ON THE MACROSCOPIC DESCRIPTION OF SLOW VISCOUS FLOW
PAST A RANDOM ARRAY OF SPHERES

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ON THE MACROSCOPIC DESCRIPTION OF SLOW VISCOUS FLOW
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Jacob Rubinstein

Abstract

We give rigorous derivation of the Brinkman's equation as the effective equation for slow viscous flow in porous media with high porosity. The medium is composed of spherical obstacles which are distributed randomly, and the microscopic flow is described by the Stokes equation. Along the road we give W_2^1 convergence estimates for the point sources approximation.

1. Introduction

The derivation of macroscopic equations for flow in porous media is a long standing problem of highly practical and theoretical importance. One of the equations commonly used is the Brinkman's equation (Brinkman [2], Saffman [15]). In this paper we show that the solution to this equation is indeed the limit (in the appropriate sense) of the flow of slow viscous incompressible fluid through a random distribution of spheres (which is our model for a porous medium).

We assume that the microscopic flow is described by Stokes equation (Happel and Brenner [7]), and that the number of spheres N goes to ∞ as their radius goes to zero under the scaling

$$(1.1) \quad R = \frac{\alpha'}{N}, \quad \alpha' = O(1).$$

In the last decade there appeared several works which analysed boundary value problems for the Laplace equation and the heat equation in a similar set up (Kac [9], Rauch and Taylor [14], Hruslov and Marchenko [8], Papanicolaou and Varadhan [13], Ozawa [12], Figari et al. [5]). Our approach is similar to the one suggested first by Ozawa [12] and then extended and clarified by Figari et al. [5]. The idea is first to approximate the full microscopic problem by point sources and then to average those singularities in order to obtain the desired smooth field. There is, however, a major difference between our proof and the works we mentioned above: Stokes equation does not have a maximum principle, nor does it have an obvious probabilistic interpretation, so that the arguments used for the elliptic problems do not go through here. Instead, we use in Section 3 variational principles to establish the convergence of the point sources approximation to the microscopic flow. Interestingly, we obtain estimates in the W_2^1 norm in contrast to the L_p estimates obtained in [12] and [5].

In Section 2 we formulate the physical problem, describe the underlying probability distributions and give a few preliminary Lemmata. The point sources are averaged out in Section 4 where we show that they converge to the smooth solution of the Brinkman's equation. This part is along the same line of Figari [5], and actually can be abstracted so that it holds for a large family of operators. The reader can consult Brinkman [2], Childress [4], Caflisch and Rubinstein [3], Lundgren [11], Saffman [15] or Tam [16] (among others) for formal derivations and various applications of the Brinkman's equation

2. Formulation and Basic Estimates

Let $D \subseteq \mathbb{R}^3$ be an open smooth domain. $\{y_j\}$ $j = 1, \dots, N$ are a collection of points in D that are independent identically distributed random variables with density function $\rho(\underline{x})$ which is continuous with compact support in D . There is a sphere B_j^N centered at each y_j with radius

$$(2.1) \quad R = \frac{\alpha^1}{N}, \quad \alpha^1 = O(1).$$

Set $D_1^N = D - \bigcup_{j=1}^N B_j^N$, and consider in D_1^N the problem:

$$(2.2) \quad \begin{aligned} \mu \Delta \underline{u}^N - \lambda \underline{u}^N &= \nabla P^N + \underline{f} \quad \text{in } D_1^N \\ \nabla \cdot \underline{u}^N &= 0 \quad \text{in } D_1^N \\ \underline{u}^N &= 0 \quad \text{on } S_j^N \quad j = 1, 2, \dots, N. \end{aligned}$$

Here μ is the viscosity, $\underline{u}$ the velocity and P is the pressure, while S_j^N is the surface of B_j^N and $\underline{f}(\underline{x})$ is a continuous vector field with compact support in D . (We concentrate here on $D = \mathbb{R}^3$ so we do not prescribe boundary conditions on ∂D . It is easy to extend our proofs to bounded domains).

Equation (2.2) is the Laplace transform of the time dependent Stokes equation. Its fundamental tensor is given by $\underline{\psi}$ where

$$(2.3) \quad (\underline{\psi})_{ij} = \frac{1}{4\pi\mu} \left[\frac{e^{-\sigma|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} \delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1 - e^{-\sigma|\underline{x}-\underline{y}|}}{\sigma^2 |\underline{x}-\underline{y}|} \right) \right],$$

so that

$\underline{w} = \underline{\psi} \underline{e}$ solves

$$(2.4) \quad \mu \Delta \underline{w} - \mu \sigma^2 \underline{w} = \nabla P - \underline{e} \delta(\underline{y})$$

(derivatives are with respect to $\underline{x}$).

