WIENER CRITERIA AND ENERGY DECAY FOR RELAXED DIRICHLET PROBLEMS

BY

GIANNI DAL MASO

AND

UMBERTO MOSCO

IMA Preprint Series # 197

November 1985

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

Recent IMA Preprints (Limited number of preprints are available on request.)

#	Author(s), Title
40	**William Ruckle**, The Strong ¢ Topology on Symmetric Sequence Spaces
41	**Charles R. Johnson**, A Characterization of Borda's Rule Via Optimization
42	**Hans Weinberger, Kazuo Kishimoto**, The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains
43	**K.A. Pericak-Spector, W.O. Williams**, On Work and Constraints in Mixtures
44	**H. Rosenberg, E. Toubiania**, Some Remarks on Deformations of Minimal Sufaces
45	**Stephan Pelikan**, The Duration of Transients
46	**V. Capasso, K.L. Cooke, M. Witten**, Random Fluctuations of the Duration of Harvest
47	**E. Fabes, D. Stroock**, The L^p-Intergrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations
48	**H. Brezis**, Semilinear Equations in R^n without Conditions at Infinity
49	**M. Slemrod**, Lax-Friedrichs and the Viscosity-Capillarity Criterion
50	**C. Johnson, W. Barrett**, Spanning Tree Extensions of the Hadamard-Fischer Inequalities
51	**Andrew Postlewaite, David Schmeidler**, Revelation and Implementation under Differential Information
52	**Paul Blanchard**, Complex Analytic Dynamics on the Riemann Sphere
53	**G. Levitt, H. Rosenberg**, Topology and Differentiability of Labyrinths in the Disc and Annulus
54	**G. Levitt, H. Rosenberg**, Symmetry of Constant Mean Curvature Hypersurfaces in Hyperbolic Space
55	**Ennio Stacchetti**, Analysis of a Dynamic, Decentralized Exchange Economy
56	**Henry Simpson, Scott Spector**, On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics
57	**Craig Tracy**, Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations
58	**Tongren Ding**, Boundedness of Solutions of Duffing's Equation
59	**Abstracts** for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles
60	**Rafael Rob**, The Coase Theorem an Informational Perspective
61	**Joseph Jerome**, Approximate Newton Methods and Homotopy for Stationary Operator Equations
62	**Rafael Rob**, A Note on Competitive Bidding with Asymmetric Information
63	**Rafael Rob**, Equilibrium Price Distributions
64	**William Ruckle**, The Linearization Projection, Global Theories
65	**Russell Johnson, Kenneth Palmer, George R. Sell**, Ergodic Properties of Linear Dynamical Systems
66	**Stanley Reiter**, How a Network of Processors can Schedule its Work
67	**R.N. Goldman, D.C. Heath**, Linear Subdivision is Strictly a Polynomial Phenomenon
68	**R. Giachett, R. Johnson**, The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients
69	**Steve Williams**, Realization and Nash Implementation: Two Aspects of Mechanism Design
70	**Steve Williams**, Sufficient Conditions for Nash Implementation
71	**Nicholas Yannelis, William R. Zame**, Equilibria in Banach Lattices Without Ordered Preferences
72	**W. Harris, Y. Sibuya**, The Reciprocals of Solutions of Linear Ordinary Differential Equations
73	**Steve Pelikan**, A Dynamical Meaning of Fractal Dimension
74	**D. Heath, W. Sudderth**, Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach a Goal
75	**J.S. Jordan**, Information Flows Intrinsic to the Stability Economic Equilibrium
76	**J. Jerome**, An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition
77	**David Schmeidler**, Integral Representation Without Additivity
78	**Abstracts** for the Workshop on Bayesian Analysis in Economics and Game Theory
79	**G. Chichilnisky, G.M. Heal**, Existence of a Competitive Equilibrium in L^p and Sobolev Spaces
80	**Thomas Seidman**, Time-dependent Solutions of a Nonlinear System in Semiconductory Theory, II: Boundedness and Periodicity
81	**Yakar Kannai**, Engaging in R&D and the Emergence of Expected Non-convex Technologies
82	**Herve Moulin**, Choice Functions over a Finite Set: A Summary
83	**Herve Moulin**, Choosing from a Tournament
84	**David Schmeidler**, Subjective Probability and Expected Utility Without Additivity
85	**I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan**, The Numerical Computation of Invariant Circles of Maps
86	**F. William Lawvere**, State Categories, Closed Categories, and the Existence Semi-Continuous Entropy Functions
87	**F. William Lawvere**, Functional Remarks on the General Concept of Chaos
88	**Steven R. Williams**, Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process
89	**Steven R. Williams**, Implementing a Generic Smooth Function
90	**Dilip Abreu**, Infinitely Repeated Games with Discounting: A General Theory
91	**J.S. Jordan**, Instability in the Implementation of Walrasian Allocations
92	**Myrna Holtz Wooders, William R. Zame**, Large Games: Fair and Stable Outcomes
93	**J.L. Noakes**, Critical Sets and Negative Bundles
94	**Graciela Chichilnisky**, Von Neumann-Morgenstern Utilities and Cardinal Preferences
95	**J.L. Ericksen**, Twinning of Crystals
96	**Anna Nagurney**, On Some Market Equilibrium Theory Paradoxes
97	**Anna Nagurney**, Sensitivity Analysis for Market Equilibrium
98	**Abstracts** for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations
99	**Millard Beatty**, A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability
100	**Filomena Pacella**, Central Configurations of the N-Body Problem via the Equivariant Morse Theory
101	**D. Carlson and A. Hoger**, The Derivative of a Tensor-valued Function of a Tensor
102	**Kenneth Mount**, Privacy Preserving Correspondence
103	**Millard Beatty**, Finite Amplitude Vibrations of a Neo-hookean Oscillator
104	**D. Emmons and N. Yannelis**, On Perfectly Competitive Economies: Loeb Economies
105	**E. Mascolo and R. Schianchi**, Existence Theorems in the Calculus of Variations
106	**D. Kinderlehrer**, Twinning of Crystals (II)
107	**R. Chen**, Solutions of Minimax Problems Using Equivalent Differentiable Equations
108	**D. Abreu, D. Pearce, and E. Stacchetti**, Optimal Cartel Equilibria with Imperfect Monitoring
109	**R. Lauterbach**, Hopf Bifurcation from a Turning Point
110	**C. Kahn**, An Equilibrium Model of Quits under Optimal Contracting
111	**M. Kaneko and M. Wooders**, The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results
112	**Haim Brezis**, Remarks on Sublinear Equations
113	**D. Carlson and A. Hoger**, On the Derivatives of the Principal Invariants of a Second-order Tensor
114	**Raymond Deneckere and Steve Pelikan**, Competitive Chaos
115	**Abstacts** for the Workshop on Homogenization and Effective Moduli of Materials and Media
116	**Abstracts** for the Workshop on the Classifying Spaces of Groups
117	**Umberto Mosco**, Pointwise Potential Estimates for Elliptic Obstacle Problems
118	**J. Rodrigues**, An Evolutionary Continuous Casting Problem of Stefan Type
119	**C. Mueller and F. Weissler**, Single Point Blow-up for a General Semilinear Heat Equation

WIENER CRITERIA AND ENERGY DECAY FOR RELAXED DIRICHLET PROBLEMS

Gianni Dal Maso and Umberto Mosco

INTRODUCTION

In its simplest form, a relaxed Dirichlet problem in an open region Ω of R^n, $n \geq 2$, can be formally written as

(1) $$-\Delta u + \mu u = 0 \quad \text{in} \quad \Omega$$

where Δ is the Laplace operator and μ is an arbitrary non-negative Borel measure in R^n. The measure μ must vanish on sets of (harmonic) capacity zero in R^n, but may take the value $+\infty$ on some large subset of R^n.

Special cases of (1) are the Dirichlet problems of the type

(2) $$-\Delta u = 0 \quad \text{in} \quad \Omega - \overline{E}, \quad u = 0 \quad \text{on} \quad E$$

as well as the stationary Schrödinger equations

(3) $$-\Delta u + q(x)u = 0 \quad \text{in} \quad \Omega$$

for a non-negative potential $q(x)$.

Problem (2) corresponds to the measure

(4) $$\mu = \infty_E$$

that takes the value $+\infty$ on every (Borel) subset of R^n intersecting the given E in a set of positive capacity and the value 0 otherwise, while equation (3) occurs when μ has a (Borel) density $q(x)$ with respect to the Lebesgue measure $\mathcal{L}_n$ in R^n, i.e.

(5) $$\mu = q(x)\mathcal{L}_n .$$

More general problems like (1) do arise naturally as asymptotic equations satisfied by the limit u of sequences of "perturbed" solutions u_h of Dirichlet problems (2), where the set $E = E_h$ may vary, or of Schrödinger equations (3) with a varying potential $q_h(x)$.

Equations such as (1) were called in [2] "relaxed Dirichlet problems" to stress the fact that homogeneous Dirichlet conditions, such as $u_h = 0$ on the set E_h, may take in the limit as $h \to \infty$ the relaxed form of the "penalization" term μu appearing in (1).

We refer to [2] for some examples of asymptotic behavior and for references to the literature on this kind of problems.

As shown in [2], the class $\mathcal{M}_0$ of all measures allowed in (1) turns out to be a natural variational closure of the class of "Dirichlet" measures (4), as well as of the class of "Schrödinger" measures (5). A convergence of functional variational type, called γ-convergence, can in fact be defined in $\mathcal{M}_0$, which fits all the relevant features of the perturbations we are interested in and is such, at the same time, to make the set $\mathcal{M}_0$ (sequentially) compact and the class of all measures (4), as that of all measures (5), dense in $\mathcal{M}_0$.

By relying on these density results, that permit us, in particular, to approximate a given equation (1) by a sequence of Dirichlet problems of the form (2), a pointwise study was carried out in [2] of the local weak solutions of (1) and stable estimates, with respect to γ-convergence, were obtained at an arbitrary given point of the domain.

We should point out in this regard that since perturbed solutions can only be expected to converge in a weak topology, allowing for wild oscillations of the u_h in their domain as $h \to +\infty$, it is not at all obvious "a priori" that significant stable pointwise properties should indeed exist.

The main goal of the present paper is to develop the pointwise analysis of [2] for a more general class of equations.

These are of the form

$$(6) \qquad Lu + \mu u = \nu \quad \text{in} \quad \Omega,$$

where

$$(7) \qquad Lu = -\sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j})$$

is any uniformly elliptic operator with bounded (Lebesgue) measurable coefficients in R^n, ν is some given (signed) Radon measure in R^n and μ is an arbitrary given Borel measure of the class $\mathcal{M}_0$.

We will consider an arbitrary <u>local weak solution</u> u of (1) in the space $H^1_{loc}(\Omega) \cap L^2_{loc}(\Omega,\mu)$ of functions of finite local μ-<u>energy</u>

$$\int_{\Omega'} |Du|^2 dx + \int_{\Omega'} u^2 d\mu, \qquad \Omega' \subset\subset \Omega,$$

and we will study u at an arbitrary point x_0 of Ω.

As in [2], our results will be expressed in terms of the <u>Wiener modulus</u> of μ at the given x_0. This is a function

$$\omega(r,R) = \omega_\mu(x_0;r,R)$$

of $0 < r \leqslant R$, whose definition relies on an appropriate notion of μ-<u>capacity</u> of a set of R^n (see Section 5).

We will first establish a <u>variational Wiener Criterion</u> for equations (6), that extends the classical Wiener's criterion of potential theory [13], as well as its generalization to operators of the form (7) given by Littman-Stampacchia-Weinberger in [8].

Such a criterion characterizes those points x_0 of R^n having the property that every local weak solution u of (6) in a neighborhood of x_0 is continuous and vanishes at x_0, as those points x_0 of R^n at which the Wiener modulus of μ vanishes as $r \to 0^+$ for some fixed $R > 0$ (see Theorem 5.5). The former are called in [2] regular Dirichlet points of μ, the latter Wiener points of μ.

We will also show that the modulus of continuity of u at any Wiener point x_0 of μ can be estimated using only the L^2 norm of u, the norm of ν in the class K_n of Kato [6] and Aizenman-Simon [1] and the Wiener modulus of μ at x_0. Moreover, the estimate is uniform with respect to all operators L sharing the same ellipticity and boundedness constants. This extends classical estimates for the boundary regularity of Dirichlet problems, due to Maz'ja [9].

Similar estimates are also given for the decay of the μ-energy

$$E_\mu(r) = \int_{B_r} |Du|^2 dx + \int_{B_r} u^2 \, d\mu$$

on balls $B_r = B_r(x_0)$ as $r \to 0^+$.

All these estimates are derived from a structural estimate of the ratio

$$V(r)/V(R)$$

on two concentric balls, $0 < r \leq R$, of the quantity

$$V(r) = \sup_{B_r} u^2 + \int_{B_r} |Du|^2 |x - x_0|^{2-n} dx + \int_{B_r} u^2 |x - x_0|^{2-n} d\mu$$

when $n \geq 3$, or

$$V(r) = \sup_{B_r} u^2 + \int_{B_r} |Du|^2 \log\left(\frac{2R}{|x - x_0|} \right) dx + \int_{B_r} u^2 \log\left(\frac{2R}{|x - x_0|} \right) d\mu$$

when $n = 2$.

This estimate has the form

(8) $$V(r) \leq k\omega(r,R)^{\beta}V(R) + k\|\nu\|^2_{K_n(B_R)}$$

for every $0 < r \leq R$, $B_R = B_R(x_0) \subseteq \Omega$, where $\omega_\mu = \omega_\mu(x_0;r,R)$ is the Wiener modulus of μ at x_0, the norm of ν is taken in the Kato space (see Section 4) and $k > 0$ and $\beta > 0$ are suitable constants that depend only on the dimension n of the space and on the ellipticity and boundedness constants of L (see Theorem 6.2).

Let us point out an important feature of the estimate (8), that is, its stability under γ-convergence of the measure μ in $\mathcal{M}_0$. In fact, μ appears in the right hand side of (8) only via its Wiener modulus ω_μ: This has been proven in [2] to have the stability property

$$\omega_{\mu_h}(x_0;r,R) \to \omega_\mu(x_0;r,R)$$

for every $x_0 \in R^n$ and every $0 < r \leq R$, whenever the sequence μ_h γ-converges to μ in $\mathcal{M}_0$ as $h \to +\infty$.

Let us also remark that this stability of (8) provides the link between the results of the present paper and the perturbation theory of [2].

If the measure μ in (6) does not charge a neighborhood of x_0, that is $\mu(B_R(x_0)) = 0$ for some $R > 0$, then the equation (6) obviously reduces, locally around x_0, to the equation

$$Lu = \nu$$

for which estimates of the oscillation of u and of the energy

$$E(u) = \int_{B_r} |Du|^2 dx$$

on a ball $B_r = B_r(x_0)$ as $r \to 0^+$ are well known. They can also be obtained by simple variants in our proofs. These estimates, however, are not stable under γ-convergence of the measure μ of (6), since the condition that μ does not charge a neighborhood of x_0 clearly is not stable.

Finally, let us mention that a more detailed analysis of the Wiener modulus will be carried out in a forthcoming paper [3] for measures μ which are _rotationally_ _invariant_ in R^n and applications will be given to Schrödinger equations (3) with a _radial_ potential $q(|x|)$.

1. NOTATION AND PRELIMINARIES

Throughout the paper we denote by n a fixed integer, with $n \geq 2$.

1.1 Let Ω be a _bounded_ open subset of R^n. For every compact set $K \subseteq \Omega$ the _capacity_ of K with respect to Ω is defined by

$$\mathrm{cap}(K,\Omega) = \inf\{\int_\Omega |D\phi|^2 dx : \phi \in C_0^\infty(\Omega), \quad \phi \geq 1 \quad \text{on} \quad K\}.$$

The definition is extended to open sets $G \subseteq \Omega$ by

$$\mathrm{cap}(G,\Omega) = \sup\{\mathrm{cap}(K,\Omega) : K \subseteq G, \quad K \text{ compact}\}$$

and to arbitrary sets $E \subseteq \Omega$ by

$$\mathrm{cap}(E,\Omega) = \inf\{\mathrm{cap}(G,\Omega) : G \supseteq E, \ G \quad \text{open}\}.$$

We say that a set $E \subseteq R^n$ has _capacity_ _zero_ if

$$\mathrm{cap}(E \cap \Omega, \Omega) = 0$$

for every bounded open set $\Omega \subseteq R^n$. It is easy to see that a bounded set $E \subseteq R^n$ has capacity zero if and only if $\mathrm{cap}(E,\Omega) = 0$ for one (hence for all) bounded open set $\Omega \subseteq R^n$ such that $E \subseteq \Omega$.

