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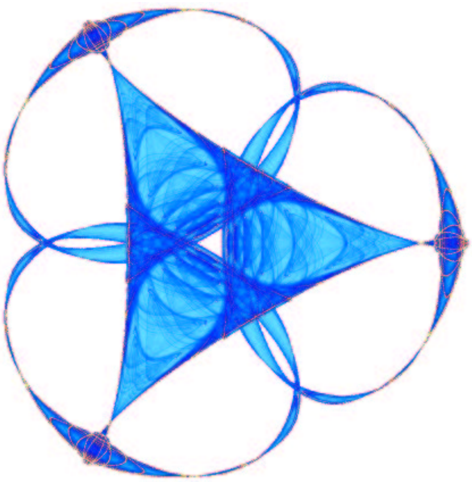

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AREA DENSITY AND REGULARITY FOR SOAP FILM-LIKE SURFACES SPANNING GRAPHS

ROBERT GULLIVER AND SUMIO YAMADA

ABSTRACT. For a given boundary Γ consisting of arcs and vertices, with two or more arcs meeting at each vertex, we treat the problem of estimating the area density of a soap film-like surface Σ spanning Γ. Σ is assumed to minimize area, or more generally, to be strongly stationary for area with respect to Γ. We introduce a notion of total curvature $\mathcal{C}_{\rm tot}(\Gamma)$ for such *graphs*, or nets, Γ. When the ambient manifold M^n has non-positive sectional curvatures, we show that 2π times the area density of Σ at any point is less than or equal to $\mathcal{C}_{\rm tot}(\Gamma)$. For $n = 3$, these density estimates imply, for example, that if $\mathcal{C}_{\rm tot}(\Gamma) \leq 22.9\pi$, then the only possible singularities of a piecewise smooth $(\mathbf{M}, 0, \delta)$-minimizing set Σ are curves, along which three smooth sheets of Σ meet with equal angles of 120°. We also extend these results to the case where M has a positive or negative upper bound on sectional curvatures.

1. INTRODUCTION

The investigation of minimal surfaces has proved extremely fruitful in a wide range of topics in geometry. One of the essential breakthroughs in the subject is the solution of the Plateau problem by Douglas and by Radó, that is, the construction of a disc type minimal surface spanned by a Jordan curve Γ in $\mathbf{R}^n$ [D1], [R]. Plateau's original motivation was, in part, to study the geometry of soap films spanned by variously shaped wires. In particular, it is natural to want to generalize the boundary condition imposed by Douglas and Radó that the wire Γ spanning the surface be a Jordan curve, or a disjoint union of Jordan curves (*cf.* [D2], where Σ is a branched immersion of higher topological type). In this paper, we will introduce a class of surfaces Σ in an ambient manifold M, having a piecewise smooth boundary Γ which is homeomorphic to a *graph*, that is, a a finite 1-dimensional polyhedron (sometimes called a "net"). Each surface is stationary for area under variations induced by one-parameter families of diffeomorphisms of

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the ambient manifold and fixing Γ. This setting allows us to consider surfaces whose induced topology is not locally Euclidean, such as the singular surfaces which may readily be observed in soap film experiments. The main theorems of this paper provide descriptions of the possible singularities and self-intersections of those minimal surfaces in terms of the geometry of the boundary set Γ. Indeed, intuition would indicate that *simple boundaries can span only simple minimal surfaces.*

In a Riemannian manifold M^n, we shall consider an embedded graph Γ which is a union of arcs a_k meeting at vertices q_j, each of which has valence at least two. The *valence* of a vertex q is the number of times q occurs as an endpoint among all of the 1-simplices a_k. Each 1-simplex a_k is assumed to be C^2, and to meet its end points with C^1 smoothness; thus there is a well-defined tangent vector T_k to each 1-simplex a_k at a vertex, pointing into a_k. At a vertex q_j of valence d, we consider the *contribution to total curvature* at q_j:

$$\mathrm{tc}(q_j) := \sup_{e \in T_{q_j}M} \left\{ \sum_{\ell=1}^{d} \left(\frac{\pi}{2} - \beta_j^\ell(e) \right) \right\} \tag{1}$$

where $\beta_j^\ell = \beta_j^\ell(e) \in [0,\pi]$ is the angle between the tangent vector T_ℓ to a_ℓ at q_j and the vector e. We define the *total curvature* of Γ as

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_{\Gamma^{\mathrm{reg}}} |\vec{k}|\, ds + \sum \{\mathrm{tc}(q) : q \text{ a vertex of } \Gamma\}. \tag{2}$$

where $\vec{k}$ is the geodesic curvature vector of a_i as a curve in M^n, and $\Gamma^{\mathrm{reg}} = \Gamma\backslash\{\text{vertices}\}$. It should be noted that our definition of total curvature coincides with the standard definition in the case when Γ is a piecewise smooth Jordan curve: the integral of the norm of geodesic curvature vector plus the sum of the exterior angles at the vertices. Namely, in that case, every vertex q of the graph Γ is of valence two, and the supremum in equation (1) is assumed at vectors e lying in the smaller angle between the tangent vectors T_1 and T_2 to Γ.

The type of surface we will consider in this paper is the immersed image Σ of a finite union of C^2-smooth open two-dimensional manifolds Σ_i with compact closure, C^1 up to the piecewise C^1 boundary $\partial\Sigma_i$. We further impose that the graph Γ is a subset of the 1-skeleton S of Σ. S is defined as the union of images of the piecewise C^1 curves $\partial\Sigma_i$. The class of such surfaces Σ will be denoted by $\mathcal{S}_\Gamma$.

Recall that the *density* of Σ at a point $p \in M$ is

$$\Theta_\Sigma(p) := \lim_{\varepsilon \to 0} \frac{\mathrm{Area}(\Sigma \cap B_\varepsilon(p))}{\pi \varepsilon^2}.$$

Note that given a surface Σ in $\mathcal{S}_\Gamma$, the density $\Theta_\Sigma(p)$ is a well defined, upper semi-continuous function of $p \in M$. Moreover, for Σ in the class $\mathcal{S}_\Gamma$, we may compute

$$\Theta_\Sigma(p) = \lim_{\varepsilon \to 0} \frac{\mathrm{Length}(\Sigma \cap \partial B_\varepsilon(p))}{2\pi \varepsilon}.$$

Note that in contrast to the special case of an immersed PL manifold Σ, there is no local topological criterion to distinguish the boundary of Σ as a specific subset of S. Further, the definition of $\partial\Sigma$ motivated by Stokes' Theorem [Fed] is not appropriate, since Σ is not orientable in general, not even modulo a globally defined integer. For these reasons, we shall define the boundary of Σ in variational terms, as follows. A surface Σ in $\mathcal{S}_\Gamma$ is said to be *strongly stationary with respect to* Γ if the first variation of the area of the surface is at most equal to the integral over Γ of the length of the component of the variation vector field normal to Γ [EWW].

In our future paper [GY2], we will obtain analogous results for surfaces which need not be of the class $\mathcal{S}_\Gamma$, but may be rectifiable varifolds which are strongly stationary with respect to Γ.

We can now state the main density estimate for the case when the ambient space is Euclidean (see Corollary 5 below):

Density Estimate: *Let Σ in the class $\mathcal{S}_\Gamma$ be a strongly stationary surface in $\mathbf{R}^n$ with respect to the graph Γ. Then*

$$2\pi\Theta_\Sigma(p) \leq \mathcal{C}_{\mathrm{tot}}(\Gamma).$$

This estimate is a consequence of two inequalities, the first being the comparison of density of Σ and of the cone $C_p(\Gamma)$. Here, and in the remainder of this paper, for a point $p \in \mathbf{R}^n$ and a set $\Gamma_0 \subset \mathbf{R}^n$, we write the *cone over* Γ_0 as

$$C_p(\Gamma_0) := \{p + t(x - p) : x \in \Gamma_0, 0 \leq t \leq 1\}.$$

In section 6, where Euclidean space $\mathbf{R}^n$ is replaced more generally by a strongly convex Riemannian manifold, $C_p(\Gamma_0)$ will denote the *geodesic* *cone* over Γ_0.

Theorem 1: *Given a strongly stationary surface* Σ *in* $\mathcal{S}_\Gamma$, *and a point* p *in* $\Sigma\backslash\Gamma$, *let* $C_p(\Gamma)$ *be the cone spanned by* Γ *with its vertex at* p. *Then we have*

$$\Theta_\Sigma(p) < \Theta_{C_p(\Gamma)}(p)$$

unless Σ *itself is a cone over* p *with planar faces.*

The second inequality follows from the Gauss-Bonnet formula applied to the *double cover* of the cone $C_p(\Gamma)$. (We have not found a useful Gauss-Bonnet formula for general 2-dimensional Riemannian polyhedra in the literature.)

Theorem 2 (and **Corollary 5**):

$$2\pi\Theta_{C_p(\Gamma)}(p) = -\sum_{k=1}^{n}\int_{a_k}\vec{k}\cdot\nu_C\,ds + \sum_{k=1}^{n}\sum_{j=1,2}\left(\frac{\pi}{2} - \beta_j^k\right) \le \mathcal{C}_{\rm tot}(\Gamma),$$

where $\vec{k}$ *is the geodesic curvature vector of* a_k *in* $\mathbf{R}^n$, ν_C *is the outward unit normal vector to* $C_p(\Gamma)$ *along* $a_k \subset \partial C_p(\Gamma)$, *and* β_j^k *is the angle between the tangent vector to* a_k *at its endpoint* q_j *and the line segment from* q_j *to* p.

The density estimate $2\pi\Theta_\Sigma(p) \le \mathcal{C}_{\rm tot}(\Gamma)$, when Γ is a rectifiable Jordan curve, is a major ingredient of the work by Ekholm, White and Wienholtz [EWW], where it was proven that if $\mathcal{C}_{\rm tot}(\Gamma) \le 4\pi$, then every stationary branched minimal surface Σ in $\mathbf{R}^n$ spanned by Γ is a smooth embedded submanifold; and that given a compactly supported rectifiable varifold Σ which is strongly stationary with respect to Γ and with density ≥ 1 on $\Sigma\backslash\Gamma$, the inequality $\mathcal{C}_{\rm tot}(\Gamma) < 3\pi$ implies that Σ is smooth in the interior. Therefore one can view the results in this paper as partial extensions of those theorems in [EWW], when the Jordan curve Γ of [EWW] is replaced, more generally, by a graph. Since a rectifiable Jordan curve Γ always bounds a minimizing immersed disk [D1], [R], a specific consequence is the 1949 theorem of Fáry and Milnor that a curve of total curvature $\le 4\pi$ is unknotted. Analogous results about the isotopy classes of graphs do not yet follow from the present paper, in part because of the lack (so far) of results about boundary behavior of a minimizing surface relative to a graph. Our future paper [GY2] will provide additional extensions of results in [EWW].

By imposing appropriate upper bounds on the total curvature of the graph Γ, we obtain the following statements. We will denote by $C_Y = 3/2$ the density at its vertex of the Y-singularity cone composed of three planes meeting at $120°$, and by $C_T = 6\cos^{-1}(-1/3) \approx 11.468$ the density at its vertex of the T-singularity cone spanned by the one-skeleton of the regular tetrahedron with vertex at the center of the tetrahedron.

Theorem 3: *Suppose Γ is a graph in $\mathbf{R}^n$ with $\mathcal{C}_{\text{tot}}(\Gamma) \leq 2\pi C_Y = 3\pi$, and let Σ be a strongly stationary surface relative to Γ in the class $\mathcal{S}_\Gamma$. Then Σ is an embedded surface or a subset of the Y singularity cone, with planar faces.*

Theorem 4: *Suppose Γ is a graph in $\mathbf{R}^3$ with $\mathcal{C}_{\text{tot}}(\Gamma) \leq 2\pi C_T$, and let $\Sigma \in \mathcal{S}_\Gamma$ be embedded as an $(\mathbf{M}, 0, \delta)$-minimizing set with respect to Γ. Then Σ has possibly Y singularities but no other singularities, unless it is a subset of the T stationary cone, with planar faces.*

For the definition of $(\mathbf{M}, \varepsilon, \delta)$-minimizing sets, see Definition 3 below.

