining to him the following construction.

Let Q be any probability measure on $B_0(\Gamma)$ satisfying composability condition (4.4) (resp. (4.3)). Then the n -fold convolution $Q^{*n} = Q * \dots * Q$ exists for each n , and satisfies composability condition (4.4) (resp. (4.3)), by Corollary 4.10. For $t > 0$, define Q_t to be the unique Radon extension

to $\mathbb{B}(\Gamma)$ of the following probability measure on $\mathbb{B}_0(\Gamma)$:

$$(6.6) \quad Q_t(A) = e^{-\lambda t} \sum_{n \geq 0} ((\lambda t)^n / n!) Q^{*n}(A), \quad A \in \mathbb{B}_0(\Gamma),$$

where $\lambda > 0$ is some constant; and take $Q_0(\{e\}) = 1$, where e is the identity mapping from M to M .

LEMMA 6.3

- (i) *Each Q_t satisfies composability condition (4.4).*
- (ii) *If $Q^{*m} * Q^{*n} = Q^{*(m+n)}$ for all $m, n \geq 0$, then $\{Q_t, t \geq 0\}$ is a convolution semigroup of probability measures on $\mathbb{B}(\Gamma)$ (Definition 4.12).*

Proof (i) since Q^{*n} satisfies (4.5) for each n by Corollary 4.10, it follows that Q_t also satisfies (4.5). Next we must establish the uniform stochastic continuity of Q_t on T (Definition 4.1), for fixed t . Given $\delta > 0$, choose an integer r such that

$$e^{-\lambda t} \sum_{n \geq r+1} ((\lambda t)^n / n!) \leq \delta/2.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such the following inequality holds for

$n = 1, 2, \dots, r :$

$$Q^{*n}(\{f: d(f(x), f(y)) > \varepsilon\}) \leq \delta/2 \text{ whenever } d(x, y) \leq \delta,$$

for x and y in T . Then whenever $d(x, y) \leq \delta$,

$$\begin{aligned} Q_t(\{f: d(f(x), f(y)) > \varepsilon\}) &\leq \\ e^{-\lambda t} \sum_{0 \leq n \leq r} ((\lambda t)^n / n!) Q^{*n}(\{f: d(f(x), f(y)) > \varepsilon\}) &+ e^{-\lambda t} \sum_{n > r} (\lambda t)^n / n! \\ &\leq (\delta/2) e^{-\lambda t} \sum_{0 \leq n \leq r} ((\lambda t)^n / n!) + \delta/2 \leq \delta. \end{aligned}$$

The uniform stochastic continuity of Q_t on T is established.

(ii) Let $(N(t), t \geq 0)$ be a Poisson process with rate λ , on some probability space $(\Omega_1, \underline{F}, P_1)$, thus

$$(6.7) \quad P_1(N(t) = j) = e^{-\lambda t} (\lambda t)^j / j!, \quad j = 0, 1, 2, \dots; t \geq 0.$$

Comparing this with (6.6), we see that

$$(6.8) \quad Q_t(A) = \sum_{n \geq 0} Q^{*n}(A) P_1(N(t) = n) = \mathbb{E}[Q^{*N(t)}(A)].$$

Thus for $s, t \geq 0$, and A in $\mathcal{B}_0(\Gamma)$,

$$\begin{aligned}
 Q_{t+s}(A) &= \mathbb{E}[Q^{*N(t+s)}(A)], \\
 (6.9) \qquad &= \mathbb{E}[\mathbb{E}[Q^{*(N(t+s) - N(t))} * Q^{*N(t)}(A) | N(t)]] ,
 \end{aligned}$$

since $Q^{*(m+n)} = Q^{*m} * Q^{*n}$. A Poisson process has independent increments, and therefore $N(t+s) - N(t)$ is independent of $N(t)$. Moreover it is easy to check from (6.6) and the time-homogeneity of $(N(t), t \geq 0)$ that for any $j \geq 0$,

$$Q_s * Q^{*j}(A) = \sum_{n \geq 0} Q^{*n} * Q^{*j} P_1(N(t+s) - N(t) = n)$$

Therefore (6.9) gives

$$\begin{aligned}
 Q_{t+s}(A) &= \mathbb{E}[Q_s * Q^{*N(t)}(A)] , \\
 &= \sum_{n \geq 0} Q_s * Q^{*n}(A) P_1(N(t) = n) = Q_s * Q_t(A).
 \end{aligned}$$

We have now verified the conditions of Definition 4.12. $\square$

COROLLARY 6.4

There exists a probability space on which is defined a time-homogeneous pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ on M , with

independent increments, such that the distribution of X_{st} is P_{t-s} , for all $0 \leq s \leq t$, where Q_t is defined in (6.6).

Proof. Combine Lemma 6.3 and Theorem 6.1.

Remark 6.5. The convolution semigroup of probability measures, constructed above, should more correctly be labelled $\{Q_{\lambda,t}, t \geq 0\}$ (see (6.6)). Suppose $\{R_t, t \geq 0\}$ is any convolution semigroup of Radon probability measures on $B(\Gamma)$, and take Q to be $R_{t/\lambda}$ in Example 6.2, thus (6.8) gives:

$$(6.10) \quad Q_{\lambda,t}(A) = \mathbb{E}[R_{t/\lambda} * N(\lambda,t)(A)], \quad A \in B_0(\Gamma),$$

where $(N(\lambda,t), t \geq 0)$ is a Poisson process with rate λ . As λ tends to ∞ , we would expect the convolution semigroup $\{Q_{\lambda,t}, t \geq 0\}$ to converge in some sense to $\{R_t, t \geq 0\}$.

Part III. Construction of a stochastic flow assuming almost no fixed points of discontinuity.

Part III is a repeat of Part II under assumptions which are stronger in some ways and weaker in others, and can only be verified easily in the case where M is one-dimensional. Theorem 9.1 is better than Theorem 6.1 in as much as for each random transformation X_{st} , the mapping $(x, \omega) \rightarrow X_{st}(x, \omega)$ is measurable with respect to a completed product measure. Proposition 8.1 would seem to be of some interest in its own right.

7. Probability measures with almost no fixed points of discontinuity

The concepts of this section are drawn from Nelson's fundamental paper [21].

Let M and Γ be as in Section 3. For the sake of brevity we will restrict ourselves to the case where M is the one-point compactification $T \cup \{\infty\}$ of an infinite separable metric space (T, d) ; the case where M is compact, without a distinguished point labelled ∞ , may be treated in a

similar way.

In order to study classes of (possibly discontinuous) functions from M to M , we introduce the following notations: for x in T and $\delta > 0$,

$$(7.1) \quad B(x, \delta) = \{y \in T: d(x, y) < \delta\}, \quad D(x, \delta) = \{y \in T: d(x, y) \leq \delta\},$$

$$(7.2) \quad F_x = \bigcap_m \bigcup_n \bigcap_{y \in D(x, 1/n)} \{f \in \Gamma : d(f(x), f(y)) \leq 1/m\}$$

Evidently F_x is the set of functions in Γ which are continuous at x ; as noted in Remark 4.2, the set $\{ \dots \}$ appearing in (7.2) is closed, and so F_x is an $F_{\sigma\delta}$ set in Γ , hence Borel.

For each f in Γ , define

$$(7.3) \quad O(f) = \bigcup_m \bigcap_n \{x \in T: \rho(f(y), f(z)) \geq 1/m \text{ for some } y, z \in B(x, 1/n)\}.$$

Evidently $O(f)$ is the set of discontinuities of f in T . The set $\{ \dots \}$ is closed in T , and hence $O(f)$ is a Borel subset of T .

Finally define

$$(7.4) \quad \Theta = \{(x, f) \in M \times \Gamma : x = \infty \text{ or } f \text{ is discontinuous at } x\},$$

$$(7.5) \quad Z: M \times \Gamma \rightarrow M, \quad Z(x, f) = f(x). \quad (\text{as in (4.1)})$$

Observe that, in terms of our previous definitions,

$$(7.6) \quad \Theta = \{(x, f) \in M \times \Gamma : x = \infty \text{ or } x \in F_x^C\},$$

$$(7.7) \quad \Theta = \{(x, f) \in M \times \Gamma : x = \{\infty\} \cup O(f)\}.$$

Henceforward let μ be some fixed Radon (hence σ -finite) measure on $\mathbb{B}(M)$, such that $\mu(\{\infty\}) = 0$. (Typically $T = \mathbb{R}^d$ and μ is Lebesgue measure on $\mathbb{R}^d$.) Since $O(f)$ is a Borel subset of T for each f in Γ , we may define

$$(7.8) \quad C_\mu = \{f \in \Gamma : \mu(O(f)) = 0\} \quad (\text{see (7.3)}).$$

LEMMA 7.1. (Nelson [21]).

- (i) *The set Θ in (7.4) is a Borel subset of $M \times \Gamma$.*
- (ii) *The mapping Z in (7.5) is Borel measurable on $(M \times \Gamma) - \Theta$.*
- (iii) *C_μ belongs to $\mathbb{B}(\Gamma)$.*

Proof. See Nelson [21], Theorems 3.2 and 3.5.

LEMMA 7.2 (Nelson [21]).

Let Q be a Radon probability measure on $\mathbb{B}(\Gamma)$, satisfying (4.5). Then

(a), (b), (c) and (d) are equivalent.

(a) $\mu \otimes Q(\Theta) = 0$ (see (7.4)).

(b) $Q(F_x) = 1$ for $[\mu]$ -almost all x in T (see (7.2)).

(c) $\mu(0(f)) = 0$ for $[Q]$ -almost all f in Γ (see (7.3)).

(d) $Q(C_\mu) = 1$ (see (7.8)).

If these equivalent conditions hold, then Z (see (4.1)) is $\mu \otimes Q$ -measurable.

Proof. The equivalence follows essentially from Fubini's theorem, (7.6), (7.7), and Lemma 7.1 (i).

Definition 7.3. Given a Radon measure μ on $\mathbb{B}(M)$ with $\mu(\{\infty\}) = 0$, we shall say that a Radon probability measure Q on $\mathbb{B}(\Gamma)$ satisfying (4.5) has **almost no fixed points of discontinuity** (with respect to μ) if the equivalent conditions of Lemma 7.2 hold.

Remark 7.4. Let Z be the canonical Γ -valued random variable (see (4.1)) on the probability space $(\Gamma, \mathbb{B}(\Gamma), Q)$. Definition 7.3 is equivalent to the condition that for $[\mu]$ -almost every x in T , the random variables

$(Z(y), y \in M)$ are almost surely continuous at x . Of course, almost sure continuity even at every fixed x is much weaker than requiring that the mapping $y \rightarrow Z(y)$ is almost surely continuous on M . On the other hand, almost sure continuity of Z at some fixed x in T implies continuity in probability at x . Thus if Q has almost no fixed points of discontinuity, then Q is stochastically continuous almost everywhere on T (see Definition 4.1).

We conclude that having almost no fixed points of discontinuity neither implies, nor is implied by, composability condition (4.4) (which requires uniform stochastic continuity everywhere on T).

Next we show how the conditions of Lemma 7.1 may be checked by studying the finite-dimensional distributions of Q .

PROPOSITION 7.5.

Given μ as in Definition 7.3, and given a Radon probability measure Q on $\mathbb{B}(T)$ satisfying (4.5), conditions (i) and (ii) are equivalent:

- (i) Q has almost no fixed points of discontinuity.
- (ii) For $[\mu]$ -almost every x in T , and for every $\varepsilon > 0$,

$$(7.9) \sup_k \sup_{y_1, \dots, y_k \in D(x, 1/n)} Q(\bigcup_{1 \leq j \leq k} \{f: d(f(x), f(y_j)) > \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof. Assume (ii) holds, and suppose x is a point in T such that (7.9) holds for every $\varepsilon > 0$. It suffices to prove that $Q(F_x) = 1$, for this verifies Lemma 7.2, condition (b). Referring to (7.2), it suffices to show that for every $\varepsilon > 0$,

$$(7.10) \quad Q(\cap_n \bigcup_{y \in D(x, 1/n)} \{f: d(f(x), f(y)) > \varepsilon\}) = 0$$

Let $Q(\cap_n A_n)$ denote this expression. Then $(A_n, n \geq 1)$ is a decreasing sequence of open subsets of Γ (compare Remark 4.2). Given $\delta > 0$, the Radon property of Q implies that there exists for each n a compact subset K_n of A_n such that $Q(A_n - K_n) \leq \delta 2^{-n}$. Then

$$Q([\cap_n A_n] \cap [\cap_m K_m]^c) \leq Q(\cup_m (A_m \cap K_m^c)) \leq \sum_m Q(A_m - K_m) \leq \delta.$$

Since δ is arbitrary, equation (7.10) will follow as soon as we prove that $Q(\cap_m K_m) = 0$.

The sets $\{\{f: d(f(x), f(y)) > \varepsilon\}, y \in D(x, 1/n)\}$ form an open cover of the

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compact set K_n , for each n . Hence for each n , there exists a finite subcover; in other words there exist $y_{n,1}, \dots, y_{n,r(n)}$ in $D(x, 1/n)$ such that

$$K_n \subseteq \bigcup_{1 \leq j \leq r(n)} \{f: d(f(x), f(y_j)) > \varepsilon\}$$

Now (7.9) implies that $Q(K_n) \rightarrow 0$ as $n \rightarrow \infty$, which completes the proof that (ii) implies (i).

Conversely, assume that (i) holds, and that x is a point in T such that $Q(F_x) = 1$. Given $\varepsilon > 0$, (7.10) holds. Hence

$$\lim_n Q\left(\bigcup_{y \in D(x, 1/n)} \{d(f(x)) > \varepsilon\}\right) = 0$$

which implies (7.9). Thus (i) implies (ii). $\square$

8. Fluid Radon probability measures and their convolution.

The goal of this section is to prepare an analogue of Theorem 6.2, with different hypothesis and stronger conclusions.

Suppose Q and R are Radon probability measures on $\mathbb{B}(\Gamma)$ with almost no fixed points of discontinuity. Their composition convolution (Definition 4.7) may not be definable in general, because the following pathology may occur: there may exist a set E in $\mathbb{B}(M)$, with $\mu(E) = 0$, such that $R(\{g : g(M) \subseteq E\}) > 0$ and $Q(\{f : O(f) \cap E \neq \emptyset\}) > 0$. To prevent this pathology, we introduce an extra condition on the measures Q and R .

Definition 8.1. Let Q be a Radon probability measure on $\mathbb{B}(\Gamma)$, and let μ be a Radon measure on $\mathbb{B}(M)$ such that $\mu(\{\infty\}) = 0$. We shall say that Q is a **fluid Radon probability measure** (abbreviated to **FRPM**), with respect to μ , if all the following conditions hold:

(i) $Q(\{f : f(\infty) = \infty\}) = 1$, and $Q(\{f : f(x) = \infty\}) = 0$ for $x \in T$. (Condition 4.5))

(ii) Q has almost no fixed points of discontinuity (Definition 7.3).

(iii) For each x in T , the distribution of $Z(x)$ is absolutely continuous

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with respect to μ , where Z is the canonical Γ -valued random variable on $(\Gamma, \mathbb{B}(\Gamma), Q)$ (see (4.1)); in other words, if E is in $\mathbb{B}(M)$ and $\mu(E)=0$, then

$$(8.1) \quad Q(\{f: f(x) \in E\}) = 0 \text{ for all } x \in T.$$

PROPOSITION 8.2

Suppose Q and R are FRPM's (Definition 8.1) on $\mathbb{B}(\Gamma)$. Then the map $c: \Gamma^2 \rightarrow \Gamma$, $c(f,g) = f \circ g$ (composition), is $Q \otimes R$ - measurable (Radon product) into $\mathbb{B}_0(\Gamma)$.

Before giving the proof, we introduce some notations and a Lemma:
for x in T , define

$$(8.2) \quad H_x = \{(f,g) \in \Gamma^2 : f \text{ is continuous at } g(x)\},$$

$$(8.3) \quad C_x = \{(f,g) \in \Gamma^2 : g \text{ is continuous at } x \text{ and } f \text{ is continuous at } g(x)\}$$

LEMMA 8.3

Suppose Q and R are FRPM's on $\mathbb{B}(\Gamma)$. Then

(i) H_x and C_x belong to $\mathbb{B}(\Gamma^2)$, for all x in T ,

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(ii) $Q \otimes R(H_x) = 1$ for all x in T , and

(iii) $Q \otimes R(C_x) = 1$ for all $[\mu]$ - almost all x in T .

Proof. (i) Define a mapping $\theta_x : \Gamma^2 \rightarrow M \times \Gamma$ by $\theta_x(f, g) = (g(x), f)$. It is easy to verify that θ_x is a continuous map. Moreover $H_x = \theta_x^{-1}(\Theta)$ (see (7.4)), and $\Theta \in \mathcal{B}(M \times \Gamma)$ by Lemma 7.1(i). Therefore $H_x^C \in \mathcal{B}(\Gamma^2)$.