If a property $A(x)$ holds for all $x \in E$ except for a subset E_0 of E with capacity zero, then we say that $A(x)$ holds _quasi_ _everywhere_ in E (q.e.

in E).

1.2 Let Ω be an arbitrary open subset of R^n. We denote by $H^{1,p}(\Omega)$ $1 < p < +\infty$, the Sobolev space of all functions $u \in L^p(\Omega)$ with distribution derivatives $D_i u \in L^p(\Omega)$, $i = 1,\dots,n$. The space $H^{1,p}(\Omega)$ is normed by

$$\|u\|_{H^{1,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)})^{1/p},$$

where $Du = (D_1u,\dots,D_nu)$ is the gradient of u. By $H^{1,p}_{loc}(\Omega)$ we denote the set of functions $u \in L^p_{loc}(\Omega)$ such that $u|_{\Omega'} \in H^{1,p}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ (i.e. $\overline{\Omega}'$ compact and $\overline{\Omega}' \subseteq \Omega$). By $H^{1,p}_c(\Omega)$ we denote the set of functions of $H^{1,p}(\Omega)$ with compact support in Ω. By $H^{1,p}_0(\Omega)$ we denote the closure of $H^{1,p}_c(\Omega)$ in $H^{1,p}(\Omega)$. As usual, if $p = 2$ we omit the exponent p in the above notation, thus $H^1(\Omega) = H^{1,2}(\Omega)$, $H^1_c(\Omega) = H^{1,2}_c(\Omega)$, etc.. By $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$. By $H^{-1}_{loc}(\Omega)$ we denote the set of linear functions f on $H^1_c(\Omega)$ such that $f|_{\Omega'} \in H^{-1}(\Omega')$ for every open set Ω' Ω. By $\langle\cdot,\cdot\rangle$ we denote the dual pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$, as well as its extension to the duality between $H^{-1}_{loc}(\Omega)$ and $H^1_c(\Omega)$.

For every $x \in R^n$ and every $r > 0$ we set

$$B_r(x) = \{y \in R^n: |y - x| < r\},$$

and we denote by $|B_r(x)|$ its Lebesgue measure.

It is well known that for every $u \in H^1_{loc}(\Omega)$ the limit

$$\lim_{r\to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy$$

exists and is finite quasi everywhere in Ω.

We make the following convention about the pointwise values of a function $u \in H^1_{loc}(\Omega)$: for every $x \in \Omega$ we always require that

$$\text{(1.1)} \qquad \liminf_{r\to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \le u(x) \le \limsup_{r\to 0_+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy.$$

With this convention, the pointwise value $u(x)$ is determined quasi everywhere in Ω and the function u is quasi continuous in Ω.

If Ω is bounded, it can be proved that

$$\mathrm{cap}(E,\Omega) = \min\{ \int_\Omega |Du|^2 dx \colon u \in H_0^1(\Omega);\ u \ge 1 \ \text{ q.e. in } E\}.$$

For the preceding capacity properties see e.g. [5].

Given two functions u and v defined in Ω, we denote by $u \wedge v$ and $u \vee v$ the functions defined in Ω by

$$(u \wedge v)(x) = \min\{u(x),v(x)\}, \qquad (u \vee v)(x) = \max\{u(x),v(x)\}.$$

It is well known that, if u and v belong to $H^1(\Omega)$ (resp. $H^1_{loc}(\Omega)$, $H^1_c(\Omega)$, $H^1_0(\Omega)$), then $u \wedge v$ and $u \vee v$ belong to $H^1(\Omega)$ (resp. $H^1_{loc}(\Omega)$, $H^1_c(\Omega)$, $H^1_0(\Omega)$).

1.3 Let Ω be an arbitrary open subset of R^n. By a <u>non-negative Borel measure</u> on Ω we mean a countably additive set function defined in the Borel σ-field of Ω and with values in $[0,+\infty]$.

If μ is a non-negative Borel measure on Ω, we denote by $L^p(\Omega,\mu)$ (resp. by $L^p_{loc}(\Omega,\mu)$), $1 \le p < +\infty$, the set of all [μ-equivalence classes of] <u>Borel functions</u> $u\colon \Omega \to R$ such that

$$\int_\Omega |u|^p d\mu < +\infty$$

(resp. such that

$$\int_K |u|^p d\mu < +\infty$$

for every compact set $K \subseteq \Omega$). If μ is the Lebesgue measure, the corresponding spaces will be denoted by $L^p(\Omega)$ and $L^p_{loc}(\Omega)$.

By $\mathcal{M}_0(\Omega)$ we denote the set of all <u>non-negative Borel measures</u> μ on Ω such that $\mu(E) = 0$ for every Borel set $E \subseteq \Omega$ with <u>capacity zero</u>.

By a <u>Radon measure</u> on Ω we mean a countably additive set function, with values in $\mathbb{R}$, defined on the δ-ring of all Borel sets $E \subseteq \Omega$ such that $\overline{E}$ is compact and $\overline{E} \subseteq \Omega$.

With every Radon measure μ on Ω we associate three non-negative Radon measures: the <u>total variation</u> $|\mu|$ and the <u>positive</u> and <u>negative parts</u> μ^+ and μ^-. We recall that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$.

If μ is a non-negative Radon measure, then μ can be extended in a unique way to a non-negative Borel measure, that will be denoted by the same symbol μ.

By a <u>bounded Radon measure</u> on Ω we mean a Radon measure on Ω such that $|\mu|(\Omega) < +\infty$.

1.4 Let Ω be a bounded open subset of $\mathbb{R}^n$. By L we denote a second order partial differential operator on Ω in divergence form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju),$$

whose coefficients a_{ij} are measurable on Ω and satisfy an ellipticity and boundedness condition

$$(1.2) \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega \quad \forall \xi \in R^n$$

$$(1.3) \qquad |a_{ij}(x)| \leq \Lambda \qquad \forall x \in \Omega$$

for some constants $0 < \lambda \leq \Lambda$.

The associated bilinear form in $H^1(\Omega)$ is denoted by

$$a(u,v) = \int_\Omega [\sum_{i,j=1} a_{ij}(x)D_j u D_i v]dx.$$

We notice that $a(u,v)$ is well defined also for $u \in H^1_{loc}(\Omega)$ and $v \in H^1_c(\Omega)$, or for $u \in H^{1,1}_{loc}(\Omega)$ and $v \in C^1_0(\Omega)$.

Suppose now that Ω is a <u>ball</u>, say $\Omega = B_R(x_0)$. By $G^y = G(\cdot,y)$ we denote the <u>Green function</u>, with singularity at y, for the Dirichlet problem in Ω relative to the operator L. This function is defined as the unique solution

$$G^y \in H_0^{1,p}(\Omega) \qquad 1 < p < \frac{n}{n-1}$$

of the equation

$$a(\phi,G^y) = \phi(y) \qquad \forall\phi \in C^1_0(\Omega).$$

It is well known that this function exists, that $G^y \in H^1(\Omega - \overline{B_r(y)})$ for every $r > 0$, and that $G(x,y)$ is continuous in (x,y) for $x \neq y$. Moreover for every $0 < q < 1$ there exist two constants $c_1 > 0$ and $c_2 > 0$ such that the following estimates hold for every $x,y \in B_{qR}(x_0)$:

$$\frac{c_1}{\Lambda}|x-y|^{2-n} \leq G(x,y) \leq \frac{c_2}{\lambda}|x-y|^{2-n}, \tag{1.4}$$

if $n \geq 3$, and

$$\frac{c_1}{\Lambda}\log\left(\frac{2R}{|x-y|}\right) \leq G(x,y) \leq \frac{c_2}{\lambda}\log\left(\frac{2R}{|x-y|}\right), \tag{1.5}$$

if $n = 2$. The constants c_1 and c_2 depend only on q,n, and the ratio $\frac{\Lambda}{\lambda}$. We point out explicitly that they are independent of R and L.

It is well known that for every bounded Radon measure μ in Ω the function

$$u(y) = \int_\Omega G(x,y)d\mu(x)$$

is the unique solution

$$u \in H_0^{1,p}(\Omega) \qquad 1 \leq p < \frac{n}{n-1}$$

of the equation

$$a(u,\phi) = \int_\Omega \phi \, d\mu \qquad \forall \phi \in C_0^1(\Omega).$$

For the preceding properties of the Green function see [8], [12].

For every $y \in \Omega$ and every $\rho > 0$ such that $B_\rho(y) \subseteq \Omega$, we denote by G_ρ^y the approximate Green function, defined as the unique solution $G_\rho^y \in H_0^1(\Omega)$ of the equation

$$a(v,G_\rho^y) = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} v(x)dx \qquad \forall v \in H_0^1(\Omega).$$

It is well known that this function exists and that $G_\rho^y \geq 0$ in Ω. Moreover, by the De Giorgi-Nash theorem, the function G_ρ^y is Hölder continuous in Ω and $G_\rho^y \to G_y$ as $\rho \to 0_+$ uniformly on every compact subset of $\Omega - \{y\}$.

1.5 Let Ω be an arbitrary open subset of R^n. We say that a Radon measure μ on Ω belongs to $H^{-1}(\Omega)$ (resp. $H_{loc}^{-1}(\Omega)$) if there exists $\lambda \in H^{-1}(\Omega)$ (resp. $H_{loc}^{-1}(\Omega)$) such that

$$\langle \lambda,\phi \rangle = \int_\Omega \phi \, d\mu \qquad \forall \phi \in C_0^\infty(\Omega).$$

In this case we identify λ and μ.

It is well known that if μ is a <u>non-negative</u> Radon measure on Ω which belongs to $H^{-1}(\Omega)$, then $\mu \in \mathcal{M}_0(\Omega)$, $H_0^1(\Omega) \subseteq L^1(\Omega,\mu)$, and

$$\langle \mu,v \rangle = \int_\Omega v \, d\mu \qquad \forall v \in H_0^1(\Omega)$$

If ν is another Radon measure on Ω such that $|\nu| \leq \mu$ in Ω, then ν, ν^+, ν^-, and $|\nu|$ belong to $H^{-1}(\Omega)$ and

$$<\nu, v> = \int_\Omega v \, d\nu \qquad \forall v \in H_0^1(\Omega).$$

Moreover

$$\|\nu\|_{H^{-1}(\Omega)} \leq \|\mu\|_{H^{-1}(\Omega)}.$$

Suppose now that Ω is a ball and let $G(x,y)$ be the Green function for the Dirichlet problem in Ω relative to the Laplace operator $-\Delta$. Let μ be a Radon measure on Ω. Then $|\mu| \in H^{-1}(\Omega)$ if and only if

$$\int_\Omega \int_\Omega G(x,y) d|\mu|(x) d|\mu|(y) < +\infty,$$

and in this case

(1.6) $$\|\mu\|_{H^{-1}(\Omega)} \leq \left(\int_\Omega \int_\Omega G(x,y) d\mu(x) d\mu(y) \right)^{1/2}.$$

2. RELAXED DIRICHLET PROBLEMS

In this section we study problems of the form

$$Lu + \mu u = f \quad \text{in} \quad \Omega$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $Lu = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u)$ is an elliptic operator as in 1.4, μ belongs to the set of the measures $\mathcal{M}_0(\Omega)$ introduced in 1.3, and $f \in H^{-1}_{loc}(\Omega)$.

These problems were called relaxed Dirichlet problems in [2], where the relationship with classical Dirichlet problems is extensively discussed.

The main goal of this section is to prove a comparison theorem for weak solutions of relaxed Dirichlet problems.

We denote by $a(u,v)$ the bilinear form associated with L as in 1.4.

DEFINITION 2.1. We say that a function u is a <u>local weak solution</u> of the equation

$$(2.1) \qquad Lu + \mu u = f \quad \text{in} \quad \Omega$$

if

(i) $$u \in H^1_{loc}(\Omega) \cap L^2_{loc}(\Omega,\mu)$$

and

(ii) $$a(u,v) + \int_\Omega uv d\mu = \langle f,v\rangle$$

for every $v \in H^1(\Omega) \cap L^2(\Omega,\mu)$ with compact support in Ω. □

For a discussion of the non trivial relationships between the definition above and the definition in the sense of distribution, see [2], Section 3.

DEFINITION 2.2 Given $g \in H^1(\Omega)$, we say that a function u is a <u>weak solution</u> of the problem

$$(2.2) \qquad \begin{cases} Lu + \mu u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases}$$

if u is a local weak solution of equation (2.1) and in addition

(iii) $$u - g \in H^1_0(\Omega).$$ □

We remark explicitly that (iii) implies that $u \in H^1(\Omega) \cap L^2_{loc}(\Omega,\mu)$.

The existence and uniqueness of the solution to problem (2.2) is given by the following theorems.

THEOREM 2.3. Problem (2.2) has at most one weak solution.

PROOF. Suppose that u_1 and u_2 are weak solutions of problem (2.2). Then the difference $u = u_1 - u_2$ belongs to $H_0^1(\Omega) \cap L_{loc}^2(\Omega,\mu)$ and

$$a(u,v) + \int_\Omega uv d\mu = 0 \tag{2.3}$$

for every $v \in H^1(\Omega) \cap L^2(\Omega,\mu)$ with compact support. There exists a sequence (u_h) in $H_c^1(\Omega) \cap L^2(\Omega,\mu)$ which converges to u strongly in $H^1(\Omega)$, such that the sequence (uu_h) is increasing and converges pointwise to u^2. By taking $v = u_h$ in (2.3) we obtain

$$a(u,u_h) + \int_\Omega uu_h d\mu = 0.$$

Passing to the limit as $h \to +\infty$ we obtain

$$a(u,u) + \int_\Omega u^2 d\mu = 0.$$

By the coerciveness assumption we have $u = 0$ a.e. in Ω, hence $u_1 = u_2$ a.e. in Ω. □

THEOREM 2.4 Suppose that $f \in H^{-1}(\Omega)$ and that there exists $w \in H^1(\Omega) \cap L^2(\Omega,\mu)$ such that $w - g \in H_0^1(\Omega)$. Then problem (2.2) has one and only one weak solution u. Moreover $u \in H^1(\Omega) \cap L^2(\Omega,\mu)$ and

$$a(u,v) + \int_\Omega uv d\mu = \langle f,v \rangle \tag{2.4}$$

for every $v \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$.

PROOF. We set $u = z + w$. Then u is a weak solution to problem (2.2) if and only if

$$z \in H_0^1(\Omega) \cap L_{loc}^2(\Omega,\mu)$$

and

$$(2.5) \qquad a(z,v) + \int_\Omega zv\,d\mu = \langle f,v\rangle - a(w,v) - \int_\Omega wv\,d\mu$$

for every $v \in H^1(\Omega) \cap L^2(\Omega,\mu)$ with compact support in Ω.

Let $H = H_0^1(\Omega) \cap L^2(\Omega,\mu)$ with the norm

$$\|v\|_H = \left(\int_\Omega |Dv|^2 dx + \int_\Omega v^2 d\mu\right)^{1/2}.$$

Since $\mu \in \mathcal{M}_0(\Omega)$, it is easy to prove that H is a Hilbert space. The left hand side of (2.5) is a continuous and coercive bilinear form on H (in the variables z and v), whereas the right hand side of (2.5) is a continuous linear form on H (in the variable v). By the Lax-Milgram theorem, there exists $z \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$ such that (2.5) holds for every $v \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$. Therefore $u = z + w$ is a weak solution to problem (2.2), $u \in H^1(\Omega) \cap L^2(\Omega,\mu)$, and (2.4) holds for every $v \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$. □

The following variational characterization of the weak solution to problem (2.2) is proved in [2], Theorem 3.13.