In $\mathbf{R}^3$, there are many known examples of strongly stationary surfaces. In particular, there are exactly ten stationary cones in $\mathbf{R}^3$ [AT]. Each stationary cone is the cone over a graph $\Gamma \subset S^2$, where Γ consists of geodesic segments on the sphere meeting in threes at angles of $120°$; this includes the planar case, where Γ is simply one great circle spanning $\mathbf{R}^2$. These ten minimal cones form a list of possible tangent cones at the interior points of an $(\mathbf{M}, \varepsilon, \delta)$-set Σ. The three cones with the smallest densities at the vertex are the plane with its density 1 where the graph Γ is a great circle; the Y-singularity cone with its density $C_Y = 3/2$ where Γ consists of three semicircles meeting at the north and south poles at angles of $120°$; and the T-singularity cone with density $C_T = 6\cos^{-1}(1/3)$, where Γ is the radial projection of the 1-skeleton of a regular tetrahedron inscribed in S^2. Recall that the other seven cones are stationary, but not minimizing, under interior deformations. Hence given one of those seven graphs Γ, there is another surface which is also strongly stationary with respect to Γ, but has strictly smaller area. Indeed when it comes to soap films, the three cones of lowest density are the only tangent cones experimentally observed in the interior of soap films. This is also true for the mathematical model in terms of 2-rectifiable sets, a result shown by Jean Taylor [T]:

Regularity Theorem for Soap Films: *Away from* Γ*, an* $(\mathbf{M}, \varepsilon, \delta)$*-minimizing set with respect to* Γ *consists of* $C^{1,\alpha}$ *surfaces meeting smoothly in threes at* 120° *angles along smooth curves, with these curves in turn meeting in fours at angles of* $\cos^{-1}(-1/3)$.

The singular curves were proved to be $C^{1,\alpha}$ by Taylor [T], and later shown to be real analytic in [KNS], for the locally area-minimizing case $\varepsilon \equiv 0$. The class $\mathcal{S}_\Gamma$ of surfaces we consider in this paper is chosen so that, given a graph Γ, it is reasonable to expect that every $(\mathbf{M}, 0, \delta)$-minimizing set relative to Γ is in the regularity class $\mathcal{S}_\Gamma$. The ten stationary cones described above are in fact in $\mathcal{S}_\Gamma$. However, due to the lack of understanding of boundary regularity of such $(\mathbf{M}, 0, \delta)$-minimizing sets, it is not yet known that in general $(\mathbf{M}, 0, \delta)$-minimizing sets are indeed elements of the class $\mathcal{S}_\Gamma$.

In **Section 6**, we turn our attention to the case where the ambient manifold is of variable curvature. The lack of homogeneity of the ambient space forces us to consider a comparison space of constant sectional curvatures, as was done previously in [CG2]. We consider two classes of Riemannian manifolds M which are strongly convex (not necessarily complete): manifolds with sectional curvature K_M bounded above by $-\kappa^2 \leq 0$, and manifolds with sectional curvature bounded above by $\kappa^2 > 0$. For a Euclidean ambient space, as seen above in Theorems 1 and 2, the density of a polyhedral minimal surface is bounded above by the total curvature of Γ. In the variable curvature case, the total curvature of Γ is not invariant under diffeomorphisms of M which mimic the homotheties of $\mathbf{R}^n$. Thus, in order to have significance for both large graphs Γ and for small ones, $\mathcal{C}_{\rm tot}(\Gamma)$ needs to be corrected in the following manner:

Density Estimate ($K_M \leq -\kappa^2$ case)**:** *Let* Σ *be a strongly stationary surface relative to* Γ *in the class* $\mathcal{S}_\Gamma$ *in a strictly convex manifold* M^n *which has sectional curvatures* $\leq -\kappa^2$*. Then for all* $p \in \Sigma\backslash\Gamma$*,*

$$2\pi\Theta_p(\Sigma) \leq \mathcal{C}_{\rm tot}(\Gamma) - \kappa^2 \mathcal{A}(\Gamma),$$

where $\mathcal{A}(\Gamma)$ *is the minimum cone area of all the cones with vertex in the convex hull of the set* Γ.

Density Estimate ($K_M \leq +\kappa^2$ case)**:** *Let* Σ *be a strongly stationary surface relative to* Γ *in the class* $\mathcal{S}_\Gamma$ *in a strictly convex manifold* M^n *which has sectional curvatures* $\leq \kappa^2$*. Assume that* Γ *has diameter* $< \pi/\kappa$*. Then for all*

p,

$$2\pi\Theta_p(\Sigma) \le \mathcal{C}_{\rm tot}(\Gamma) + \kappa^2 \widehat{\mathcal{A}}(\Gamma),$$

where $\hat{\mathcal{A}}(\Gamma)$ is the maximum spherical area of all the cones with vertex in the convex hull of the set Γ.

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2. Density and the Regularity of Strongly Stationary Surfaces

Let $\Gamma \subset \mathbf{R}^n$ be a graph, consisting of immersed arcs a_i, which are C^2 in the interior and C^1 up to their vertices, as in Section 1; we assume that each vertex has valence at least two. Let the class $\mathcal{S}_\Gamma$ of singular surfaces be defined as in Section 1: for $\Sigma \in \mathcal{S}_\Gamma$, Γ is a subset of the one-dimensional part $S \subset \Sigma$. Within the class $\mathcal{S}_\Gamma$, we will consider the surfaces satisfying the following property.

Definition 1. [EWW] *A rectifiable varifold Σ in $\mathbf{R}^n$ is called* strongly stationary with respect to Γ *if for all smooth $\phi : \mathbf{R} \times \mathbf{R}^n$ with $\phi(0, x) \equiv x$, we have*

$$\frac{d}{dt}\Big(\mathrm{Area}(\phi(t,\Sigma)) + \mathrm{Area}(\phi([0,t]\times\Gamma))\Big)\Big|_{t=0} \ge 0.$$

The regularity condition on each Σ_i guarantees that at almost every point p of S, there exists a unit vector ν_{Σ_i}, normal to Γ, tangential to Σ_i, pointing *out* of Σ_i. Hence on each Σ_i we have the divergence theorem

$$\int_{\Sigma_i} \mathrm{div}_{\Sigma_i} X^T dA = \int_{\partial\Sigma_i} \langle X, \nu_i\rangle ds,$$

where X^T is the tangential (to Σ_i) component of X. Note that the strong stationarity condition implies that $\Sigma \in \mathcal{S}_\Gamma$ is stationary, i.e. the first variation of the area vanishes under deformations supported away from Γ:

$$\frac{d}{dt}\Big(\mathrm{Area}(\phi(t,\Sigma))\Big)\Big|_{t=0} = 0.$$

If in addition the vector field $X = \phi'(0, x)$ is supported away from the singular set S, then the stationarity condition implies that the interior of each Σ_i is minimal, i.e. its mean curvature vector $\vec{H}$ vanishes.

If the vector field X is supported away from Γ, the stationarity condition implies

$$\int_S \sum_{j\in J\subset I} \langle \nu_{\Sigma_j}(p), X^\perp(p)\rangle\, ds(p) - \sum \int_{\Sigma_i} \langle \vec{H}, X^\perp\rangle\, dA = 0$$

where $J = J(p)$ indexes the collection of surfaces Σ_j which meet at a point p in $S\backslash\Gamma$. Note that the second term vanishes since $\vec{H} \equiv 0$. Since the choice of X is arbitrary, it follows that the vector

$$\nu_\Sigma(p) := \sum_{j\in J(p)\subset I} \nu_{\Sigma_j}(p) = 0 \tag{3}$$

almost everywhere on $S\backslash\Gamma$, which we call the *balancing* of ν_{Σ_i} along the singular curves of Σ, away from Γ.

The strong stationarity condition of a varifold with respect to Γ is equivalent to the existence of an $\mathcal{H}^1$-measurable normal (to Γ) vector field ν on Γ with $\sup|\nu| \le 1$ such that

$$\int_\Sigma \mathrm{div}_\Sigma X\, dA = \int_\Gamma \langle X, \nu\rangle\, ds \tag{4}$$

for all smooth vector fields X on $\mathbf{R}^n$ (see section 7 of [EWW]). Note that since X is an ambient vector field along Σ, $\mathrm{div}_\Sigma X$ is the trace on Σ of the ambient covarant derivative of X.

In our context, that is, when Σ is in $\mathcal{S}_\Gamma$, the $\mathcal{H}^1$-measurable vector field $\nu = \nu_\Sigma$ arises as

$$\nu_\Sigma(p) = \sum_{j\in J(p)} \nu_{\Sigma_j}(p) \tag{5}$$

for each $p \in \Gamma$, where $j \in J(p)$ whenever $p \in \overline{\Sigma_j}$.

First we define a surface Σ in $\mathcal{S}_\Gamma$ to be *locally minimizing relative to* Γ at p if for a neighborhood U of p, there exists a smaller neighborhood V of p such that for any $\tilde{\Sigma}$, if $\tilde{\Sigma}\backslash V = \Sigma\backslash V$ and $\partial\tilde{\Sigma} = \Gamma$, then $\mathrm{Area}(\tilde{\Sigma}) \ge \mathrm{Area}(\Sigma)$. We are particularly interested in the case p is a point on Γ.

For intuition, it is useful to understand the relation between strong stationarity and the local minimizing property within the class $\mathcal{S}_\Gamma$. First define an $\mathcal{H}^1$ measurable vector field ν_Σ defined on Γ, as in (5) above. The following proposition may be proved using well-known methods of the calculus of variations.

Proposition 1. *Suppose that Σ is a surface in $\mathcal{S}_\Gamma$. Then Σ is locally minimizing relative to Γ at each point of Γ if and only if $|\nu_\Sigma| \leq 1 \ \ \mathcal{H}^1$-almost everywhere on Γ, $\nu_\Sigma = 0 \ \mathcal{H}^1$-almost everywhere on a neighborhood of Γ in $S\backslash\Gamma$ and the regular parts of Σ have vanishing mean curvature vector $\vec{H}$ in some neighborhood of Γ.*

This proposition says that within the class $\mathcal{S}_\Gamma$, the local minimizing property relative to Γ and stationarity away from Γ imply strong stationarity with respect to Γ. We remark here that strong stationarity is strictly weaker than the locally area minimizing condition. In particular, there are surfaces which are strongly stationary but not locally area minimizing at certain interior points. One such example is the cone $\Sigma \subset \mathbf{R}^3$ spanned by the 1-skeleton Γ of a cube, with its vertex at the center of the cube. It is strongly stationary relative to Γ, but is not locally minimizing at the cone vertex. Namely, there exists a one parameter family of polyhedral surfaces of strictly smaller area, in which an arbitrarily small neighborhood of the vertex at the center is replaced by the 2-skeleton of a small cube [T].

Next we introduce the following definition, which, for surfaces in the class $\mathcal{S}_\Gamma$, allows us to isolate the two independent parts of the strong stationarity condition. In fact, strong stationarity for surfaces in the class $\mathcal{S}_\Gamma$ is equivalent to stationarity in $\mathbf{R}^n\backslash\Gamma$ plus the following boundary condition.

Definition 2. *Γ is said to be a* variational boundary *of a surface Σ if there exists an $\mathcal{H}^1$ measurable vector field ν_Σ along Γ which is orthogonal to Γ, with $|\nu_\Sigma| \leq 1$ a.e., such that for all smooth vector fields X defined on $\mathbf{R}^n$, $\int_\Sigma \mathrm{div}_\Sigma X^T \, dA = \int_\Gamma \langle X, \nu_\Sigma \rangle \, ds$.*

Observe that Definition 1 of strong stationarity refers to ambient derivatives of X, in contrast with Definition 2, which is intrinsic to Σ.