Also

$$(8.4) \quad C_x = H_x \cap (\Gamma \times F_x) \quad (\text{see (7.2)}),$$

and F_x is in $\mathcal{B}(\Gamma)$; thus C_x is in $\mathcal{B}(\Gamma^2)$.

(ii) Since Q is fluid, it follows from Lemma 7.2 (b) that there exists a subset J of T with $\mu(J^C) = 0$, such that $Q(F_y) = 1$ for y in J (see (7.2)). Since R is fluid, (8.1) implies that $R(A_x) = 1$, where $A_x = \{g \in \Gamma : g(x) \in J\}$.

Now apply Fubini's theorem, noting that $\{f : (f, g) \in H_x\} = F_{g(x)}$ (see (7.2), (8.2));

$$Q \otimes R(H_x) = \int_{\Gamma} Q(F_{g(x)}) R(dg) = \int_{A_x} Q(F_{g(x)}) R(dg) = 1.$$

(iii) The final assertion follows from combining Lemma 7.2(b), the result of part (ii), and equation (8.4). □

Proof of Proposition 8.2.

By the same reasoning as was used in the proof of Proposition 4.5, it suffices to prove that $c^{-1}(A)$ is $Q \otimes R$ - measurable, whenever A is a Baire subset of Γ of the form

$$(8.5) \quad A = \{h : h(x) \in G\}$$

for some x in M and some open subset of G of M .

Case 1 : $x \in T$. Define a subset W of Γ^2 as follows:

$$(8.6) \quad W = \bigcap_k \bigcup_{y \in T} (\{f : f(y) \in G\} \times \{g : d(g(x), y) < 1/k\})$$

Each of the sets $\{ \dots \}$ is open in Γ , so the union over y is open in Γ^2 .

Hence W is a Borel subset of Γ^2 . Moreover $c^{-1}(A)$ is contained in W ,

for if (f,g) belongs to $c^{-1}(A)$, then $f(y) \in G$ and $d(g(x), y) = 0$ when

$y = g(x)$. To show that $c^{-1}(A)$ is $Q \otimes R$ - measurable, it suffices to prove

that $W - c^{-1}(A)$ is a $Q \otimes R$ - nullset. Now

$$W - c^{-1}(A) = \{(f,g) : \text{for all } k, \text{ there exists } y \text{ such that}$$

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$d(g(x), y) < 1/k$, $f(y) \in G$, but $f(g(x)) \in G^c$.

$\subseteq \{(f, g) : f \text{ is discontinuous at } g(x)\} = H_x^c$.

However $Q \otimes R(H_x^c) = 0$ by Lemma 8.3, (ii); thus $W = c^{-1}(A)$ is contained in a $Q \otimes R$ - nullset, which completes the proof for Case I.

Case II: $x = \infty$. Let $L = \{h \in \Gamma : h(\infty) = \infty\}$. Now

$$\begin{aligned} c^{-1}(A) &= \{(f, g) : f(g(\infty)) \in G\} \\ (8.7) \quad &= \{(f, g) : g \in L, f(\infty) \in G\} \cup \{(f, g) : g(\infty) \in T, f(g(\infty)) \in G\} \end{aligned}$$

Now $R(L) = 1$ and $Q(\{f : f(\infty) \in G\}) = 1_G(\infty)$, by (4.5). If $\infty \in G$, then the first of the two sets $\{ \dots \}$ in (8.7) has full $Q \otimes R$ - measure, so $c^{-1}(A)$ is $Q \otimes R$ - measurable. If ∞ is not in G , then both sets in (8.7) have zero $Q \otimes R$ measure, and so $c^{-1}(A)$ is $Q \otimes R$ - measurable. $\square$

Definition 8.4. Given FRPM's Q and R on $\mathbb{B}(\Gamma)$, define their

composition convolution (or simply convolution) $Q * R$ to be the unique Radon extension to $\mathbb{B}(\Gamma)$ (see Appendix A.4) of the probability measure $(Q \otimes R) \circ c^{-1}$ on $\mathbb{B}_0(\Gamma)$.

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Remark 8.5 Suppose that Q and R also satisfy composability condition (4.4). Then the convolution of Definition 8.4 is identical to that of Definition 4.7, since $Q \otimes_L R$ equals $Q \otimes R$ on $\mathbb{B}_0(\Gamma^2)$ (and hence their unique Radon extensions to $\mathbb{B}(\Gamma^2)$ are equal also).

PROPOSITION 8.6.

If Q and R are FRPM's on $\mathbb{B}(\Gamma)$, then $Q * R$ (Definition 8.4) is also an FRPM on $\mathbb{B}(\Gamma)$.

Proof. We shall verify the conditions of Definition 8.1. First, $Q * R$ is a Radon probability measure by definition. Condition (i) is verified as in the proof of Proposition 4.9. As for (ii); for x in T ,

$$Q * R(F_x) = Q \otimes R(\{(f, g) : f \circ g \text{ is continuous at } x\})$$

$$\geq Q \otimes R(C_x), \quad (\text{see (8.3)}),$$

Hence $Q * R(F_x) = 1$ for $[\mu]$ - almost all x , by Lemma 8.3, (iii). By Lemma 7.2(b), this proves that $Q * R$ has almost no fixed points of discontinuity.

For (iii), let E be a Borel set in M with $\mu(E) = 0$, and let x be a point in M . Using Fubini's theorem,

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$$\begin{aligned} Q * R(\{h : h(x) \in E\}) &= Q \otimes R(\{(f,g) : f \circ g(x) \in E\}) \\ &= \int Q(\{f : f(g(x)) \in E\}) R(dg) = 0 \end{aligned}$$

using (8.1) and the fact that $R(\{g : g(x) \in T\}) = 1$. $\square$

The analogue of Corollary 4.10 is as follows:

COROLLARY 8.7.

Suppose $Q_1, \dots, Q_n$ are FRPM's on $\mathbb{B}(\Gamma)$. By virtue of Proposition 8.6, we may define the convolution $Q_1 * \dots * Q_{n-1}$, for $n \geq 3$, inductively as follows:

$$(8.8) \quad Q_1 * \dots * Q_n = (Q_1 * \dots * Q_{n-1}) * Q_n.$$

Moreover

(i) Every such composition convolution is an FRPM.

(ii) Let $c_n : \Gamma^n \rightarrow \Gamma$ be the mapping $c_n(f_1, \dots, f_n) = f_1 \circ \dots \circ f_n$.

Then c_n is $Q_1 \otimes \dots \otimes Q_n$ - measurable into the $Q_1 * \dots * Q_n$ - measurable sets, and

$$(8.9) \quad Q_1 * \dots * Q_n = (Q_1 \otimes \dots \otimes Q_n) \circ c_n^{-1}.$$

(iii) Among FRPM's $$ is associative.*

Proof. (i) is immediate from Proposition 8.6. For (ii), notice first that $c: \Gamma^2 \rightarrow \Gamma$ is $Q \otimes R$ - measurable into the $Q * R$ - measurable sets; for by Appendix A.7, for each $Q * R$ - measurable subset E of Γ , there exists E_0 in $\mathbb{B}_0(\Gamma)$ such that $E = E_0$ a.e. $[Q * R]$, and so $c^{-1}(E) = c^{-1}(E_0)$ a.e. $[Q \otimes R]$. Abusing notation slightly, define $c: \Gamma^n \rightarrow \Gamma^{n-1}$, for $n \geq 2$, by $c(f_1, \dots, f_n) = (f_1 \circ f_2, f_3, f_4, \dots, f_n)$. By the previous remark; we see that:

$c: \Gamma^n \rightarrow \Gamma^{n-1}$ is $Q_1 \otimes \dots \otimes Q_n$ - measurable into the $(Q_1 * Q_2) \otimes Q_3 \otimes \dots \otimes Q_n$ - measurable sets,

$c: \Gamma^{n-1} \rightarrow \Gamma^{n-2}$ is $(Q_1 * Q_2) \otimes Q_2 \otimes \dots \otimes Q_n$ - measurable into the

$(Q_1 * Q_2 * Q_3) \otimes Q_4 \otimes \dots \otimes Q_n$ - measurable sets,

..

$c: \Gamma^2 \rightarrow \Gamma_1$ is $(Q_1 * \dots * Q_{n-1}) \otimes Q_n$ - measurable into the

$Q_1 * \dots * Q_n$ - measurable sets.

Hence the composition of all these mappings, which is $c_n: \Gamma^n \rightarrow \Gamma$, is

$Q_1 \otimes \dots \otimes Q_n$ - measurable into the $Q_1 * \dots * Q_n$ - measurable sets.

Equation (8.9) is immediate from the construction above. The equation

$(Q_1 * Q_2) * Q_3 = Q_1 * (Q_2 * Q_3)$ follows from pushing forward the

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measure $Q_1 \otimes Q_2 \otimes Q_3$ both ways in the following commutative diagram:

$$\begin{array}{ccc}
 (f_1, f_2, f_3) & \xrightarrow{\quad\quad\quad} & (f_1, f_2 \circ f_3) \\
 \downarrow & & \downarrow \\
 (f_1 \circ f_2, f_3) & \xrightarrow{\quad\quad\quad} & (f_1 \circ f_2 \circ f_3)
 \end{array}$$

Thus assertion (iii) follows.

□

Definition 8.8. A family $\{Q_t, t \geq 0\}$ of FRPM's on $\mathbb{B}(\Gamma)$ is called a **convolution semigroup of FRPM's** if $Q_s * Q_t = Q_{s+t}$ for all $s, t \geq 0$, and if $Q_0(\{e\}) = 1$, where $e : M \rightarrow M$ is the identity map.

9. Existence theorem for pure stochastic flows assuming almost no fixed points of discontinuity.

The following theorem is analogous to Theorem 6.1.

THEOREM 9.1

Let M be an infinite compact Hausdorff space with a countable base, with a distinguished point labelled ∞ (this is not essential). Let μ be a Radon measure on $\mathbb{B}(M)$ such that $\mu(\{\infty\}) = 0$, and let Γ be M^M with the product topology. Suppose $\{Q_t, t \geq 0\}$ is a convolution semigroup of FRPM's (Definitions 8.1, 8.8) on $\mathbb{B}(\Gamma)$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ on M , with independent increments (Definition 3.2), such that (a), (b) and (c) hold :

- (a) $X_{st}(\omega)$ belongs to C_μ (see (7.8)) for all ω in Ω and all $0 \leq s \leq t$; i.e. the set of discontinuities of each mapping $x \rightarrow X_{st}(x, \omega)$ is a $[\mu]$ -nullset.
- (b) The distribution of X_{st} is Q_{t-s} for all $0 \leq s \leq t$.
- (c) The map $M \times \Omega \rightarrow M$ given by $(y, \omega) \rightarrow X_{st}(y, \omega)$ is $\mu \otimes P$ -measurable, for all $0 \leq s \leq t$.

Proof. Construct a projective system of Radon topological probability spaces as in Section 5, with minor alterations as follows: product measure instead of left product measure may be used on the right side of (5.5), and since $Q_t(C_\mu) = 1$ for all $t \geq 0$, we may take S_α to be $\Psi_\alpha(\Gamma^{n-1}) \cap C_\mu^{3m}$ in (5.4). The latter ensures that property (a) holds when we recapitulate the proof of Theorem 6.1. Indeed the proof of Theorem 6.1 verifies all the assertions of Theorem 9.1 except (c).

To prove (c), suppose $0 < s < t$ and let $\alpha = (0, s, t)$. Abbreviate $\omega_\alpha = (X_{st}(\omega), X_{0s}(\omega), X_{0t}(\omega))$ to $(f, h, f \circ h)$. The mapping $(y, \omega) \rightarrow X_{st}(y, \omega)$ may be factored as follows:

$$(9.1) \quad (y, \omega) \rightarrow (y, \omega_\alpha) = (y, (f, h, f \circ h)) \rightarrow (y, f) \rightarrow f(y).$$

By the definition of the σ -algebra $\underline{E}$ (see Appendix B), the first factor is $\mathbb{B}(M) \times \underline{E}$ - measurable into $\mathbb{B}(M) \times \mathbb{B}(S_\alpha)$. The second factor is $\mathbb{B}(M) \times \mathbb{B}(S_\alpha)$ - measurable into $\mathbb{B}(M) \times (\mathbb{B}(\Gamma) \cap C_\mu)$. The third factor is $\mu \otimes Q_{t-s}$ - measurable by Lemma 7.2 (last part). Since $\mu \otimes Q_{t-s}$ is the image of the measure $\mu \otimes P$ under the mapping $(y, \omega) \rightarrow (y, f)$, it follows that the mapping $(y, \omega) \rightarrow h(y)$ is $\mu \otimes P$ -measurable. A similar argument shows that

$(y, \omega) \rightarrow X_{0s}(y, \omega)$ is $\mu \otimes P$ -measurable. The assertion is trivially true when $s = t$, and so (c) follows. $\square$

EXAMPLE 9.2

Example 6.2 may be repeated in this case, taking Q to be an FRPM. Each convolution Q^{*n} is an FRPM by Corollary 8.7 (i), and Lemma 6.3 becomes:

LEMMA 9.3.

- (i) *Each Q_t , as defined by (6.6), is an FRPM.*
- (ii) *$\{Q_t, t \geq 0\}$ is a convolution semigroup of FRPM's on $\mathbb{B}(\Gamma)$ (Definition 8.8) (Note that the auxiliary condition of Lemma 6.3 is automatically satisfied for FRPM's, by Corollary 8.7 (iii)).*

Proof. For (i), we shall verify each of the three conditions of Definition 8.1 for Q_t . Condition (4.5) holds for each Q^{*n} by Corollary 8.7 (i), and hence it holds for Q_t . $Q^{*n}(C_\mu) = 1$ for each n by Corollary 8.7 (i) and Lemma 7.2 (d), and so $Q_t(C_\mu) = 1$; hence Q_t has almost no fixed points of

discontinuity (Definition 7.3). Finally for x in T , and E in $\mathbb{B}(M)$ with $\mu(E) = 0$, $Q^{*n}(\{f: f(x) \in E\}) = 0$ for each n by Corollary 8.7 (i), and so Q_t satisfies (8.2). We have now established that Q_t is an FRPM.

For (ii), repeat the proof of Lemma 6.3 (ii). $\square$

COROLLARY 9.4

There exists a probability space on which is defined a time-homogeneous pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ on M , with independent increments, such that assertions (a), (b) and (c) of Theorems 9.1 hold.

Remark 9.5 In this case, a more direct realization of a pure stochastic flow may be obtained as follows. Let $(\Omega, \mathcal{E}, P)$ be a complete probability space on which the following random variables are defined:

$(N(t), t \geq 0)$, a Poisson process of rate $\lambda > 0$;

$Y_1, Y_1, \dots$, independent, identically, distributed $(\Gamma, \mathbb{B}_0(\Gamma))$ -valued random variables with distribution Q , independent of $(N(t), t \geq 0)$. Then

$V_n = Y_n \circ \dots \circ Y_1 = c(Y_n, \dots, Y_1)$ is an Γ -valued random variable,

by Corollary 8.7 (ii). Take

$$(9.2) \quad X_{st}(x, \omega) = V_{N(t, \omega) - N(s, \omega)}(x, \omega), \quad 0 \leq s \leq t, \quad \omega \in \Omega, \quad x \in T$$

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$$X_{st}(\infty, \omega) = \infty.$$

Then $(X_{st}, 0 \leq s \leq t < \infty)$ is a pure stochastic flow on M , with the desired distribution.

Part IV.

10. Construction of a convolution semigroup of probability measures from finite dimensional Markov processes.

The aim of this section is to describe a natural way in which the convolution semigroup $\{Q_t, t \geq 0\}$, required for Theorem 6.1 or 9.1 may be obtained.

Let M be the one-point compactification $T \cup \{\infty\}$ of a separable metric space (T, d) . As usual $\Gamma = M^M$.

Definition 10. Suppose that for $k = 1, 2, \dots$, for each set of distinct points $z_1, \dots, z_k$ in T and for each $t \geq 0$, we are given a T^k -valued random variable $(Z_t(z_1), \dots, Z_t(z_k))$ with the following properties:

$$(A.1) \quad (Z_0(z_1), \dots, Z_0(z_k)) = (z_1, \dots, z_k).$$

(A.2) Consistency: the marginal distribution of the first k components of $(Z_t(z_1), \dots, Z_t(z_{k+j}))$ is the distribution of $(Z_t(z_1), \dots, Z_t(z_k))$, for every $t \geq 0$, every $j = 1, 2, \dots$, and every $(z_1, \dots, z_{k+j})$.

Such a system is called a **consistent system of finite-dimensional motions** on T . By defining $Z_t(\infty) = \infty$ for all t , we obtain a consistent

system of finite-dimensional motions on M .

Let $Q_{t,\alpha}$ denote the probability measure on $\mathbb{B}(M^k)$ induced by $(Z_t(z_1), \dots, Z_t(z_k))$, for each $t \geq 0$ and each $\alpha = (z_1, \dots, z_k)$ in M^k . As noted in Appendix A.5, there is a unique Radon probability measure Q_t on $\mathbb{B}(\Omega)$ with the $\{Q_{t,\alpha}, \alpha \text{ a finite subset of } M\}$ as its finite-dimensional distributions.