PROPOSITION 2.5. Suppose that $f \in H^{-1}(\Omega)$ and that there exists $w \in H^1(\Omega) \cap L^2(\Omega,\mu)$ such that $w - g \in H_0^1(\Omega)$. If $a_{ij} = a_{ji}$ for $i,j = 1,\dots,n$, then the weak solution of problem (2.2) is the unique minimum point of the functional

$$F(v) = a(v,v) + \int v^2 d\mu - 2\langle f,v\rangle$$

in the set $H(g) = \{v \in H^1(\Omega):\ v - g \in H_0^1(\Omega)\}$. □

The following result will be frequently used in the sequel.

PROPOSITION 2.6. Let ν be a Radon measure in Ω such that $|\nu|$ belongs to $H^{-1}(\Omega)$, and let u be a local weak solution of the equation

$$Lu + \mu u = \nu \quad \text{in} \quad \Omega.$$

Then

$$a(|u|,v) \leq \int_\Omega v\,d|\nu|$$

for every $v \in H_0^1(\Omega)$ with $v \geq 0$ a.e. in Ω.

PROOF. First of all we remark that, by 1.5, we have

$$\langle \nu, v\rangle = \int_\Omega v\,d\nu \qquad \forall v \in H_0^1(\Omega).$$

Let (ψ_h) be a sequence in $C^2(\mathbb{R})$ such that for every $t \in \mathbb{R}$

$$\lim_{h\to\infty} \psi_h(t) = |t|, \quad 0 \leq \psi_h(t) \leq |t|, \qquad \psi_h(-t) = \psi_h(t)$$

$$|\psi_h'(t)| \leq 1, \qquad \psi_h'(t)t \geq 0 \qquad 0 \leq \psi_h''(t) \leq h$$

We put $u_h = \psi_h(u)$. Then $u_h \in H^1_{loc}(\Omega)$ and $Du_h = \psi_h'(u)Du$.

Let v be a function of $H_c^1(\Omega) \cap L^2(\Omega,\mu)$, such that $0 \leq v \leq 1$ a.e. in Ω. Then $\psi_h'(u)v \in H_c^1(\Omega)$, for $\psi_h'(u) \in H^1(\Omega) \cap L^\infty(\Omega)$ and $v \in H_c^1(\Omega) \cap L^\infty(\Omega)$. Moreover $\psi_h'(u)v \in L^2(\Omega,\mu)$, for $|\psi_h'(u)| \leq 1$. We now use $\psi_h'(u)v$ as test function for our equation and we obtain

$$a(u,\psi_h'(u)v) + \int_\Omega u\psi_h'(u)v\,d\mu = \int_\Omega \psi_h'(u)v\,d\nu,$$

hence

$$(2.6) \quad \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(x)D_juD_iv\Big]\psi_h'(u)dx + \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(x)D_juD_iu\Big]\psi_h''(u)v\,dx +$$

$$+ \int_\Omega \psi_h'(u)uv\,d\mu = \int_\Omega \psi_h'(u)v\,d\nu$$

Since $\psi_h''(u) \geq 0$ we have

$$\int_\Omega [\sum_{i,j=1}^{n} a_{ij}(x)D_j u D_i u]\psi_h''(u)v\ dx \geq 0.$$

Since $\psi_h'(u)u \geq 0$ we have

$$\int_\Omega \psi_h'(u)uv\ d\mu \geq 0.$$

Since $|\psi_h'(u)| \leq 1$ we have

$$\int_\Omega \psi_h'(u)v\ d\nu \leq \int_\Omega v\ d|\nu|.$$

Therefore we obtain from (2.6)

$$\int_\Omega [\sum_{i,j=1}^{n} a_{ij}(x)\psi_h'(u)D_j u D_i v]dx \leq \int_\Omega vd|\nu|$$

Since $\psi_h'(u)D_j u = D_j u_h$, we have

(2.7) $$a(u_h,v) \leq \int_\Omega vd|\nu|.$$

Since (u_h) converges to $|u|$ in $L^2_{loc}(\Omega)$ and $|Du_h| \leq |Du|$, the sequence (u_h) converges to $|u|$ weakly in $H^1_{loc}(\Omega)$. Since v has compact support in Ω, we can pass to the limit in (2.7) as $h \to +\infty$ and obtain

(2.8) $$a(|u|,v) \leq \int_\Omega vd|\nu|$$

for every $v \in H^1_c(\Omega) \cap L^2(\Omega,\mu)$ such that $0 \leq v \leq 1$ a.e. in Ω.

It remains to prove that inequality (2.8) holds for every $v \in H^1_0(\Omega)$ with $v \geq 0$ a.e. in Ω. Let $\phi \in C^\infty_0(\Omega)$ with $\phi \geq 0$ in Ω, and let $\phi_h = (\frac{1}{h}\phi) \wedge |u|$. Then $\phi_h \in H^1_c(\Omega)$, and $0 \leq \phi_h \leq 1$ a.e. in Ω for h large enough. Since $0 \leq \phi_h \leq |u|$ a.e. in Ω, $u \in L^2_{loc}(\Omega,\mu)$, and ϕ_h has compact support in Ω, then we have $\phi_h \in L^2(\Omega,\mu)$. We now take $v = \phi_h$ in (2.8) and we obtain

$$a(|u|,\phi_h) \leq \int_\Omega \phi_h d|\nu|.$$

Since $D\phi_h = \frac{1}{h} D\phi$ on $\{\phi < h|u|\}$ and $D\phi_h = D|u|$ on $\{\phi \geq h|u|\}$, we obtain

$$\frac{1}{h} \int_{\{\phi<h|u|\}} [\sum_{i,j=1} a_{ij}(x)D_j|u|D_i\phi]dx + \int_{\{\phi\geq h|u|\}} [\sum_{i,j=1} a_{ij}(x)D_j|u|D_i|u|]dx \leq$$

$$\leq \int_\Omega \phi_h d|\nu| \leq \frac{1}{h}\int_\Omega \phi d|\nu|.$$

By neglecting the second term in the left hand side, which is non-negative by the ellipticity assumption, we obtain

$$\int_{\{\phi<h|u|\}} [\sum_{i,j=1}^n a_{ij}(x)D_j|u|D_i\phi]dx \leq \int_\Omega \phi d|\nu|.$$

By taking the limit as $h \to \infty$ we obtain

$$\int_{\{|u|>0\}} [\sum_{i,j=1}^n a_{ij}(x)D_j|u|D_i\phi]dx \leq \int_\Omega \phi d|\nu|.$$

Since $D_j|u| = 0$ a.e. on $\{|u| = 0\}$, we get

$$a(|u|,\phi) \leq \int_\Omega \phi \, d|\nu|$$

for every $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$ in Ω. The extension of this inequality to $H_0^1(\Omega)$ is trivial, since both sides are continuous in $H^1(\Omega)$. □

In order to state the comparison theorem we need the following definitions.

DEFINITION 2.7. Let $u,v \in H^1_{loc}(\Omega)$. We say that $u \leq v$ on $\partial\Omega$, or equivalently that $v \geq u$ on $\partial\Omega$, if $(v - u) \wedge 0 \in H_0^1(\Omega)$. □

It is easy to see that if $u \geq 0$ on $\partial\Omega$ and $v \geq 0$ on $\partial\Omega$, then $u + v \geq 0$ on $\partial\Omega$ and $\lambda u \geq 0$ on $\partial\Omega$ for every constant $\lambda \geq 0$. This implies easily that the relation $u \leq v$ on $\partial\Omega$ is transitive. Moreover it is clearly reflexive, and $v - u \in H_0^1(\Omega)$ if and only if both inequalities $u \leq v$ and $v \leq u$ hold on $\partial\Omega$.

If $u,v \in H^1(\Omega)$ and Ω has a Lipschitzian boundary, it is easy to see that the previous definition coincides with the classical definition in Sobolev spaces (see [8], Definition (1.2')).

DEFINITION 2.8. Let $f, g \in H^{-1}_{loc}(\Omega)$. We say that $f \leq g$ in Ω, or equivalently that $g \geq f$ in Ω, if $\langle g - f, v\rangle \geq 0$ for every $v \in H^1_c(\Omega)$ with $v \geq 0$ a.e. in Ω. $\square$

PROPOSITION 2.9 Let u be a local weak solution of equation (2.1). If $f \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ a.e. in Ω.

PROOF. Let $v = -(u \wedge 0)$. Since v is a non-negative function in $H^1_0(\Omega)$, there exists a sequence (v_h) of non-negative functions of $H^1_0(\Omega)$ with compact support in Ω which converge to v strongly in $H^1(\Omega)$ and such that $0 \leq v_h \leq v$ q.e. in Ω. Since $v \in L^2_{loc}(\Omega,\mu)$, we have $v_h \in L^2(\Omega,\mu)$. Therefore we can take v_h as a test function for the equation (2.1) and we obtain

$$a(u,v_h) + \int_\Omega uv_h d\mu = \langle f, v_h\rangle.$$

Since $uv_h \leq 0$ q.e. in Ω and $\langle f, v_h\rangle \geq 0$, we have

$$a(u,v_h) \geq 0.$$

Passing to the limit as $h \to +\infty$ we obtain

$$a(u,v) \geq 0.$$

Since $Dv = -Du$ on $\{v > 0\}$ and $Dv = 0$ on $\{v = 0\}$, we obtain

$$a(v,v) \leq 0.$$

By the coerciveness assumption we have $v = 0$ a.e. in Ω, hence $u \geq 0$ a.e. in Ω. $\square$

We now come to the main result of this section: the comparison theorem.

THEOREM 2.10 Let $\mu_1,\mu_2 \in \mathcal{M}_0(\Omega)$, let $f_1,f_2 \in H^{-1}_{loc}(\Omega)$, and let u_1,u_2 by local weak solutions of the equations

$$(2.9) \qquad Lu_1 + \mu_1 u_1 = f_1 \quad \text{in} \quad \Omega$$

$$(2.10) \qquad Lu_2 + \mu_2{}_2 = f_2 \quad \text{in} \quad \Omega.$$

If $\mu_1 \le \mu_2$ in Ω, $0 \le f_2 \le f_1$ in Ω, and $0 \le u_2 \le u_1$ on $\partial\Omega$, then $0 \le u_2 \le u_1$ a.e. in Ω.

PROOF. By Proposition 2.9 we have $u_1 \ge 0$ and $u_2 \ge 0$ a.e. in Ω. Let $v = (u_2 - u_1) \vee 0 = -[(u_1 - u_2) \wedge 0]$. Since $u_1 \ge u_2$ on $\partial\Omega$, we have $v \in H^1_0(\Omega)$. Since $u_1 \ge 0$ and $u_2 \ge 0$ a.e. in Ω, we have $0 \le v \le u_2$ q.e. in Ω, therefore $v \in L^2_{loc}(\Omega,\mu_2) \subseteq L^2_{loc}(\Omega,\mu_1)$. Since v is a non-negative function in $H^1_0(\Omega)$, there exists a sequence (v_h) of non-negative functions of $H^1_0(\Omega)$ with compact support in Ω which converge to v strongly in $H^1(\Omega)$ and such that $0 \le v_h \le v$ q.e. in Ω. Since $v \in L^2_{loc}(\Omega,\mu_2)$, we have $v_h \in L^2(\Omega,\mu_2) \subseteq L^2(\Omega,\mu_1)$. By taking v_h as test function in equations (2.9) and (2.10) we obtain

$$(2.11) \qquad a(u_1,v_h) + \int u_1 v_h d\mu_1 = \langle f_1,v_h\rangle$$

$$(2.12) \qquad a(u_2,v_h) + \int u_2 v_h d\mu_2 = \langle f_2,v_h\rangle.$$

Since $u_2 v_h \ge 0$ q.e. in Ω and $\mu_1 \le \mu_2$ in Ω, we obtain from (2.12)

$$(2.13) \qquad a(u_2,v_h) + \int_\Omega u_2 v_h d\mu_1 \le \langle f_2,v_h\rangle.$$

By subtracting (2.11) from (2.13) we get

$$a(u_2 - u_1,v_h) + \int_\Omega (u_2 - u_1) v_h d\mu_1 \le \langle f_2 - f_1,v_h\rangle.$$

Since $(u_2 - u_1)v_h \ge 0$ q.e. in Ω and $\langle f_2 - f_1,v_h\rangle \le 0$, we have

$$a(u_2 - u_1, v_h) \leq 0.$$

Passing to the limit as $h \to +\infty$ we obtain

$$a(u_2 - u_1, v) \leq 0.$$

Since $D_j(u_2 - u_1)D_i v = D_j v D_i v$ a.e. in Ω, we obtain

$$a(v,v) \leq 0.$$

By the coerciveness assumption we have $v = 0$ a.e. in Ω, hence $u_2 \leq u_1$ a.e. in Ω. □

3. A POINCARÉ INEQUALTY FOR THE μ-CAPACITY

In this section we study the properties of the variational μ-capacity defined below and of the corresponding capacitary potentials. These properties will be the basic tools for establishing the necessity of the Wiener condition in Section 5, as well as its sufficiency in Section 6.

The main result of this section is a Poincaré type inequality involving the μ-capacity, which will be essential in the proof of the energy estimates of Section 6.

Let Ω be a bounded open subset of R^n, let $\mu \in \mathcal{M}_0(\Omega)$, let $Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u)$ be an elliptic operator as in 1.4, and let $a(u,v)$ be the corresponding bilinear form on $H^1(\Omega)$. For every Borel set $E \subseteq \Omega$ we denote by μ_E the Borel measure on Ω defined by $\mu_E(B) = \mu(B \cap E)$ for every Borel set $B \subseteq \Omega$. We notice that $\mu_E \in \mathcal{M}_0(\Omega)$ for every Borel set $E \subseteq \Omega$.

DEFINITION 3.1. We say that a set E is μ-admissible in Ω if E is a Borel subset of Ω and there exists $w \in H^1(\Omega) \cap L^2(\Omega,\mu_E)$ such that $w - 1 \in H^1_0(\Omega)$.

If E is μ-admissible in Ω, we define the <u>μ-capacitary potential</u> of E in Ω, relative to the operator L, as the weak solution w_E of the problem

$$\begin{aligned} Lw_E + \mu_E w_E &= 0 \quad \text{in } \Omega, \\ w_E &= 1 \quad \text{on } \partial\Omega. \end{aligned}$$

The <u>μ-capacity</u> of E in Ω, relative to the operator L, is defined by

$$\text{cap}^L_\mu(E,\Omega) = a(w_E,w_E) + \int_\Omega w_E^2 d\mu_E.$$

If E is a Borel subset of Ω which is not μ-admissible in Ω, we define

$$\text{cap}^L_\mu(E,\Omega) = +\infty.$$

If $L = -\Delta$, the corresponding capacity is denoted by $\text{cap}_\mu(E,\Omega)$. □

REMARK 3.2 If E is μ-admissible in Ω, then the μ-capacitary potential w_E exists and is unique by Theorem 2.4. Moreover $w_E \in L^2(\Omega,\mu_E)$, hence $0 \leq \text{cap}^L_\mu(E,\Omega) < +\infty$. By the comparison theorem (Theorem 2.10) we have $0 \leq w_E \leq 1$ a.e. in Ω. □

REMARK 3.3 Suppose that $\mu(E) = 0$ if E has capacity zero, and $\mu(E) = +\infty$ otherwise. Then $\text{cap}^L_\mu(E,\Omega)$ coincides with the capacity, associated with the operator L, introduced by G. Stampacchia in [12], Definition 3.1. If, in addition, $L = -\Delta$, then $\text{cap}^L_\mu(E,\Omega) = \text{cap}(E,\Omega)$. □

REMARK 3.4 If L is symmetric, i.e. $a_{ij} = a_{ji}$ for $i,j = 1,\dots,n$, then by Proposition 2.5 we have

$$\text{cap}^L_\mu(E,\Omega) = \min\{a(v,v) + \int_\Omega v^2 d\mu_E : v - 1 \in H^1_0(\Omega)\}$$

for every μ-admissible $E \subseteq \Omega$. □

PROPOSITION 3.5. If E is μ-admissible in Ω, then there exists a non-negative Radon measure $\nu \in H^{-1}(\Omega)$ such that

$$a(w_E,v) + \int_\Omega v \, d\nu = 0$$

for every $v \in H_0^1(\Omega)$. The measure ν has support in $\overline{E}$, and $\mathrm{cap}_\mu^L(E,\Omega) = \nu(\Omega)$.