Now we are ready to state and prove the main result of this section.

Theorem 1. *Given a strongly stationary surface Σ in $\mathcal{S}_\Gamma$, and a point $p \in M\backslash\Gamma$ of $\Sigma\backslash\Gamma$, let $C_p(\Gamma)$ be the cone spanned by Γ with its vertex at p. Then we have the inequality:*

$$\Theta_\Sigma(p) < \Theta_{C_p(\Gamma)}(p),$$

unless Σ is a cone over p with planar faces, in which case we have equality.

Proof. Let $G(x)$ be the test function $\log \rho(x)$, where $\rho(x) = |x - p|$. $G(x)$ is the Green's function for the Laplace operator defined on two-dimensional subspaces of $\mathbf{R}^n$ which contain the point p. On the other hand, on an immersed minimal surface in $\mathbf{R}^n$, the function $G(x)$ is subharmonic, as a consequence of the trace formula:

$$\triangle_\Sigma G = \sum_{\alpha=1}^{2} \overline{\nabla}^2 G(e_\alpha, e_\alpha) + dG(\vec{H}), \tag{6}$$

where $\overline{\nabla}$ is the covariant derivative for the ambient manifold $\mathbf{R}^n$ (see [CG1]). Thus, the divergence theorem implies the following integral estimate:

$$0 \le \int_{\Sigma_i \setminus B_\varepsilon(p)} \triangle_{\Sigma_i} G \; dA = \int_{\partial(\Sigma_i \setminus B_\varepsilon(p))} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma_i}} \; ds$$

for each i, where $\Sigma = \cup_{i \in I} \Sigma_i$ is a surface in the class $\mathcal{S}_\Gamma$. Note that each boundary $\partial(\Sigma_i \setminus B_\varepsilon(p))$ consists of three parts:

$$\partial(\Sigma_i \setminus B_\varepsilon(p)) = \Big(\partial\Sigma_i \cap \Gamma\Big) \cup \Big(\partial B_\varepsilon(p) \cap \Sigma_i\Big) \cup \Big(\partial\Sigma_i \setminus (\Gamma \cup \overline{B_\varepsilon})\Big),$$

since $S = \cup \partial\Sigma_i$. Now we sum the inequality above over i and reorganize the boundary terms:

$$0 \le \int_\Gamma \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_\Sigma} \, ds + \int_{\partial B_\varepsilon \cap \Sigma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_\Sigma} \, ds + \sum_{i \in I} \int_{\partial \Sigma_i \setminus (\Gamma \cup \overline{B_\varepsilon})} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma_i}} \, ds,$$

where ν_Σ is as in equation (5). The last term vanishes, since we have the balancing condition among the unit vectors ν_{Σ_i} normal to the edges of $\Sigma_i \cap (S \setminus \Gamma)$, and tangent to Σ_i, pointing outward from Σ_i, as a consequence of the (interior) stationarity (3) of Σ:

$$\sum_{j \in J(p)} \frac{\partial \rho}{\partial \nu_{\Sigma_j}} = \Big\langle \overline{\nabla} \rho, \sum_{j \in J(p)} \nu_{\Sigma_j} \Big\rangle = 0,$$

for each $p \in S \setminus \Gamma$, where $J(p)$ is the collection of $j \in I$ with $p \in \overline{\Sigma_j}$.

As for the second term, note that as ε goes to zero, $\dfrac{\partial \rho}{\partial \nu_{\Sigma_i}}$ approaches -1 uniformly, and hence

$$\int_{\partial B_\varepsilon(p) \cap \Sigma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_\Sigma} ds$$

converges to

$$\lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon}\right) \mathrm{Length}(\Sigma \cap \partial B_\varepsilon(p)) = -2\pi \Theta_\Sigma(p).$$

Therefore we have obtained the following upper bound for the density of Σ at p:

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_\Sigma} ds. \tag{7}$$

We repeat the argument for the surface Laplacian of $G(x)$, this time replacing Σ with the cone $C_p(\Gamma)$ spanned by Γ with vertex p. Recall that $\Gamma = \cup_j a_j$ where each arc a_j is C^2-regular, C^1 up to the end points. Denote by A_j the cone $C_p(a_j)$ spanned by a_j with its vertex at p. Thus the cone $C_p(\Gamma)$ is the union of all the *fans* $\overline{A_j} = A_j \cup \partial A_j$. Observe using (6) that away from the vertex p, $G(x)$ is harmonic on A_j [CG1]. Hence we have

$$0 = \int_{A_j \setminus B_\varepsilon(p)} \triangle_C G(x)\ dA = \int_{\partial(A_j \setminus B_\varepsilon(p))} \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\ ds.$$

As seen above for Σ, each boundary $\partial(A_j \setminus B_\varepsilon(p))$ consists of three parts; we sum the equation above over j and reorganize the boundary terms, and find:

$$0 = \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\, ds + \int_{\partial B_\varepsilon(p)\cap C} \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\, ds + \sum_j \int_{C_p(\partial a_j)\setminus B_\varepsilon(p)} \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_{A_j}}\, ds,$$

where $\nu_C = \nu_{C_p(\Gamma)}$ is defined to be $\sum_j \nu_{A_j}$, with ν_{A_j} being the unit vector normal to the boundary ∂A_j, and tangent to the fan A_j, pointing out of A_j.

The last term vanishes since the vector ν_{A_j} and $\overline{\nabla}\rho$ are perpendicular, which makes $\partial\rho/\partial\nu_{A_j}$ identically zero on $C_p(\partial a_j)$. The second term is equal to $-\mathrm{Length}(C_p(\Gamma) \cap \partial B_\varepsilon(p))/\varepsilon$ which in turn is equal to $-2\pi\Theta_C(p)$, independent of sufficiently small $\varepsilon > 0$. Therefore we have obtained

$$2\pi\Theta_C(p) = \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\ ds. \tag{8}$$

Now observe that ν_C is the unit vector normal to Γ most closely aligned with the gradient of ρ along Γ, while ν_Σ is normal to Γ with $|\nu_\Sigma| \leq 1$, since Γ is a variational boundary of Σ. Hence we have the following inequality:

$$\frac{\partial\rho}{\partial\nu_C} \geq \frac{\partial\rho}{\partial\nu_\Sigma}$$

almost everywhere along Γ. By integrating, we have

$$\int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\ \ ds \geq \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_\Sigma}\ \ ds \tag{9}$$

Combining the inequalities (7), (9) and the equality (8), we finally get

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_\Sigma}\ ds \leq \int_\Gamma \frac{1}{\rho}\frac{\partial\rho}{\partial\nu_C}\ ds = 2\pi\Theta_C(p). \tag{10}$$

If equality occurs in (10), then $\triangle_\Sigma G \equiv 0$, and the trace formula (6), along with a computation of $\overline{\nabla}^2 G$, implies that $\overline{\nabla}\rho$ is tangent to Σ. Thus each two-dimensional face of Σ is both a regular minimal surface and a stationary cone in $\mathbf{R}^n$, and therefore is part of a plane passing through p. ∎

3. Total Curvatures of Graphs

Let Γ be a graph in $\mathbf{R}^n$, as in Sections 1 and 2 above, consisting of immersed arcs $a_1, a_2, \ldots, a_n$ and vertices $q_1, q_2, \ldots, q_m$. Recall the definition (2) of total curvature $\mathcal{C}_{\mathrm{tot}}(\Gamma)$ of a graph Γ, and esp. the definition (1) of $\mathrm{tc}(q)$ for a vertex q of valence d:

$$\mathrm{tc}(q) := \sup_{e\in T_qM}\left\{\sum_{\ell=1}^{d}\left(\frac{\pi}{2}-\beta^\ell(e)\right)\right\}$$

where $\beta^\ell = \beta^\ell(e) \in [0,\pi]$ is the angle between the tangent vector T_ℓ to a_ℓ at q and the (unit) vector e.

The usefulness of definition (1) will become clear in section 4 below; see esp. Theorem 2.

It might be noted that even though the geodesic curvature in Σ at a smooth point of Γ is given by the tangential component of the curvature vector of Γ, there is no such appropriate vector at a vertex. This is true already at a vertex of degree $d = 2$, that is, for a piecewise smooth Jordan curve.

In this section, we shall collect some observations about $\mathcal{C}_{\mathrm{tot}}(\Gamma)$ for specific cases of a graph $\Gamma \subset \mathbf{R}^n$. These will be used for the examples below, but will not be referred to in the proofs of the theorems. As the results of this section are elementary, and some of them previously known, we include brief proofs for the sake of completeness (see [MY] and references therein for more general discussion on minimal network problems.)

Consider a vertex q of Γ of valence d, and let $T_1, \ldots, T_d$ be the unit tangent vectors to Γ at q. For a given unit vector e, we write $\beta_\ell(e)$ for the angle between e and T_ℓ, as above. Further, we write $e = e_0 \in S^2$ for a point where the sum $\sum_{\ell=1}^d \left(\frac{\pi}{2} - \beta_\ell(e)\right)$ assumes its maximum value $\mathrm{tc}(q)$. Note here that e_0 is also the minimizer of $\sum_{\ell=1}^d \beta_\ell(e)$, the total spherical distance to $T_1, \dots, T_d$. The existence, but not uniqueness, of such an e_0 follows from compactness of S^2 [MY].

3.1. **Valence three.**

Proposition 2. *For all T_1, T_2 and $T_3 \in S^2$, there exists $e \in \{T_1, T_2, T_3\}$ so that $\beta_1(e) + \beta_2(e) + \beta_3(e) \leq 4\pi/3$.*

Proof. T_1, T_2 and T_3 lie in a small (or great) circle γ of S^2. Each spherical distance $d(T_i, T_{i+1})$ ($i = 1, 2, 3 \bmod 3$) is less than or equal to the length of the smaller arc of γ between T_i and T_{i+1}, so their sum is at most the length of γ, hence $\leq 2\pi$. Renumber T_1, T_2, T_3 so that $d(T_2, T_3)$ is the largest of the three distances, and choose $e = T_1$. Then $\beta_1(e) = 0$, while $\beta_2(e), \beta_3(e) \leq \frac{2\pi}{3}$. ■

Corollary 1. *For any vertex q of valence $d = 3$, $\mathrm{tc}(q) \geq \pi/6$, with equality if and only if the three unit tangent vectors T_1, T_2 and T_3 at q are* balanced: $T_1 + T_2 + T_3 = 0$.

Proof. By Proposition 2, $\sup_e \sum_{\ell=1}^{3} \left(\frac{\pi}{2} - \beta_\ell(e)\right) \geq \frac{3\pi}{2} - \inf_i \sum_{\ell=1}^{3} \beta_\ell(T_i) \geq \frac{\pi}{6}$.

Now suppose that $\mathrm{tc}(q) = \pi/6$. As in the proof of Proposition 2, the unit tangent vectors T_1, T_2, T_3 lie on a circle $\gamma \subset S^2$. But $\beta_2(T_1) + \beta_3(T_1) = \sum_{\ell=1}^{3} \beta_\ell(T_1) \geq \sum_{\ell=1}^{3} \beta_\ell(e_0) = \frac{3\pi}{2} - \mathrm{tc}(q) = \frac{4\pi}{3}$, while $d(T_2, T_3) \geq \beta_\ell(T_1)$, $\ell = 2, 3$, which implies that γ has length 2π. Thus γ is a great circle and all of the $d(T_i, T_{i+1}) = \frac{2\pi}{3}$. ■

In specific situations, it is of interest to compute $\mathrm{tc}(q)$ exactly, or even to identify the spherical total distance-minimizing point e_0. The following lemma is not difficult to prove, using the first variation of the sum of distances on S^2.