PROPOSITION 10.2

Besides (A.1) and (A.2) above, suppose that the measures $\{Q_t, t \geq 0\}$ above satisfy (A.3) and (A.4) (resp. (A.3') and (A.4')) below:

(A.3) Markov Property: for each set $\alpha = \{z_1, \dots, z_k\}$ of distinct points in T , $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$ is a time-homogeneous Markov process on some probability space $(\Omega_\alpha, \mathcal{E}_\alpha, P_\alpha)$.

(A.4) Composability: each Q_t satisfies composability condition (4.4).

(A.4') Fluidity: each Q_t is fluid (Definition 8.1).

Then $\{Q_t, t \geq 0\}$ is a convolution semigroup, in the sense of Definition 4.12

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(resp. a convolution semigroup of FRPM's in the sense of Definition 8.8), and so Theorem 6.1 (resp. 9.1.) applies.

Remarks

1. Without (A.4) (resp. (A.4')), the convolution of Q_s and Q_t is not well-defined.
2. Notice that (unlike Harris [12, 2.8]) we do not insist that our Markov processes have continuous trajectories in (A.3). This allows Theorems 6.1 and 9.1 to be applied to construct pure stochastic flows of jump type, such as Harris' "random stirrings" [12].

Proof. Assume (A.3) and (A.4). (A.1) ensures $Q_0(\{e\}) = 1$, where e is the identity map, and it remains to check that $Q_s * Q_t = Q_{s+t}$. According to Remarks 4.8, 3., it suffices to check that for all ψ in $C(\Gamma)$ of the form:

$$(10.1) \quad \psi(h) = \varphi_1(h(x_1)) \cdot \varphi_2(h(x_2)) \dots \varphi_m(h(x_m)) , \text{ where } x_i \in M, \varphi_i \in C(M),$$

that

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$$(10.2) \quad \int_{\Gamma} \psi(h) (Q_t * Q_s)(dh) = \int_{\Gamma} \psi(h) Q_{s+t}(dh), \quad \text{for all } s, t \geq 0.$$

By (4.13), the left side of (10.2) is

$$(10.3) \quad \int \left[\int \varphi(f \circ g) Q_t(df) \right] Q_s(dg);$$

$$= \int \left[\int \varphi_1(f \circ g(x_1)) \dots \varphi_m(f \circ g(x_m)) Q_t(df) \right] Q_s(dg).$$

For any bounded, Borel measurable function $u : T^m \rightarrow \mathbb{R}$, define

$$(10.4) \quad P_t u(z_1, \dots, z_m) = \mathbb{E} [u(Z_t(z_1), \dots, Z_t(z_m))] , (z_1, \dots, z_m) \in T^m$$

Then $P_t u$ is Borel measurable, and the Markov property implies that

$$P_s(P_t u) = P_{s+t} u.$$

Take $z_1 = g(x_1)$ and $u(y_1, \dots, y_m) = \varphi_1(y_1) \dots \varphi_m(y_m)$. Then (10.3)

equals

$$\int P_t u(g(x_1), \dots, g(x_m)) Q_s(dg),$$

$$= P_s(P_t u)(x_1, \dots, x_m) = P_{s+t} u(x_1, \dots, x_m),$$

$$= \int u(h(x_1), \dots, h(x_m)) Q_{s+t}(h) = \int \psi(h) Q_{s+t}(dh).$$

This verifies (10.2). The same proof works, assuming (A.4') instead of (A.4). $\square$

EXAMPLE 10.3 (The one-dimensional case)

In [13], Harris constructs a pure stochastic flow when $T = \mathbb{R}^1$, under some special assumptions about the generators of the Markov processes mentioned in (A.3). We shall now show that these special assumptions are not needed.

Consider the following properties of a consistent system of finite-dimensional motions on $\mathbb{R}^1$.

(S.1). For each $t > 0$, and each z in $\mathbb{R}^1$, $\mathbb{E} |Z_t(z)| < \infty$ and $\mathbb{E} [Z_t(z)] = z$.

(S.2) If $\alpha = (z_1, \dots, z_k)$ and $z_1 < z_2 < \dots < z_k$, then for each $t > 0$, it is true that

$$P_\alpha(Z_t(z_1) \leq Z_t(z_2) \leq \dots \leq Z_t(z_k)) = 1.$$

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(S.3) For each $t \geq 0$ and each i , the distribution of the real-valued random variable $Z_t(z_i)$ is absolutely continuous with respect to Lebesgue measure m .

PROPOSITION 10.4

Suppose that we are given a consistent system of finite-dimensional motions as in Definition 10.1, having the Markov property (A.3).

(i) If, in addition, (S.1) and (S.2) hold, then (A.4) holds, and we obtain a convolution semigroup of probability measures, and a corresponding pure stochastic flow as in Theorem 6.1.

(ii) If, in addition, (S.1) to (S.3) all hold, then (A.4') holds, and we obtain a convolution semigroup of FRPMs, and a corresponding pure stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ as in Theorem 9.1; in particular, $(x, \omega) \rightarrow X_{st}(x, \omega)$ is $m \otimes P$ - measurable.

Proof (i) Assume (S.1) and (S.2). Fix $t > 0$, $\epsilon > 0$ and $\delta > 0$. If

$y < z \leq y + \epsilon\delta$, then

$$P(|Z_t(z) - Z_t(y)| > \epsilon) = P(Z_t(z) - Z_t(y) > \epsilon) \text{ by (S.2)}$$

(10.5)

$$\leq \varepsilon^{-1} \mathbb{E} [Z_t(z) - Z_t(y)] = \varepsilon^{-1} (z-y) \leq \delta \quad \text{by (S.1).}$$

Hence Q_t is uniformly stochastically continuous on $\mathbb{R}^1$, for each t (Definition 4.1). Condition (4.5) is automatic, so (A.4) holds: now apply Proposition 10.2 and Theorem 6.1.

(ii) Assume (S.1) to (S.3). Fix x in $\mathbb{R}^1$, $t > 0$, and $\varepsilon > 0$. Suppose $x - 1/n \leq z_1 < z_2 < \dots < z_l = x < \dots < z_k \leq x + 1/n$. From (S.2), it follows that

$$\begin{aligned} & Q_t \left(\bigcup_{1 \leq j \leq k} \{f : |f(x) - f(z_j)| > \varepsilon\} \right) \\ &= P_\alpha (|Z_t(x) - Z_t(z_j)| > \varepsilon \text{ for some } j, 1 \leq j \leq k) \\ &\leq P_\alpha (Z_t(z_k) - Z_t(x) \geq \varepsilon) + P_\alpha (Z_t(x) - Z_t(z_1) > \varepsilon) \text{ by (S.2)} \\ &\leq 2/(\varepsilon n) \text{ by (10.5).} \end{aligned}$$

Hence Q_t has no fixed points of discontinuity by Proposition 7.5.

Condition (S.2) implies the property (8.1) in Definition 8.1, and (4.5) is automatic. Hence each Q_t is fluid: Now apply Proposition 10.2 and Theorem 9.1. $\square$

Part VCovariance functions and the corresponding sets of finite-dimensional motions.

In section 10 we gave assumptions on a set of finite-dimensional motions that allow the construction of a convolution semigroup of FRPM's. Now we restrict to the case of $T = \mathbb{R}^d$, and show in sections 11 and 12 that a general class of covariance functions (for random fields on $\mathbb{R}^d$) can be used to construct such systems of finite-dimensional motions. Chapter 13 concerns the additional properties the covariance must possess to ensure composability (assumption A.4 of section 10). Most of the techniques used here were learned from Harris [13].

11. Algebraic properties of the covariance function

We now take $T = \mathbb{R}^d$ (see section 10) and consider the problem of constructing a consistent system of finite-dimensional motions on $\mathbb{R}^d$ with a given (spatially homogeneous) "covariance" (see Corollary 12.2).

Definition 11.1. A function $z \mapsto b(z) = (b^{pq}(z))$, $1 \leq p, q \leq d$, from $\mathbb{R}^d$ into the space of real $d \times d$ matrices, will be called a **spatially homogeneous covariance function** on $\mathbb{R}^d$ if it is expressible in the form

$$b^{pq}(z) = \int_{\mathbb{R}^d} e^{iu \cdot z} F^{pq}(du) \quad (11.1)$$

where $F = (F^{pq})$ is a $d \times d$ matrix of complex-valued measures on $\mathbb{R}^d$ with the following properties:

- (a) $F(C)$ is Hermitean non-negative definite for all measurable sets C in $\mathbb{R}^d$, and

(b) $F^{pq}(-C) = F^{qp}(C)$ (which implies $b^{pq}(-z) = b^{qp}(z)$).

Remark. Yaglom [28] shows that such an F is the spectral measure of a spatially homogeneous $\mathbb{R}^d$ -valued random field U on $\mathbb{R}^d$, with mean zero and covariance $b^{pq}(z) = E[U^p(y) U^q(y+z)]$ for y, z in $\mathbb{R}^d$ and $1 \leq p, q \leq d$.

We shall make further assumptions as follows:

(C.1) (Normalization) $b^{pq}(0) (= F^{pq}(\mathbb{R}^d)) = \delta^{pq}$ (= identity matrix).

(C.2) (A positive definiteness condition). Let G be the absolutely continuous part of F with respect to Lebesgue measure $m(\cdot)$ on $\mathbb{R}^d$. Assume that there exists a measurable subset A of $\mathbb{R}^d$ with $m(A) > 0$, on which the Radon-Nikodym derivative dG/dm is Hermitean strictly positive definite.

Equation (11.1) and Definition 11.1 ensure that $z \mapsto b^{pq}(z)$ is continuous. We strengthen this as follows:

(C.3) (Local Lipschitz continuity away from 0). For each $1 \leq p, q \leq d$, the mapping $z \mapsto b^{pq}(z)$ is Lipschitz continuous on the complement of every open ball in $\mathbb{R}^d$ centered at the origin.

For any collection of k points $(z_1, \dots, z_k)$ in $\mathbb{R}^d$, we define (following Harris [13]) a $dk \times dk$ matrix $B^{(k)}$, called the **k -point covariance matrix**, as follows:

$$B^{(k)} = B^{(k)}(z_1, \dots, z_k) = \begin{bmatrix} B_{11} & \cdot & \cdot & \cdot & B_{1k} \\ \vdots & & & & \vdots \\ B_{k1} & \cdot & \cdot & \cdot & B_{kk} \end{bmatrix} \quad (11.2)$$

where $B_{ij} = b(z_j - z_i)$. From (C.1) and property (b) of F , $B_{ii} = I$ (= $d \times d$ identity matrix) and $B_{ji} = B_{ij}'$ (transpose). Therefore $B^{(k)}$ is symmetric.

Notation. Let D_k denote the set of k -tuples of **distinct** points $(z_1, \dots, z_k)$ in $(\mathbb{R}^d)^k$ and let H_k denote the set of k -tuples with two or more points the same.

LEMMA 11.2

The greatest eigenvalue of $B^{(k)}$ is $\leq k$, and the least eigenvalue is ≥ 0 . If $(z_1, \dots, z_k)$ belongs to D_k , then the least eigenvalue is > 0 , so $B^{(k)}$ is strictly positive definite. In general, the rank of $B^{(k)}$ is rd , where r is the number of distinct points among $\{z_1, \dots, z_k\}$.

Proof. Let $\underline{c}^1, \dots, \underline{c}^k$ be arbitrary elements of $\mathbb{C}^d$, and let $\underline{c}$ be the vector in $(\mathbb{C}^d)^k$ with $\underline{c}' = ((\underline{c}^1)', \dots, (\underline{c}^k)').$ Let $\underline{c}^*$ denote the conjugate transpose of $\underline{c}$. Then

$$\begin{aligned} \underline{c}^* B^{(k)} \underline{c} &= \sum_{1 \leq r, s \leq k} (\underline{c}^r)^* B_{rs} \underline{c}^s \\ &= \sum_{r, s} \sum_{p, q} \overline{c_p^r} b^{pq} (z_s - z_r) c_q^s \\ &= \int_{\mathbb{R}^d} \sum_{r, s} \sum_{p, q} (\overline{c_p^r} e^{-iu \cdot z_r}) (c_q^s e^{iu \cdot z_s}) f^{pq}(du) \end{aligned}$$

$$\therefore \underline{c}^* B^{(k)} \underline{c} = \int_{\mathbb{R}^d} \underline{a}(u)^* F(du) \underline{a}(u) \quad (11.3)$$

where $\underline{a}' = (a_1, \dots, a_d)$, and $a_q(u) = \sum_{s=1}^k c_q^s e^{iu \cdot z_s}$ for u in $\mathbb{R}^d$. The hermitean non-negative definiteness of F ensures that $\underline{c}^* B^{(k)} \underline{c} \geq 0$, proving that the least eigenvalue of $B^{(k)}$ is ≥ 0 . Now suppose that all the z_i are distinct. Assume that $\underline{c}$ is not zero. Then there is at most a finite subset N of $\mathbb{R}^d$ (depending on $\underline{c}$ and on $z_1, \dots, z_k$ distinct) on which all

the a_q are zero. We wish to prove that $\mu_a(\mathbb{R}^d) > 0$, where μ_a is the non-negative real-valued measure

$$\mu_a(C) = \int_C \underline{a}(u)^* F(du) \underline{a}(u) .$$

Let the complex measure G and the set A be as in assumption (C.2). Let J be the support of the singular part of F , and let $A_1 = A \cap (N \cup J)^c$. Then $m(A_1) = m(A) > 0$, and $\mu_a(\mathbb{R}^d) \geq \mu_a(A_1) = \int_{A_1} (\underline{a}(u)^* (dG/dm)(u) \underline{a}(u)) m(du) > 0$. This proves that $\underline{c}^* B^{(k)} \underline{c} > 0$ for $\underline{c} \neq 0$.

Next, let λ_1 denote the real-valued measure representing the highest eigenvalue of F . Equation (11.3) and the Cauchy-Schwarz inequality show that

$$\begin{aligned} \underline{c}^* B^{(k)} \underline{c} &\leq \int_{\mathbb{R}^d} \sum_{q=1}^d |a_q(u)|^2 \lambda_1(du) \\ &\leq \int_{\mathbb{R}^d} \sum_{q=1}^d k \sum_{s=1}^k |c_q^s|^2 \lambda_1(du) = k \underline{c}^* \underline{c} \lambda_1(\mathbb{R}^d) \end{aligned}$$

However $F^{pq}(\mathbb{R}^d) = \delta^{pq}$, so $\lambda_1(\mathbb{R}^d) = 1$. Hence $\underline{c}^* B^{(k)} \underline{c} \leq k \underline{c}^* \underline{c}$. proving that the greatest eigenvalue of $B^{(k)}$ is always $\leq k$.

Next we rewrite the matrix $B^{(k)}$ as (β_{mn}) , where $m, n = 1, 2, \dots, kd$, and for $1 \leq i, j \leq k$, $1 \leq p, q \leq d$

$$\beta_{(i-1)d+p, (j-1)d+q} = b^{pq}(z_j - z_i) = (B_{ij})^{pq} . \quad (11.4)$$

The next result generalizes Lemma 2.3 of Harris [13].

PROPOSITION 11.3

Under conditions (C.1) - (C.3), there exists a unique matrix

$V = V^{(k)}(z_1, \dots, z_k) = (\sigma_{mn})$, $1 \leq m, n \leq kd$, with the following properties:

(a) $B^{(k)} = VV'$,

(b) V is lower triangular with all diagonal entries > 0 .

The matrix V has the following additional properties:

(i) If z_i is not one of the elements of the set $\{z_1, \dots, z_{i-1}\}$, then $\sigma_{(i-1)d+p, (i-1)d+p} > 0$ for $p = 1, 2, \dots, d$.

(ii) If z_i is one of the elements of the set $\{z_1, \dots, z_{i-1}\}$, then $\sigma_{m, (i-1)d+q} = 0$ for $m = 1, 2, \dots, kd$ and $q = 1, 2, \dots, d$.

(iii) $\sigma_{pq} = \delta_{pq}$ for $1 \leq p, q \leq d$, and $\sigma_{(i-1)d+p, q} = b^{pq}(z_i - z_1)$ for $1 \leq i \leq k$, $1 \leq p, q \leq d$.

(iv) Each $\sigma_{mn} = \sigma_{mn}(z_1, \dots, z_k)$ is bounded in $(\mathbb{R}^d)^k$, continuous in D_k , and Lipschitz in each compact subset of D_k .

The remainder of this section is devoted to the proof. We begin with:

LEMMA 11.4

Let G be a Gramian, i.e. a non-negative definite symmetric matrix, of order k and of rank r . Then:

(i) There exists a unique lower triangular matrix T of rank r with all diagonal entries > 0 , such that $G = TT'$.

(ii) If the j^{th} column of G is linearly dependent on columns 1 through $(j-1)$ of G , then the j^{th} column of T is zero.

(iii) If the j^{th} column of G is linearly independent of columns 1 through $(j-1)$ of G , then $T_{jj} > 0$.