PROOF. Since $w_E \geq 0$ by Remark 3.2, Proposition 2.6 implies that

$$a(w_E,v) \leq 0$$

for every $v \in H_0^1(\Omega)$ with $v \geq 0$ a.e. in Ω. By the Riesz representation theorem, there exists a non-negative measure $\nu \in H^{-1}(\Omega)$ such that

$$a(w_E,v) = -\int_\Omega v d\nu \tag{3.1}$$

for every $v \in H_0^1(\Omega)$. If $v \in H_0^1(\Omega)$ and $(\mathrm{supp}\, v) \cap \overline{E} = \emptyset$, then $v \in L^2(\Omega,\mu_E)$, hence

$$a(w_E,v) = a(w_E,v) + \int_\Omega w_E v d\mu_E = 0. \tag{3.2}$$

From (3.1) and (3.2) it follows that

$$\int_\Omega v d\nu = 0,$$

hence $\mathrm{supp}\, \nu \subseteq \overline{E}$.

It remains to prove that $\mathrm{cap}_\mu^L(E,\Omega) = \nu(\Omega)$. Let Ω' be an open set with $\Omega' \subset\subset \Omega$, and let $\psi \in C_0^1(\Omega)$ be such that $0 \leq \psi \leq 1$ in Ω and $\psi = 1$ in Ω'. Since $w_E\psi \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$, by Theorem 2.4 we have

$$a(w_E,w_E\psi) + \int_\Omega w_E^2 \psi d\mu_E = 0.$$

Therefore, since $1 - w_E(1 - \psi) \in H_0^1(\Omega)$, we have

$$\begin{aligned}\mathrm{cap}_\mu^L(E,\Omega) &= a(w_E,w_E) + \int_\Omega w_E^2 d\mu_E = \\ &= a(w_E,w_E(1-\psi)) + \int_\Omega w_E^2(1-\psi)d\mu_E = \\ &= -a(w_E,1-w_E(1-\psi)) + \int_\Omega w_E^2(1-\psi)d\mu_E = \\ &= \int_\Omega [1-w_E(1-\psi)]d\nu + \int_\Omega w_E^2(1-\psi)d\mu_E.\end{aligned}$$

This implies

$$\nu(\Omega') \leq \mathrm{cap}_\mu^L(E,\Omega) \leq \nu(\Omega) + \int_{\Omega-\Omega'} w_E^2 d\mu_E.$$

Since $w_E \in L^2(\Omega,\mu_E)$, by taking the limit as $\Omega' \uparrow \Omega$ we obtain $\mathrm{cap}_\mu^L(E,\Omega) = \nu(\Omega)$. □

REMARK 3.6 If $\mu(E) < +\infty$, then it is easy to prove that

$$\nu(B) = \int_B w_E d\mu_E$$

for every Borel set $B \subseteq \Omega$. This equality does not hold, in general, if $\mu(E) = +\infty$, as one can see easily by considering the measure μ of Remark 3.3. Indeed in this case the condition $w_E \in L^2(\Omega,\mu_E)$ implies that $w_E = 0$ q.e on E, hence $\int w_E d\mu_E = 0$, whereas $\nu(\Omega) = \mathrm{cap}_\mu^L(E,\Omega)$ by Proposition 3.5. □

The following propositions single out some properties of the μ-capacitary potentials.

PROPOSITION 3.7 Let E and F be two μ-admissible disjoint subsets of Ω. Then $E \cup F$ is μ-admissible in Ω and

$$w_E + w_F \leq w_{E\cup F} + 1 \quad \text{a.e. in } \Omega.$$

PROOF. The function $w_E \wedge w_F$ belongs to $L^2(\Omega,\mu_{E\cup F})$ and $(w_E \wedge w_F) - 1 \in H_0^1(\Omega)$. Therefore $E \cup F$ is μ-admissible in Ω. Since $\mu_E \leq \mu_{E\cup F}$ and $\mu_F \leq \mu_{E\cup F}$, by the comparison theorem we have

$$0 \leq w_{E\cup F} \leq w_E \leq 1 \qquad \text{a.e. in } \Omega,$$
$$0 \leq w_{E\cup F} \leq w_F \leq 1 \qquad \text{a.e. in } \Omega.$$

Let $v = (w_E + w_F - w_{E\cup F} - 1) \vee 0$. It is evident that $v \in H_0^1(\Omega)$. From the above inequalities it follows that $0 \leq v \leq w_E$ and $0 \leq v \leq w_F$ q.e. in Ω, hence $v \in L^2(\Omega,\mu_E) \cap L^2(\Omega,\mu_F) = L^2(\Omega,\mu_{E\cup F})$. By Theorem 2.4 we have

$$a(w_E,v) + \int_E w_E v d\mu = 0$$

$$a(w_F,v) + \int_F w_F v d\mu = 0$$

$$a(w_{E\cup F},v) + \int_{E\cup F} w_{E\cup F} v d\mu = 0.$$

By adding the first two equalities, and by subtracting the third one, we obtain

$$a(w_E + w_F - w_{E\cup F},v) + \int_E (w_E - w_{E\cup F})v\, d\mu + \int_F (w_F - w_{E\cup F})v\, d\mu = 0.$$

Since $a(w_E + w_F - w_{E\cup F},v) = a(v,v)$, $(w_E - w_{E\cup F})v \geq 0$ q.e. in Ω and $(w_F - w_{E\cup F})v \geq 0$ q.e. in Ω, we obtain

$$a(v,v) \leq 0.$$

By the coerciveness assumption we have $v = 0$ a.e. in Ω, hence $w_E + w_F - w_{E\cup F} - 1 \leq 0$ a.e. in Ω. □

PROPOSITION 3.8 Let $(E_i)_{i\in I}$ be a finite family of pairwise disjoint μ-admissible subsets of Ω, and let $E = \bigcup_{i\in I} E_i$. Then E is μ-admissible and

$$1 - w_E \leq \sum_{i\in I} (1 - w_{E_i}).$$

PROOF. It follows from Proposition 3.7 by induction on the number of elements of the family $(E_i)_{i\in I}$. □

PROPOSITION 3.9 Suppose that Ω is a ball, say $\Omega = B_R(x_0)$. Let $0 < q < 1$, and let E be a Borel set contained in $B_{qR}(x_0)$. Then there exists a constant $k > 0$ such that

$$w_E(x) \geq 1 - \frac{k}{\lambda}\,\mathrm{cap}_\mu^L(E,\Omega)\mathrm{dist}(x,E)^{2-n} \quad \text{a.e. in } B_{qR}(x_0) - \overline{E}$$

if $n \geq 3$, and

$$w_E(x) \geq 1 - \frac{k}{\lambda}\,\mathrm{cap}_\mu^L(E,\Omega)\log\left(\frac{2R}{\mathrm{dist}(x,E)}\right) \quad \text{a.e. in } B_{qR}(x_0) - \overline{E}$$

if $n = 2$. The constant k depends only on q,n and the ratio $\frac{\Lambda}{\lambda}$ of the ellipticity constants in 1.4.

PROOF. We consider only the case $n \geq 3$, the case $n = 2$ being analogous. Let ν be the measure given by Proposition 3.5. Then $1 - w_E \in H_0^1(\Omega)$ and

$$a(1 - w_E,\phi) = \int_\Omega \phi\, d\nu$$

for every $\phi \in C_0^1(\Omega)$. Let G^y be the Green function of the operator L in Ω (see 1.4). Then

$$1 - w_E(x) = \int_\Omega G^x(y)d\nu(y) \quad \text{a.e. in } \Omega.$$

Since $\mathrm{supp}\,\nu \subseteq \overline{E}$ and, by (1.4),

$$G^x(y) \leq \frac{c_2}{\lambda}\,|x - y|^{2-n}$$

for every $x,y \in B_{qR}(x_0)$, we have

$$1 - w_E(x) \leq \frac{c_2}{\lambda}\,\nu(\Omega)\mathrm{dist}(x,E)^{2-n} \quad \text{a.e. in } B_{qR}(x_0) - \overline{E}$$

The proposition follows now from the equality $\nu(\Omega) = \mathrm{cap}_\mu^L(E,\Omega)$, proved in Proposition 3.5. $\square$

We compare now the capacity $\mathrm{cap}_\mu^L(E,\Omega)$ with the capacity $\mathrm{cap}_\mu(E,\Omega)$ corresponding to the Laplace operator $-\Delta$.

THEOREM 3.10. There exist two constants $k_1 > 0$ and $k_2 > 0$, depending only on n, λ, Λ such that

$$k_1 \mathrm{cap}_\mu(E,\Omega) \le \mathrm{cap}_\mu^L(E,\Omega) \le k_2 \mathrm{cap}_\mu(E,\Omega)$$

for every μ-admissible $E \subseteq \Omega$.

PROOF. Let w_E be the μ-capacitary potential relative to L and let v_E be the μ-capacitary potential relative to $-\Delta$. Since

$$-\Delta v_E + \mu_E v_E = 0 \quad \text{in} \quad \Omega$$

and $v_E - w_E \in H_0^1(\Omega) \cap L^2(\Omega,\mu_E)$, by Theorem 2.4 we have

$$\int_\Omega Dv_E(Dv_E - Dw_E)dx + \int_\Omega v_E(v_E - w_E)d\mu_E = 0,$$

hence

$$\begin{aligned}
\mathrm{cap}_\mu(E,\Omega) &= \int_\Omega |Dv_E|^2 dx + \int_\Omega v_E^2 d\mu_E = \\
&= \int_\Omega Dv_E Dw_E dx + \int_\Omega v_E w_E d\mu_E \le \\
&= \Big(\int_\Omega |Dv_E|^2 dx + \int_\Omega v_E^2 d\mu_E \Big)^{1/2} \Big(\int_\Omega |Dw_E|^2 dx + \int_\Omega w_E^2 d\mu_E \Big)^{1/2} \le \\
&\le [\mathrm{cap}_\mu(E,\Omega)]^{1/2} \Big[\frac{1}{\lambda} a(w_E,w_E) + \int_\Omega w_E^2 d\mu_E \Big]^{1/2} \le \\
&\le \Big[\frac{1}{\lambda \wedge 1} \Big]^{1/2} [\mathrm{cap}_\mu(E,\Omega)]^{1/2} [\mathrm{cap}_\mu^L(E,\Omega)]^{1/2}.
\end{aligned}$$

Therefore

$$(\lambda \wedge 1)\mathrm{cap}_\mu(E,\Omega) \leq \mathrm{cap}_\mu^L(E,\Omega)$$

and the first inequality is proved.

Since

$$Lw_E + \mu_E w_E = 0 \quad \text{in} \quad \Omega,$$

using again $v_E - w_E$ as test function we obtain

$$a(w_E, v_E - w_E) + \int w_E(v_E - w_E)d\mu_E = 0,$$

hence

$$\begin{aligned}
\mathrm{cap}_\mu^L(E,\Omega) &= a(w_E,w_E) + \int_\Omega w_E^2 d\mu_E = \\
&= a(w_E,v_E) + \int_\Omega w_E v_E d\mu_E \leq \\
&= n\Lambda \int_\Omega Dw_E Dv_E dx + \int_\Omega w_E v_E d\mu_E \leq \\
&\leq [n\Lambda \int_\Omega |Dw_E|^2\, dx + \int_\Omega w_E^2 d\mu_E]^{1/2}\, [n\Lambda \int_\Omega |Dv_E|^2 dx + \int_\Omega v_E^2 d\mu_E]^{1/2} \leq \\
&\leq [\frac{n\Lambda}{\lambda} a(w_E,w_E) + \int_\Omega w_E^2 d\mu_E]^{1/2}\, [(n\Lambda) \wedge 1]^{1/2}[\mathrm{cap}_\mu(E,\Omega)]^{1/2} \leq \\
&\leq [\frac{n\Lambda}{\lambda}]^{1/2}[(n\Lambda) \vee 1]^{1/2}[\mathrm{cap}_\mu^L(E,\Omega)]^{1/2}[\mathrm{cap}_\mu(E,\Omega)]^{1/2}.
\end{aligned}$$

Therefore

$$\mathrm{cap}_\mu^L(E,\Omega) \leq \frac{n\Lambda}{\lambda} [(n\Lambda)\vee 1]\mathrm{cap}_\mu(E,\Omega). \qquad \square$$

The main properties of the set function cap_μ are summarized in the following proposition (see [2], Proposition 5.3).

PROPOSITION 3.11 Let $\mu,\nu \in \mathcal{M}_0(\Omega)$, let E, F, Ω' be Borel subsets of Ω, with Ω' open. Then

(a) $$0 = \mathrm{cap}_\mu(\emptyset,\Omega) \leq \mathrm{cap}_\mu(E,\Omega) \leq \mathrm{cap}(E,\Omega)$$

(b) $$E \subseteq F \Rightarrow \mathrm{cap}_\mu(E,\Omega) \leq \mathrm{cap}_\mu(F,\Omega),$$

(c) $$\mathrm{cap}_\mu(E \cup F,\Omega) + \mathrm{cap}_\mu(E \cap F,\Omega) \leq \mathrm{cap}_\mu(E,\Omega) + \mathrm{cap}_\mu(F,\Omega),$$

(d) $$E \subseteq \Omega' \subseteq \Omega \Rightarrow \mathrm{cap}_\mu(E,\Omega) \leq \mathrm{cap}_\mu(E,\Omega')$$

(e) $$\mu \leq \nu \Rightarrow \mathrm{cap}_\mu(E,\Omega) \leq \mathrm{cap}_\nu(E,\Omega).$$ □

REMARK 3.12 The same properties, with same proof, hold for the capacity cap_μ^L provided the operator L is symmetric, i.e. $a_{ij} = a_{ji}$ for $i,j = 1,\dots,n$. □

We now come to the main result of this section: the Poincaré inequality.

THEOREM 3.13 For every $0 \leq q < 1$ there exists a constant $k > 0$, depending only on q and n, such that

$$\int_{B_r - B_{qr}} u^2 dx \leq \frac{kr^n}{\mathrm{cap}_\mu(B_r - B_{qr}, B_{2r})} \left[\int_{B_r - B_{qr}} |Du|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right]$$

for every triple of concentric balls $B_{qr} = B_{qr}(x_0)$, $B_r = B_r(x_0)$, $B_{2r} = B_{2r}(x_0)$, for every $u \in H^1(B_r)$, and for every $\mu \in \mathcal{M}_0(B_r)$.