Lemma 1. *Suppose a vertex q of Γ has valence three, with unit tangent vectors T_1, T_2, T_3 to Γ at q. Let e_0 be a total distance minimizing point for T_1, T_2, T_3. For $\ell = 1, 2, 3$ choose a minimizing geodesic (great circle) in S^2 from e_0 to T_ℓ, and let $\xi_\ell \in T_{e_0}S^2$ be the unit tangent vector at e_0 to the geodesic. Then either (1) $\xi_1 + \xi_2 + \xi_3 = 0$, that is, the geodesics make equal angles $2\pi/3$ at e_0; or (2) $e_0 = T_\ell$ for some $\ell = 1, 2, 3$, and the remaining two vectors $\xi_{\ell+1}, \xi_{\ell+2}$ form an angle $\geq 2\pi/3$ (subscripts* modulo 3*).*

For equilateral spherical triangles, one might expect the total distance minimizing point e_0 of the vertices to be the center of the triangle; however, if the triangle is too large, e_0 can only be one of the corners of the triangle:

Corollary 2. *If the vertex q of Γ has valence 3 and its unit tangent vectors T_1, T_2, T_3 make equal angles with each other, then*

$$\mathrm{tc(q)} = \begin{cases} 3\left(\frac{\pi}{2} - \beta\right) & \text{if} \beta \le R_0, \\ \frac{3\pi}{2} - 4\sin^{-1}(\frac{1}{2}\sqrt{3}\sin\beta) & \text{if} \beta \ge R_0; \end{cases} \tag{11}$$

where $0 \le \beta \le \pi/2$ is the circumradius, the common spherical distance from T_ℓ to the closer center N, of the triangle formed by T_1, T_2, T_3; and where $R_0 \approx 1.33458$ radians is the value of β which makes the two options in formula (11) equal.

Proof. It follows from Lemma 1 that a minimizer of $\sum \beta_\ell$ must be one of the five points $N, -N, T_1, T_2$ or T_3. But $\sum \beta_\ell(-N) \ge \sum \beta_\ell(N) = 3\beta$, and $\sum \beta_\ell(T_i) = 4s$, $i = 1, 2, 3$, where $2s$ is the side of the equilateral triangle: $\sin s = \sin\beta \sin(\pi/3)$. But $3\beta - 4s$ has the same sign as $\beta - R_0$. ■

3.2. **Even valence.**

Proposition 3. *If T_1, T_2, T_3 and T_4 are points on S^2, then any of the total distance minimizing points e_0 must be one of the T_ℓ or one of the six (or more) points of intersection of the two great circles passing through disjoint pairs of the four points T_ℓ.*

The proof of Proposition 3 will be immediate from the following lemma.

Lemma 2. *Let e_0 be a total distance minimizing point for $T_1, T_2, T_3, T_4 \in S^2$, and write $\xi_\ell \in T_{e_0}S^2$ for the initial unit tangent vector to the minimizing geodesic from e_0 to T_ℓ. If e_0 is not equal to any of the T_ℓ, then after reindexing $\xi_1, \xi_2, \xi_3, \xi_4$ in circular order around the unit circle of $T_{e_0}S^2$, we have $\xi_1 = -\xi_3$ and $\xi_2 = -\xi_4$.*

Proof. We compute the first variation of $\sum_{\ell=1}^4 \beta_\ell(e)$, and find that $0 = -\sum_{\ell=1}^4 \langle \xi_\ell, \xi \rangle$ for any $\xi \in T_{e_0}S^2$. We conclude that the ξ_ℓ are balanced:

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0. \tag{12}$$

Write η_ℓ for the oriented angle from ξ_ℓ to $\xi_{\ell+1}$, ℓ *modulo* 4, with $0 \le \eta_\ell \le 2\pi$.

If $\xi_1 = -\xi_3$, then also $\xi_2 = -\xi_4$ according to (12), and we are done. Otherwise, the sum $\xi_1 + \xi_3$ makes the oriented angle $\frac{1}{2}(\eta_1 + \eta_2)$ *modulo* π with ξ_1, while the sum $\xi_2 + \xi_4$ makes the angle $\frac{1}{2}(\eta_2 + \eta_3)$ *modulo* π with ξ_2. But $\xi_2 + \xi_4 = -(\xi_1 + \xi_3)$, hence $\eta_1 + \frac{1}{2}(\eta_2 + \eta_3) = \frac{1}{2}(\eta_1 + \eta_2) + \pi$ *modulo* π, implying that $\eta_1 + \eta_3 = 0$ *modulo* 2π. But $\eta_1 + \eta_2 + \eta_3 + \eta_4 = 2\pi$ and $\eta_\ell \ge 0$, so this forces either $\eta_1 = \eta_3 = 0$, implying $\xi_1 = \xi_2$ and $\xi_3 = \xi_4$; or

$\eta_2 = \eta_4 = 0,$ implying $\xi_2 = \xi_3$ and $\xi_4 = \xi_1$. The conclusion now follows from equation (12) in this case as well. ■

The following addition lemma has a complicated statement but a straightforward demonstration.

Lemma 3. *Let* $\widetilde{\Gamma}$ *and* $\widehat{\Gamma}$ *be graphs with a common vertex* $\widetilde{q} = \widehat{q}$. *Write* Γ *for the union of* $\widetilde{\Gamma}$ *and* $\widehat{\Gamma}$, *and write* q *for the common vertex when considered as a vertex of* Γ. *Write* $\{\widetilde{T}_1, \ldots, \widetilde{T}_k\}$ *for the unit tangent vectors to* $\widetilde{\Gamma}$ *at* $\widetilde{q}$, *and let* $\{\widehat{T}_1, \ldots, \widehat{T}_{d-k}\}$ *be the unit tangent vectors to* $\widehat{\Gamma}$ *at* $\widehat{q}$. *Then* $\mathrm{tc}(q) \leq \mathrm{tc}(\widetilde{q}) + \mathrm{tc}(\widehat{q})$. *If further* $\{\widetilde{T}_1, \ldots, \widetilde{T}_k\}$ *and* $\{\widehat{T}_1, \ldots, \widehat{T}_{d-k}\}$ *share the same total distance minimizing point* $\widetilde{e}_0 = \widehat{e}_0$, *then the total distance minimizing point* e_0 *of* $\{\widetilde{T}_1, \ldots, \widetilde{T}_k, \widehat{T}_1, \ldots, \widehat{T}_{d-k}\}$ *is equal to both, and* $\mathrm{tc}(q) = \mathrm{tc}(\widetilde{q}) + \mathrm{tc}(\widehat{q})$.

Corollary 3. *If a vertex* q *of* Γ *has an even valence* d *and the tangent vectors at* q *occur in antipodal pairs, then* $\mathrm{tc}(q) = 0$.

Proof. Observe that a vertex q of degree 2 in a straight edge, that is, with $T_2 = -T_1$, has $\mathrm{tc}(q) = 0$, with *any* point of S^2 as a total distance minimizing point. The conclusion then follows from Lemma 3 by induction on $d/2$. ■

In contrast with Corollary 2, even valence makes computations easier:

Corollary 4. *For a regular polygon in* S^2 *with an* even *number* d *of sides, the closer center in* S^2 *of the polygon is a total distance minimizing point of the corners* $T_1, \ldots, T_d$.

Proof. Let N be the closer center (closer than $-N$) of the regular polygon of $d =: 2k$ sides, with vertices $T_1, \ldots, T_{2k}$ in order. A total distance minimizing point of two opposite vertices $\{T_i, T_{k+i}\}$ is any point along the minimizing geodesic arc joining them, in particular the midpoint N. Now apply Lemma 3 via induction on k. ■

Proposition 4. *For a vertex* q *of a graph* $\Gamma \subset \mathbf{R}^3$ *with unit tangent vectors* $T_1, \ldots, T_d$ *all lying in a plane through* 0 *and making equal angles, an orthogonal unit vector* N *is a total distance minimizing point if and only if* d *is even.*

Proof. If d is even, the conclusion is given by Corollary 4. If $d = 2k+1$ is odd, then the sum $\sum_{\ell=1}^{d} \beta_\ell(e)$ equals $(2k+1)\pi/2$ for $e = N$, and equals $(1+2+\cdots+k)4\pi/(2k+1)$ for $e = T_1$, which is smaller by a difference of $\frac{\pi}{2(2k+1)}$. Thus N does not minimize the total spherical distance. ■

4. Gauss-Bonnet formula for Cones

In this section we will prove a Gauss-Bonnet formula for two-dimensional cones in $\mathbf{R}^n$. First we recall the following classical result.

Euler's Theorem (see[O]) *For a connected graph* Γ' *with even valence at each vertex, there is a continuous mapping of the circle to* Γ' *which traverses each edge exactly once.*

An immediate consequence of this result is that *any* connected finite graph Γ has a continuous mapping of the circle which traverses each edge exactly *twice.* Namely, we may apply Euler's theorem to the graph Γ' obtained from Γ by doubling each edge and leaving the vertices alone. Note that the new graph Γ' has even valence at each vertex.

We shall derive the density formula of Theorem 2 below in three steps, beginning from a well known case.

Suppose first that Γ_0 is a **smooth closed curve** in $\mathbf{R}^n$, not necessarily simple, and p a point not on Γ_0. Without loss of generality (after a suitable scaling centered at p), we may assume that Γ_0 lies outside the unit ball $B_1(p)$ centered at p.

Define Π_p to be the radial projection to the unit sphere centered at p:

$$\Pi_p : \mathbf{R}^n \backslash \{p\} \to \partial B_1(p);$$

$$\Pi_p(x) = p + \frac{x-p}{|x-p|}.$$

Let $A = C_p(\Gamma_0) \backslash B_1(p)$ be the annular region between Γ_0 and $\Pi_p \Gamma_0$. By the Gauss-Bonnet formula, we have

$$-\int_{\partial A} \vec{k} \cdot \nu_C \, ds + \int_A K \, dA = 2\pi \chi(A) \tag{13}$$

where $\vec{k}$ is the curvature vector of ∂A in $\mathbf{R}^n$, ν_C is the outward normal to ∂A, K is the Gauss curvature of A, and $\chi(A)$ is the Euler characteristic of A. For A, $K \equiv 0$ and $\chi(A) = 0$. Hence

$$\begin{aligned} 0 &= \int_{\partial A} \vec{k} \cdot \nu_C \, ds \\ &= \int_{\Pi_p \Gamma_0} \vec{k} \cdot \nu_C \, ds + \int_{\Gamma_0} \vec{k} \cdot \nu_C \, ds \end{aligned}$$

For $q \in \Pi_p\Gamma_0$, $\vec{k}(q) = \nu_C(q)$ is the unit vector from q to p, so that the first integral on the last line is equal to the length of $\Pi_p\Gamma_0$, which is also equal to $2\pi\Theta_{C_p(\Gamma_0)}(p)$. Therefore we have for the cone $C_p(\Gamma_0)$ the following equation:

$$2\pi\Theta_{C_p(\Gamma_0)}(p) = \mathrm{Length}(\Pi_p\Gamma_0) = -\int_{\Gamma_0} \vec{k}\cdot\nu_C\,ds, \tag{14}$$

where $\nu_C(q)$ is the unit normal vector to Γ_0 in the plane spanned by the tangent vector at q and the vector $p-q$, and pointing away from the cone vertex p.