We want to prove that $\underline{u}^N$ converges (in a sense which will be made precise in the sequel) to the macroscopic vector field $\underline{v}(\underline{x})$ which is defined as the solution of

$$(2.5) \quad \mu \Delta \underline{v} - \lambda \underline{v} - 6\pi\mu\alpha'\rho(\underline{x})\underline{v} = \nabla \overline{p} + \underline{f} \quad \text{in } D,$$

$$\underline{\nabla} \cdot \underline{v} = 0 \quad \text{in } D.$$

(2.5) is sometimes referred to as the Brinkman's equation (Saffman [15], Lundgren [11], Tam [16]).

We start with a few preliminary propositions:

Lemma 1

Let Ω be the configurations ensemble of particle centers, and $\Omega_1^N \subseteq \Omega$ be that part of Ω for which

$$(2.6) \quad \min_{i \neq j} |y_i - y_j| > cN^{-1+\nu} \quad \nu < \frac{1}{3},$$

$$(2.7) \quad \frac{1}{N^2} \sum'_{i,j} |y_i - y_j|^{-3+\xi} < c < \infty \quad \xi > 0.$$

and

$$(2.8) \quad \frac{1}{N^3} \sum_{\substack{i,j,k \\ i \neq j \neq k}} |\underline{y}_i - \underline{y}_j|^{-2} |\underline{y}_j - \underline{y}_k|^{-2} < c < \infty.$$

(c will denote a generic constant, and $\sum'_j \equiv \sum_{j \neq i}$).

Then

$$\lim_{N \rightarrow \infty} P_{\Omega}(\Omega - \Omega_1^N) = 0.$$

A proof is given by Ozawa [12]. (He treats only (2.6) and (2.7), but (2.8) can be included using the same method).

From now on we will assume $\{y_j\}$ satisfy (2.6)-(2.8), and hence our result will hold in probability. Notice that (2.6) excludes the possibility of overlapping.

Next, let us define a $3N \times 3N$ matrix $\tilde{\Psi}$ which will play a major role in the analysis. $\tilde{\Psi}$ will be the $N \times N$ blocks matrix where the (i,j) block is $\underline{\Psi}(\underline{y}_i, \underline{y}_j)$ for $i \neq j$, and the 3×3 zero matrix for $i = j$. $i, j: 1, 2, \dots, N$.

Lemma 2

$$\| \frac{1}{N} \tilde{\Psi} \| \leq c \sigma^{-\beta} \quad \forall \beta < \frac{1}{2}.$$

$\| \cdot \|$ denotes matrix (or vectors) norm, $\sigma^2 = \frac{\lambda}{\mu}$.

Proof

Rewriting $\underline{\Psi}(\underline{x}, 0)$

$$(\underline{\Psi})_{ij} = \frac{e^{-\sigma|\underline{x}|}}{|\underline{x}|} \delta_{ij} + \frac{x_i x_j}{|\underline{x}|^3} e^{-\sigma|\underline{x}|} + \frac{1 - e^{-\sigma|\underline{x}|} - \sigma|\underline{x}| e^{-\sigma|\underline{x}|}}{\sigma^2 |\underline{x}|^2} \left(-\frac{\delta_{ij}}{|\underline{x}|} + \frac{3x_i x_j}{|\underline{x}|^3} \right),$$

and observing that

$$e^{-\sigma|\underline{x}|} \leq (\sigma|\underline{x}|)^{-\beta}, \quad \beta < \frac{1}{2}$$

$$\frac{1 - e^{-\sigma|\underline{x}|} - \sigma|\underline{x}| e^{-\sigma|\underline{x}|}}{\sigma^2 |\underline{x}|^2} \leq (\sigma|\underline{x}|)^{-\beta}$$

we get

$$\| \underline{\Psi}(\underline{y}_i, \underline{y}_j) \|^2 \leq c \frac{\sigma^{-2\beta}}{|\underline{y}_i - \underline{y}_j|^{2+2\beta}}.$$

So,

$$\| \frac{1}{N} \tilde{\psi} \| \leq c(N^{-2} \sum_{i,j} \frac{\sigma^{-2\beta}}{|\underline{y}_i - \underline{y}_j|^{2+2\beta}})^{1/2} \leq c\sigma^{-\beta}$$

where we used (2.7).