PROOF. Throughout the proof, the letter k will denote various positive constants which depend only on q and n and whose value may change from one line to the other. Let us fix $0 \leq q < 1$, $B_{qr} = B_{qr}(x_0)$, $B_r = B_r(x_0)$, $B_{2r} = B_{2r}(x_0)$, $u \in H^1(B_r)$, and $\mu \in \mathcal{M}_0(B_r)$. There exists a function $v \in H^1(B_{2r})$ such that $v = u$ q.e. in $B_r - B_{qr}$ and

$$\int_{B_{2r}} |Dv|^2 dx \leq k \int_{B_r - B_{qr}} |Du|^2 dx.$$

By the classical Poincaré inequality

$$\int_{B_{2r}} |v - v_{2r}|^2 dx \leq kr^2 \int_{B_{2r}} |Dv|^2 dx$$

where v_{2r} denotes the average of v on B_{2r}. Therefore

$$(3.3) \qquad \int_{B_r - B_{qr}} u^2 dx \leq \int_{B_{2r}} v^2 dx \leq 2 \int_{B_{2r}} |v - v_{2r}|^2 dx + kr^n |v_{2r}|^2 \leq$$
$$\leq kr^2 \int_{B_{2r}} |Dv|^2 dx + kr^n |v_{2r}|^2 \leq$$
$$\leq kr^2 \int_{B_r - B_{qr}} |Du|^2 dx + kr^n |v_{2r}|^2 .$$

Let us prove that

$$(3.4) \qquad |v_{2r}|^2 \leq \frac{k}{cap_\mu(B_r - B_{qr}, B_{2r})} \left[\int_{B_r - B_{qr}} |Du|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right].$$

If $v_{2r} = 0$ the inequality is trivial. Let us suppose that $v_{2r} \neq 0$. Let $\tau \in C_0^1(B_{2r})$ with $\tau = 1$ on B_r, $0 \leq \tau \leq 1$ on B_{2r}, and $|D\tau| \leq \frac{2}{r}$ on B_{2r}. We set

$$w = 1 + \tau \frac{v - v_{2r}}{v_{2r}} .$$

Since $w - 1 \in H_0^1(B_{2r})$, from the minimizing property of cap_μ (see Remark 3.4) we obtain

$$cap_\mu(B_r - B_{qr}, B_{2r}) \leq \int_{B_{2r}} |Dw|^2 dx + \int_{B_r - B_{qr}} w^2 d\mu \leq$$
$$\leq \frac{2}{|v_{2r}|^2} \int_{B_{2r}} |D\tau|^2 |v - v_{2r}|^2 dx + \frac{2}{|v_{2r}|^2} \int_{B_{2r}} |Dv|^2 dx +$$
$$+ \frac{1}{|v_{2r}|^2} \int_{B_r - B_{qr}} v^2 d\mu \leq$$
$$\leq \frac{1}{|v_{2r}|^2} \left[\frac{8}{r^2} \int_{B_{2r}} |v - v_{2r}|^2 dx + 2 \int_{B_{2r}} |Dv|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right] \leq$$
$$\leq \frac{1}{|v_{2r}|^2} \left[k \int_{B_{2r}} |Dv|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right] \leq$$
$$\leq \frac{k}{|v_{2r}|^2} \left[\int_{B_r - B_{qr}} |Du|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right]$$

and (3.4) is proved.

There exists a constant k such that

$$\mathrm{cap}_\mu(B_r - B_{qr}, B_{2r}) \leq \mathrm{cap}(B_r, B_{2r}) = kr^{n-2},$$

hence

$$r^2 \leq \frac{kr^n}{\mathrm{cap}_\mu(B_r - B_{qr}, B_{2r})} . \tag{3.5}$$

From (3.3), (3.4), (3.5) it follows that

$$\int_{B_r - B_{qr}} u^2 dx \leq \frac{kr^n}{\mathrm{cap}_\mu(B_r - B_{qr}, B_{2r})} \left[\int_{B_r - B_{qr}} |Du|^2 dx + \int_{B_r - B_{qr}} u^2 d\mu \right],$$

and the theorem is proved. □

4. THE SPACES $K_n(\Omega)$ AND $K_n^{loc}(\Omega)$.

In this section we introduce two spaces of Radon measures which generalize the spaces K_n and K_n^{loc} studied in [1], Section 4, see also [6].

Let Ω be a bounded open subset of R^n.

DEFINITION 4.1 We denote by $K_n(\Omega)$ the set of all Radon measures ν on Ω such that

$$\lim_{r \to 0_+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |y - x|^{2-n} d|\nu|(y) = 0,$$

if $n \geq 3$, and

$$\lim_{r \to 0_+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} \log\left(\frac{1}{|y - x|} \right) d|\nu|(y) = 0,$$

if $n = 2$. By $K_n^{loc}(\Omega)$ we denote the set of all Radon measures ν on Ω such that $\nu \in K_n(\Omega')$ for every open set $\Omega' \subset\subset \Omega$. □

It is easy to see that $K_n(\Omega)$ and $K_n^{loc}(\Omega)$ are vector spaces. If the function $f\colon \mathbb{R}^n \to \mathbb{R}$ belongs to the space K_n^{loc} defined in [1], Section 4, then the measure $f(x)dx$ belongs to $K_n(\Omega)$ for every bounded open set $\Omega \subseteq R^n$.

PROPOSITION 4.2. If $\nu \in K_n(\Omega)$ then $|\nu|(\Omega) < +\infty$.

PROOF. We consider only the case $n = 2$, the case $n \geq 3$ being analogous. Let $\nu \in K_2(\Omega)$. By the definition there exists $0 < r < \frac{1}{2}$ such that

$$|\nu|(\Omega \cap B_r(x))\log 2 \leq \int_{\Omega\cap B_r(x)} \log\left(\frac{1}{|y - x|}\right)d|\nu|(y) \leq 1$$

for every $x \in \Omega$. Since $\overline{\Omega}$ is compact, and

$$\overline{\Omega} \subseteq \bigcup_{x\in\Omega} B_r(x),$$

there exists $x_1,\ldots,x_k \in \Omega$ such that

$$\Omega \subseteq \bigcup_{i=1}^{k} B_r(x_i),$$

hence,

$$|\nu|(\Omega) \leq \sum_{i=1}^{k} |\nu|(\Omega \cap B_r(x_i)) \leq \frac{k}{\log 2}\,. \qquad \square$$

PROPOSITION 4.3 If $\nu \in K_n(\Omega)$, then

$$\sup_{x\in\Omega} \int_{\Omega} |y - x|^{2-n}d|\nu|(y) < +\infty$$

if $n \geq 3$, and

$$\sup_{x\in\Omega} \int_{\Omega} \log\left(\frac{\operatorname{diam}(\Omega)}{|y - x|}\right)d|\nu|(y) < +\infty$$

if $n = 2$.

PROOF. We consider only the case $n = 2$, the case $n \geq 3$ being analogous. Let $\nu \in K_2(\Omega)$. By the definition there exists $0 < r \leq \frac{1}{2}$ such that

$$\int_{\Omega \cap B_r(x)} \log\left(\frac{1}{|y - x|} \right) d|\nu|(y) < 1$$

for every $x \in \Omega$. Since

$$\int_{\Omega - B_r(x)} \log\left(\frac{1}{|y - x|} \right) d|\nu|(y) \leq |\nu|(\Omega) \log\left(\frac{1}{r} \right)$$

we have

$$\int_{\Omega} \log\left(\frac{1}{|y - x|} \right) d|\nu|(y) \leq 1 + |\nu|(\Omega) \log\left(\frac{1}{r} \right)$$

for every $x \in \Omega$, therefore

$$\sup_{x \in \Omega} \int_{\Omega} \log\left(\frac{\mathrm{diam}(\Omega)}{|y - x|} \right) d|\nu|(y) \leq 1 + |\nu|(\Omega) \log\left(\frac{\mathrm{diam}(\Omega)}{r} \right).$$

This concludes the proof because $|\nu|(\Omega) < +\infty$ by Proposition 4.2. □

DEFINITION 4.4 Let $\nu \in K_n(\Omega)$. If $n \geq 3$, we define

$$\|\nu\|_{K_n(\Omega)} = \sup_{x \in \Omega} \int_{\Omega} |y - x|^{2-n} d|\nu|(y).$$

If $n = 2$, we define

$$\|\nu\|_{K_2(\Omega)} = \sup_{x \in \Omega} \int_{\Omega} \log\left(\frac{\mathrm{diam}(\Omega)}{|y - x|} \right) d|\nu|(y) + |\nu|(\Omega).$$ □

REMARK 4.5 It is easy to see that $\|\cdot\|_{K_n(\Omega)}$ is a norm in $K_n(\Omega)$ and that

(4.1) $$|\nu|(\Omega) \leq \mathrm{diam}(\Omega)^{n-2} \|\nu\|_{K_n(\Omega)}.$$

From the definition of $K_n(\Omega)$ it follows that

(4.2)
$$\lim_{r \to 0_+} \|\nu\|_{K_n(B_r(x))} = 0$$

for every $\nu \in K_n(\Omega)$ and every $x \in \Omega$. □

PROPOSITION 4.6. The space $K_n(\Omega)$ with the norm $\|\cdot\|_{K_n(\Omega)}$ is a Banach space.

PROOF. Let (ν_h) be a Cauchy sequence in $K_n(\Omega)$. By the inequality (4.1) of Remark 4.5 we have

$$\lim_{h,k \to \infty} |\nu_h - \nu_k|(\Omega) = 0.$$

By the completeness of the space of all bounded Radon measures, there exists a bounded Radon measure ν on Ω such that $|\nu_h - \nu|(\Omega) \to 0$ as $h \to +\infty$.

Suppose now that $n \geq 3$ (the case $n = 2$ is analogous). Since (ν_h) is a Cauchy sequence in $K_n(\Omega)$, for every $\varepsilon > 0$ there exists h_ε such that

$$\int_\Omega |y - x|^{2-n} d|\nu_h - \nu_k|(y) \leq \varepsilon$$

for every $x \in \Omega$ and every $h,k \geq h_\varepsilon$. By taking the limit as $k \to +\infty$ we obtain

(4.3)
$$\int_\Omega |y - x|^{2-n} d|\nu_h - \nu|(y) \leq \varepsilon$$

for every $x \in \Omega$ and every $h \geq h_\varepsilon$. Let us fix $h \geq h_\varepsilon$. Since $\nu_h \in K_n(\Omega)$, there exists $r > 0$ such that

(4.4)
$$\int_{\Omega \cap B_r(x)} |y - x|^{2-n} d|\nu_h|(y) \leq \varepsilon$$

for every $x \in \Omega$. From (4.3) and (4.4) it follows that

$$\sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} |y - x|^{2-n} d|\nu|(y) \leq 2\varepsilon,$$

hence $\nu \in K_n(\Omega)$. From (4.3) we obtain that

$$\|\nu_h - \nu\|_{K_n(\Omega)} \leq \varepsilon$$

for every $h > h_\varepsilon$, hence (ν_h) converges to ν in $K_n(\Omega)$. □

Some examples of measures of the class $K_n(\Omega)$ are given by the following two propositions.

PROPOSITION 4.7. If $f \in L^p(\Omega)$ with $p > \frac{n}{2}$, then the measure $d\nu = fdx$ belongs to $K_n(\Omega)$ and

$$\|\nu\|_{K_n(\Omega)} \leq k\|f\|_{L^p(\Omega)}$$

where $k > 0$ is a constant which depends on n, p, and Ω.

PROOF. Use Hölder inequality. □

PROPOSITION 4.8 Let S be a compact $(n - 1)$-dimensional manifold of class C^1 contained in Ω, and let σ be the $(n - 1)$-dimensional measure on S. If $f \in L^p(S,\sigma)$ with $p > n - 1$, then the measure $\nu(E) = \int_{S \cap E} fd\sigma$ belongs to $K_n(\Omega)$ and

$$\|\nu\|_{K_n(\Omega)} \leq k\|f\|_{L^p(S,\sigma)},$$

where $k > 0$ is a constant which depends on n, p, Ω and on the geometry of S.

PROOF. Use Hölder inequality in local coordinates on S. □

THEOREM 4.9 If $\nu \in K_n(\Omega)$, then $\nu \in H^{-1}(\Omega)$ and

$$\|\nu\|_{H^{-1}(\Omega)} \leq k \text{ diam } (\Omega)^{n/2 - 1}\|\nu\|_{K_n(\Omega)},$$

where $k > 0$ is a constant which depends only on the dimension n of the space.

PROOF. Let $x_0 \in \Omega$ and let $\Omega' = B_R(x_0)$ with $R = 2\,\mathrm{diam}(\Omega)$. Let $\nu \in K_n(\Omega)$ and let ν' be the Radon measure on Ω' defined by

$$\nu'(E) = \nu(E \cap \Omega)$$

for every Borel set $E \subseteq \Omega'$. Let G be the Green function for the Dirichlet problem in Ω' relative to the Laplace operator $-\Delta$. By (1.4) and (1.5) there exists a constant $k > 0$ such that for every $x,y \in \Omega$ we have

$$G(x,y) \leq k|x - y|^{2-n}$$

if $n \geq 3$, and

$$G(x,y) \leq k \log\left(\frac{4\,\mathrm{diam}\,\Omega}{|x - y|} \right)$$

if $n = 2$. By (4.1) we have

$$\int_{\Omega'}\int_{\Omega'} G(x,y)d|\nu'|(x)d|\nu'|(y) = \int_{\Omega}\int_{\Omega} G(x,y)d|\nu|(x)d|\nu|(y) \leq$$

$$\leq k \int_{\Omega} [\int_{\Omega} |x - y|^{2-n} d|\nu|(y)] d|\nu|(x) \leq k\|\nu\|_{K_n(\Omega)}|\nu|(\Omega) \leq$$

$$\leq k\,\mathrm{diam}(\Omega)^{n-2}\|\nu\|^2_{K_n(\Omega)},$$

if $n \geq 3$, and

$$\int_{\Omega'}\int_{\Omega'} G(x,y)d|\nu'|(x)d|\nu'|(y) \leq k \int_{\Omega} [\int_{\Omega} \log\left(\frac{4\,\mathrm{diam}\,\Omega}{|x - y|} \right) d|\nu|(y)] d|\nu|(x) \leq$$

$$\leq k(\log 4)\|\nu\|_{K_2(\Omega)}|\nu|(\Omega) \leq k(\log 4)\|\nu\|^2_{K_2(\Omega)},$$

if $n = 2$.

In any case, by 1.5 we have $|\nu'| \in H^{-1}(\Omega')$ and by (1.6) we have

$$\||\nu'|\|_{H^{-1}(\Omega')} \leq k'\,\mathrm{diam}(\Omega)^{n/2-1}\|\nu\|_{K_n(\Omega)}$$

where $k' = k^{1/2}$, if $n \geq 3$, and $k' = (k \log 4)^{1/2}$, if $n = 2$. This implies easily that $\nu \in H^{-1}(\Omega)$ and

$$\|\nu\|_{H^{-1}(\Omega)} \leq \||\nu'|\|_{H^{-1}(\Omega')} \leq k' \operatorname{diam}(\Omega)^{n/2-1} \|\nu\|_{K_n(\Omega)}$$

(see 1.5). □

Let $Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u)$ be an elliptic operator as in 1.4 and let $a(u,v)$ be the associated bilinear form on $H^1(\Omega)$. According to Definition 2.1, a local weak solution of the equation

$$Lu = \nu \quad \text{in} \quad \Omega,$$

with $\nu \in K_n^{loc}(\Omega)$, is a function $u \in H^1_{loc}(\Omega)$ such that

$$a(u,v) = \int_\Omega v d\nu$$

for every $v \in H^1_c(\Omega)$.

THEOREM 4.11 If $\nu \in K_n^{loc}(\Omega)$ and u is a local weak solution of the equation

$$Lu = \nu \quad \text{in} \quad \Omega,$$

then $u \in C^\circ(\Omega)$.

PROOF. We consider only the case $n \geq 3$, the case $n = 2$ being analogous.

Let $x_0 \in \Omega$ and let $R > 0$ such that $B_R(x_0) \subset\subset \Omega$. We set $\Omega' = B_R(x_0)$ and, for every $y \in \Omega'$, we denote by G^y the Green function of the operator L in Ω' with singularity at y. By (1.4) we have

$$G^y(x) \leq \frac{c_2}{\lambda} |x - y|^{2-n}$$

for every $x,y \in B_{qR}(x_0)$ $(0 < q < 1)$.

For every $x \in \Omega'$ we put

$$v(x) = \int_{\Omega'} G^x(y)d\nu(y).$$

Let us prove that v is continuous at x_0. Let (x_h) be a sequence converging to x_0 in Ω'. For every $0 < r < \frac{R}{3}$ and for every $x_h \in B_r(x_0)$ we have

$$|v(x_h) - v(x_0)| \leq \int_{B_r(x_0)} G(y,x_h)d|\nu|(y) + \int_{B_r(x_0)} G(y,x_0)d|\nu|(y) +$$

$$+ \int_{\Omega'-B_r(x_0)} |G(y,x_h) - G(y,x_0)|d|\nu|(y) \leq$$

$$\leq \int_{B_{2r}(x_h)} G(y,x_h)d|\nu|(y) + \int_{B_r(x_0)} G(y,x_0)d|\nu|(y) +$$

$$+ \int_{\Omega'-B_r(x_0)} |G(y,x_h) - G(y,x_0)|d|\nu|(y) \leq$$

$$\leq 2\frac{c_2}{\lambda} \sup_{x\in\Omega'} \int_{\Omega' \cap B_{2r}(x)} |y - x|^{2-n}d|\nu|(y) +$$

$$+ \int_{\Omega'-B_r(x_0)} |G(y,x_h) - G(y,x_0)|d|\nu|(y).$$

Since $G^{x_h}(y) \to G^{x_0}(y)$ on $\Omega' - B_r(x_0)$ as $h \to +\infty$, we have

$$\limsup_{h\to\infty} |v(x_h) - v(x_0)| \leq 2\frac{c_2}{\lambda} \sup_{x\in\Omega'} \int_{\Omega' \cap B_{2r}(x)} |y - x|^{2-n}d|\nu|(y).$$

Since $\nu \in K_n(\Omega)$, the right hand side tends to 0 as r tends to 0^+, hence

$$\lim_{h\to\infty} |v(x_h) - v(x_0)| = 0$$

and v is continuous at x_0.