Next, when Γ' is a **piecewise smooth** immersion of the circle, we generalize the formula above as follows. Let Γ' be a union of smooth segments a_i, each of which is C^2 in the interior and C^1 up to the end points $q_{i,0}, q_{i,1}$. We denote $q_{i,j} \sim q_{i',j'}$ if they represent the same point where a_i and $a_{i'}$ meet. Then the cone $C_p(\Gamma')$ can be thought as a union of *fans* $A_i(p) = C_p(a_i)$, which is the part of the cone $C_p(\Gamma')$ spanned by a_i, with radial edges $\overline{pq_{i,0}}$ and $\overline{pq_{i,1}}$. The right hand side of the equation (14) then generalizes as

$$\begin{aligned} 2\pi\Theta_{C_p(\Gamma')}(p) &= \mathrm{Length}(\Pi_p\Gamma') \\ &= -\sum_i\int_{a_i}\vec{k}\cdot\nu_C\,ds + \sum_i\sum_{j=1,2}\left(\frac{\pi}{2}-\beta^i_j\right) \end{aligned} \tag{15}$$

where β^i_j is the angle between a_i and $\overline{pq_{i,j}}$ as they meet at $q_{i,j}$. To see how the last term arises, suppose that a_i and a_k are the consecutive edges in Γ' joined at $q_{i,j} \sim q_{k,j'}$. Then the quantity $(\pi/2-\beta^i_j)+(\pi/2-\beta^k_{j'}) = \pi-(\beta^i_j+\beta^k_{j'})$ is the amount the curve $a_i \cup a_k$ turns at $q_{i,j} \sim q_{k,j'}$, when considered as a locally isometrically embedded curve in $\mathbf{R}^2$.

Finally, coming back to the **original graph** Γ, Euler's theorem says that the graph Γ with each edge traced twice while its vertices are left intact, which we denoted by Γ', can be parameterized by a copy of S^1. Write Γ' as the union of a'_i where each a_k $(k=1,\dots,n)$ arises twice as one of the a'_i $(i=1,\dots,2n)$, as one goes *around* Γ' once.

Applying the generalized equation (15) to $\Gamma' = \cup_i^{2n} a'_i$, we obtain the following description of the density of the cone $C_p(\Gamma)$ at p.

Theorem 2. *With the notations as above we have the following cone density formula:*

$$2\pi\Theta_{C_p(\Gamma)}(p) = -\sum_{k=1}^{n}\int_{a_k}\vec{k}\cdot\nu_C\,ds + \sum_{k=1}^{n}\sum_{j=1,2}\left(\frac{\pi}{2}-\beta^k_j\right). \tag{16}$$

Proof. From the preceding discussion, we have

$$\begin{aligned} 2\pi\Theta_{C_p(\Gamma')}(p) &= \operatorname{Length}(\Pi_p\Gamma') \\ &= -\sum_{i=1}^{2n}\int_{a'_i}\vec{k}\cdot\nu_C\,ds + \sum_{i=1}^{2n}\sum_{j=1,2}\left(\frac{\pi}{2}-\beta'^{i}_{j}\right). \end{aligned} \tag{17}$$

Note that the length of Γ' is twice the length of Γ. Also note that when the edges a'_{i_1} and a'_{i_2} of Γ' represent the same edge a_k of Γ, we have

$$\int_{a_k}\vec{k}\cdot\nu_C\,ds = \int_{a'_{i_1}}\vec{k}\cdot\nu_C\,ds = \int_{a'_{i_2}}\vec{k}\cdot\nu_C\,ds$$

independent of the orientations imposed by the Euler circuit. Lastly, over the whole circuit Γ', the quantity $\pi/2-\beta_i^j,\ (i=1,\dots,n;\ j=1,2)$ appears twice. The statement of the theorem then follows by dividing both sides of equation (17) by two. ■

5. Regularity of Stationary Surfaces

Using the notations from section 2 above, we have the following immediate consequence to the density comparison (Theorem 1) between the density of a strongly stationary surface Σ with respect to Γ and that of the cone $C_p(\Gamma)$ over Γ with vertex p; and the Gauss-Bonnet formula (Theorem 2), which estimates the density of the cone in terms of the total curvature of the graph Γ:

Corollary 5. *The following inequality holds between the density of a strongly stationary surface Σ and the total curvature $\mathcal{C}_{\rm tot}$ of Γ:*

$$2\pi\Theta_\Sigma(p) \le \mathcal{C}_{\rm tot}(\Gamma).$$

Proof. We need only observe that in the conclusion of Theorem 2, the right-hand side of equation (16) is bounded above by $\mathcal{C}_{\rm tot}(\Gamma)$. ■

Theorem 3. *Suppose Γ is a graph in $\mathbf{R}^n$ with $\mathcal{C}_{\rm tot}(\Gamma) \le 2\pi C_Y = 3\pi$, and let Σ be a strongly stationary surface relative to Γ in the class $\mathcal{S}_\Gamma$. Then Σ is an embedded surface or a subset of the Y singular cone.*

Proof. At a point p on Σ, the proof of the above Corollary 5 to the Gauss-Bonnet formula says that

$$\Theta_\Sigma(p) \le \Theta_{C_p(\Gamma)}(p) \le \frac{1}{2\pi}\mathcal{C}_{\rm tot}(\Gamma) \le C_Y,$$

where the last inequality is the hypothesis. If $\Theta_\Sigma(p) < C_Y$, Σ is regular at p by the proof of Theorem 7.1 of [EWW]. For the sake of completeness, we reproduce their argument here.

Let $T_p\Sigma$ be the tangent cone at p, whose existence and uniqueness is guaranteed by the regularity assumption we impose on the class of surfaces $\mathcal{S}_\Gamma$. Then $\Theta_{T_p\Sigma}(0) = \Theta_\Sigma(p) < 3/2$, and thus $\Theta_{T_p\Sigma}(x) < 3/2$ for all x in the cone since in any minimal cone, the highest density occurs at the vertex. This is because the density function $\Theta_{T_p\Sigma}(x)$ is upper semi-continuous ([Si] §17.8) and constant along open radial segments. Now the intersection of $T_p\Sigma$ with the unit sphere is a collection of geodesic arcs [AA], which means that the cone is a polyhedron. At most two faces of the polyhedron $T_p\Sigma$ can meet along a radial edge, since otherwise the density at points along the edge would be $\geq 3/2$. This means $T_p\Sigma \cap S^{n-1}$ is a union of complete great circles. Since the density is $< 3/2$, there is only one great circle and it has multiplicity 1. By Allard's regularity theorem ([Al] or [Si]), this means that Σ is regular at p.

On the other hand, if $\Theta_\Sigma(p) = 3/2 = C_Y$, then equality holds in Theorem 1, implying that Σ itself is a cone with vertex p and planar faces. But the Y cone is the unique (up to rotation in $\mathbf{R}^n$) stationary cone having density $3/2$. ■

As seen above, $3/2$ is the first nontrivial upper bound for the density above 1, for the class of surfaces we are studying. As for a larger upper bound, we will restrict our attention to the case when the ambient Euclidean space is $\mathbf{R}^3$. There are exactly ten stationary cones in $\mathbf{R}^3$ [AT], where a cone is stationary when its intersection with the unit sphere is a net of geodesics meeting in threes at angles of $120°$. Ordered with respect to the density Θ at the vertices of the cones, the first three on the list are the plane with $\Theta = 1$; Y = three half-planes meeting at $120°$ with $\Theta = C_Y = 3/2$; and the cone T spanned by the regular tetrahedron with $\Theta = C_T = 6\cos^{-1}(-1/3) \approx 11.4638$.

In order to state the next result, we need to introduce the following definition [Alm].

Definition 3. *Let ε be a bound of the form $\varepsilon(r) = Cr^\alpha$ for some $\alpha > 0$, and choose $\delta > 0$. We define $\Sigma \subset \mathbf{R}^n$ to be an* $(\mathbf{M}, \varepsilon, \delta)$-minimal set *with respect to $\Gamma \subset \mathbf{R}^n$ if Σ is 2-rectifiable and if, for every Lipschitz mapping $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ with the diameter r of the support W of $\Phi - \mathrm{id}$ less than δ,*

$$\mathcal{H}^2(S \cap W) \leq (1 + \varepsilon(r))\, \mathcal{H}^2\Big(\Phi(S \cap W)\Big).$$

We have the following partial regularity statement in $\mathbf{R}^3$ for Γ with small total curvature.

Theorem 4. *Suppose Γ is a graph in $\mathbf{R}^3$ with $\mathcal{C}_{\rm tot}(\Gamma) \leq 2\pi C_T$, and let $\Sigma \in \mathcal{S}_\Gamma$ be embedded as an $(\mathbf{M}, 0, \delta)$-minimizing set with respect to Γ. Then Σ has possibly Y singularities but no other singularities, unless it is a subset of the T stationary cone, with planar faces.*

Proof. As in the proof of the previous theorem, for each point p in Σ, we have a series of inequalities

$$\Theta_\Sigma(p) < \Theta_{C_p(\Gamma)}(p) \leq \frac{1}{2\pi}\mathcal{C}_{\rm tot}(\Gamma) \leq C_T,$$

unless Σ is a cone over p with planar faces. We now use results in [T](II.2 and II.3), which imply that the tangent cone of an $(\mathbf{M}, \varepsilon, \delta)$-minimal set S at p is area-minimizing with respect to the intersection with the unit sphere centered at p, and that the plane, the Y-cone and the T-cone are the only possibilities for the tangent cone. The inequality above implies that if there is a point p where $T_p(\Sigma)$ is any other cone than the plane or Y, then it can only be the T stationary cone. But in this case, after rotation, $\Theta_\Sigma(p) = C_T$, and Σ itself is a cone over p. It follows that $\Sigma =$ T. ∎

Remark 1. *A surface Σ in the class $\mathcal{S}_\Gamma$ which is $(\mathbf{M}, 0, \delta)$-minimal with Γ as its variational boundary is in particular strongly stationary with respect to Γ (See the remark preceding* **Definition 2**.*) However note that a cone over the one-skeleton Γ of the cube is strongly stationary w.r.t. Γ, but is not an $(\mathbf{M}, 0, \delta)$-minimal set.*

Remark 2. *The previous papers* [EWW] *and* [CG2] *had consequences for the knot class of a curve in a 3-dimensional manifold satisfying an inequality on its total curvature. Similar consequences for the isotropy class of a graph would follow from Theorems 3 and 4 if the boundary regularity of an area-minimizing rectifiable set bounded by a graph could be proved.*

Example 1. *This example shows that the hypothesis $\mathcal{C}_{\rm tot}(\Gamma) \leq 3\pi$ of Theorem 3 is sharp. Specifically, we construct a graph Γ in $\mathbf{R}^3$ with $\mathcal{C}_{\rm tot}(\Gamma) = 3\pi$, such that a subset of the minimal cone Y, including a nonempty segment of the singular line, is strongly stationary with respect to Γ.*

Recall the description of Y in Section 2 above: Y consists of three half-planes P_1, P_2, P_3 meeting along a line S, and making equal angles $2\pi/3$ at each point of S. Recall also the angle $R_0 = 1.33458$ radians $= 76.466^o$ of Corollary 2.

We choose two points $q^\pm$ along S, and construct Γ as the union of three C^2 convex plane arcs a_ℓ, where a_ℓ joins q^- to q^+ in the half-plane P_ℓ, $\ell = 1, 2, 3$, all making an angle $\alpha^\pm$ with S at the endpoint $q^\pm$, where $0 < \alpha^\pm \leq R_0$. Since a_ℓ is a convex plane arc, the integral of $|\vec{k}|$ along a_ℓ equals $\alpha^+ + \alpha^-$. Using Corollary 2, we may compute that the contribution at $q^\pm$ to the total curvature of Γ is $\mathrm{tc}(q^\pm) = 3(\pi/2 - \alpha^\pm)$. Thus $\mathcal{C}_{\mathrm{tot}}(\Gamma) = 3(\alpha^+ + \alpha^-) + 3(\pi/2 - \alpha^+) + 3(\pi/2 - \alpha^-) = 3\pi$, as claimed. ■

In Example 1, intuition might lead the reader to expect that every case, with a skinny or fat angle, would give rise to a sharp inequality. In fact, for the case $\alpha^\pm > R_0$, the inequality is *not* sharp, as follows using Corollary 2.

Example 1 illustrates that the upper bound 3π for $\mathcal{C}_{\mathrm{tot}}(\Gamma)$ is achieved for certain graphs Γ. The next proposition in turn says that among all the embedded graphs Γ which are homeomorphic to the graph of Example 1, 3π is the sharp lower bound for the total curvature $\mathcal{C}_{\mathrm{tot}}(\Gamma)$.