Proof. Refer to Rao [24, pp. 20, 69], for the following facts: G has a unique Gramian square root H , and H is of rank r ; moreover there exists an orthogonal matrix U such that $UH = S$, an upper triangular matrix of rank r . Let $N = \text{diag}(n(1), \dots, n(k))$, where $n(i) = \text{sgn}(S_{ii})$, taking

$\text{sgn}(0) = 1$. Then $N^2 = I$, and

$$G = H^2 = HH = H'U'UH = S'S = S'N'NS = TT'$$

where $T = (NS)'$. Then T is lower triangular, of rank r , with all diagonal entries > 0 . The existence part of (i) is proved. As for the uniqueness part, suppose $G = WW' = TT'$ gives two such decompositions. For $1 \leq j \leq i \leq r$, $\sum_{r=1}^j W_{ir}W_{jr} = G_{ij} = \sum_{r=1}^j T_{ir}T_{jr}$. This implies that $W_{11} = G_{11}^{1/2} = T_{11}$. By induction on i and j it can be shown that $W_{ij} = T_{ij}$ for all $j \leq i$. Hence $W = T$, and the uniqueness part of (i) follows.

As for (ii) and (iii), relabel the rows and columns such that the first r columns of G are linearly independent, and the remaining $(k-r)$ columns are linearly dependent on the first r . Then

$$G = \begin{bmatrix} K & A \\ A' & A'K^{-1}A \end{bmatrix}$$

where K is a non-singular $r \times r$ Gramian, and A is $r \times (k-r)$. By part (i), there is a unique decomposition $K = WW'$ as in (i). Moreover W is of rank r with non-zero determinant by (i). Since W is triangular, the determinant is the product of the diagonal entries, which are all > 0 . Hence all the diagonal entries are > 0 , proving (iii).

Now define a lower triangular matrix T by

$$T = \begin{bmatrix} W & 0 \\ A'(W')^{-1} & 0 \end{bmatrix}$$

Then

$$\begin{aligned}
TT' &= \begin{bmatrix} W & 0 \\ A'(W')^{-1} & 0 \end{bmatrix} \begin{bmatrix} W' & W^{-1}A \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} WW' & A \\ A' & A'K^{-1}A \end{bmatrix} = G
\end{aligned}$$

By the uniqueness assertion in (i), $G = TT'$ is the unique decomposition of G as in (i). Since columns $(r+1)$ through k of T are zero, (ii) is proved.

Proof of Proposition 11.3

The existence and uniqueness of $v^{(k)}$ satisfying (a) and (b) follows from Lemma 11.4. To prove assertions (i) and (ii), remember that columns $(i-1)d+1$ through id of $B^{(k)}$ are linearly independent, and independent of columns 1 through $(i-1)d$, if z_i does not belong to $\{z_1, \dots, z_{i-1}\}$, while if z_i does belong to $\{z_1, \dots, z_{i-1}\}$, then columns $(i-1)d+1$ through id of $B^{(k)}$ are linearly dependent on columns 1 through $(i-1)d$; then apply Lemma 11.4 parts (ii) and (iii).

To prove (iii); for $1 \leq q \leq p \leq d$, $\sum_{r=1}^q \sigma_{pr} \sigma_{qr} = b^{pq}(0) = \delta^{pq}$. Let $T = (\sigma_{pr})$, $1 \leq p, r \leq d$. Then T is lower triangular, with all diagonal entries ≥ 0 , and $TT' = I$. By the uniqueness part of Lemma 11.4, part (i), $T = I$. Thus $\sigma_{pq} = \delta_{pq}$ for $1 \leq p, q \leq d$.

For $1 \leq p, q \leq d$, $\sum_{r=1}^q \sigma_{qr} \sigma_{(i-1)d+p,r} = \beta_{q,(i-1)d+p} = b^{qp}(z_i - z_1)$. However $\sigma_{qr} = \delta_{qr}$ by the previous paragraph, so $\sigma_{(i-1)d+p,q} = b^{qp}(z_i - z_1)$ as desired. This completes (iii).

As for (iv), notice that

$$\sum_r \sigma_{(i-1)d+q,r}^2 = \beta_{(i-1)d+p,(i-1)d+q} = b^{qq}(z_i - z_i) = 1$$

proving that each σ_{mn} is bounded. Before proving the local Lipschitz assertion, we state the following Lemma.

LEMMA 11.5

Suppose that w, h, f, g_i, u_i and $v_i, i = 1, 2, \dots, n$, are functions from $\mathbb{R}^m$ (or any metric space) to $\mathbb{R}$ with the following properties:

- (a) Their absolute values are bounded above by 1.
- (b) All the functions except f are Lipschitz, with Lipschitz constant L , when restricted to a certain compact subset K of $\mathbb{R}^m$.
- (c) Either (i) or (ii) holds:
 - (i) $M = \inf\{f(x) : x \in K\} > 0$ and $f(x)^2 = 1 - \sum_{i=1}^n g_i(x)^2$
 - (ii) $N = \inf\{h(x) : x \in K\} > 0$ and $f(x)h(x) = w(x) - \sum_{i=1}^n u_i(x)v_i(x)$

Then f is Lipschitz on K .

Proof. First suppose that (i) holds. Then $f(x)^2 - f(y)^2 = \sum_{i=1}^n (g_i(x)^2 - g_i(y)^2)$.

It follows that for x, y in K ,

$$|f(x) - f(y)| \leq \frac{2 \sum_{i=1}^n |g_i(x) - g_i(y)|}{2 \min(f(x), f(y))} \leq \frac{nL}{M} |x-y|.$$

On the other hand if (ii) holds, then

$$h(x)|f(x) - f(y)| = |h(x)f(x) - h(x)f(y) + h(y)f(y) - h(y)f(y)|$$

$$\leq f(y)|h(x) - h(y)| + |w(x) - w(y)| + \sum_{i=1}^n |u_i(x)v_i(x) - u_i(y)v_i(x) + u_i(y)v_i(x) - u_i(y)v_i(y)|$$

$$\leq (1 + f(y))L|x-y| + \sum_{i=1}^n (|v_i(x)|L|x-y| + |u_i(y)|L|x-y|)$$

$$\leq (2n + 2) L|x-y|.$$

Hence $|f(x) - f(y)| \leq \frac{2(n+1)L}{N} |x-y|$, for x, y in K .

Proof of Proposition 11.3, concluded.

Let K be a compact subset of D_k . Part (iii) shows that entries in columns 1 through d of V are Lipschitz on K . Make the inductive hypothesis that all entries in columns 1 through $(j-1)d + q - 1$ are Lipschitz on K , where $1 \leq q \leq d$. If z_j belongs to the set $\{z_1, \dots, z_{j-1}\}$, then the $(j-1)d + q$ column of V is zero by part (ii), and therefore all entries in columns 1 through $(j-1)d + q$ are Lipschitz on K . On the other hand if z_j does not belong to $\{z_1, \dots, z_{j-1}\}$, then $\inf\{\sigma_{(j-1)d+q, (j-1)d+q}(z_1, \dots, z_k) : (z_1, \dots, z_k) \text{ in } K\} > 0$ by part (i). Moreover the construction of V and the fact that $\beta_{(j-1)d+q, (j-1)d+q} = 1$ imply that

$$(\sigma_{(j-1)d+q, (j-1)d+q})^2 = 1 - \sum_{r=1}^{(j-1)d+q-1} (\sigma_{(j-1)d+q, r})^2$$

The inductive hypothesis, the lower bound on the left side, and the upper bound on all the entries of V together imply that the conditions of Lemma 11.5 are satisfied. Hence $\sigma_{(j-1)d+q, (j-1)d+q}$ is Lipschitz on K .

Now perform another induction down the $(j-1)d+q$ column of V . The inductive hypothesis is that $\sigma_{m, (j-1)d+q}$ is Lipschitz on K for

$m = (j-1)d + q, (j-1)d + q + 1, \dots, n - 1$, where $n \geq (j-1)d + q + 1$, and σ_{rs} is Lipschitz on K for all $s \leq (j-1)d + q - 1$ and all r . By part (a), $B^{(k)} = VV'$, so

$$\sigma_{n,(j-1)d+q} \sigma_{(j-1)d+q,(j-1)d+q} = \beta_{n,(j-1)d+q} - \sum_{r=1}^{(j-1)d+q-1} \sigma_{n,r} \sigma_{(j-1)d+q,r}$$

By assumption (C.3), $\beta_{n,(j-1)d+q}$ is Lipschitz on K . We have seen that all entries of V are bounded above by 1, and $\sigma_{(j-1)d+q,(j-1)d+q}$ is bounded below on K by a constant > 0 . The inductive hypothesis implies that all terms in the sum on the right side are Lipschitz on K . Thus all the conditions of Lemma 11.5 are in force, so $f = \sigma_{n,(j-1)d+q}$ is Lipschitz on K . This completes the induction down the $(j-1)d+q$ column of V , which in turn completes the original induction across the columns of V . Thus the Lipschitz continuity on K is proved, and continuity on D_k follows.

12. Constructing the finite-dimensional motions

Definitions and notations are continued from Section 11; we assume that $b(\cdot)$ is a spatially homogeneous covariance function on $\mathbb{R}^d$ satisfying assumptions (C.1), (C.2) and (C.3), with associated k -point covariance matrix $B^{(k)}(z_1, \dots, z_k)$ for $(z_1, \dots, z_k)$ in $(\mathbb{R}^d)^k$. As in Proposition 11.3, $B^{(k)} = V^{(k)} V^{(k)'} where $V^{(k)} = (\sigma_{mn}^{(k)})$, $1 \leq m, n \leq kd$.$

We now wish to study solutions of the follow systems of stochastic differential equations in $(\mathbb{R}^d)^k$, for $k = 1, 2, \dots$;

$$Z_i(t) = (Z_i^1(t), \dots, Z_i^d(t)), \quad i = 1, 2, \dots, k,$$

$$Z_i^p(t) = z_i^p + \sum_{n=1}^{(i-1)d+p} \int_0^t \sigma_{(i-1)d+p,n}^{(i-1)d+p,n}(Z_1(s), \dots, Z_k(s)) dW^n(s), \quad p = 1, 2, \dots, d, \quad (12.1)$$

where $(W^1(t), \dots, W^{kd}(t), t \geq 0)$ is a kd -dimensional Wiener process.

Let $C_0^\infty((\mathbb{R}^d)^k)$ denote the C^∞ functions from $(\mathbb{R}^d)^k$ to $\mathbb{R}$ with compact support. Let Ω_{kd} denote the space of continuous mappings $\omega : [0, \infty) \rightarrow (\mathbb{R}^d)^k$

with the topology of uniform convergence on compact sets. Putting

$Z_t(\omega) = \omega(t) = \omega_t$, let $\underline{G}_t^k = \underline{G}_t$ be the σ -field generated by

$\{Z_s, 0 \leq s \leq t\}$, giving a right-continuous filtration of $\underline{G} = \bigvee_t \underline{G}_t$.

Define $A = A^{(k)}$ operating on $C_0^\infty((\mathbb{R}^d)^k)$ by

$$(12.2) \quad Af(Z_1, \dots, Z_k) = \frac{1}{2} \sum_{1 \leq i, j \leq k} \sum_{1 \leq p, q \leq d} b^{pq}(Z_j - Z_i) D_i^p D_j^q f(Z_1, \dots, Z_k)$$

where $(Z_1, \dots, Z_k)$ is any point of $(\mathbb{R}^d)^k$, and D_i^p means $\partial/\partial z_i^p$. Notice that the matrix of coefficients is as in (11.4). A probability measure P on $\underline{G}$ governing a process (ξ_t) in $(\mathbb{R}^d)^k$, is said to be a solution to the martingale problem (MP) for A from $(Z_1, \dots, Z_k)$ in $(\mathbb{R}^d)^k$

if $P(\xi_0 = (z_1, \dots, z_k)) = 1$ and $(f(\xi_t) - \int_0^t Af(\xi_s)ds, \mathcal{G}_t, P)$ is a martingale for each f in $C_0^\infty((\mathbb{R}^d)^k)$.

Define the shift operator $S_t: \Omega_{kd} \rightarrow \Omega_{kd}$ by $(S_t \omega)(s) = \omega(t+s)$, $s, t \geq 0$.

Definition (following Harris [13]).

If C belongs to $\mathcal{G}$, if $S_t C$ is contained in C for all $t \geq 0$, if P solves the MP for A , and if $P(C) = 1$, then we call P a C -solution.

Henceforward we specialize to the case where C is the set of ω in Ω_{kd} such that if $\omega(t) = (\omega_1(t), \dots, \omega_k(t))$ and if $\omega_i(t) = \omega_j(t)$ in $\mathbb{R}^d$ for some $1 \leq i < j \leq k$ and $t \geq 0$, then $\omega_i(t+s) = \omega_j(t+s)$ for all $s \geq 0$; in other words, if paths collide in $\mathbb{R}^d$, they coalesce.

The next result is nearly identical to Lemma 3.2 of Harris [13].

PROPOSITION 12.1

For $\underline{z} = (z_1, \dots, z_k)$ in $(\mathbb{R}^d)^k$, there is a unique C -solution $P_{\underline{z}}$ to the MP for A from $\underline{z}$. Moreover:

- (i) If E belongs to $\mathcal{G}$, then $P_{\underline{z}}(E)$ is measurable in $\underline{z}$.
- (ii) (Consistency) If $1 \leq r < k$, if π_{kr} is the projection of Ω_{kd} onto Ω_{rd} which maps $(\omega_1, \dots, \omega_k)$ to $(\omega_1, \dots, \omega_r)$, and if $p^{(k)}$ is a C -solution of the MP for $A^{(k)}$, then $\pi_{kr}(p^{(k)}) \equiv p^{(k)} \cdot \pi_{kr}^{-1}$ is a C -solution of the MP for $A^{(r)}$.
- (iii) Let $(Z(t), t \geq 0) = (Z_1(t), \dots, Z_k(t), t \geq 0)$ be the canonical process representing a C -solution for $A^{(k)}$. Fix $i \neq j$, and let $V_t = Z_i(t) - Z_j(t)$. Then $(V_t, t \geq 0)$ is a diffusion in $\mathbb{R}^d$ with absorbing state 0 and operator

$$Lf(y) = \frac{1}{2} \sum_{p,q} (2\delta^{pq} - b^{pq}(y) - b^{qp}(y)) \frac{\partial^2 f}{\partial x_p \partial x_q}(y), \quad f \text{ in } C_0^\infty(\mathbb{R}^d) \quad (12.3)$$

(iv) The family $\{P_z, z \text{ in } (\mathbb{R}^d)^k\}$ is strong Markov and Feller.

Proof. The proof of the existence and uniqueness assertion is essentially the same as that of Lemma 3.2 in Harris [13], and involves induction on k . The crucial point in the proof is that the coefficient matrix (σ_{mn}) appearing in (12.1) is Lipschitz on every compact subset of D_k , as was proved in Proposition 11.3. Harris then applies the methods of Stroock and Varadhan [25]. Assertion (i) is essentially the same as Lemma 2.2 of Harris [13], and the proof depends on Stroock and Varadhan [25, 6.7.4]. The consistency assertion (ii), which is essentially trivial, is like Lemma 3.3 of Harris [13]. To prove assertion (iii), take $i = 2$ and $j = 1$ without loss of generality, and consider the stochastic differential equations satisfied by $(Z_1(t), Z_2(t))$ and by $V_t = Z_2(t) - Z_1(t)$, using (12.1) and Proposition 11.3. If $B^{(2)}$ is as in Proposition 11.3, then on $[0, \tau)$, where $\tau = \inf\{t : V_t = 0\}$,

$$dV_t^p = \sum_{n=1}^d (b^{np}(V_t) - \delta^{np}) dW_t^n + \sum_{n=d+1}^{d+p} \sigma_{d+p,n}^{(0, V_t)} dW_t^n. \quad (12.4)$$

Using Ito's formula, it is easy to check that $(V_t, t \geq 0)$ has the required generator. The fact that 0 is an absorbing state follows from the choice of C in the notion of 'C-solution'. The strong Markov assertion in part (iv) is proved using Stroock and Varadhan [25, 6.2.2], in the context of C-solutions. To prove the Feller property, proceed as follows: suppose $(Z_1(t), \dots, Z_k(t), Z'_1(t), \dots, Z'_k(t), t \geq 0)$ is the C-solution for $A^{(2k)}$ from $(\underline{z}, \underline{z}') = (z_1, \dots, z_k, z'_1, \dots, z'_k)$. Then by part (ii) (consistency), Z and Z' are the C-solutions for $A^{(k)}$ from $\underline{z}$ and $\underline{z}'$ respectively. Let

$V_t = Z_i(t) - Z'_i(t)$ for some $1 \leq i \leq k$. Then $(V_t, t \geq 0)$ satisfies the stochastic differential equation (12.4) on $[0, \tau)$, and equals zero on $[\tau, \infty)$. For fixed t , $Z_i(t) - Z'_i(t) \rightarrow 0$ in law as $z_i \rightarrow z'_i$, which implies the Feller property.