The function $w = u - v$ is a local weak solution of the equation

$$Lw = 0 \quad \text{in} \quad \Omega',$$

therefore w is continuous in Ω' by De Giorgi-Nash theorem. Thus the function $u = v + w$ is continuous at x_0. Since x_0 is arbitrary in Ω, we have $u \in C^0(\Omega)$. $\square$

5. THE WIENER CRITERION

Let Ω be a bounded open subset of R^n, let $Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u)$ be an elliptic operator on Ω as in 1.4, let $\mu \in {}_0(\Omega)$, and let $x_0 \in \Omega$.

DEFINITION 5.1 We say that x_0 is a <u>regular</u> <u>Dirichlet</u> <u>point</u> for the measure μ and the operator L if every local weak solution u of the equation

$$Lu + \mu u = 0$$

in an arbitrary small neighborhood of x_0 is continuous at x_0 and satisfies $u(x_0) = 0$. $\square$

For the definition of the pointwise values of u we refer to the convention (1.1).

We shall prove that the notion of regular Dirichlet point is independent of L, and can be characterized by means of a Wiener criterion involving the μ-capacity cap_μ of arbitrarily small balls $B_r(x_0)$ around x_0. Moreover, as we shall see in the next section, if x_0 is a regular Dirichlet point for the measure $\mu \in \mathcal{M}_0(\Omega)$, and $\nu \in K_n^{loc}(\Omega)$, then every local weak solution u of the equation

$$Lu + \mu u = \nu \quad \text{in} \quad \Omega$$

is continuous at x_0 and satisfies $u(x_0) = 0$.

In order to state the Wiener condition, we need the following definition. We recall that cap_μ is the μ-capacity relative to the Laplace operator $-\Delta$ introduced in Definition 3.1.

Let us fix a radius $R_0 > 0$ such that $\overline{B}_{R_0} \subseteq \Omega$. Here and henceforth we put $B_\rho = B_\rho(x_0)$ for every $\rho > 0$.

DEFINITION 5.2 For every $0 < \rho \leq R_0$ we put

$$\delta(\rho) = \frac{cap_\mu(B_\rho, B_{2\rho})}{cap(B_\rho, B_{2\rho})}$$

and we define the <u>Wiener modulus</u> $\omega(r,R)$ of μ at x_0 by

$$\omega(r,R) = \exp(-\int_r^R \delta(\rho)\frac{d\rho}{\rho})$$

for every $0 < r \leq R \leq R_0$. □

REMARK 5.3 It is easy to see that

$$0 \leq \delta(\rho) \leq 1$$

for every $0 < \rho \leq R_0$ (see Propositin 3.11(a)), and that

$$\frac{r}{R} \leq \omega(r,R) \leq 1$$

for every $0 < r \leq R \leq R_0$. □

DEFINITION 5.4 We say that x_0 is a <u>Wiener point</u> of the measure μ if

(5.1) $$\lim_{r \to 0_+} \omega(r,R) = 0$$

for some (hence for all) $0 < R \leq R_0$. □

Let us notice that (5.1) is obviously equivalent to the condition

(5.2) $$\int_0^R \delta(\rho)\frac{d\rho}{\rho} = +\infty$$

which is called the <u>Wiener condition</u> for the measure μ at the point x_0.

THEOREM 5.5. The point x_0 is a regular Dirichlet point for the measure μ and the operator L if and only if x_0 is a Wiener point of μ. □

Since the notion of Wiener point is independent of L, Theorem 5.5 shows that the notion of regular Dirichlet point is independent of L.

In order to prove Theorem 5.5 we need the following lemma. For every $0 < r < R < R_0$ we denote by w_R (resp. $w_{R,r}$) the μ-capacitary potential of B_R (resp. $B_R - B_r$) in B_{R_0} with respect to the operator L.

LEMMA 5.6 If x_0 is a regular Dirichlet point for the measure μ and the operator L, then

$$\lim_{r \to 0_+} w_{R,r}(x_0) = 0$$

for every $0 < R < R_0$.

PROOF. Suppose that x_0 is a regular Dirichlet point. Let us fix $0 < R < R_0$ and $\varepsilon > 0$. Since w_R is a local weak solution of the equation

$$Lw_R + \mu w_R = 0 \quad \text{in } B_R,$$

and x_0 is a regular Dirichlet point, there exists $\eta > 0$ such that

$$w_R \leq \varepsilon \quad \text{a.e. in } B_{2\eta}. \tag{5.3}$$

By Theorem 3.10 and Proposition 3.11 there exists a constant $k > 0$ such that

$$\text{cap}_\mu^L(B_r, B_{R_0}) \leq k \; \text{cap}(B_r, B_{R_0})$$

for every $0 < r < R_0$, hence

$$\lim_{r\to 0_+} \operatorname{cap}_\mu^L(B_r, B_{R_0}) = 0.$$

Therefore it follows easily from Proposition 3.9 that there exists $r_0 > 0$ such that

(5.4) $$w_r \geq 1 - \varepsilon \quad \text{a.e. in } B_{2\eta} - B_\eta$$

for every $0 < r < r_0$. By Proposition 3.7 we have

$$w_r + w_{R,r} \leq w_R + 1 \quad \text{a.e. in } B_{R_0}.$$

From (5.3) and (5.4) it follows that

$$w_{R,r} \leq 2\varepsilon \quad \text{a.e. in } B_{2\eta} - B_\eta$$

for every $0 < r < r_0$. We now apply the comparison theorem (Theorem 2.10) with $\Omega = B_{2\eta}$, $\mu_1 = 0$, $\mu_2 = \mu_{B_{2\eta} - B_r}$, $f_1 = f_2 = 0$, $u_1 = 2\varepsilon$, $u_2 = w_{R,r}$, and obtain

$$w_{R,r} \leq 2\varepsilon \quad \text{a.e. in } B_{2\eta},$$

hence

$$w_{R,r}(x_0) \leq \limsup_{\rho\to 0_+} \frac{1}{B_\rho} \int_{B_\rho} w_{R,r}(x)dx \leq 2\varepsilon$$

for every $0 < r < r_0$. □

PROOF OF THEOREM 5.5 The sufficiency of the Wiener condition is a consequence of Theorem 6.4 of the next section.

Let us prove its necessity. Suppose that x_0 is a regular Dirichlet point for the measure μ and for the operator L. Suppose, by contradiction, that

(5.5) $$\int_0^{R_0} \delta(\rho) \frac{d\rho}{\rho} < +\infty$$

For every $0 < \rho < R_0$ we put $\gamma(\rho) = \mathrm{cap}_\mu(B_\rho, B_{R_0})$. By Proposition 3.11(b) the function $\gamma(\rho)$ is non decreasing for $0 < \rho < R_0$.

Let us fix $0 < q < \frac{1}{2}$. By Proposition 3.11(d) we have

$$\gamma(\rho) \leq \mathrm{cap}_\mu(B_\rho, B_{2\rho})$$

for every $0 < \rho \leq qR_0$. Since there exists a constant $k > 0$ such that

$$\mathrm{cap}(B_\rho, B_{2\rho}) = k\rho^{n-2},$$

we have

$$k\gamma(\rho)\rho^{2-n} \leq \delta(\rho) \tag{5.6}$$

for every $0 < \rho \leq qR_0$.

For every $i \in N$ we define $r_i = R_0 q^i$, if $n \geq 3$, and $r_i = R_0 q^{2^i}$, if $n = 2$. From (5.6) it follows that

$$\int_0^{R_0} \delta(\rho)\,\frac{d\rho}{\rho} \geq k \int_0^{qR_0} \gamma(\rho)\rho^{1-n} d\rho \geq k \sum_{i=1}^{\infty} \int_{r_{i+1}}^{r_i} \gamma(\rho)\rho^{1-n} d\rho \geq \tag{5.7}$$

$$\geq k \sum_{i=1}^{\infty} \gamma(r_{i+1}) \int_{r_{i+1}}^{r_i} \rho^{1-n} d\rho.$$

For $n \geq 3$ we have

$$\int_{r_{i+1}}^{r_i} \rho^{1-n} d\rho = r_{i+1}^{2-n}\,\frac{1 - q^{n-2}}{n - 2} = r_{i+2}^{2-n}\, q^{n-2}\,\frac{1 - q^{n-2}}{n - 2},$$

whereas for $n = 2$ we have

$$\int_{r_{i+1}}^{r_i} \rho^{1-n} d\rho = 2^i \log\left(\frac{1}{q}\right) \geq \frac{1}{5} \log\left(\frac{2R_0}{r_{i+2}}\right).$$

Therefore from (5.5) and (5.7) it follows that

$$\sum_{i=1}^{\infty} \gamma(r_i) r_{i+1}^{2-n} < +\infty,$$

if $n \geqslant 3$, and

$$\sum_{i=1}^{\infty} \gamma(r_i) \log\left(\frac{2R_0}{r_{i+1}} \right) < +\infty,$$

if $n = 2$.

By Theorem 3.10 and by Proposition 3.11(b) there exists a constant $k > 0$ such that

$$\mathrm{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) \leqslant k\gamma(r_i)$$

for every $i \in N$, therefore

$$\sum_{i=1}^{\infty} \mathrm{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0})\, r_{i+1}^{2-n} < +\infty, \tag{5.8}$$

if $n \geqslant 3$, and

$$\sum_{i=1}^{\infty} \mathrm{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) \log\left(\frac{2R_0}{r_{i+1}} \right) < +\infty, \tag{5.9}$$

if $n = 2$.

For every $1 \leqslant h < j$ we denote by $w_{h,j}$ the μ-capacitary potential of $B_{r_h} - B_{r_j}$ in B_{R_0} relative to the operator L. By Proposition 3.8 we have

$$1 - w_{h,j}(x_0) \leqslant \sum_{i=h}^{j-1} (1 - w_{i,i+1}(x_0)).$$

By Proposition 3.9 there exists a constant $K > 0$ such that

$$1 - w_{i,i+1}(x_0) \leqslant K\, \mathrm{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) r_{i+1}^{2-n}$$

if $n \geqslant 3$, and

$$1 - w_{i,i+1}(x_0) \leqslant K\, \mathrm{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) \log\left(\frac{2R_0}{r_{i+1}} \right)$$

if $n = 2$. Therefore

$$1 - w_{h,j}(x_0) \leq K \sum_{i=h}^{j-1} \operatorname{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) r_{i+1}^{2-n}$$

if $n \geq 3$, and

$$1 - w_{h,j}(x_0) \leq K \sum_{i=h}^{j-1} \operatorname{cap}_\mu^L(B_{r_i} - B_{r_{i+1}}, B_{R_0}) \log\left(\frac{2R_0}{r_{i+1}}\right)$$

if $n = 2$. By (5.8) and (5.9), there exists $h \in N$ such that

(5.10) $$1 - w_{h,j}(x_0) < \frac{1}{2}$$

for every $j \geq h$. By Lemma 5.6 we have

(5.11) $$\lim_{j\to\infty} w_{h,j}(x_0) = 0.$$

The contradiction between (5.10) and (5.11) proves that (5.5) is false and concludes the proof of the theorem. □

6. ENERGY ESTIMATES

In this section we consider a local weak solution of the equation

(6.1) $$Lu + \mu u = \nu \quad \text{in } \Omega,$$

where Ω is a bounded open subset of R^n, $Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}x)D_j u)$ is an elliptic operator on Ω as in 1.4, $\mu \in \mathcal{M}_0(\Omega)$, and $\nu \in K_n^{loc}(\Omega)$. We study the behavior of u at a given point $x_0 \in \Omega$. In particular we prove in Theorem 6.4 that, if x_0 is a Wiener point of μ, then u is continuous at x_0 and $u(x_0) = 0$.

Let us fix a radius $R_0 > 0$ such that $\overline{B}_{R_0} \subseteq \Omega$. Here and henceforth we put $B_\rho = B_\rho(x_0)$ for every $\rho > 0$.

DEFINITION 6.1. For every $0 < R < R_0$ we put

$$V(R) = \sup_{x \varepsilon B_R} u(x)^2 + \int_{B_R} |Du(x)|^2 |x - x_0|^{2-n} dx + \int_{B_R} u(x)^2 |x - x_0|^{2-n} d\mu(x)$$

if $n \geq 3$, and

$$V(R) = \sup_{x \varepsilon B_R} u(x)^2 + \int_{B_R} |Du(x)|^2 \log\left(\frac{2R}{|x-x_0|} \right) dx + \int_{B_R} u(x)^2 \log\left(\frac{2R}{|x-x_0|} \right) d\mu(x),$$

if $n = 2$. □

In this section we estimate $V(R)$ in terms of the Wiener modulus of μ introduced in Definition 5.2, of the K_n-norm of ν introduced in Definition 4.4, and of the L^2-norm of u.

THEOREM 6.2. There exist two constants $k > 0$ and $\beta > 0$, depending only on the dimension n of the space and on the ellipticity constants λ and Λ, such that

$$V(r) \leq k\omega(r,R)^{\beta} V(R) + k\|\nu\|^2_{K_n(B_R)}$$

for every $0 < r \leq R \leq R_0$. □

The term $V(R)$ in the inequality above is estimated by the following theorem

THEOREM 6.3 For every $0 < q < 1$ there exists a constant $k > 0$, depending only on q, n, λ, and Λ such that

$$V(R) \leq k\ (1/R_0^n) \int_{B_{R_0}} u^2 dx + k\|\nu\|^2_{K_n(B_{R_0})}$$

for every $0 < R \leq qR_0$.

PROOF. Theorem 6.3 follows directly from Lemma 6.6, that will be proved later. □

The following theorem follows easily from Theorems 6.2 and 6.3.

THEOREM 6.4. If x_0 is a Wiener point of the measure μ, then

$$\lim_{r\to 0_+} V(r) = \lim_{x\to x_0} u(x) = u(x_0) = 0.$$

PROOF. Suppose that x_0 is a Wiener point of μ. Let us fix $0 < q < 1$. By Theorems 6.2 and 6.3, there exists two constants $k > 0$ and $\beta > 0$ such that

$$V(r) \leqslant k\omega(r,R)^{\beta}\left[\frac{1}{R_0^n}\int_{B_{R_0}} u^2dx + k\|\nu\|^2_{K_n(B_{R_0})}\right] + k\|\nu\|^2_{K_n(B_R)}$$

for every $0 < r \leqslant R \leqslant qR_0$. Since

$$\lim_{r\to 0_+} \omega(r,R) = 0,$$

we have

$$\limsup_{r\to 0_+} V(r) \leqslant k\|\nu\|^2_{K_n(B_R)}$$

for every $0 < R \leqslant qR_0$. By (4.2) of Remark 4.5 we have

$$\lim_{R\to 0_+} \|\nu\|^2_{K_n(B_R)} = 0,$$

hence

$$\lim_{r\to 0_+} V(r) = 0.$$

Since $|u(x)| \leqslant V(r)^{1/2}$ a.e. in B_r, by convention (1.1) we have

$$|u(x)| \leqslant V(r)^{1/2} \quad \forall x \in B_r,$$

hence

$$\lim_{x\to x_0} u(x) = u(x_0) = 0,$$

which concludes the proof of the theorem. □

Moreover, Theorems 6.2 and 6.3 also provide an estimate of the "μ-energy"

$$\mathcal{E}_\mu(r) = \int_{B_r} |Du(x)|^2 dx + \int_{B_r} u(x)^2 d\mu, \quad 0 < r \leq R_0.$$

In fact, we have

THEOREM 6.5. There exist two constants $k > 0$ and $\beta > 0$, depending only on n, λ and Λ, such that

$$\mathcal{E}_\mu(r) \leq k\omega(r,R)^\beta \frac{r^{n-2}}{\mathrm{cap}_\mu(B_{2R},B_{4R})} \mathcal{E}_\mu(2R) + kr^{n-2}\|\nu\|^2_{K_n(B_{2R})}$$

for every $0 < r \leq R \leq R_0/2$.