Proposition 5. *Let Γ be an embedding into $\mathbf{R}^3$ of the topological graph with exactly two vertices $q^\pm$ and three edges a_1, a_2 and a_3, each of which has endpoints q^+ and q^-. Then $\mathcal{C}_{\mathrm{tot}}(\Gamma) \geq 3\pi$. Moreover, equality holds if and only if each a_ℓ is a convex plane arc with unit tangent vectors $T_\ell^\pm$ at $q^\pm$ satisfying the condition that $\pm e := \pm \frac{q^- - q^+}{|q^- - q^+|}$ is a total distance minimizing point for the three points $T_1^\pm, T_2^\pm, T_3^\pm$ on S^2, at both q^- and q^+.*

Proof. The "if" part of the equality conclusion follows essentially from the discussion of Example 1 above. We have adapted the notation introduced there; further, let $\alpha_\ell^\pm$ be the angle between $T_\ell^\pm$ and the unit tangent vector $\pm e$ at $q^\pm$ to the closed line segment L joining $q^\pm$ to $q^\mp$. Then $a_\ell \cup L$ is a closed curve in $\mathbf{R}^3$, so by Fenchel's theorem

$$2\pi \leq \mathcal{C}_{\mathrm{tot}}(a_\ell \cup L) = \int_{a_\ell} |\vec{k}|\, ds + (\pi - \alpha_\ell^+) + (\pi - \alpha_\ell^-).$$

Thus $\int_{a_\ell} |\vec{k}|\, ds \geq \alpha_\ell^+ + \alpha_\ell^-$, with equality if and only if a_ℓ is a convex planar arc.

Meanwhile, $\mathrm{tc}(q^\pm) := \sup_{e'} \sum_{\ell=1}^3 \left(\frac{\pi}{2} - \beta_\ell^\pm(e')\right) \geq \sum_{\ell=1}^3 \left(\frac{\pi}{2} - \alpha_\ell^\pm\right)$. Further, equality holds if and only if $\pm e$ is a total distance minimizing point on S^2 for the three points $T_1^\pm, T_2^\pm, T_3^\pm$. Therefore,

$$\begin{aligned} \mathcal{C}_{\rm tot}(\Gamma) &:= \sum_{\ell=1}^{3} \int_{a_\ell} |\vec{k}|\, ds + {\rm tc}(q^+) + {\rm tc}(q^-) \\ &\geq \sum_{\ell=1}^{3} \Big[(\alpha_\ell^+ + \alpha_\ell^-) + (\frac{\pi}{2} - \alpha_\ell^+) + (\frac{\pi}{2} - \alpha_\ell^-) \Big] = 3\pi, \end{aligned}$$

with equality if and only if a_ℓ is a convex planar arc and $\pm e$ is the total distance minimizing point. ■

There is a second combinatorial structure for a connected graph Γ with two trivalent vertices and three edges: the "handcuff" consisting of two loops plus an arc joining the vertices of the loops. Similarly to Proposition 5, it may be shown that an embedding of such Γ in $\mathbf{R}^3$ must have total curvature at least 3π. In fact, it appears likely that the hypothesis of Theorem 3 can hold strictly only for the embedded circle or the two-leafed rose, that is, two circles connected at a point.

Example 2. *In this example, we shall show that the hypothesis* $\mathcal{C}_{\rm tot}(\Gamma) \leq 2\pi C_T$ *of Theorem 4 is sharp. In fact, the cone* Σ *over the one-skeleton* Γ *of the regular tetrahedron itself provides an example.*

Let α_T be the angle between an edge $\overline{q_k q_i}$ of Γ and $\overline{q_k p}$, $1 \leq k < i \leq 4$, where p is the center of the tetrahedron. Then $\cos(\alpha_T) = \sqrt{2/3}$, so $\alpha_T = 0.61548$ radians, which is less than $R_0 = 1.33458$ radians. This shows, using Corollary 2, that $\mathcal{C}_{\rm tot}(\Gamma) = 6\pi - 12\alpha_T$.

On the other hand, we may apply Theorem 2 above to compare the total curvature of Γ with the density of Σ at the interior singular point p. Namely, away from the vertices, the curvature vector $\vec{k} \equiv 0$. In the notation of Theorem 2, all twelve of the interior angles β_k^j ($1 \leq k \leq 6,\ j = 1, 2$) are equal to α_T. Therefore the density $2\pi C_T$ of the cone at p equals

$$\sum_{k=1}^{6} \sum_{j=1,2} \left(\frac{\pi}{2} - \alpha_T \right) = 6\pi - 12\alpha_T = \mathcal{C}_{\rm tot}(\Gamma).$$

■

The next example will be much more complex than those above.

Example 3. *In this example, we shall construct a graph* Γ *with* $\mathcal{C}_{\rm tot}(\Gamma) = 44\pi < 2\pi C_T$*, which is sufficiently complicated that the presence of a T* *singularity in a strongly stationary surface Σ might appear likely, although this would be excluded by Theorem 4 above.*

Let Γ be the union of eleven convex plane ovals. Γ will consist of six horizontal ovals in planes $\{z = c_k\}$, $1 \leq k \leq 6$, obtained from each other by translation in the z-direction; and five ovals in vertical planes $\{y = c_k\}$, $7 \leq k \leq 11$, obtained from each other by translation in the y-direction. For clarity, we assume that each of the eleven ovals includes two unit line segments tangent to the faces $\{x = 0\}$ and $\{x = 1\}$ of the unit cube. It follows that that each of the five vertical ovals meets each of the six horizontal ovals twice.

Then Γ has 60 vertices $q_1, \ldots, q_{60}$, each of valence $d = 4$, and at each vertex, the unit tangent vectors T_1, T_2, T_3, T_4 satisfy $T_3 = -T_1$ and $T_4 = -T_2$. Corollary 3 implies that $\mathrm{tc}(q_i) = 0$ ($i = 1, \ldots, 60$). Each of the eleven ovals contributes 2π to the total curvature of Γ_{reg}. Therefore $\mathcal{C}_{\mathrm{tot}}(\Gamma) = 44\pi < 2\pi C_T$. ■

6. NONZERO AMBIENT CURVATURE

In this section, we shall indicate the modifications which need to be made to generalize Theorems 1, 2, 3 and 4 above to the case where the ambient space $\mathbf{R}^n$ is replaced by a manifold M^n having variable sectional curvatures. In the case of an immersed minimal surface (or a branched immersion) with smooth boundary, the proof was carried out in [CG2]. The conclusions in subsection 6.2, however, are more general than those of [CG2], even in the case of a Jordan curve Γ, since [CG2] requires constant sectional curvature in the positive curvature case. This greater generality is obtained at the cost of a less geometric hypothesis involving the spherical area of cones in place of the area induced from M. Many of the proofs of [CG2] can be adapted with little change to the present context of singular minimal surfaces which are strongly stationary with respect to a graph Γ.

For the rest of this section, let M^n be a strongly convex Riemannian manifold having sectional curvatures bounded above by either (subsection 6.1) a non-positive constant $-\kappa^2$; or (subsection 6.2) a positive constant κ^2. M^n is said to be *strongly convex* if any two points are connected by a unique, minimizing, geodesic. For example, M^n might be a complete, simply connected Hadamard-Cartan manifold, or a locally convex open subset of such a complete manifold, or a locally convex open subset of a ball of radius π/κ in a complete, simply connected manifold M^n with sectional curvatures $K_M \leq \kappa^2$.

6.1. **Nonpositively Curved Manifold.** Throughout this subsection, let M^n be a strongly convex Riemannian manifold whose sectional curvatures are bounded above by a non-positive constant $-\kappa^2$. We consider a graph $\Gamma \subset M^n$ and a surface Σ in the class $\mathcal{S}_\Gamma$ which is strongly stationary with respect to Γ.

Choose a point p of Σ. We shall *assume* that Γ is nowhere tangent to the geodesic from p; the general cases of Theorems 5, 6, 7 and 8 below then follow by approximating Γ in $C^1 \cap W^{2,1}$ (cf. pp. 351–352 of [CG2]).

We shall compare Σ with the *geodesic cone* $C = C_p(\Gamma)$, which is formed from the minimizing geodesics joining p to points of Γ. C may naturally be given the Riemannian metric ds^2 induced from M^n. However, it should be observed that C with the metric ds^2 is not likely to be relevant to the strongly stationary surface Σ. In fact, Σ and the cone C over its boundary inhabit different regions of M^n, whose geometries are not related except by an upper bound on curvatures, so that one should not expect any useful comparison between them. For these reasons, we shall endow C with a second metric $d\widehat{s}^2$ of constant Gauss curvature $-\kappa^2$, such that the unit-speed geodesics from p to points of Γ, which generate $C = C_p(\Gamma)$, remain unit-speed geodesics in the metric $d\widehat{s}^2$, and so that $d\widehat{s}^2$ agrees with ds^2 on the tangent space to C at points of Γ [CG2]. For clarity, we shall refer to the cone with this hyperbolic metric as $\widehat{C} = \widehat{C}_p(\Gamma)$.

More precisely, let $a_j,\ 1 \le j \le m,$ be the smooth arcs of Γ, and let $A_j = C_p(a_j),\ 1 \le j \le m,$ be the two-dimensional fans of $C_p(\Gamma)$. On each A_j, let θ be a coordinate which is constant along each of the radial geodesics through p, and such that $\rho = \operatorname{dist}(\cdot, p)$ and θ form a local system of coordinates. We have assumed that Γ is nowhere tangent to the radial geodesic, which implies that θ may be used as a regular parameter along the closed arc a_j. Write $\rho =: r(\theta)$ for the corresponding values of $\rho := \operatorname{dist}_M(p, \cdot)$ along a_j, and let $r(\theta)$ be extended to $C_p(\Gamma)$ so that it is constant along each radial geodesic. Then $\rho < r(\theta)$ elsewhere on A_j. Note that under our assumption, there holds $|dr/d\theta| < ds/d\theta$ along Γ. We may now define the metric $d\widehat{s}^2$ on A_j by

$$d\widehat{s}^2 = d\rho^2 + \left[\left(\frac{ds}{d\theta}\Big|_\Gamma \right)^2 - \left(\frac{dr(\theta)}{d\theta} \right)^2 \right] \frac{\sinh^2 \kappa\rho}{\sinh^2 \kappa r(\theta)}\, d\theta^2.$$

We may observe that, along any radial geodesic, we have $d\widehat{s}^2 = d\rho^2 = ds^2$. In particular, if arcs a_j and a_k of Γ share a common endpoint q, then the hyperbolic metrics $d\widehat{s}^2$ defined on the fan A_j and $d\widehat{s}^2$ defined on A_k agree along their common edge, which is the minimizing geodesic from p to q. That is, $d\widehat{s}^2$ makes $\widehat{C}$ into a Riemannian polyhedron.

Another description of the hyperbolic metric $d\widehat{s}^2$ may be useful. The metric $d\widehat{s}^2$ has constant Gauss curvature $-\kappa^2$; each radial geodesic from p in the induced metric ds^2 remains a geodesic of equal length under $d\widehat{s}^2$; the length of any arc of Γ remains the same; and the angles formed by Γ and the radial geodesic from P remain the same.

Theorem 5. *Given a strongly stationary surface Σ in M^n of class $\mathcal{S}_\Gamma$, and a point $p \in M\backslash\Gamma$ of $\Sigma\backslash\Gamma$, the densities at p of Σ and of $\widehat{C}_p(\Gamma)$ may be compared:*

$$\Theta_\Sigma(p) \leq \Theta_{\widehat{C}_p(\Gamma)}(p).$$

Moreover, equality implies that Σ is a cone with totally geodesic faces of constant Gauss curvature $-\kappa^2$.