COROLLARY 12.2

Denote the canonical process in $(\mathbb{R}^d)^k$ representing a C-solution for $A^{(k)}$ from $(z_1, \dots, z_k)$ by $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$, instead of by $(Z_1(t), \dots, Z_k(t), t \geq 0)$. Then the following properties hold:

$$(i) \quad d \langle Z^p(z_i), Z^q(z_j) \rangle_t = b^{pq}(Z_t(z_j) - Z_t(z_i)) dt \quad (12.5)$$

$$(ii) \quad \lim_{t \downarrow 0} (1/t) \mathbb{E}[(Z_t^p(z_i) - z_i^p)(Z_t^q(z_j) - z_j^q)] = b^{pq}(z_j - z_i) \quad (12.6)$$

(iii) $(Z_t(z_i) - z_i, t \geq 0)$ is Brownian motion in $\mathbb{R}^d$, for each z_i .

COROLLARY 12.3

The processes $(Z_t(z_1), \dots, Z_t(z_k), t \geq 0)$, as $\{z_1, \dots, z_k\}$ runs through the finite subsets of $\mathbb{R}^d$, constitute a consistent system of finite-dimensional motions on $\mathbb{R}^d$, in the sense of Definition 10.1; moreover they satisfy assumption (A.3) of Proposition 10.2. For each $t \geq 0$, let Q_t be the unique Radon probability measure on $(\Omega, \underline{B}(\Omega))$ which is consistent with all the $(Z_t(z_1), \dots, Z_t(z_k))$ (t fixed), as constructed in Section 10 (here $T = \mathbb{R}^d$). If, furthermore, condition (A.4) (resp. A.4') of Proposition 10.2 holds, then $\{Q_t, t \geq 0\}$ satisfies the conditions of Theorem 6.1 (resp. 9.1) for the existence of a pure stochastic flow.

13. Stochastic continuity in the non-isotropic case

The purpose of this section is to derive a sufficient condition on the covariance function for the measures Q_t (see Corollary 12.3) to be composable (see (4.4)). The condition will be weak enough to admit some covariance functions which are not differentiable at zero, and for which the two-point motions may coalesce (see sections below). The importance of composability is that it enables us to construct a pure stochastic flow via Theorem 6.1.

Given a covariance $b(\cdot)$ as described in section 11, and assuming that the dimension $d \geq 2$, define functions G_L and G_N from $\mathbb{R}^d$ to $\mathbb{R}$ as follows:

$$G_L(z) = 1 - |z|^{-2} \sum_{p,q=1}^d z^p b^{pq}(z) z^q \quad (13.1)$$

$$G_N(z) = \sum_{p=1}^d (1 - b^{pp}(z)) - G_L(z) \quad (13.2)$$

The definitions are modelled on the definitions of functions V_L and V_R which occur in the isotropic case: see (14.5).

LEMMA 13.1

The functions G_L and G_N above satisfy:

$$0 < G_L(z) < 2, \quad z \text{ in } \mathbb{R}^d, \quad |z| \neq 0; \quad (13.3)$$

$$-2 < G_N(z) < 2d, \quad z \text{ in } \mathbb{R}^d, \quad |z| \neq 0; \quad (13.4)$$

$$G_L(0) = 0 = G_N(0). \quad (13.5)$$

Proof. According to Lemma 11.2, for non-zero y and z in $\mathbf{R}^d$,

$$0 < |y', -y'| \begin{bmatrix} I & B \\ B' & I \end{bmatrix} \begin{bmatrix} y \\ -y \end{bmatrix} \leq 2|y', -y'| \begin{bmatrix} y \\ -y \end{bmatrix} = 4|y|^2 \quad (13.6)$$

where $B = (b^{pq}(z))$. An elementary calculation yields $-1 \leq |y|^{-2} y' B y < 1$.

If we remove the minus signs in (13.6), we obtain $-1 < |y|^{-2} y' B y \leq 1$.

Now take $y = z$ to obtain (13.3). On the other hand, by taking y to be e_p (the p^{th} basis vector of $\mathbf{R}^d$) we see that $-1 < b^{pp}(z) < 1$ for all $z \neq 0$, and so by (13.3),

$$-2 < \sum_{p=1}^d (1 - b^{pp}(z)) - G_L(z) < 2d$$

which yields (13.4).

Fix y in $\mathbf{R}^d$, and let $(Z_t(0), Z_t(y), t \geq 0)$ represent a C-solution for $A^{(2)}$ from $(0, y)$, as in Corollary 12.2. (Since the covariance is spatially homogenous, there is no loss of generality in taking the starting point to be $(0, y)$ instead of (x, y) .)

PROPOSITION 13.2

Suppose there exist constants $c > 0$ and $\alpha > 1$ such that $G_L(z) \geq \alpha G_N(z)$ whenever $0 \leq |z| \leq c$. Then for all $t \geq 0$ and $\varepsilon > 0$,

$$P[|Z_t(y) - Z_t(0)| > \varepsilon] \leq e^{\gamma t} \varepsilon^{-2\beta} |y|^{2\beta}, \quad (13.7)$$

where

$$\beta = (1 - 1/\alpha)/2 \quad (\text{so } 0 < \beta < 0.5), \quad (13.8)$$

$$\gamma = \sup_{|z| \geq c} \{2\beta |y|^{-2} |G_N(z) - \frac{1}{\alpha} G_L(z)|\} \vee 0 \quad (13.9)$$

(It follows from Lemma 13.1 that $0 \leq \gamma \leq \infty$).

COROLLARY 13.3

If the condition of Proposition 13.2 holds, in addition to the conditions of Section 11, then each probability measure Q_t (see Corollary 12.3) satisfies composability condition (4.4), and $\{Q_t, t \geq 0\}$ is a convolution semigroup of probability measures (Definition 4.12).

Remark. See Remarks 14.4 for a counterexample to show that the condition on G_L and G_N is not necessary for fluidity.

Proof of the Corollary. Let M be the one-point compactification of the Euclidean metric space $\mathbb{R}^d$. Now (13.7) implies that for any x and y in $\mathbb{R}^d$,

$$Q_t(\{f \in \Gamma : |f(x) - f(y)| > \varepsilon\}) \leq e^{\gamma t} \varepsilon^{-2\beta} |x - y|^{2\beta}.$$

Hence for fixed t , the left side converges to 0 as y tends to x , uniformly in x . Condition (4.5) is immediate, so each Q_t satisfies (4.4), as desired. Now apply Proposition 10.2. $\square$

Proof of Proposition 13.2

Step I. For $m = 1, 2, \dots$, let h_m be a C^∞ function from $\mathbb{R}^d$ to $\mathbb{R}$, with compact support, such that

$$h_m(y) = (m^{-1} + |y|^2)^\beta, \text{ when } 0 \leq |y| \leq m \quad (13.10)$$

where β is the constant defined in (13.8). For $m = 1, 2, \dots$ define non-negative semimartingales $(R_m(t), t \geq 0)$ by

$$R_m(t) = e^{-\gamma t} h_m(Z_t(y) - Z_t(0)) \quad (13.11)$$

We wish to apply Ito's formula to (13.11). Notice that

$$\begin{aligned} D_p h_m(y) &= 2\beta y^p (m^{-1} + |y|^2)^{\beta-1}, \\ D_p D_q h_m(y) &= 2\beta \delta^{pq} (m^{-1} + |y|^2)^{\beta-1} - 4\beta(1-\beta) y^p y^q (m^{-1} + |y|^2)^{\beta-2} \end{aligned} \quad (13.12)$$

$$\text{where } D_p = \frac{\partial}{\partial y^p}.$$

Since h_m has compact support, the process $(N_m(t), t \geq 0)$ defined as follows is a martingale:

$$N_m(t) = h_m(V_t(y)) - h_m(y) - \int_0^t Lh_m(V_s(y)) ds \quad (13.13)$$

where

$$V_t(y) = Z_t(y) - Z_t(0) \quad (13.14)$$

and L is the operator defined in (12.3). Now apply Ito's formula to (13.11) to obtain:

$$dR_m(t) = e^{-\gamma t} dN_m(t) + e^{-\gamma t} [Lh_m(V_t(y)) - \gamma h_m(V_t(y))] dt. \quad (13.15)$$

According to (12.3),

$$Lh_m(y) - \gamma h_m(y) = \frac{1}{2} \sum_{p,q=1}^d (2\delta^{pq} - b^{pq}(y) - b^{qp}(y)) D_p D_q h_m(y) - \gamma(m^{-1} + |y|^2)^\beta$$

When $0 \leq |y| \leq m$, this may be evaluated using (13.12), and equals:

$$2\beta(m^{-1} + r^2)^{\beta-1} \left\{ \sum_p (1 - b^{pp}(y)) - (1-\beta)(m^{-1} + r^2)^{-1} (r^2 - \sum_{p,q} y^p b^{pq}(y) y^q) \right\} - \gamma(m^{-1} + r^2)^\beta$$

where $r = |y|$. Comparing this with (13.1) and (13.2), we see that

$$\begin{aligned} \{...\} &= G_N(y) + G_L(y) - 2(1-\beta)r^2(m^{-1} + r^2)^{-1}G_L(y) \\ &= G_N(y) - (1/\alpha)G_L(y) + 2(1-\beta)m^{-1}(m^{-1} + r^2)^{-1}G_L(y). \end{aligned}$$

Therefore for $0 \leq r = |y| \leq m$,

$$Lh_m(y) - \gamma h_m(y) = 2\beta(m^{-1} + r^2)^{\beta-1} [G_N(y) - (1/\alpha)G_L(y)] - \gamma(m^{-1} + r^2)^\beta + \lambda(r, m) \quad (13.16)$$

where

$$\lambda(r, m) = 4\beta(1-\beta)m^{-1}(m^{-1} + r^2)^{\beta-2}G_L(y). \quad (13.17)$$

Since $\alpha > 1$, it follows that $0 < \beta < 0.5$ (see (13.8)) and so $4\beta(1-\beta) < 1$; moreover $G_L(y) < 2$ by (13.3). Hence

$$\lambda(r, m) \leq 2(1/m)^{1-\beta} \quad (13.18)$$

When $0 \leq r \leq c$, we have $G_N(y) - (1/\alpha)G_L(y) \leq 0$, by assumption. When $r > c$, (13.9) implies that $2\beta r^{-2}[G_N(y) - (1/\alpha)G_L(y)] - \gamma \leq 0$; hence for $r > c$

$$(m^{-1} + r^2)^\beta \{2\beta(m^{-1} + r^2)^{-1}[G_N(y) - (1/\alpha)G_L(y)] - \gamma\} \leq 0.$$

Combined with (13.18), this shows that

$$Lh_m(y) - \gamma h_m(y) \leq 2(1/m)^{1-\beta}, \quad 0 \leq |y| \leq m \quad (13.19)$$

Step II.

Let $a > 0$ be a fixed time, and define bounded stopping-times

$(\tau(m), m = 1, 2, \dots)$ by

$$\tau(m) = \inf\{t : |Z_t(y) - Z_t(0)| \geq m\} \wedge a \quad (13.20)$$

Now apply Markov's inequality and Doob's martingale inequality as follows:

$$P(\tau(m) < a) = P\left(\sup_{0 \leq t \leq a} |Z_t(y) - Z_t(0)|^2 \geq m^2\right)$$

$$\begin{aligned} &\leq m^{-2} \mathbb{E} \left[\sup_{0 \leq t \leq a} |Z_t(y) - Z_t(0)|^2 \right] \\ &\leq 4m^{-2} \mathbb{E} [|Z_a(y) - Z_a(0)|^2]. \end{aligned}$$

Since $(Z_t(x) - x, t \geq 0)$ is Brownian motion in $\mathbb{R}^d$, for each x in $\mathbb{R}^d$, the last inequality shows that

$$\lim_{m \rightarrow \infty} P(\tau(m) < a) = 0. \quad (13.21)$$

Inequality (13.19) implies that

$$Lh_m(V_s(y)) - \gamma h_m(V_s(y)) \leq 2(1/m)^{1-\beta}, \quad 0 \leq s \leq \tau(m).$$

It follows from (13.15) that

$$\mathbb{E}[R_m(\tau(m))] \leq h_m(y) + 2(1/m)^{1-\beta} \int_0^a e^{-\gamma s} ds. \quad (13.22)$$

Given $\varepsilon > 0$, for each $m = 1, 2, \dots$ it is true that

$$\begin{aligned} P(|V_a(y)| > \varepsilon) &= P(e^{-\gamma a} |V_a(y)|^{2\beta} > e^{-\gamma a} \varepsilon^{2\beta}) \\ &\leq P(e^{-\gamma a} (m^{-1} + |V_a(y)|^2)^\beta > e^{-\gamma a} \varepsilon^{2\beta}) \\ &= P(R_m(a) > e^{-\gamma a} \varepsilon^{2\beta}), \text{ by (13.11),} \\ &\leq P(\{R_m(a) > e^{-\gamma a} \varepsilon^{2\beta}\} \cap \{\tau(m) = a\}) + P(\tau(m) < a) \\ &= P(\{R_m(\tau(m)) > e^{-\gamma a} \varepsilon^{2\beta}\} \cap \{\tau(m) = a\}) + P(\tau(m) < a) \\ &\leq e^{\gamma a} \varepsilon^{-2\beta} \mathbb{E}[R_m(\tau(m))] + P(\tau(m) < a) \\ &\leq e^{\gamma a} \varepsilon^{-2\beta} [m^{-1} + |y|^2]^\beta + s(1/m)^{1-\beta} \int_0^a e^{-\gamma s} ds + P(\tau(m) < a) \end{aligned}$$

Now let m tend to infinity. Using (13.21) we see that

$$P(|V_a(y)| > \varepsilon) \leq e^{\gamma_a \varepsilon^{-2\beta}} |y|^{2\beta}$$

which completes the proof.

14. Stochastic continuity and coalescence in the isotropic case

A more refined analysis of the distance between $Z_t(y)$ and $Z_t(0)$ is possible in the isotropic case, because the distance process is a one-dimensional diffusion. We shall also give a necessary and sufficient condition for two trajectories to coalesce in finite time, with positive probability.

Suppose $d \geq 2$, and the covariance function $b(\cdot)$ is not only spatially homogeneous but also isotropic (i.e., for all $d \times d$ orthogonal matrices G , $b(z) = G'b(Gz)G$). Yaglom [28] shows that isotropic correlation depends on two scalar functions B_L and B_N , the longitudinal and transverse correlation functions:

$$B_L(r) = b^{pp}(re_p), \quad r \geq 0; \quad (14.1)$$

$$B_N(r) = b^{pp}(re_q), \quad r \geq 0, \quad q \neq p. \quad (14.2)$$

Isotropy implies that B_L and B_N do not depend on the choice of p and q or of the basis vectors $e_1, \dots, e_d$. As usual we normalize so that $B_L(0) = 1 = B_N(0)$. It is more convenient in the sequel to work not with B_L and B_N , but with the functions V_L and V_N defined by

$$V_L(r) = 1 - B_L(r), \quad V_N(r) = 1 - B_N(r), \quad r \geq 0 \quad (14.3)$$

We may extend V_L and V_N to continuous functions on $(-\infty, \infty)$ by taking $V_L(-r) = V_L(r)$, $V_N(-r) = V_N(r)$. We have

$$b^{pq}(z) = (V_N(|z|) - V_L(|z|))z^p z^q / |z|^2 + (1 - V_N(|z|))\delta^{pq}, \quad z \neq 0, \quad b^{pq}(0) = \delta^{pq} \quad (14.4)$$

Comparing this expression with (13.1) and (13.2), we see that

$$V_L(|z|) = G_L(z), V_N(|z|) = G_N(z)/(d-1) \quad (14.5)$$

It follows from Lemma 13.1 that for $r > 0$,

$$0 < V_L(r) < 2, -2/(d-1) < V_N(r) < 2d/(d-1), \quad (14.6)$$

and evidently $V_L(0) = V_N(0) = 0$. From Proposition 13.2 and Corollary 13.3, we obtain immediately:

COROLLARY 14.1

In the isotropic case, suppose that there exist constants $c > 0$ and $\alpha > 1$ such that $1 - B_L(r) \geq \alpha(d-1)(1 - B_N(r))$ whenever $0 \leq r \leq c$. Then each probability measure Q_t (see Corollary 12.3) is fluid with respect to Lebesgue measure.