REMARK 6.6 In the special case $\mu = \infty_E$, Theorem 6.5 gives an estimate of the "energy"

$$\mathcal{E}(r) = \int_{B_r} |Du|^2 dx, \quad 0 < r \leq R_0,$$

namely

$$\mathcal{E}(r) \leq k\mathcal{E}(2R) \frac{r^{n-2}}{\mathrm{cap}(E \cap B_{2R},B_{4R})} \exp\left(-\beta \int_r^R \mathrm{cap}(E\cap B_\rho,B_{2\rho})\rho^{1-n}d\rho\right)$$
$$+ kr^{n-2} \|\nu\|^2_{K_n(B_{2R})}$$

for every $0 < r \leq R \leq R_0/2$, see also [10], section 5. □

PROOF of Theorem 6.5. By Theorem 6.3 and Poincaré inequality of Theorem 3.13, we have

$$V(R) \leq k \frac{1}{\mathrm{cap}_\mu(B_{2R},B_{4R})} \mathcal{E}_\mu(2R) + k\|\nu\|^2_{K_n(B_{2R})}.$$

On the other hand, we have

$$V(r) \geq kr^{2-n} \mathcal{E}_\mu(r)$$

Therefore Theorem 6.5 follows immediately from Theorem 6.2. □

LEMMA 6.7 For every $0 < q < 1$ there exist a constant $k > 0$, depending only on q, n, λ, and Λ such that

$$\sup_{B_{qR}} |u| \leq k\Big(\frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx\Big)^{1/2} + k\|\nu\|_{K_n(B_R)}$$

for every $0 < R \leq R_0$.

PROOF. Let $0 < R \leq R_0$, $s = \frac{2q+1}{3}$, $t = \frac{q+2}{3}$, $r = \frac{1-q}{6} R$, and let G be the Green function for the Dirichlet problem in B_R relative to the operator L. For every $x \in B_{tR}$ we define

$$w(x) = \int_{B_{tR}} G(y,x) d|\nu|(y).$$

By (1.4) and (1.5) there exists a constant $c_2 > 0$ such that

$$0 \leq w(x) \leq \frac{c_2}{\lambda} \|\nu\|_{K_n(B_R)}$$

for every $x \in B_{tR}$. Since $|\nu| \in H^{-1}(B_{tR})$ (see Theorem 4.9) we have $w \in H^1(B_{tR})$ and

$$a(w,v) = \int_{B_{tR}} v d|\nu|$$

for every $v \in H_0^1(B_{tR})$ (see 1.4). By Proposition 2.6 we have

$$a(|u|,v) \leq \int_{B_{tR}} v d|\nu|$$

for every $v \in H_0^1(B_{tR})$ with $v \geq 0$ a.e. in B_{tR}. Let $z = |u| - w$. Then $z \in H^1(B_{tR})$ and

$$a(z,v) \leq 0$$

for every $v \in H_0^1(B_{tR})$ with $v \geq 0$ a.e. in B_{tR}, therefore z is a local subsolution of the operator L in B_{tR}. By the maximum principle (see [12], Theorem 3.6) we have

$$\sup_{B_{sR}} z \leq \sup_{\partial B_{sR}} z.$$

By the local estimates for subsolutions of elliptic operators (see [12], Theorem 5.1), there exists a constant $k > 0$ such that for every $y \in \partial B_{sR}$ we have

$$\sup_{B_r(y)} z \leq k\left(\frac{1}{r^n} \int_{B_{2r}(y)} |z|^2 dx\right)^{1/2}.$$

Since $r = \frac{1-q}{6} R$ and $B_{2r}(y) \subseteq B_{tR} - B_{qR}$ for every $y \in \partial B_{sR}$, we obtain

$$\sup_{B_{sR}} z \leq k'\left(\frac{1}{R^n} \int_{B_{tR}-B_{qR}} |z|^2 dx\right)^{1/2},$$

where $k' = k6^{n/2}(1-q)^{-n/2}$, hence

$$\sup_{B_{qR}} |u| \leq \sup_{B_{qR}} z + \sup_{B_{qR}} w \leq$$

$$\leq k'\left(\frac{1}{R^n} \int_{B_{tR}-B_{qR}} (|u| - w)^2 dx\right)^{1/2} + \sup_{B_{tR}} w \leq$$

$$\leq k'\left(\frac{1}{R^n} \int_{B_R-B_{qR}} u^2 dx\right)^{1/2} + (1 + k')\sup_{B_{tR}} w \leq$$

$$\leq k'\left(\frac{1}{R^n} \int_{B_R-B_{qR}} u^2 dx\right)^{1/2} + (1 + k')\frac{c_2}{\lambda} \|\nu\|_{K_n(R)},$$

which is the estimate to be proved. □

LEMMA 6.8 For every $0 < q < 1$ there exists a constant $k > 0$, depending only on q, n, λ, and Λ such that

$$V(qR) \leq k \frac{1}{R^n} \int_{B_R-B_{qR}} u^2 dx + k\|\nu\|^2_{K_n}(B_R)$$

for every $0 < R \leq R_0$.

PROOF. We consider only the case $n \geq 3$, the case $n = 2$ being analogous. Let $0 < R \leq R_0$, $s = \frac{2q+1}{3}$, and $t = \frac{q+2}{3}$. For every $y \in B_R$ let $G^y = G(\cdot, y)$ be the Green function with singularity at y for the Dirichlet problem in B_R relative to the operator L, and let G^y_ρ, $\rho > 0$, be the corresponding approximate Green function (see 1.4). Let $\tau \in C_0^\infty(B_{tR})$ with $0 \leq \tau \leq 1$ in B_{tR}, $\tau = 1$ in B_{sR}, and

$$(6.2) \qquad |D\tau| \leq \frac{6}{(1-q)R} \quad \text{in } B_{tR}.$$

For every $0 < \rho < qR$ we define $v_\rho = u\tau^2 G_\rho^{x_0}$. Since $u \in H^1(B_{tR}) \quad L^\infty(B_{tR})$ by Lemma 6.5 and $G_\rho^{x_0} \in H^1(B_{tR} \cap L^\infty(B_{tR})$, we have $v_\rho = H^1(B_{tR})$. Since $u \in L^2(B_{tR},\mu)$ and $\tau^2 G_\rho^{x_0}$ is bounded in B_{tR}, we have $v_\rho \in L^2(B_{tR},\mu)$. Since τ has compact support in B_{tR}, the function v_ρ has compact support in B_{tR}. Therefore we can use v_ρ as test function for equation (6.1). From condition (ii) of Definition 2.1 we obtain

$$\int_{B_{tR}} [\sum_{i,j=1}^n a_{ij} D_j u D_i (u\tau^2 G_\rho^{x_0})] dx + \int_{B_{tR}} u^2 \tau^2 G_\rho^{x_0} d\mu = \int_{B_{tR}} u\tau^2 G_\rho^{x_0} \, d\nu,$$

which we rewrite as

$$(6.3) \qquad I_1 + I_2 + I_3 + I_4 = I_5$$

where

$$I_1 = \int_{B_{tR}} [\sum_{i,j=1}^n a_{ij} D_j u D_i u] \tau^2 G_\rho^{x_0} dx,$$

$$I_2 = 2\int_{B_{tR}} [\sum_{i,j=1}^n a_{ij} D_j u D_i \tau] u\tau G_\rho^{x_0} dx,$$

$$I_3 = \int_{B_{tR}} [\sum_{i,j=1}^n a_{ij} D_j u D_i G_\rho^{x_0}] u\tau^2 dx,$$

$$I_4 = \int_{B_{tR}} u^2\tau^2 G_\rho^{x_0} d\mu,$$

$$I_5 = \int_{B_{tR}} u\tau^2 G_\rho^{x_0} d\nu.$$

The term I_1 is easily estimated from below by the ellipticity condition:

(6.4) $$\lambda \int_{B_{tR}} |Du|^2 \tau^2 G_\rho^{x_0} dx \leq I_1.$$

The term I_2 can be estimated in absolute value from above by the boundedness of the coefficients and Young's inequality:

(6.5) $$|I_2| \leq 2n\Lambda \int_{B_{tR}-B_{sR}} |Du||D\tau||u|\tau G_\rho^{x_0} dx \leq$$

$$\leq \varepsilon n\Lambda \int_{B_{tR}-B_{sR}} |Du|^2\tau^2 G_\rho^{x_0} dx + \frac{n\Lambda}{\varepsilon} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 G_\rho^{x_0} dx$$

where $\varepsilon > 0$ is to be chosen later.

The term I_3 can be rewritten as

(6.6) $$I_3 = I_{31} + I_{32},$$

where

$$I_{31} = \int_{B_{tR}} [\sum_{i,j=1}^{n} a_{ij} D_j(\tfrac{1}{2} u^2\tau^2) D_i G_\rho^{x_0}] dx$$

and

$$I_{32} = -\int_{B_{tR}} [\sum_{i,j=1}^{n} a_{ij} D_j\tau D_i G_\rho^{x_0}] u^2\tau dx.$$

The term I_{31} is evaluated by taking the definition of $G_\rho^{x_0}$ into account (see 1.4):

(6.7)
$$I_{31} = |B_\rho|^{-1} \int_{B_\rho} \frac{1}{2} u^2 \tau^2 dx \geq 0.$$

The term I_{32} can be estimated in absolute value from above by

(6.8)
$$|I_{32}| \leq n\Lambda \int_{B_{tR}-B_{sR}} |D\tau|\,|DG_\rho^{x_0}|\,u^2 \tau dx \leq$$

$$\leq \frac{n\Lambda}{2\varepsilon\eta} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 dx + \frac{\varepsilon\eta n\Lambda}{2} \int_{B_{tR}-B_{sR}} |DG_\rho^{x_0}|^2 u^2 \tau^2 dx,$$

where $\eta > 0$ is to be chosen later.

In order to estimate the right hand side of (6.8) we rely on the following lemma, proved for example in [11], Lemma 6.2.

LEMMA 6.9 For every $0 < \rho \leq r \leq R$ and every $v \in H_0^1(B_R) \cap L^\infty(B_R)$, such that $v = 0$ a.e. in B_r, we have

$$\int_{B_R} |DG_\rho^{x_0}|^2 v^2 dx \leq 2n^2 \left(\frac{\Lambda}{\lambda} \right)^2 \int_{B_R} |G_\rho^{x_0}|^2 |Dv|^2 dx. \qquad \square$$

In order to apply Lemma 6.9, we introduce a function $\sigma \in C_0^\infty(B_R)$ such that $0 \leq \sigma \leq 1$ in B_R, $\sigma = 1$ in $B_{tR} - B_{sR}$, $\sigma = 0$ in B_{qR} and

(6.9)
$$|D\sigma| \leq \frac{6}{(1-q)R} \quad \text{in } B_{tR}.$$

We now apply Lemma 6.9 with $v = u\tau\sigma$ and $r = qR$ and we obtain

(6.10)
$$\int_{B_{tR}-B_{sR}} |DG_\rho^{x_0}|^2 u^2 \tau^2 dx \leq 2\alpha^2 \int_{B_{tR}-B_{qR}} |G_\rho^{x_0}|^2 |D(u\tau\sigma)|^2 dx \leq$$

$$\leq 6\alpha^2 \int_{B_{tR}-B_{qR}} |Du|^2 \tau^2 |G_\rho^{x_0}|^2 dx +$$

$$+ 6\alpha^2 \int_{B_{tR}-B_{qR}} (|D\tau|^2 + |D\sigma|^2) u^2 |G_\rho^{x_0}|^2 dx,$$

where $\alpha = \frac{n\Lambda}{\lambda}$. Therefore, the term I_{32} can be estimated by (6.8) and (6.10) as

(6.11)
$$|I_{32}| \leq \frac{\lambda\alpha}{2\varepsilon\eta} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 dx + $$
$$+ 3\varepsilon\eta\alpha^3\lambda \int_{B_{tR}-B_{qR}} |Du|^2 \tau^2 |G_\rho^{x_0}|^2 dx +$$
$$+ 3\varepsilon\eta\alpha^3\lambda \int_{B_{tR}-B_{qR}} (|D\tau|^2 + |D\sigma|^2) u^2 |G_\rho^{x_0}|^2 dx.$$

The term I_5 can be estimated in absolute value from above

(6.12)
$$|I_5| \leq \sup_{B_{tR}} |u| \int_{B_{tR}} G_\rho^{x_0} d|\nu|.$$

In order to estimate the right hand side of (6.12), we introduce the function

$$w(y) = \int_{B_{tR}} G(x,y) d|\nu|(x).$$

Since $|\nu|_{B_{tR}}$ belongs to $H^{-1}(B_R)$, we have that $w \in H_0^1(B_R)$ and

(6.13)
$$\int_{B_R} [\sum_{i,j=1}^n a_{ij} D_j w D_i v] dx = \int_{B_{tR}} v d|\nu|$$

for every $v \in H^1(B_R)$ (see 1.4). By putting $v = G_\rho^{x_0}$ in (6.13), and by taking the definition of $G_\rho^{x_0}$ into account (see 1.4), we obtain

(6.14)
$$\int_{B_{tR}} G_\rho^{x_0} d|\nu| = |B_\rho|^{-1} \int_{B_\rho} w(y) dy =$$
$$= \int_{B_{tR}} [|B_\rho|^{-1} \int_{B_\rho} G(x,y) dy] d|\nu|(x).$$

By (1.4) we have

$$G(x,y) \leq \frac{c_2}{\lambda} |x - y|^{2-n} \tag{6.15}$$

for every $x,y \in B_{tR}$. Since $y \to |x - y|^{2-n}$ is superharmonic, for every $x \in B_{tR}$ we have

$$|B_\rho|^{-1} \int_{B_\rho} G(x,y)dy \leq \frac{c_2}{\lambda} |B_\rho|^{-1} \int_{B_\rho} |x - y|^{2-n}dy \leq \frac{c_2}{\lambda} |x - x_0|^{2-n}. \tag{6.16}$$

From (6.12), (6.14), (6.16) we obtain

$$|I_5| \leq \frac{c_2}{\lambda} \sup_{B_{tR}} |u| \int_{B_{tR}} |x - x_0|^{2-n} d|\nu|(x) \leq \tag{6.17}$$

$$\leq \frac{c_2}{\lambda} \|\nu\|_{K^n(B_R)} \sup_{B_{tR}} |u|.$$

From (6.3), (6.4), (6.5), (6.6), (6.7), (6.11), and (6.17) we obtain the estimate:

$$\lambda \int_{B_{tR}} |Du|^2 \tau^2 G_\rho^{x_0} dx + \int_{B_{tR}} u^2 \tau^2 G_\rho^{x_0} d\mu \leq$$

$$\leq \varepsilon\lambda\alpha \int_{B_{tR}-B_{sR}} |Du|^2 \tau^2 G_\rho^{x_0} dx + \frac{\lambda\alpha}{\varepsilon} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 G_\rho^{x_0} dx +$$

$$+ \frac{\lambda\alpha}{2\varepsilon\eta} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 dx + 3\varepsilon\eta\alpha^3\lambda \int_{B_{tR}-B_{qR}} |Du|^2 \tau^2 |G_\rho^{x_0}|^2 dx +$$

$$+ 3\varepsilon\eta\alpha^3\lambda \int_{B_{tR}-B_{qR}} (|D\tau|^2 + |D\sigma|^2) u^2 |G_\rho^{x_0}|^2 dx +$$

$$+ \frac{c_2}{\lambda} \|\nu\|_{K_n(B_R)} \sup_{B_{tR}} |u|.$$

We pass to the limit in this inequality as $\rho \to 0_+$ and we obtain

$$\lambda \int_{B_{tR}} |Du|^2 \tau^2 G^{x_0} dx + \int_{B_{tR}} u^2 \tau^2 G^{x_0} d\mu \leq$$