Proof. The proof is similar to the proof of Theorem 1, with certain modifications. The test function $G(x)$ is taken to be $\log\tanh(\kappa\rho(x)/2)$, rather than $\log\rho(x)$. Since the fans $\widehat{A}_j$ of $\widehat{C}_p(\Gamma)$ are locally isometric to the hyperbolic plane of constant Gauss curvature $-\kappa^2$, with $\rho(x)$ corresponding to the hyperbolic distance from a point, we may readily verify that $G(x)$ is harmonic on the fans of $\widehat{C}_p(\Gamma)$ away from p. If $e_n = \overline{\nabla}\rho$ and $e_1, \dots, e_{n-1}$ form an orthonormal frame on $M^n\backslash\{p\}$, then by the Hessian comparison theorem $\overline{\nabla}^2_{e_i,e_i}G \geq \frac{\kappa^2\cosh\kappa\rho}{\sinh^2\kappa\rho}$ for $i = 1, \dots, n-1$, and $\overline{\nabla}^2_{e_n,e_n}G = -\frac{\kappa^2\cosh\kappa\rho}{\sinh^2\kappa\rho}$ (See p. 4 of [SY] and [CG2]). It follows using the trace formula (6) that $G(x)$ is subharmonic on the faces Σ_i of the minimal polyhedral surface Σ. The factor $\frac{1}{\rho}$ appearing in boundary integrals in the proof of Theorem 1 is replaced by $\frac{\kappa}{\sinh(\kappa\rho)}$, which is the derivative of G with respect to ρ. Note that $-\kappa\,\mathrm{Length}(\widehat{C}\cap\partial B_\varepsilon(p))/\sinh(\kappa\varepsilon)$ is equal to $-2\pi\Theta_{\widehat{C}}(p)$, independent of sufficiently small $\varepsilon > 0$.

The remainder of the proof is as in the proof of Theorem 1. ∎

Theorem 6. *Let Γ be a graph in M^n, and choose $p \in M\backslash\Gamma$. Then the cone $\widehat{C} = \widehat{C}_p(\Gamma)$, with the hyperbolic metric $d\widehat{s}^2$, satisfies the density estimate*

$$2\pi\Theta_{\widehat{C}}(p) \leq -\sum_{k=1}^{n}\int_{a_k}\vec{k}\cdot\nu_C\,ds - \kappa^2\mathrm{Area}\,(C_p(\Gamma)) + \sum_k\sum_j\left(\frac{\pi}{2} - \beta_k^j\right),$$

where ν_C is the outward unit normal vector to $C_p(\Gamma)$; and at a vertex q_j of Γ, β_k^j is the angle, in the metric of M, between the edge a_k of Γ and the minimizing geodesic from $q_j \in \partial a_k$ to p.

Proof. The proof resembles the proof of Theorem 2 above, with appropriate modifications. We apply the Gauss-Bonnet formula (13) to the double $\widehat{C}_p(\Gamma')$ of the hyperbolic cone $\widehat{C} = \widehat{C}_p(\Gamma)$, and find

$$\int_{\widehat{C}\setminus B_\varepsilon(p)} K_{\widehat{C}}\, dA_{\widehat{C}} + \int_{\widehat{C}\cap\partial B_\varepsilon(p)} \widehat{k}\, d\widehat{s} + \int_{\Gamma_{\rm reg}} \widehat{k}\, d\widehat{s} + \sum_k \sum_j \left(\frac{\pi}{2} - \widehat{\beta}_j^k\right) = 0,$$

where $K_{\widehat{C}} \equiv -\kappa^2$ is the Gauss curvature of the fans of $\widehat{C}$; $\widehat{k}$ is the inward geodesic curvature along $\partial\left(\widehat{C}\setminus B_\varepsilon(p)\right)$; and $\widehat{\beta}_j^k$ is the angle formed by the edge a_k of Γ and the geodesic edge joining p to $q_j \in \partial a_k$, in the metric $d\widehat{s}^2$. But along $\partial B_\varepsilon(p) \cap \widehat{C}$, we have $\widehat{k} \equiv -\kappa \coth \kappa\varepsilon$ by a standard computation in the hyperbolic plane. Along Γ, $d\widehat{s}^2 = ds^2$, so that $\widehat{\beta}_j^k = \beta_j^k$, the corresponding angle in the induced metric ds^2. Further, for each $q \in \Gamma_{\rm reg}$, there holds $\widehat{k}(q) \le k(q)$, the geodesic curvature of Γ in the cone $C_p(\Gamma)$ with the induced metric ds^2 (see Proposition 4 of [CG2]). Thus

$$\kappa \coth \kappa\varepsilon\ \ \mathrm{Length}(\partial B_\varepsilon(p) \cap \widehat{C}) \le \tag{18}$$
$$-\kappa^2 \mathrm{Area}(\widehat{C}\setminus B_\varepsilon(p)) + \int_{\Gamma_{\rm reg}} k\, ds + \sum_k \sum_j \left(\frac{\pi}{2} - \beta_j^k\right).$$

Taking the limit as $\varepsilon \to 0$, we find

$$2\pi\Theta_{\widehat{C}}(p) \le -\int_{\Gamma_{\rm reg}} \nu_C \cdot \vec{k}\, ds + \sum_k \sum_j \left(\frac{\pi}{2} - \beta_j^k\right) - \kappa^2 \mathrm{Area}(\widehat{C}),$$

since for all $q \in \Gamma_{\rm reg}$, $k(q) = -\nu_C \cdot \vec{k}(q)$. Finally, $\mathrm{Area}(\widehat{C}) \ge \mathrm{Area}(C)$, as may be proved by applying Proposition 5 of [CG2] to each fan A_k of C. ∎

In order to state the following corollary and the next two theorems, it will be useful to make the following

Definition 4. $\mathcal{A}(\Gamma)$ *is the* minimum cone area *of* Γ*:*

$$\mathcal{A}(\Gamma) := \min_{p \in \mathcal{H}_{\rm cvx}(\Gamma)} \mathrm{Area}(C_p(\Gamma)).$$

Here, the convex hull $\mathcal{H}_{\rm cvx}(\Gamma)$ *of* Γ *in* M *is the intersection of closed, geodesically convex subsets of* M^n *which contain* Γ.

Corollary 6. *For a strongly stationary surface $\Sigma \in \mathcal{S}_\Gamma$ in a strongly convex manifold M^n with sectional curvatures $K_M \leq -\kappa^2$, the density estimate holds:*

$$2\pi\Theta_\Sigma(p) \leq \mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2\mathcal{A}(\Gamma).$$

Moreover, equality may only hold when Σ is itself a cone over p with totally geodesic faces of constant Gauss curvature $-\kappa^2$.

Proof. Theorem 6 estimates the hyperbolic cone density:
(19)

$$2\pi\Theta_{\widehat{C}}(p) \leq -\sum_{k=1}^{n}\int_{a_k}\vec{k}\cdot\nu_C\,ds + \sum_k\sum_j\left(\frac{\pi}{2}-\beta_k^j\right) - \kappa^2\text{Area}\Big(C_p(\Gamma)\Big).$$

Since Σ must lie in the convex hull $\mathcal{H}_{\text{cvx}}(\Gamma)$ by the maximum principle, we have $\text{Area}\,(C_p(\Gamma)) \geq \mathcal{A}(\Gamma)$. Also, $|\int_{\Gamma_{\text{reg}}}\vec{k}\cdot\nu_C\,ds| + \sum_k\sum_j(\frac{\pi}{2}-\beta_j^k) \leq \mathcal{C}_{\text{tot}}(\Gamma)$. Therefore, the right-hand side of inequality (19) is $\leq \mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2\mathcal{A}(\Gamma)$, while according to Theorem 5, the left-hand side is $\geq 2\pi\Theta_\Sigma(p)$. Moreover, if equality holds, then we must have equality in the conclusion of Theorem 5, implying that Σ must be a cone over p with totally geodesic faces of constant Gauss curvature $-\kappa^2$. ■

In the following two theorems, the total curvature of Γ is "corrected" by subtracting $\kappa^2\mathcal{A}(\Gamma)$. Without this improved hypothesis, Theorems 7 and 8 would have only extremely limited application for Γ of large diameter in manifolds M^n of uniformly negative sectional curvature (see Example 2 of [CG2]).

Theorem 7. *Suppose Γ is a graph in M^n with $\mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2\mathcal{A}(\Gamma) \leq 3\pi$, and let Σ in the class $\mathcal{S}_\Gamma$ be a strongly stationary surface relative to Γ. Then Σ is either an embedded minimal surface; or, a subset of a singular minimal cone with an interior edge where three convex, totally geodesic faces, of constant Gauss curvature $-\kappa^2$, meet at equal angles.*

Proof. Given $p \in \Sigma$, Corollary 6 above implies that

$$2\pi\Theta_\Sigma(p) \leq \mathcal{C}_{\text{tot}}(\Gamma) - \kappa^2\mathcal{A}(\Gamma).$$

Thus, the present hypothesis implies that $\Theta_\Sigma(p) \leq \frac{3}{2}$, and that equality may only hold when Σ is a geodesic cone over p with totally geodesic faces of Gaussian curvature $-\kappa^2$ (see Corollary 6). If $\Theta_\Sigma(p) < 3/2$, then Σ is embedded near p. If $\Theta_\Sigma(p) = 3/2$, then Σ is a geodesic cone, with tangent cone at p congruent to the Y stationary cone, and its faces are totally geodesic with Gauss curvature $\equiv -\kappa^2$. Since Σ is a totally geodesic cone of class $\mathcal{S}_\Gamma$, it is the exponential image of its tangent cone at p. It follows that the exponential map of M at p maps a subset of the Y cone in T_pM onto Σ. Finally, equality in the inequality $-\vec{k}\cdot\nu_C \le |\vec{k}\cdot\nu_C|$ implies convexity of $\Gamma_{\rm reg}$. ∎

Theorem 8. *Suppose Γ is a graph in M^3 with $\mathcal{C}_{\rm tot}(\Gamma) - \kappa^2\mathcal{A}(\Gamma) \le 2\pi C_T$, and let Σ be an embedded polyhedral surface of the class $\mathcal{S}_\Gamma$, which is strongly stationary and an $(\mathbf{M},\varepsilon,\delta)$-minimal set with respect to Γ. Then Σ is a surface with possibly Y singularities but no other singularities p, unless it is a geodesic cone over p with convex, totally geodesic faces of constant Gauss curvature $-\kappa^2$, and having tangent cone at p equal to the T stationary cone.*

Proof. Choose a point $p \in \Sigma$. Then with respect to a local geodesic coordinate chart centered at p, the surface Σ is an $(\mathbf{M},\varepsilon,\delta)$-minimal set with $\varepsilon(r) = Cr^\alpha$ for some sufficiently large $C > 0$ and some $\alpha > 0$. Here we again apply the set of results [T](II.2 and II.3) to conclude that the tangent cone $T_p\Sigma \subset T_pM^3 \cong \mathbf{R}^3$ is area minimizing and that the tangent cone can only be the plane, the Y-cone or the T-cone.

As in the proof of Theorem 7, we apply Corollary 6 to show that either $\Theta_\Sigma(p) < C_T$; or that $\Theta_\Sigma(p) = C_T$, Σ is a geodesic cone over p with convex, totally geodesic faces of constant Gauss curvature $-\kappa^2$, and Σ is the image under the exponential map of M at p of the T-cone. If $\Theta_\Sigma(p) < C_T$, then the tangent cone to Σ at p is either a plane or the Y stationary cone. If $T_p\Sigma$ is a plane, then Σ is an embedded surface in a neighborhood of p. If $T_p\Sigma$ is the Y stationary cone, then there are Y-type singularities along a curve passing through p. ∎

Remark 3. *In Theorems 7 and 8, the minimum cone area $\mathcal{A}(\Gamma)$ may be replaced by*

$$\inf_{\mathcal{H}_{\rm cvx}(\Gamma)} \mathrm{Area}(\widehat{C}_p(\Gamma)),$$

which may be larger (and thus better). See the proof of Theorem 10 below. We have chosen to write Theorems 7 and 8 in terms of the minimum cone area $\mathcal{A}(\Gamma)$, since this quantity is more closely related to the geometry of M. (If M has constant sectional curvature $-\kappa^2$, they are equal.)