We shall now study the distance process afresh using more specialized techniques. For $y \neq 0$ in $\mathbb{R}^d$, consider the two-point process $(Z_t(0), Z_t(y), t \geq 0)$ as in Corollary 12.2, and define the distance process as follows:

$$\rho_t(y) = |Z_t(y) - Z_t(0)|, t \geq 0 \quad (14.7)$$

LEMMA 14.2 (due to Baxendale and Harris)

In the case of isotropic covariance, satisfying the conditions of section 11, the following hold:

- (i) $(\rho_t(y), t \geq 0)$ is a one-dimensional diffusion process, with 0 as an absorbing state, and with generator R , where

$$Rg(u) = V_L(u)g''(u) + (d-1)(V_N(u)/u)g'(u), \quad g \text{ in } C_b^2(\mathbb{R}'). \quad (14.8)$$

- (ii) Define a function $H : (0, \infty) \rightarrow (0, \infty)$ by

$$H(r) = \exp\left\{-(d-1) \int_1^r [V_N(s)/sV_L(s)]ds\right\} \quad (14.9)$$

($H(r)$ is well-defined and finite by (14.6) and the continuity of V_N and V_L .) Any function $S : (0, \infty) \rightarrow (0, \infty)$ such that $S'(r) = H(r)$ for all $r > 0$ is a natural scale function for $(\rho_t(y), t \geq 0)$.

Proof. These results follow essentially from (12.3); see Baxendale and Harris [3, (3.11) and (3.14)].

PROPOSITION 14.3

Suppose that there exists a function $S : (0, \infty) \rightarrow (0, \infty)$ such that:

$$S'(r) = H(r), \text{ for } 0 < r < \infty, \text{ and } S(0+) = 0. \quad (14.10)$$

Then for fixed t , $Z_t(y)$ is uniformly stochastically continuous in y (Definition 4.1)

and the probability measure Q_t (see Corollary 12.3) satisfies (4.4). Thus

$\{Q_t, t \geq 0\}$ forms a convolution semigroup of probability measures, in the sense of Definition 4.12.

Remarks 14.4

(i) Suppose there exist constants $c > 0$ and $\alpha > 1$ such that $V_L(r) \geq \alpha(d-1)V_N(r)$ for $0 \leq r \leq c$, as in Corollary 14.1. By (14.6) and the continuity of V_L ,

$$\inf\{V_L(r) : r \geq c\} = \delta > 0.$$

It is easy to check that

$$H(r) \leq c_1 r^{-1/\alpha}, \text{ when } 0 < r \leq c$$

where c_1 is a constant depending on α , δ , c and d . Hence the equation $S'(r) = H(r)$ does have a solution with $S(0+) = 0$ in this case. Therefore Proposition 14.3 is more general than Corollary 14.1.

(ii) Baxendale and Harris [3] give many examples of C^4 covariances where $S(0+) = -\infty$; however the corresponding stochastic flow is a flow of homeomorphisms, and each probability measure Q_t is composable, because it is concentrated on the continuous functions. Therefore the condition given in Proposition 14.3 is certainly not necessary for the composability of Q_t .

The proof of Proposition 14.3 depends on the following Lemma.

LEMMA 14.5

Suppose that (14.10) holds. Extend S to $[0, \infty)$ by defining $S(0) = S(0+) = 0$. Then

- (i) S is C^2 on $(0, \infty)$ with $S'(r) > 0$ and $S''(r) < 0$ for all $r > 0$;
hence S is strictly increasing, continuous, and concave on $[0, \infty)$.

(ii) $E[S(\rho_t(y))] < \infty$ for all $t \geq 0$, all y in $\mathbb{R}^d$.

Proof. It follows from (14.6) and (14.9) that $0 < H(r) < \infty$ for all $r > 0$, and $H(r)$ is strictly decreasing in r ; this proves (i). It follows that for all $r > 1$, $S(r) \leq S(1) + (r-1)S'(1)$.

Since $S'(1) = H(1) = 1$, we see that $S(r) \leq S(1) + r$, for all $r > 0$.

Therefore by Corollary 12.2 (iii), we see that

$$\begin{aligned} E[S(\rho_t(y))] &\leq S(1) + E[\rho_t(y)] \leq S(1) + E[|Z_t(y) - y|] + |y| + E[|Z_t(0)|] \\ &\leq S(1) + |y| + 2t^{1/2}. \end{aligned}$$

Proof of Proposition 14.3

Define $N_t(y) = S(\rho_t(y))$. By Lemma 14.2 (ii) and Lemma 14.5 (ii), $(N_t(y), t \geq 0)$ is a non-negative martingale for each y in $\mathbb{R}^d$. Therefore

$$E[N_t(y)] = E[N_0(y)] = S(|y|) \text{ for all } t \geq 0.$$

So for any $\varepsilon > 0$,

$$P(|Z_t(y) - Z_t(0)| > \varepsilon) = P(N_t(y) > S(\varepsilon)) \leq S(\varepsilon)^{-1} E[N_t(y)] = S(\varepsilon)^{-1} S(y)$$

which tends to zero as y tends to zero in $\mathbb{R}^d$, by Lemma 14.5 (i). This proves that $Z_t(y)$ converges in probability to $Z_t(0)$ as y tends to zero. The proof that Q_t is fluid is the same as in Corollary 13.3.

The condition of Proposition 14.3 has other important consequences, as we shall now see.

LEMMA 14.6

Suppose that the covariance is isotropic, and (14.10) holds. Then for all $x \neq y$ in $\mathbb{R}^d$,

$$\lim_{t \rightarrow \infty} |Z_t(y) - Z_t(x)| = \begin{cases} 0 & \text{with probability } 1 - (S(|y-x|)/S(\infty)) \\ \infty & \text{with probability } S(|y-x|)/S(\infty) \end{cases}$$

where $S(\infty) = \lim_{r \rightarrow \infty} S(r)$.

Proof. There is no loss of generality in taking $x = 0$, since the covariance is spatially homogeneous. Let $N_t(y) = S(\rho_t(y))$; in the proof of Proposition 14.3 we saw that $N_t(y)$, $t \geq 0$ is a non-negative martingale. Therefore it has a limit $N_\infty(y)$ almost surely, and $E[N_\infty(y)] < \infty$.

On the other hand, it follows from Lemma 14.2 (i) and Lemma 14.5 (i) that $N_t(y)$, $t \geq 0$ is a one-dimensional diffusion process, with 0 as an absorbing state, and with generator G , where

$$Gh(u) = (S'(S^{-1}(u)))^2 V_L(S^{-1}(u)) h''(u), \quad h \text{ in } C_b^2(\mathbb{R}) \quad (14.11)$$

Let $\sigma(u) = S'(S^{-1}(u)) [2V_L(S^{-1}(u))]^{1/2}$, for $u > 0$. If $0 < a < b < S(\infty)$, then it follows from (14.6) and Lemma 14.5 (i) that for some $\delta > 0$, $\sigma(u) \geq \delta$ for all u in $[a, b]$. By a result of Gihman and Skorohod [10, p. 108], the first passage time for the process $(N_t(y), t \geq 0)$ to the boundaries of the interval $[a, b]$ is finite almost surely. It follows that the limit $N_\infty(y)$ is equal to 0 or $S(\infty)$ almost surely. There are two cases to consider:

Case I. $S(\infty) = \infty$.

The fact that $E[N_\infty(y)] < \infty$ implies that $P(N_\infty(y) = 0) = 1$. Hence for all y ,

$$\lim_{t \rightarrow \infty} |Z_t(y) - Z_t(0)| = 0 \text{ a.s.} \quad (14.12)$$

Case II. $S(\infty) < \infty$.

In this case the random variables $(N_t(y), t \geq 0)$ are uniformly integrable, and therefore converge to $N_\infty(y)$ in L^1 . Therefore

$$P(N_\infty(y) = S(\infty)) = E[N_\infty(y)]/S(\infty) = S(|y|)/S(\infty) \quad (14.13)$$

In this case

$$P(\lim_{t \rightarrow \infty} |Z_t(y) - Z_t(0)| = 0) = 1 - S(|y|)/S(\infty) \quad (14.14)$$

This completes the proof.

Now we go on to consider the question of coalescence of trajectories. For each x and y in $\mathbb{R}^d$, there is a well-defined random variable $\tau(x,y)$, which is a stopping-time with respect to the two-point process $(Z_t(x), Z_t(y), t \geq 0)$, such that

$$\tau(x,y) = \inf \{t \geq 0 : Z_t(x) = Z_t(y)\} \quad (14.15)$$

According to the construction of C-solutions in Section 12

$$Z_t(x,\omega) = Z_t(y,\omega) \quad \text{for all } t \geq \tau(x,y,\omega),$$

for those ω in Ω_{2d} (see section 12) for which $\tau(x,y,\omega) < \infty$.

PROPOSITION 14.1

Suppose (14.10) holds; (the covariance is still assumed to be isotropic). We have the following dichotomy:

$$(a) \quad \text{If } \int_0^a S(u)/[S'(u)V_L(u)]du < \infty \quad \text{for some } a > 0, \quad (14.16)$$

then (i) and (ii) hold:

$$(i) \quad P(\tau(x,y) < \infty) = 1 - [S(|x-y|)/S(\infty)] > 0$$

for all x, y in $\mathbf{R}^d$, where $S(\infty) = \lim_{r \rightarrow \infty} S(r)$ Thus if

$$S(\infty) = \infty, \quad \text{then } P(\tau(x,y) < \infty) = 1$$

(ii) For each x in $\mathbf{R}^d$,

$$\lim_{y \rightarrow x} E[e^{-\tau(x,y)}] = 1$$

(b) On the other hand if (14.16) is false, then

$$P(\tau(x,y) < \infty) = 0, \quad \text{all } x \neq y \text{ in } \mathbf{R}^d.$$

Remark. An example where (14.16) occurs in dimension 2 is given in section 16.

Proof. By spatial homogeneity, there is no loss of generality in taking $x = 0$. We will abbreviate $\tau(0,y)$ to $\tau(y)$, which may be regarded as the first time that $(N_t(y), t \geq 0)$ hits zero, where $N_t(y) = S(\rho_t(y))$ as in the proof of Proposition 14.3.

It was noted in the proof of Lemma 14.6 that $((N_t(y), t \geq 0))$ is a one-dimensional diffusion, whose generator is as in (14.11). According to Feller's criterion for accessibility, the boundary point 0 is accessible (i.e., passage time is finite with positive probability) if and only if

$$\int_0^1 t\{S'(S^{-1}(t))^2 V_L(S^{-1}(t))\}^{-1} dt < \infty \quad (14.17)$$

Changing variables to $u = S^{-1}(t)$, this is found to be equivalent to (14.16) with $a = S^{-1}(1)$. (Actually (14.6) and Lemma 14.5 show that if the integral in (14.16) is finite for any $a > 0$, then it is finite for all $a > 0$.) According to Ito and McKean [15, p. 125],

$$\lim_{y \rightarrow 0} \mathbb{E}[e^{-\tau(y)}] = \begin{cases} 1 & \text{if zero is accessible} \\ 0 & \text{if zero is inaccessible.} \end{cases} \quad (14.18)$$

Therefore if (14.16) is false, then this limit is zero, and so $P(\tau(y) = \infty) = 1$ for all $y \neq 0$; this proves (b). Assertion (ii) of (a) is contained in (14.18), so only assertion (i) of (a) remains to be proved.

Since $N_t(y) = 0$ for all $t \geq \tau(y)$, we see that the event $\{N_\infty(y) = S(\infty)\}$ is contained in the event $\{\tau(y) = \infty\}$. By (14.14), it follows that

$$P(\tau(y) = \infty) \geq P(N_\infty(y) = S(\infty)) = S(|y|)/S(\infty) \quad (14.19)$$

We must show that this inequality is in fact an equality in the case where zero is accessible. In other words, we wish to prove that if zero is accessible, then

$$P(\{\tau(y) = \infty\} \cap \{N_\infty(y) = 0\}) = 0. \quad (14.20)$$

Observe that for any $k \geq 1$,

$$\mathbb{E}[e^{-\tau(y)}] \leq e^{-k}P(\tau(y) > k) + P(\tau(y) \leq k) \leq e^{-k} + P(\tau(y) \leq k).$$

Therefore

$$P(\tau(y) > k) \leq e^{-k} + 1 - E[e^{-\tau(y)}].$$

According to (14.18), if zero is accessible, then

$$\lim_{y \rightarrow 0} P(\tau(y) > k) \leq e^{-k}, \quad \text{for all } k \geq 1. \quad (14.21)$$

The truth of (14.20) now follows from the following Lemma (with $Y_t = N_t(y)$, $\tau = \tau(y)$).

LEMMA 14.8

Suppose $(Y_t, t \geq 0)$ is a Markov diffusion process on $[0, \infty)$, and let τ denote the first passage time to 0, which is an absorbing state. Assume that

$$\lim_{k \rightarrow \infty} \lim_{u \downarrow 0} P_u(\tau > k) = 0 \quad (14.22)$$

where P_u means that $Y_0 = u$. Then for all $a > 0$

$$P_u(\{\tau = \infty\} \cap A) = 0, \quad (14.23)$$

where $A = \{Y_t \rightarrow 0 \text{ as } t \rightarrow \infty\}$.

Proof.

Let $\sigma(m)$ denote the first passage time to $1/m$. Clearly $P_a(\{\sigma(m) = \infty\} \cap A) = 0$ for all $m \geq 1/a$. So for $m \geq 1/a$,

$$P_a(\{\tau = \infty\} \cap A) = P_a(\{\tau = \infty\} \cap A | \sigma(m) < \infty) P_a(\sigma(m) < \infty) \leq P_a(\tau = \infty | \sigma(m) < \infty)$$

Using the strong Markov property, the law of the process started at time $\sigma(m)$ is $P_{1/m}$. Hence this is $= P_{1/m}(\tau=\infty)$, for all $m \geq 1/a$. Given $\varepsilon > 0$, (14.22) implies that there exists $k(\varepsilon) \geq 1$ such that

$$\lim_{u \downarrow 0} P_u(\tau > k(\varepsilon)) \leq \varepsilon/2.$$

Hence there exists $m(\varepsilon)$ such that

$$P_{1/m}(\tau > k(\varepsilon)) \leq \varepsilon \quad \text{if } m \geq m(\varepsilon).$$

Putting all this together, if $m \geq \max(m(\varepsilon), 1/a)$,

$$P_a(\{\tau=\infty\} \cap A) \leq P_{1/m}(\tau=\infty) \leq P_{1/m}(\tau > k(\varepsilon)) \leq \varepsilon.$$

Since ε is arbitrary, (14.23) follows.

15. The one-dimensional case

In [13], Harris treats the problem of constructing a pure stochastic flow in the one-dimensional case. Here the covariance function b is a function from $\mathbb{R}$ to $\mathbb{R}$, satisfying the assumptions of section 11. By methods entirely different to those of this paper, Harris is able to construct a pure stochastic flow in the following special cases.

(i) $b''(0)$ is finite, in which case each X_{st} is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$;

or (ii) $1 - b(x) \geq c|x|^{2-\epsilon}$ near 0, for some $c, \epsilon > 0$, in which case $X_{0t}(\mathbb{R})$ is a countable set with no limit points.

These categories exclude C^1 functions such as those for which

$$1 - b(x) = O(x^2 \ln(1/x)) \text{ as } x \downarrow 0. \quad (15.1)$$

The methods of the present paper remove these restrictions:

PROPOSITION 15.1

Let b be a real, continuous non-negative definite function from $\mathbb{R}$ to $\mathbb{R}$, whose spectral distribution is not of pure jump type, and which is Lipschitz outside each interval $(-c, c)$, for $c > 0$. Adopt the normalization $b(0) = 1$. Then there exists a pure stochastic flow $(X_{st}, 0 \leq s \leq t \leq \infty)$ with this covariance, such that $(x, w) \rightarrow X_{st}(x, w)$ is $m \otimes P$ -measurable (m is Lebesgue measure.)

Remark. The conditions of the Proposition are precisely those of section 11, as formulated in the one-dimensional case.

Proof. Construct a consistent system of finite-dimensional motions (Definition 10.1) as in Section 12. By the definition of C-solution in Section 12, it follows that whenever $t > 0$ and $z_1 < z_2 < \dots < z_k$ in $\mathbb{R}^1$, we have

$$Z_t(z_1, \omega) \leq Z_t(z_2, \omega) \leq \dots \leq Z_t(z_k, \omega).$$

This verifies assumption (S.2) in Example 10.3. Properties (S.1) and (S.3) of Example 10.3 follow from Corollary 12.2 (iii). Now apply Proposition 10.4 (ii). $\square$

In the one-dimensional case, the function $S(x) = x$ is a natural scale function for $(p_t(y), t \geq 0)$. The methods of Proposition 14.7 have the following Corollary, which is due to Harris, but is included for the sake of completeness:

COROLLARY 15.2 (Harris[13])

Suppose the dimension d is 1, and b is assumed to have the properties described in Proposition 15.1. Then for $x \neq y$ in $\mathbb{R}$, $P(\tau(x, y) < \infty) = 0$ or 1 according as $\int_{[0,1]} u du / (1 - b(u))$ is ∞ or $< \infty$.

16. An example in dimension two (due to T.E. Harris)

The work of Baxendale and Harris [3] on isotropic stochastic flows with C^4 covariances is devoted to situations where

$$B_L(r) = 1 - \frac{1}{2} \beta_L r^2 + o(r^4) \quad , \quad r \downarrow 0$$

$$B_N(r) = 1 - \frac{1}{2} \beta_N r^2 + o(r^4) \quad , \quad r \downarrow 0$$

(see (14.1), (14.2)). T.E. Harris has kindly allowed the author to reproduce here some unpublished work of his concerning a class of examples where $b(z)$ is not differentiable at zero, and

$$B_L(r) = 1 - c_L r^{\delta-1} + o(r^2) \quad , \quad r \downarrow 0$$

$$B_N(r) = 1 - c_N r^{\delta-1} + o(r^2) \quad , \quad r \downarrow 0$$

for some constant δ , $1 < \delta < 2$. For simplicity, we take the dimension d to be 2.