(6.18)
$$\leq \varepsilon\lambda\alpha \int_{B_{tR}-B_{sR}} |Du|^2 \tau^2 G^{x_0} dx + \frac{\lambda\alpha}{\varepsilon} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 G^{x_0} dx +$$
$$+ \frac{\lambda\alpha}{2\varepsilon\eta} \int_{B_{tR}-B_{sR}} |D\tau|^2 u^2 dx + 3\varepsilon\alpha^3\lambda \int_{B_{tR}-B_{sR}} |Du|^2 \tau^2 G^{x_0}(\eta G^{x_0}) dx +$$
$$+ 3\varepsilon\alpha^3\lambda \int_{B_{tR}-B_{qR}} (|D\tau|^2 + |D\sigma|^2) u^2 G^{x_0}(\eta G^{x_0}) dx +$$
$$+ \frac{c_2}{\lambda} \|\nu\|_{K_n(B_R)} \sup_{B_{tR}} |u|.$$

By (6.15), for every $x \in B_{tR} - B_{qR}$ we have

$$G^{x_0}(x) \leq \frac{c_2}{\lambda} |x - x_0|^{2-n} \leq \frac{c_2}{\lambda} q^{2-n} R^{2-n},$$

hence, by choosing

$$\eta = \frac{\lambda}{c_2} q^{n-2} R^{n-2}$$

we have

$$\eta G^{x_0} \leq 1 \quad \text{on} \quad B_{tR} - B_{qR}.$$

By taking (6.2) and (6.9) into account, from (6.18) we obtain

$$\lambda(1 - \varepsilon\alpha - 3\varepsilon\alpha^3) \int_{B_{tR}} |Du|^2 \tau^2 G^{x_0} dx + \int_{B_{tR}} u^2 \tau^2 G^{x_0} d\mu \leq$$

$$\leq \left(\frac{54\alpha}{\varepsilon} + 216\varepsilon\alpha^3 \right) c_2 q^{2-n} (1 - q)^{-2} \frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx + \frac{c_2}{\lambda} \|\nu\|_{K_n(B_R)} \sup_{B_{tR}} |u|$$

By choosing $\varepsilon = [2\alpha(1 + 3\alpha^2)]^{-1}$ we obtain

(6.19)
$$\frac{\lambda}{2} \int_{B_{qR}} |Du|^2 G^{x_0} dx + \int_{B_{qR}} u^2 G^{x_0} d\mu \leq$$

$$\leq 648\, a^4 c_2 q^{2-n} (1 - q)^{-2} \frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx + \frac{c_2}{\lambda} \|\nu\|_{K_n(B_R)} \sup |u|.$$

By (1.4) there exists a constant $c_1 > 0$ such that

$$G^{x_0}(x) \geq \frac{c_1}{\Lambda} |x - x_0|^{2-n}$$

for every $x \in B_{qR}$, therefore by (6.19) there exists a constant $k > 0$ such that

(6.20)
$$\int_{B_{qR}} |Du|^2 |x - x_0|^{2-n} dx + \int_{B_{qR}} u^2 |x - x_0|^{2-n} d\mu \leq$$

$$\leq k \frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx + k \| \nu \|_{K_n(B_R)} \sup_{B_{tR}} |u|$$

By Lemma 6.7, with $q = t$, there exists a constant $c > 0$ such that

(6.21)
$$\sup_{B_{tR}} |u| \leq c\left(\frac{1}{R^n} \int_{B_R - B_{tR}} u^2 dx\right)^{1/2} + c \| \nu \|_{K_n(B_R)}$$

hence

(6.22)
$$\sup_{B_{qR}} u^2 \leq 2c^2 \frac{1}{R^n} \int_{B_R - B_{tR}} u^2 dx + 2c^2 \| \nu \|^2_{K_n(B_R)} .$$

By adding (6.20) and (6.22), and by using (6.21) to estimate the right hand side of (6.20) we obtain

$$V(qR) \leq (k + 2c^2) \frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx + (2c^2 + kc) \| \nu \|^2_{K_n(B_R)} +$$

$$+ kc \| \nu \|_{K_n(B_R)} \left(\frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx\right)^{1/2} \leq$$

$$\leq \left(k + 2c^2 + \frac{kc}{2}\right) \frac{1}{R^n} \int_{B_R - B_{qR}} u^2 dx + \left(2c^2 + \frac{3}{2} kc\right) \| \nu \|^2_{K_n(B_R)},$$

which is the estimate to be proved. □

We now state a lemma which reproduces an intergration argument from [9]. For a proof of this lemma, see [4] or [11].

LEMMA 6.10 Let $R > 0$, $0 < q < 1$, $0 < r \leq qR$. Let $\gamma: [r,R] \to [0,1]$ be a measurable function and let $\eta: [r,R] \to [0,+\infty[$ be a non decreasing function. Suppose that there exists a constant $k > 0$ such that

$$\eta(q\rho) \leq \frac{\eta(\rho)}{1 + k\gamma(\rho)}$$

for every $rq^{-1} < \rho < R$. Then

$$\eta(r) \leq c\eta(R) \exp(-\beta \int_r^R \gamma(\rho) \frac{d\rho}{\rho}),$$

where $c = \exp(\frac{k}{1 + k})$ and $\beta = \frac{k}{1 + k} \frac{1}{|\log q|}$.

PROOF OF THEOREM 6.2 We consider only the case $n \geq 3$, the case $n = 2$ being analogous. Let us fix $0 < q < 1$ and $0 < r \leq R \leq R_0$. Let us consider first the case $r \leq qR$. By Lemma 6.8 there exists a constant $k_0 > 0$ such that

$$V(q\rho) \leq k_0 \frac{1}{\rho^n} \int_{B_\rho - B_{q\rho}} u^2 dx + k_0 \|\nu\|^2_{K_n(B_\rho)} \tag{6.23}$$

for every $0 < \rho \leq R$. Suppose that

$$V(r) \geq 2k_0 \|\nu\|^2_{K_n(B_R)}. \tag{6.24}$$

Since $V(\rho)$ and $\|\nu\|^2_{K_n(B_\rho)}$ are non decreasing functions of ρ we have

$$k_0 \|\nu\|^2_{K_n(B_\rho)} \leq \frac{1}{2} V(q\rho)$$

for every $rq^{-1} < \rho < R$. Therefore from (6.23) we obtain

$$V(q\rho) \leq 2k_0 \frac{1}{\rho^n} \int_{B_\rho - B_{q\rho}} u^2 dx \tag{6.25}$$

for every $rq^{-1} < \rho < R$.

By Theorem 3.13 (Poincaré inequality) there exists a constant $k > 0$ such that

$$(6.26) \qquad \frac{1}{\rho^n} \int_{B_\rho - B_{q\rho}} u^2 dx \leq \frac{k}{\mathrm{cap}_\mu(B_\rho - B_{q\rho}, B_{2\rho})} \left[\int_{B_\rho - B_{q\rho}} |Du|^2 dx + \int_{B_\rho - B_{q\rho}} u^2 d\mu \right].$$

For every $0 < \rho \leq R_0$ we define

$$\delta_q(\rho) = \frac{\mathrm{cap}_\mu(B_\rho - B_{q\rho}, B_{2\rho})}{\mathrm{cap}(B_\rho, B_{2\rho})} .$$

Since there exists a constant $k > 0$ such that

$$\mathrm{cap}(B_\rho, B_{2\rho}) = k\rho^{n-2},$$

by (6.26) there exists a constant $k > 0$ such that

$$(6.27) \qquad \frac{1}{\rho^n} \int_{B_\rho - B_{q\rho}} u^2 dx \leq \frac{k\ \rho^{2-n}}{\delta_q(\rho)} \left[\int_{B_\rho - B_{q\rho}} |Du|^2 dx + \int_{B_\rho - B_{q\rho}} u^2 d\mu \right] \leq$$

$$\leq \frac{k}{\delta_q(\rho)} \left[\int_{B_\rho - B_{q\rho}} |Du|^2 |x - x_0|^{2-n} dx + \int_{B_\rho - B_{q\rho}} u^2 |x - x_0|^{2-n} d\mu \right].$$

By (6.25) and (6.27) there exists a constant $k > 0$ such that

$$k\delta_q(\rho)\ V(q\rho) \leq \int_{B_\rho - B_{q\rho}} |Du|^2 |x - x_0|^{2-n} dx + \int_{B_\rho - B_{q\rho}} u^2 |x - x_0|^{2-n} d\mu.$$

By adding $V(q\rho)$ to both sides we obtain

$$(1 + k\delta_q(\rho))V(q\rho) \leq V(\rho)$$

for every $rq^{-1} < \rho < R$.

We now apply Lemma 6.10 with $\eta(\rho) = V(\rho)$ and $\gamma(\rho) = \delta_q(\rho)$, and we obtain that

$$(6.28) \qquad V(r) \leq cV(R) \exp\left(-\alpha \int_r^R \delta_q(\rho) \frac{d\rho}{\rho}\right)$$

where c and α are positive constants which depends only on n, q, λ and Λ.

If condition (6.24) is not satisfied, then

$$V(r) \leq 2k_0 \|\nu\|^2_{K_n(B_R)}. \tag{6.29}$$

In any case, from (6.28) or (6.29) we obtain the estimate

$$V(r) \leq cV(R) \exp(-\alpha \int_r^R \delta_q(\rho) \frac{d\rho}{\rho}) + 2k_0 \|\nu\|^2_{K_n(B_R)}. \tag{6.30}$$

In order to replace $\delta_q(\rho)$ with $\delta(\rho)$ we use the following lemma.

LEMMA 6.11. For every $0 < r < R \leq R_0$ and every $0 < q < 1$ we have

$$\int_r^R \delta_q(\rho) \frac{d\rho}{\rho} \geq (1 - q^{n-2}) \int_r^R \delta(\rho) \frac{d\rho}{\rho} - q^{n-2}|\log q|.$$

PROOF. By Proposition 3.11 (c) and (d) we have

$$\begin{aligned} \mathrm{cap}_\mu(B_\rho, B_{2\rho}) &\leq \mathrm{cap}_\mu(B_{q\rho}, B_{2\rho}) + \mathrm{cap}_\mu(B_\rho - B_{q\rho}, B_{2\rho}) \leq \\ &\leq \mathrm{cap}_\mu(B_{q\rho}, B_{2q\rho}) + \mathrm{cap}_\mu(B_\rho - B_{q\rho}, B_{2\rho}) \end{aligned}$$

for every $r \leq \rho \leq R$. By dividing by $\mathrm{cap}(B_\rho, B_{2\rho})$, and by remarking that

$$\mathrm{cap}(B_{q\rho}, B_{2q\rho}) = q^{n-2}\, \mathrm{cap}(B_\rho, B_{2\rho})$$

we obtain

$$\delta(\rho) \leq q^{n-2}\delta(q\rho) + \delta_q(\rho)$$

for every $r \leq \rho \leq R$, hence

$$\begin{aligned} \int_r^R \delta(\rho) \frac{d\rho}{\rho} &\leq q^{n-2} \int_r^R \delta(q\rho) \frac{d\rho}{\rho} + \int_r^R \delta_q(\rho) \frac{d\rho}{\rho} = \\ &= q^{n-2} \int_{qr}^{qR} \delta(\rho) \frac{d\rho}{\rho} + \int_r^R \delta_q(\rho) \frac{d\rho}{\rho}. \end{aligned}$$

Since $0 \leq \delta(\rho) \leq 1$, we obtain

$$\int_r^R \delta_q(\rho)\,\frac{d\rho}{\rho} \geq (1 - q^{n-2}) \int_r^R \delta(\rho)\,\frac{d\rho}{\rho} - q^{n-2} \int_{qr}^r \delta(\rho)\,\frac{d\rho}{\rho} \geq$$

$$\leq (1 - q^{n-2}) \int_r^R \delta(\rho)\,\frac{d\rho}{\rho} - q^{n-1}|\log q|,$$

which is the inequality to be proved. □

PROOF OF THEOREM 6.2: CONCLUSION. If $r \leq qR$, from (6.30) and from Lemma 6.11 we obtain

$$V(r) \leq k_1 V(R)\exp(-\beta \int_r^R \delta(\rho)\,\frac{d\rho}{\rho}) + 2k_0 \|v\|^2_{K_n(B_R)} \tag{6.31}$$

where $k_1 = c\exp(q^{n-2}|\log q|)$ and $\beta = \alpha(1 - q^{n-2})$. If $qR \leq r \leq R$, then

$$\int_r^R \delta(\rho)\,\frac{d\rho}{\rho} \leq \int_{qR}^R \delta(\rho)\,\frac{d\rho}{\rho} = \log\frac{1}{q}$$

hence

$$\exp(-\beta \int_r^R \delta(\rho)\,\frac{d\rho}{\rho}) \geq q^\beta.$$

Therefore from $V(r) \leq V(R)$ it follows that

$$V(r) \leq q^{-\beta} V(R)\exp(-\beta \int_r^R \delta(\rho)\,\frac{d\rho}{\rho}). \tag{6.32}$$

In any case from (6.31) or (6.32) we obtain the estimate

$$V(r) \leq k\omega(r,R)^\beta V(R) + k\|v\|^2_{K_n(B_R)}$$

where $k = \max\{k_1, q^{-\beta}, 2k_0\}$. □

ACKNOWLEDGMENT

The first author was partially supported by the Gruppo Nazionale per l'Analisi Funzionale le sue Applicazioni (Consiglio Nazionale delle Ricerche) and by a National Research Project of the Ministero della Pubblica Istruzione.

The second author wishes to thank the Institute for Mathematics and its Applications for hospitality and support at the University of Minnesota, where this paper was partially written.

REFERENCES

[1] M. Aizenman and B. Simon: Brownian motion and Harnack inequality for Schrödinger operators. Comm. Pure Appl. Math. 35 (1982), 209-273.

[2] G. Dal Maso and U. Mosco: Wiener's criterion and Γ-convergence. IMA Preprint Series # , University of Minnesota (1985).

[3] G. Dal Maso and U. Mosco: The Wiener modulus of a radial measure. IMA Preprint Series # , University of Minnesota (1985).

[4] J. Frehse and U. Mosco: Wiener obstacles. In "Nonlinear partial differential equations and their applications", Collège de France Seminar, Vol. 6, Ed. H. Brezis and J.L. Lions, Research Notes in Math., Pitman (1984).

[5] H. Federer and W. Ziemer: The Lebesgue set of a function whose distribution derivatives are p-th power summable. Indiana Univ. Math. J. 22 (1972), 139-158.

[6] T. Kato: Schrödinger operators with singular potentials. Israel J. Math. 13 (1973), 135-148.

[7] N.S. Landkof: Foundations of modern potential theory, Springer-Verlag, Berlin, 1972.

[8] W. Littman, G. Stampacchia and H. Weinberger: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa 17 (1963), 45-79.

[9] V.G. Maz'ja: Behavior near the boundary of solutions of the Dirichlet problem for a second order elliptic equation in divergence form. Mat. Zametki 2 (1967), 209-220; Math. Notes 2 (1967), 610-617.

[10] U. Mosco: Pointwise potential estimates for elliptic obstacle problems. IMA Preprint Series, No. 117, 1984; to appear in Proc. of AMS Symposia in Pure Mathematics 44 (1985).

[11] U. Mosco: Wiener criterion and potential estimates for the obstacle problem. IMA Preprint Series, No. 135 (1985).

[12] G. Stampacchia: Le problème de Dirichlet pour les équations elliptiques du second order à coefficients discontinus. Ann. Inst. Fourier (Grenoble), 15 (1965), 189-258.

[13] N. Wiener: The Dirichlet problem. J. Math and Phys. 3 (1924), 127-146.

Gianni Dal Maso
Istituto di Matematica
Università di Udine
33100 UDINE (Italy)

Umberto Mosco
Dipartimento di Matematica
Università di Roma
00185 ROMA (Italy)