6.2. **Ambient Curvature with Positive Upper Bound.** Throughout this subsection, we shall assume that M^n is a strongly convex Riemannian manifold whose sectional curvatures are bounded above by a positive constant

κ^2. Consider a graph $\Gamma \subset M^n$ with diameter $< \pi/\kappa$ and a surface Σ of the regularity class $\mathcal{S}_\Gamma$ which is strongly stationary with respect to Γ.

Choose a point p of Σ, $p \notin \Gamma$. As in subsection 6.1, we shall assume that Γ is nowhere tangent to the minimizing geodesic from p. The general cases of the results of this subsection follow by $C^2 \cap W^{2,1}$ approximation to Γ.

Since M^n is strongly convex, the unique minimizing geodesic joining p to q varies smoothly as a function of q. Therefore, the geodesic cone $C = C_p(\Gamma)$, with the Riemannian metric ds^2 induced from M, is a Riemannian polyhedron enjoying the same smoothness as Γ. Further, Σ lies in $\mathcal{H}_{\mathrm{cvx}}(\Gamma)$. Since Γ has diameter $< \pi/\kappa$, Γ lies in the open ball $B_{\pi/\kappa}(p)$. The cone C will be given a second Riemannian metric $d\widehat{s}^2$, the spherical metric, so that the fans of the cone have constant Gauss curvature κ^2, so that the ambient distance ρ to the point p remains equal to the distance in either metric ds^2 or $d\widehat{s}^2$, and so that at points of Γ, $d\widehat{s}^2 = ds^2$. We may describe the spherical metric at a point q of C as

$$d\widehat{s}^2 = d\rho^2 + \frac{\sin^2 \kappa\rho}{\sin^2 \kappa r(q)} \left[\left(ds\Big|_\Gamma \right)^2 - \Big(dr(q) \Big)^2 \right].$$

As in subsection 6.1, $r(q) < \pi/\kappa$ denotes $\rho(Q)$, the distance in M from p to the point Q of Γ along the radial geodesic from p passing through q; also, the one-form $ds\Big|_\Gamma$ has been extended to the cone so that it is invariant under radial deformations. Note that $ds\Big|_\Gamma(\partial/\partial\rho) = dr(\partial/\partial\rho) = 0$. We use the notation $\widehat{C} = \widehat{C}_p(\Gamma)$ for the cone C with this spherical metric $d\widehat{s}^2$.

In this section, it will be useful to state theorems in terms of a *maximum* cone area, rather than the *minimum* cone area which was of use in subsection 6.1. To account for the positive sectional curvature which may occur in M, we will need to add a term $\kappa^2 \widehat{\mathcal{A}}(\Gamma)$ to the total curvature $\mathcal{C}_{\mathrm{tot}}(\Gamma)$. When the sectional curvatures of M are nearly equal to the constant κ^2, the theorems below are nearly sharp.

Definition 5. $\widehat{\mathcal{A}}(\Gamma)$ *is the* maximum spherical cone area *of* Γ*:*

$$\widehat{\mathcal{A}}(\Gamma) := \sup_{p \in \mathcal{H}_{\mathrm{cvx}}(\Gamma)} \mathrm{Area}(\widehat{C}_p(\Gamma)).$$

Theorem 9. *Suppose* $K_M \leq +\kappa^2$ *and that* Γ *has diameter* $< \pi/\kappa$*. For any strongly stationary surface* Σ *in* M^n *of class* $\mathcal{S}_\Gamma$*, and a point* p *of* $\Sigma \backslash \Gamma$*, the following inequality holds:*

$$\Theta_\Sigma(p) \leq \Theta_{\widehat{C}_p(\Gamma)}(p).$$

Moreover, equality implies that Σ *is a cone with totally geodesic faces of constant Gauss curvature* κ^2*.*

Proof. Analogous to the proof of Theorem 5, but replacing hyperbolic functions with circular functions throughout. Specifically, the test function $G(x) = \log\tan(\kappa\rho(x)/2)$ in place of $\log\tanh(\kappa\rho(x)/2)$. Note that for $p \in \Sigma$, $x \in \Sigma$ or $x \in C = C_p(\Gamma)$, we have $\rho(x) < \pi/\kappa$. This implies that the metric $\widehat{ds}^2$ exists on C. Also, the test function G is smooth, with $\partial G/\partial\rho > 0$, on both Σ and C. ∎

Theorem 10. *Let* Γ *be a graph in* M^n *of diameter* $< \pi/\kappa$*, and choose* $p \in \mathcal{H}_{\rm cvx}(\Gamma)\backslash\Gamma$*. Then the cone* $\widehat{C} = \widehat{C}_p(\Gamma)$*, with the spherical metric* $\widehat{ds}^2$*, satisfies the density estimate*

$$2\pi\Theta_{\widehat{C}}(p) \le -\sum_{k=1}^{n}\int_{a_k}\vec{k}\cdot\nu_C\,ds + \kappa^2\mathrm{Area}\left(\widehat{C}_p(\Gamma)\right) + \sum_k\sum_j\left(\frac{\pi}{2} - \beta_k^j\right),$$

where $\nu_C = \nu_{\widehat{C}}$ *is the outward unit normal vector to* $C_p(\Gamma)$*; and* β_k^j *is the angle, in the metric of* M*, between the edge* a_k *of* Γ *and the minimizing geodesic in* M *from the vertex* $q_j \in \partial a_k$ *to* p*.*

Proof. The demonstration, which is based on the Gauss-Bonnet formula on $\widehat{C}$, is analogous to the proof of Theorem 6; the *statement* has been modified, however, since in the middle term on the right-hand side of equation (18), $\mathrm{Area}(\widehat{C})$ was multiplied by the non-positive $-\kappa^2$ and could therefore be replaced in the conclusion of Theorem 6 with the smaller quantity $\mathrm{Area}(C)$. Here, however, the Gauss curvature of $\widehat{C}$ is κ^2, which is positive, so that the spherical area $\mathrm{Area}(\widehat{C})$ of the cone must remain on the right-hand side of the inequality. ∎

Corollary 7. *Let* M^n *be a manifold with sectional curvatures* $K_M \le +\kappa^2$*. The density of a surface* Σ *in* M*, strongly stationary with respect to a graph* Γ *of diameter* $< \pi/\kappa$*, satisfies the inequality:*

$$2\pi\Theta_\Sigma(p) \le \mathcal{C}_{\rm tot}(\Gamma) + \kappa^2\widehat{\mathcal{A}}(\Gamma).$$

Moreover, equality may only hold when Σ *is itself a cone over* p *with totally geodesic faces of constant Gauss curvature* κ^2*.*

Proof. Theorem 9 estimates the density $\Theta_\Sigma(p) \leq \Theta_{\widehat{C}_p(\Gamma)}(p)$. Meanwhile, by Theorem 10,

$$2\pi\Theta_{\widehat{C}}(p) \leq -\sum_{k=1}^{n} \int_{a_k} \vec{k}\cdot \nu_C \, ds \tag{20}$$

$$+ \sum_k \sum_j \left(\frac{\pi}{2} - \beta_k^j\right) + \kappa^2 \mathrm{Area}\left(\widehat{C}_p(\Gamma)\right). \tag{21}$$

Since Σ lies in the convex hull $\mathcal{H}_{\mathrm{cvx}}(\Gamma)$ by the maximum principle, we have $\mathrm{Area}\left(\widehat{C}_p(\Gamma)\right) \leq \widehat{\mathcal{A}}(\Gamma)$. Also, by definition of total curvature, $|\int_{\Gamma_{\mathrm{reg}}} \vec{k}\cdot\nu_C\, ds| + \sum_k \sum_j (\frac{\pi}{2} - \beta_j^k) \leq \mathcal{C}_{\mathrm{tot}}(\Gamma)$. Therefore, $2\pi\Theta_\Sigma(p) \leq \mathcal{C}_{\mathrm{tot}}(\Gamma) + \kappa^2\mathcal{A}(\Gamma)$. Moreover, if equality holds, then we must have equality in the conclusion of Theorem 9, implying that Σ must be a geodesic cone over p with totally geodesic faces of constant Gauss curvature $+\kappa^2$. ■

The proofs of our final two theorems are completely analogous to the proofs of Theorems 7 and 8.

Theorem 11. *Suppose Γ is a graph in M^n of diameter $< \pi/\kappa$, with $\mathcal{C}_{\mathrm{tot}}(\Gamma) + \kappa^2\widehat{\mathcal{A}}(\Gamma) \leq 3\pi$, and let Σ be a strongly stationary surface relative to Γ in the class $\mathcal{S}_\Gamma$. Then Σ is either an embedded minimal surface or a subset of a singular minimal cone with an interior edge where three convex, totally geodesic faces, of constant Gauss curvature κ^2, meet at equal angles.*

Theorem 12. *Suppose Γ is a graph in M^3 with $\mathcal{C}_{\mathrm{tot}}(\Gamma) + \kappa^2\widehat{\mathcal{A}}(\Gamma) \leq 2\pi C_T$ and diameter $< \pi/\kappa$, and let Σ be an embedded polyhedral surface of the class $\mathcal{S}_\Gamma$, which is strongly stationary and an $(\mathbf{M}, \varepsilon, \delta)$-minimal set with respect to Γ. Then Σ is a surface with possibly Y singularities but no other singularities p, unless it is a geodesic cone over p with convex, totally geodesic faces of constant Gauss curvature κ^2, and having tangent cone at p equal to the T stationary cone.*

REFERENCES

[AA] W. Allard and F. Almgren, *The structure of stationary one dimensional varifolds with positive density*, Invent. Math **34** (1976), 83–97.

[Al] W. Allard, *On the first variation of a varifold,* Annals of Math. **95** (1972), 417–491.

[Alm] F. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints,* Memoirs AMS **165** (1976).

[AT] F. Almgren and J. Taylor, *Geometry of soap films,* Sci. Am. **235** (1976), 82–93.

[CG1] J. Choe and R. Gulliver, *Isometric inequalities on minimal submanifolds of space forms*, Manuscripta Math. **77** (1992), 168–189.

[CG2] J. Choe and R. Gulliver, *Embedded minimal surfaces and total curvature of curves in a manifold*, Math. Research Letters **10** (2003), 343–362.

[D1] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33**(1931), 263–321.

[D2] J. Douglas, *Minimal surfaces of higher topological structure.*, Annals of Math. **40** (1939), 205–298.

[EWW] T. Ekholm, B. White, and D. Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most* 4π, Annals of Mathematics **155** (2002), 109–234.

[Fed] H. Federer, *Geometric Measure Theory*. Springer-Verlag, New York 1969.

[GY2] R. Gulliver and S. Yamada, *Density of area-minimizing varifolds spanning graphs*, in preparation 2004.

[KNS] D. Kinderlerer, L. Nirenberg and J. Spruck. *Regularity in elliptic free boundary problems, I*, J. Amer. Math. Soc. **34** (1978), 86–119.

[MY] C. Mese and S. Yamada, *The parameterized Steiner problem and the singular Plateau problem via energy*, preprint 2004.

[N3] J. C. C. Nitsche, *Lectures on Minimal Surfaces*, vol. 1. Cambridge Univ. Press, Cambridge 1989.

[O] Øystein Ore, *Graphs and their Uses*. Random House, New York 1963.

[R] T. Radó, *On the Problem of Plateau*. Springer, 1971.

[SY] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*. International Press, Boston 1994.

[Si] L. Simon, *Lectures on Geometric Measure Theory*. Proceedings of the Centre for Math. Anal., vol.3, Australian National University Centre for Mathematical Analysis, Canberra, Australia, 1983.

[T] J. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like surfaces*, Annals of Mathematics **103** (1976), 489–539.

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