According to Yaglom [28, p. 305], an isotropic covariance function of potential (irrotational) type on $\mathbb{R}^2$ is specified by a spectral measure M on $[0, \infty)$, from which the functions B_L and B_N (see (14.1), (14.2)) are calculated as follows:

$$B_L(r) = \int_{[0, \infty)} [J_1(rs)/rs - J_2(rs)] M(ds) \quad (16.1)$$

$$B_N(r) = \int_{[0, \infty)} (J_1(rs)/rs) M(ds) \quad (16.2)$$

where $J_0, J_1, J_2, \dots$ are the Bessel functions. Well-known formulae (see Watson [27]) concerning Bessel functions include the following:

$$J_2(u) = (2 J_1(u)/u) - J_0(u), \quad J_1(u) = -J_0'(u),$$

$$J_0(u) + (J_0'(u)/u) + J_0''(u) = 0, \quad J_0(u) = \sum_{m=0}^{\infty} (-1)^m (u/2)^{2m} / (m!)^2.$$

Consequently the equations for B_L and B_N when $d = 2$ are:

$$B_L(r) = -\int_{|0, \infty)} J_0''(rs) M(ds) \quad (16.3)$$

$$B_N(r) = -\int_{|0, \infty)} (J_0'(rs)/rs) M(ds). \quad (16.4)$$

T.E. Harris suggests taking δ to be a constant, $1 < \delta < 2$, and

$$M(ds) = m(s)ds, \quad m(s) = \begin{cases} 0 & \text{for } 0 \leq s < 1 \\ 2(\delta-1)/s^\delta, & s \geq 1 \end{cases}. \quad (16.5)$$

Assumptions (C.1), (C.2) and (C.3) of Section 11 are satisfied for this choice of spectral measure, as we shall see.

The integrals in (16.3) and (16.4) are well-defined for this choice of $M(\cdot)$, since the functions $J_N(u)$, $n = 0, 1, 2, \dots$, and their derivatives are $O(u^{-1/2})$ as $u \rightarrow \infty$. Also observe that the normalizing constant in (16.5) makes $B_L(0) = 1 = B_N(0)$, since $J''(0) = -1/2$ and $\lim_{u \rightarrow 0} (J'(u)/u) = -1/2$; so (C.1) of Section 11 is satisfied. Thus

$$1 - B_L(r) = (\delta-1) \int_{|1, \infty)} s^{-\delta} (1 + 2 J_0''(rs)) ds \quad (16.6)$$

$$1 - B_N(r) = (\delta-1) \int_{[1,\infty)} s^{-\delta} [1 + 2(J'_0(rs)/rs)] ds.$$

To study the behavior of these functions near $r = 0$, change variables to $u = rs$. Then

$$1 - B_L(r) = (\delta-1) r^{\delta-1} \int_{[r,\infty)} (1 + 2 J''_0(u))/u^{\delta} du. \quad (16.7)$$

Since $J''_0(u) = -1/2 + 3u^2/16 + O(u^4)$, it follows that $1 + 2 J''_0(u) = O(u^2)$ near 0. Hence

$$\begin{aligned} 1 - B_L(r) &= (\delta-1)r^{\delta-1} \int_{[0,\infty)} (1 + 2 J''_0(u))/u^{\delta} du - (\delta-1)r^{\delta-1} \int_{[0,r)} (1 + 2 J''_0(u))/u^{\delta} du \\ 1 - B_L(r) &= C_L r^{\delta-1} + O(r^2) \quad \text{as } r \downarrow 0. \end{aligned} \quad (16.8)$$

Similarly $1 + 2(J'(u)/u) = O(u^2)$ near 0, and

$$1 - B_N(r) = C_N r^{\delta-1} + O(r^2) \quad \text{as } r \downarrow 0, \quad (16.9)$$

where

$$C_L = (\delta-1) \int_{[0,\infty)} (1 + 2 J''_0(u))/u^{\delta} du, \quad C_N = (\delta-1) \int_{[0,\infty)} [1 + 2(J'_0(u)/u)]/u^{\delta} du. \quad (16.10)$$

Weber's integral formula (see Watson [27, p. 391]) says

$$\int_{[0,\infty)} J_n(t)/t^{n-a+1} dt = \Gamma(a/2)/[2^{n-a+1} \Gamma(n - \frac{1}{2}a + 1)], \quad 0 < a < n + \frac{1}{2} \quad (16.11)$$

where $\Gamma(\cdot)$ denotes the Gamma function. Repeated use of (16.11) and the relations between the various Bessel functions yield

$$C_L > C_N = 2^{-\delta} \Gamma\left(\frac{2-\delta}{2}\right) / \Gamma\left(\frac{4+\delta}{2}\right) > 0 \quad (16.12)$$

using the assumption that $1 < \delta < 2$. We shall now calculate the derivative of the scale function as in (14.9):

$$S'(t) = \exp\left(-\int_1^t (1 - B_N(r)) / |r(1 - B_L(r))| dr\right)$$

since $d = 2$. By (16.8) and (16.9)

$$\begin{aligned} S'(t) &= \exp\left(-\int_1^t \frac{C_N r^{\delta-1} + O(r^2)}{C_L r^{\delta} + O(r^3)} dr\right) \quad \text{as } t \downarrow 0, \\ &= \exp\left(-\frac{C_N}{C_L} [\ln(t) + O(t^{3-\delta})]\right) \quad \text{as } t \downarrow 0 \\ &= \psi(t) t^{-C_N/C_L}, \quad 0 < t < 1 \end{aligned} \quad (16.13)$$

where $\psi(t)$ is $O(1)$. There exists a solution $S(t)$ to this differential equation such that $S(0+) = 0$ and

$$a_1 t^{1-C_N/C_L} \leq S(t) \leq a_2 t^{1-C_N/C_L}, \quad 0 \leq t \leq 1 \quad (16.14)$$

for some constants $0 < a_1 < a_2$; note that $1 - C_N/C_L > 0$ by (16.12). This proves that assumption (14.10) of Proposition 14.3 is satisfied for this isotropic covariance function.

The formulas for $B_L(r)$ and $B_N(r)$ given above, and the properties of the spectral measure M (see (16.5)) show that the assumptions of section 11 hold true. Thus from Proposition 14.3 and Theorem 9.1 we obtain:

COROLLARY 16.1

There exists a pure stochastic flow on the one-point compactification of $\mathbb{R}^2$ with the isotropic covariance specified by (16.3) - (16.5).

We shall now proceed to verify the coalescence criterion given in Proposition 14.7.

It follows from (16.8), (16.13) and (16.14) that there is a constant $a_3 > 0$ such that

$$\int_0^1 S(t)/[S'(t)(1 - B_L(t))] dt \leq a_3 \int_0^1 t^{2-\delta} dt < \infty. \quad (16.15)$$

Thus condition (14.16) of Proposition 14.7 is satisfied, and we obtain:

PROPOSITION 16.2

For the isotropic covariance specified by (16.3)-(16.5), the stopping time $\tau(x,y)$ (see (14.15)) is finite almost surely, for each x and y in $\mathbb{R}^2$; in other words the trajectories starting from x and y respectively are almost certain to coalesce in finite time.

Proof.

It follows from Proposition 14.7 that $P(\tau(x,y) = \infty) = 1 - (S(|x-y|)/S(\infty))$.

It suffices to prove that $S(\infty) = \infty$.

According to (14.9), with $d=2$,

$$S'(r) = \exp\left\{-\int_1^r [1 - B_N(s)]/s[1 - B_L(s)]ds\right\}.$$

Using (16.7) and the corresponding expression for B_N , we find that:

$$\begin{aligned} 1 - B_N(s) &= 1 + O(s^{-3/2}) \quad \text{as } s \uparrow \infty, \\ 1 - B_L(s) &= 1 - B_N(s) + 2(\delta-1)s^{-1}J_1(s), \end{aligned}$$

and furthermore

$$J_1(s) = \left(\frac{2}{\pi s}\right)^{1/2} \left[\cos\left(s - \frac{3\pi}{4}\right) + O(s^{-1})\right], \quad \text{as } s \uparrow \infty.$$

Hence

$$[1 - B_N(s)]/s[1 - B_L(s)] = s^{-1} + O(s^{-5/2}), \quad \text{as } s \uparrow \infty.$$

It follows that there exists a constant K such that for all $r \geq 1$,

$$\int_1^r [1 - B_N(s)]/s[1 - B_L(s)]ds \leq \log r + K.$$

Hence $S'(r) \geq e^{-K}r^{-1}$, and $S(r) - S(1) \geq e^{-K}\log r$. Therefore $S(\infty) = \infty$ as desired.

Acknowledgement. The author thanks Willard Miller (Minnesota) for his assistance in the preceding proof.

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Part VI. The Geometry of Coalescence

In Part V we proved that there exist in dimension 2 (and perhaps higher dimensions) isotropic covariances with the following two properties:

1. There exists a pure stochastic flow with this covariance.
2. Trajectories starting from two distinct points have a positive probability of coalescing in finite time.

Thanks to the special properties of a pure stochastic flow, it is now possible to study the action of these 'coalescing' flows on the whole space.

17. Coalescence times and the coalescent set process

Let $(X_{st}, 0 \leq s \leq t < \infty)$ be a pure stochastic flow (Definition 3.1) on a compact metrizable space M , defined on a probability space $(\Omega, \mathcal{F}, P)$.

Definition 17.1. For each x and y in M and each ω in Ω , define

$$(17.1) \quad v(x, y, \omega) = \inf\{t : X_{0t}(x, \omega) = X_{0t}(y, \omega)\}.$$

Define a process $(J_t(x), t \geq 0)$ with values in the set of subsets of M , by

$$(17.2) \quad J_t(x, \omega) = \{z : X_{ot}(x, \omega) = X_{ot}(z, \omega)\}.$$

We call $v(x, y, \omega)$ the **coalescence time** of the trajectories $t \rightarrow X_{ot}(x, \omega)$ and $t \rightarrow X_{ot}(y, \omega)$, and we call $(J_t(x), t \geq 0)$ the **coalescent set process** for x .

LEMMA 17.2.

- (i) $X_{ot}(x, \omega) = X_{ot}(y, \omega)$ for all $t \geq v(x, y, \omega)$, for all x, y and ω .
- (ii) $J_t(x, \omega) = \{z : v(x, z, \omega) \leq t\}$. Consequently $J_s(s, \omega)$ is contained in $J_t(x, \omega)$ for $s \leq t$, and $\lim_{t \rightarrow \infty} J_t(x, \omega) = \{z : v(x, z, \omega) < \infty\}$.

Proof. Abbreviate $v(x, y, \omega)$ to v . Then for $t \geq v$,

$$X_{ot}(x, \omega) = X_{vt}(X_{ov}(x, \omega), \omega) = X_{vt}(X_{ov}(y, \omega), \omega) = X_{ot}(y, \omega)$$

by (3.2'). Part (ii) follows immediately. $\square$

LEMMA 17.3

For each x, y in M , each subset E of M , and each $t \geq 0$, the sets $\{\omega : v(x, y, \omega) \leq t\}$, $\{\omega : v(x, y, \omega) \leq t \text{ for all } y \in E\}$, and $\{\omega : v(x, y, \omega) < \infty\}$ are in $\mathcal{E}$ (the σ -algebra on Ω).

Proof. The first assertion is a special case of the second, taking $E = \{y\}$.

The set $\{f : f(x) = f(y)\}$ is closed in Γ , by the reasoning of Remark 4.2 .

Now by Lemma 17.2 (i),

$$\begin{aligned} \{\omega : v(x, y, \omega) \leq t \text{ for all } y \in E\} &= \bigcap_{y \in E} \{\omega : X_{0t}(x, \omega) = X_{0t}(y, \omega)\} \\ &= X_{0t}^{-1}(\bigcap_{y \in E} \{f : f(x) = f(y)\}). \end{aligned}$$

which belongs to $\mathcal{E}$ since the intersection of closed sets is closed, and

$X_{0t} : \Omega \rightarrow \Gamma$ is $(\mathcal{E}, \mathcal{B}(\Gamma))$ -measurable.

As for the second assertion,

$$\{\omega : v(x, y, \omega) < \infty\} = \bigcup_{n \geq 1} \{\omega : v(x, y, \omega) \leq n\}. \quad \square$$

COROLLARY 17.4

The following function is well-defined for all $t \geq 0$ and all x, y in M .

$$(17.3) \quad q(t, x, y) = P(\{\omega: v(x, y, \omega) \leq t\}) = P(\{\omega: X_{0t}(x, \omega) = X_{0t}(y, \omega)\})$$

(the equality of the last two expressions follows from Lemma 17.2 (i)).

LEMMA 17.5

Suppose that the following condition holds for each y in T :

$$(17.4) \quad q(t, y, z) \rightarrow 1 \text{ as } z \rightarrow y, \text{ for each } t > 0.$$

Then for every countable dense subset D of T , every y in T , and every

$$0 \leq s < t,$$

$$(17.5) \quad P(X_{st}(y) \in X_{st}(D)) = 1.$$

Proof. Let $D = \{x_1, x_2, \dots\}$ be any countable dense subset of T , and let

$t > 0$. Then for any y in T ,

$$\begin{aligned}
 P(\cap_n \{v(y, x_n) > t\}) &\leq \inf_n P(v(y, x_n) > t) \\
 &= \inf_n (1 - q(t, y, x_n)) = 0
 \end{aligned}$$

by (17.4), since there exists a subsequence of the $\{x_n\}$ which converges to y . In other words

$$P(\cup_n \{X_{0t}(y) = X_{0t}(x_n)\}) = 1.$$

The same result holds on replacing X_{0t} by $X_{s, s+t}$, by time homogeneity. $\square$

COROLLARY 17.6

Suppose that $b(\cdot)$ is an isotropic covariance on $\mathbb{R}^d$ satisfying (14.10) and (14.16), and let $(X_{st}, 0 \leq s \leq t < \infty)$ be a pure stochastic flow with this covariance (which exists by Proposition 14.3 and Theorem 6.1). Then for every countable dense subset D of $\mathbb{R}^d$, every y in $\mathbb{R}^d$, and every $0 \leq s < t$, (17.5) holds.

Proof. It suffices to show that (17.4) holds in this case. This follows

from the reasoning of Harris [13, p. 199]. □

It seems to be difficult to discuss properties of the range of X_{st} without any spatial measurability properties. Therefore we shall use henceforward one or other of the following hypotheses:

(F.1) (Spatial stochastic continuity). The stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ is constructed as in Theorem 6.1 from a convolution semigroup $\{Q_t, t \geq 0\}$ of probability measures satisfying (4.4).

(F.2) (Fluidity). The stochastic flow $(X_{st}, 0 \leq s \leq t < \infty)$ is constructed as in Theorem 9.1 from a convolution semigroup $\{Q_t, t \geq 0\}$ of FRPM's (see Definition 8.1) with respect to μ .

PROPOSITION 17.7.

(i) *If (F.1) holds, then the mapping $(x,y) \rightarrow q(t,x,y)$ (see (17.3)) is $\mu \otimes \mu$ measurable on M^2 , and the mapping $y \rightarrow q(t,x,y)$ is μ -measurable on M for each x in M , for every Radon measure μ on $\mathbb{B}(M)$ with $\mu(\{\infty\})=0$, and every $t \geq 0$.*

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(ii) If (F.2) holds, then the mapping $(x,y,\omega) \rightarrow v(x,y,\omega)$ is

$\mu \otimes \mu \otimes P$ -measurable.

Proof (i) Fix $t \geq 0$, and let Z be the canonical Γ -valued random field on $(\Gamma, \mathcal{B}(\Gamma), Q_t)$, as in (4.1). By composability condition (4.4), the mapping $y \rightarrow Z(y)$ is stochastically continuous on T . Hence by Doob's theorem on the existence of measurable versions (see, for example, M.M. Rao [24], p. 180), given μ as above there exists a version Y_t of Z which is $\mu \otimes Q_t$ -measurable; in other words,

(17.6) $(x,f) \rightarrow Y_t(x,f)$ is $\mu \otimes Q_t$ - measurable, and

(17.7) $Q_t(\{f : Y_t(x,f) = Z(x,f)\}) = 1$ for all x in M .

Define $\lambda_t : M^2 \times \Gamma \rightarrow M^2$ by:

(17.8) $\lambda_t(x,y,f) = (Y_t(x,f), Y_t(y,f))$.

It follows from (17.6) that λ_t is $\mu \otimes \mu \otimes Q_t$ - measurable. Now

$$q(t,x,y) = P(\{\omega : X_{ot}(x,\omega) = X_{ot}(y,\omega)\}) \text{ by (17.3),}$$

$$= Q_t(\{f : Z(x,f) = Z(y,f)\}) \text{ by Theorem 6.1,}$$

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$$= Q_t(\{f: Y_t(x,f) = Y_t(y,f)\}) \text{ by (17.7),}$$

$$= Q_t(\{f: \lambda_t(x,y,f) \in \Lambda\}) \text{ by (17.8),}$$

where Λ is the diagonal in M^2 , which is in $\mathcal{B}(M^2)$. The measurability of λ_t now shows that $(x,y) \rightarrow q(t,x,y)$ is $\mu \otimes \mu$ -measurable. A similar argument proves the assertion about $y \rightarrow q(t,x,y)$.

(ii) Assume (F.2), and fix $t \geq 0$. According to Theorem 9.1 (c), the map $(y,\omega) \rightarrow X_{ot}(y,\omega)$ is $\mu \otimes P$ - measurable. Therefore $\gamma_t: M^2 \times \Omega \rightarrow M^2$ is $\mu \otimes \mu \otimes P$ - measurable, where

$$\gamma_t(x,y,\omega) = (X_{ot}(x,\omega), X_{ot}(y,\omega))$$

If Λ denotes the diagonal in M^2 , then (17.3) shows that

$$\{(x,y,\omega): v(x,y,\omega) \leq t\} = \gamma_t^{-1}(\Lambda)$$

which is $\mu \otimes \mu \otimes P$ - measurable since Λ is in $\mathcal{B}(M^2)$. $\square$

COROLLARY 17.8

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If (F.2) holds, then for each $t \geq 0$, $J_t(x, \omega)$ (see (17.2)) is a μ -measurable subset of M for $[\mu \otimes P]$ - almost all (x, ω) ; moreover

$$(17.9) \quad \mathbb{E}[\mu(J_t(x))] = \int_M q(t, x, y) \mu(dy).$$

Proof. The measurability assertion follows from Lemma 17.2 (ii) and Proposition 17.7. To prove (17.9), note that $\{(y, \omega) : v(x, y, \omega) \leq t\}$ is $\mu \otimes P$ -measurable by Proposition 17.7; then integrate using Fubini's theorem. $\square$

Remarks 17.9.

1. The author is indebted to Richard Durrett (Cornell) for (17.9). This equation is interesting because it proves that if, for some fixed $t > 0$ and $\delta > 0$, $q(t, x, y) \geq \varepsilon > 0$ whenever $d(x, y) \leq \delta$, then $\mu(J_t(x)) > 0$ with positive probability. The major open question is: under what assumptions does $J_t(x, \omega)$ contain a neighborhood of x for (almost) all ω ?
2. By Proposition 17.6, the right side of (17.9) is well-defined whenever (F.1) holds. However $J_t(x, \omega)$ is perhaps non-measurable with respect to μ .
3. Even if (F.2) holds, we cannot infer from (17.4) that $\mu(X_{st}(T)) = 0$ if

$0 \leq s < t$. The difficulty is as follows. Let $D = \{x_1, x_2, \dots\}$ be a countable dense subset of t , and define

$$H_{st}(\omega) = \{y : X_{st}(y, \omega) \neq X_{st}(x_n, \omega) \text{ for all } n\}$$

We have seen (Lemma 17.5) that $\mu(H_{st}(\omega)) = 0$ for almost all ω , but there is no guarantee that $X_{st}(H_{st})$ is even a μ -measurable set; it may have positive μ -measure, as the following example shows.

EXAMPLE 17.10

We begin by constructing a continuous, monotone function $h: [0,1] \rightarrow [0,1]$ which maps a Lebesgue nullset on to a set of positive Lebesgue measure. Define h on the interval $(1/3, 2/3)$ to be the unique linear, increasing, one-to-one function which maps $(1/3, 2/3)$ on to $(2/5, 3/5)$. Let h map $(1/9, 2/9)$ onto $(4/25, 6/25)$, and $(7/9, 8/9)$ onto $(19/25, 21/25)$ in the same way. Continuing in this way, h is defined on $[0,1] - C_3$, where C_3 denotes the usual Cantor set in $[0,1]$. The image of $[0,1] - C_3$ under h is $[0,1] - C_5$, where C_5 is the Cantor set obtained by removing middle fifths instead of middle thirds. Since $m(C_3) = 0$ (m denotes Lebesgue measure), and since h is monotone on $[0,1] - C_3$, there is a unique way of

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defining $h(x)$ for x in C_3 , such that h is monotonic on $[0,1]$. Evidently $h(C_3) = C_5$, so $m(h(C_3)) = m(C_5) = 2/3$. So h maps a Lebesgue nullset onto a set of positive Lebesgue measure, as desired.

We shall now define a Lebesgue measurable function $\psi:[0,1] \rightarrow [0,1]$ as follows. Let $\psi(x) = h(x)$ if x is in C_3 , and

$$\begin{aligned}\psi(x) &= 1/2 \text{ for } 1/3 < x < 2/3 \\ &= 1/5 \text{ for } 1/9 < x < 2/9 \\ &= 4/5 \text{ for } 7/9 < x < 8/9 \\ &= 2/25 \text{ for } 1/27 < x < 2/27 \text{ etc.}\end{aligned}$$

Thus the image of $[0,1] - C_3$ under ψ is a countable set, and the image of C_3 under ψ is C_5 .

Let $M = [0,1]$ and $\Gamma = M^M$. We shall define an FRPM Q on $B(\Gamma)$ as follows. Let Y be a Uniform $(0,1)$ random variable defined on some probability space $(\Omega_0, \mathcal{E}_0, P_0)$. Suppose $x_1, \dots, x_n$ are points in $[0,1]$ and $G_1, \dots, G_n$ are open sets in $[0,1]$. If $A = \{f \in \Gamma: f(x_1) \in G_1, \dots, f(x_n) \in G_n\}$, let

$$Q(A) = P_0((\psi(x_1) + Y) \in G_1, \dots, (\psi(x_n) + Y) \in G_n).$$

where the addition is modulo 1. This defines a unique probability measure on $\mathbb{B}_0(\Gamma)$, which extends to a unique Radon probability measure Q on $\mathbb{B}(\Gamma)$, by Appendix A.4. For all x outside C_3 , which is a Lebesgue nullset, $Q(F_x) = 1$ (see (7.2)); hence Q has almost no fixed points of discontinuity. Also the distribution of $\psi(x) + Y$ is absolutely continuous with respect to Lebesgue measure, for each x . Hence Q is an FRPM (Definition 8.1).

Finally we construct a convolution semigroup $\{Q_t, t \geq 0\}$ of FRPM's on $\mathbb{B}(\Gamma)$ by the method of Example 9.3. Let $(X_{st}, 0 \leq s \leq t < \infty)$ be the corresponding pure stochastic flow defined on $(\Omega, \mathcal{E}, P)$ as in Remark 9.5. Then for $0 \leq s < t$,

$$P(m(X_{st}(C_3)) = 2/3 \mid N(t) - N(s) = 1) = 1,$$

where $(N(t), t \geq 0)$ is a Poisson process as in Example 9.3. Since $P(N(t) - N(s) = 1) > 0$, this shows that $P(m(X_{st}(C_3)) > 0) > 0$, for $0 \leq s < t$, although $m(C_3) = 0$. However for y in $[0,1] - C_3$, there exists $\varepsilon > 0$ such that $\psi(x) = \psi(y)$ whenever $y - \varepsilon < x < y + \varepsilon$. Consequently (17.7) holds.

This example shows that although almost every point in M has coalesced almost surely to the image of a fixed countable set by time

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$t > 0$, the image $X_{ot}(M)$ may have positive μ -measure almost surely.

Appendix A. Baire sets, Borel sets, and Radon probability measures.

Let Y, Y' denote compact Hausdorff spaces, let M denote a compact Hausdorff space with a countable base, and let $\Gamma = M^M$ with the product topology.

Definitions. The **Borel sigma-algebra** $\mathbb{B}(Y)$ is the sigma-algebra generated by the open sets in Y ; the **Baire sigma-algebra** $\mathbb{B}_0(Y)$ is the sigma-algebra generated by the compact G_δ sets. (A set is a G_δ if it is the intersection of a decreasing sequence of open sets.) Elements of $\mathbb{B}(Y)$ and $\mathbb{B}_0(Y)$ are referred to as Borel sets and Baire sets, respectively. A real-valued function on Y is called Borel (resp. Baire) measurable if the inverse image of every Borel set in $\mathbb{R}$ is Borel (resp. Baire).

Suppose $(E, \underline{E}, \mu)$ is a measure space and $(F, \underline{F})$ is a measurable space; a mapping $\sigma: E \rightarrow F$ is called μ -measurable if $\sigma^{-1}(A)$ belongs to the μ -completion of $\underline{E}$ for every A in $\underline{F}$.

A measure Q on $\mathbb{B}(Y)$ is called Radon if $Q(A) = \sup \{Q(K): K \text{ compact, } K \text{ contained in } A\}$ for every A in $\mathbb{B}(Y)$, and if every point has a

neighbourhood of finite Q - measure.

Let $\mathbb{B}_0^r(Y) = \mathbb{B}_0(Y) \times \dots \times \mathbb{B}_0(Y)$, (r factors), $= \mathbb{B}_0(Y^r)$

$\mathbb{B}^r(Y) = \mathbb{B}(Y) \times \dots \times \mathbb{B}(Y)$, (r factors), ($\neq \mathbb{B}(Y^r)$ in general).

Properties (from Nelson [21] and Halmos [11]).

- A.1. $\mathbb{B}_0(M) = \mathbb{B}(M)$, but generally $\mathbb{B}_0(\Gamma)$ is strictly contained in $\mathbb{B}(\Gamma)$.
- A.2. $\mathbb{B}_0(\Gamma)$ is generated by the sets of the form $\{f \in \Gamma : f(x) \in G\}$, for x in M and G open in M .
- A.3. Let I be the collection of all finite subsets of M , and suppose P_α is a probability measure on $\mathbb{B}(M^\alpha)$ for each α in I . If the family $\{P_\alpha, \alpha \text{ in } I\}$ is consistent, then there is a unique probability measure P_0 on $\mathbb{B}_0(\Gamma)$ such that $P_\alpha = P_0 \cdot \pi_\alpha^{-1}$, for all α , where $\pi_\alpha : \Gamma \rightarrow M^\alpha$ is the canonical projection.
- A.4. For each probability measure P_0 on $\mathbb{B}_0(\Gamma)$, there is a unique Radon probability measure P_1 on $\mathbb{B}(\Gamma)$ such that $P_1(E) = P_0(E)$ for every Baire set E .
- A.5. (Combine 3 and 4). Each consistent family $\{P_\alpha, \alpha \text{ in } I\}$ as in 3. induces a unique Radon probability measure P_1 on $\mathbb{B}(\Gamma)$.
- A.6. $\mathbb{B}_0(Y \times Y') = \mathbb{B}_0(Y) \times \mathbb{B}_0(Y')$. However, in general $\mathbb{B}(Y \times Y')$ strictly

includes $\mathbb{B}(Y) \times \mathbb{B}(Y')$.

A.7. If P is a Radon probability measure on $\mathbb{B}(\Gamma)$, then for each

P -measurable set E , there exists a Baire set E_0 such that $E = E_0$

a.e. $[P]$. In particular, $\mathbb{B}(\Gamma)$ is contained in the P -completion of $\mathbb{B}_0(\Gamma)$.

A.8. Suppose Q is a Radon probability measure on $\mathbb{B}(Y)$, $\underline{G}$ is a

sub-sigma-algebra of $\mathbb{B}(Y)$, $\underline{G}_1$ is the Q -completion of $\underline{G}$, and

$\chi: Y \rightarrow \Gamma$ is a $(\underline{G}_1, \mathbb{B}_0(\Gamma))$ -measurable map. Let P_0 be the

measure $Q \circ \chi^{-1}$ on $\mathbb{B}_0(\Gamma)$, and let P be the unique Radon probability

measure on $\mathbb{B}(\Gamma)$ which is an extension of P_0 (see A.4). Then χ is

$(\underline{G}_1, \mathbb{B}_1(\Gamma))$ -measurable, where $\mathbb{B}_1(\Gamma)$ is the P -completion of $\mathbb{B}(\Gamma)$.

(Proof. For any E in $\mathbb{B}_1(\Omega)$, there exist E_0 and E_1 in $\mathbb{B}_0(\Gamma)$ with

$E_0 \subseteq E \subseteq E_1$ and $P_0(E_1 - E_0) = Q(\chi^{-1}(E_1) - \chi^{-1}(E_0)) = 0$, by A.7).

A.9. Suppose Q and Q' are Radon probability measures on $\mathbb{B}(Y)$ and $\mathbb{B}(Y')$

respectively. There is a unique extension of $Q \otimes Q'$ from $\mathbb{B}(Y) \times \mathbb{B}(Y')$

to a Radon probability measure, also denoted $Q \otimes Q'$, on $\mathbb{B}(Y \otimes Y')$.

(Proof: $Q \otimes Q'$ induces a measure on $\mathbb{B}_0(Y) \times \mathbb{B}_0(Y')$, which is the same

as $\mathbb{B}_0(Y \times Y')$ by A.6. By A.4, this measure extends uniquely to a Radon

probability measure on $\mathbb{B}(Y \times Y')$. It follows that the completions of

$\mathbb{B}(Y \times Y')$ and $\mathbb{B}(Y) \times \mathbb{B}(Y')$ are the same.)

A.10. A Baire regular probability measure on Y means a probability

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measure P on $\mathcal{B}_0(Y)$ such that $P(G) = \sup \{P(C) : C \text{ a compact Baire set contained in } G\}$, for all Baire sets G .

Appendix B. Projective systems of topological measure spaces

The construction of a pure stochastic flow requires the subtle use of projective limits. The terminology we use here is from Bochner [4] and Rao [24].

Definition B.1. Let $(I, \leq)$ be a directed set, and for each α in I let S_α be a Hausdorff topological space with Borel σ -algebra $\mathcal{B}(S_\alpha)$ on which a Radon probability measure P_α is defined. Suppose that for $\alpha \leq \beta$, there is a map $g_{\alpha\beta} : S_\beta \rightarrow S_\alpha$ with the following properties:

- (i) $g_{\alpha\beta}$ is continuous,
- (ii) $g_{\alpha\alpha}$ is the identity map,
- (iii) $g_{\alpha\beta}(S_\beta) = S_\alpha$,
- (iv) $g_{\alpha\gamma} = g_{\alpha\beta} \circ g_{\beta\gamma}$ whenever $\alpha \leq \beta \leq \gamma$,
- (v) $P_\beta(g_{\alpha\beta}^{-1}(C)) = P_\alpha(C)$ whenever $\alpha \leq \beta$ and C is in $\mathcal{B}(S_\alpha)$.

Then the system $\{(S_\alpha, \mathbb{B}(S_\alpha), P_\alpha, g_{\alpha\beta})_{\alpha \leq \beta} : \alpha, \beta \text{ in } I\}$ is called a **projective system of Radon topological probability spaces**. The **projective limit** of the spaces $(S_\alpha, \alpha \text{ in } I)$ with respect to $(g_{\alpha\beta})$ means the subset S of $\prod_\alpha S_\alpha$ consisting of the elements $\omega = \{\omega_\alpha : \alpha \text{ in } I\}$ such that for $\alpha \leq \beta$, it is true that $g_{\alpha\beta}(\omega_\beta) = \omega_\alpha$.

Define $g_\alpha : S \rightarrow S_\alpha$ by $g_\alpha(\omega) = \omega_\alpha$; then $g_\alpha = g_{\alpha\beta} \circ g_\beta$ for $\alpha \leq \beta$. Let $\mathcal{S}$ be the σ -algebra in S generated by $\{g_\alpha^{-1}(C), C \text{ in } \mathbb{B}(S_\alpha), \alpha \text{ in } I\}$.

Such a projective system is said to have the **sequential maximality property** if for every sequence $\alpha(1) \leq \alpha(2) \leq \dots$ in I and every collection of elements $\omega_{\alpha(i)}$ of $S_{\alpha(i)}$ such that $g_{\alpha(i), \alpha(i+1)}(\omega_{\alpha(i+1)}) = \omega_{\alpha(i)}$ for each i , there exists ω in S (the projective limit) such that $g_{\alpha(i)}(\omega) = \omega_{\alpha(i)}$ for every i .

PROPOSITION B.2 (Bochner [4])

Let $\{(S_\alpha, \mathbb{B}(S_\alpha), P_\alpha, g_{\alpha\beta})_{\alpha \leq \beta} : \alpha, \beta \text{ in } I\}$ be a projective system of Radon topological probability spaces with the sequential maximality property. Then there exists a probability measure P on $(S, \mathcal{S})$ (necessarily unique), such that

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$$P_\alpha(C) = P(g_\alpha^{-1}(C)), \alpha \text{ in } I, C \text{ in } \underline{B}(S_\alpha).$$

We call $(S, \mathcal{S}, P)$ the **projective limit** of this projective system of Radon probability measure spaces.

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