Q.E.D.

Now, the Stokes equation is not an elliptic problem, and in particular it does not satisfy the maximum principle. There are, however, certain a priori estimates and variational principles which hold for it. The following proposition will be used in the next section:

Lemma 3

Let A be a bounded domain in R^3 ,

$$\mu \Delta \underline{z} - \lambda \underline{z} = \nabla p \quad \text{in } A$$

$$\nabla \cdot \underline{z} = 0 \quad \text{in } A$$

$$\underline{z}|_{\partial A} = \underline{u}.$$

Set

$$F(\underline{u}) = \mu \int_A u_{i,j} u_{i,j} + \lambda \int_A u_i u_i$$

and let G be the class of piecewise differentiable divergence free vector fields in A , satisfying $\underline{u}|_{\partial A} = \underline{u}$.

Then

$$F(\underline{z}) = \min_{\underline{u} \in G} F(\underline{u})$$

Proof

Let $\underline{w} \in A$, and write $\underline{w} = \underline{z} + \underline{u}$.

$$F(\underline{w}) = F(\underline{z}) + F(\underline{u}) + 2\mu \int_A u_{i,j} z_{i,j} + 2\lambda \int_A u_i z_i$$

but $u_{i,j}z_{i,j} = \partial_j(u_i z_{i,j}) - u_i z_{i,jj}$, so

$$F(\underline{w}) = F(\underline{z}) + F(\underline{u}) + 2 \int_{\partial A} u_i z_{i,j} n_j ds - 2 \int u_i (\mu z_{i,jj} - \lambda z_i)$$

using $u_i|_{\partial A} = 0$ and $u_{i,j} = 0$ we conclude

$$F(\underline{w}) = F(\underline{z}) + F(\underline{u}) > F(\underline{z}).$$

Q.E.D.

Note that for finite μ and λ , $F(\underline{u})$ is equivalent to the W_2^1 norm, where

$$\|g\|_{2,1}(A) = \int_A |\nabla g|^2 + |g|^2.$$

The main result we are going to prove is:

Theorem 1

Let $\underline{u}^N, \underline{v}$ be the solution of (2.2), (2.5), respectively, under the conditions specified above. Then

$$\lim_{N \rightarrow \infty} P_{\Omega} \{N^{\gamma} \|\underline{u}^N - \underline{v}\|_2(D) < \epsilon\} = 1 \quad \forall \gamma < \frac{1}{6}, \quad \epsilon > 0$$

$(\underline{u}^N_N)$ is extended from D_1^N to D by defining it to be identically zero in $\bigcup_{j=1}^N B_j$.

3. Point Sources Approximation

Let $\underline{w}_e^N$ be the Green's function associated with (2.2). (A source of strength $\underline{e}$ is located at $\underline{x} \in D_1^N$). The idea is to approximate $\underline{w}_e^N$ by point sources distributed at $\{\underline{y}_j\}$ with suitable strengths. For this purpose we let

$$(3.1) \quad \underline{z}_e^N = \underline{\psi}(\underline{x}, \underline{y}) \underline{e} + \sum_j \underline{\psi}(\underline{y}_j, \underline{y}) \underline{q}_x^j, \quad \underline{q}_x^j = (q_{1x}^j, q_{2x}^j, q_{3x}^j).$$

Point sources do not suffice to account for the full boundary conditions. We ask instead that the average value of $\underline{z}_e^N$ on each S_i will be zero. Averaging $\underline{\psi}(\underline{x}, \underline{y})$ for $\underline{y} \in S_i$, neglecting the exponential factor (recall that N is large) and using

$$\langle \frac{\partial^2}{\partial x_i \partial x_j} | \underline{x} \rangle = \frac{1}{3} \delta_{ij} \cdot \frac{1}{R},$$

where $\langle \cdot \rangle$ means surface average over a sphere of radius R , we arrive at the following system of equations for $\{\underline{q}_x^j\}$:

$$(3.2) \quad \underline{\psi}(\underline{x}, \underline{y}_i) \underline{e} + \sum_j' \underline{\psi}(\underline{y}_j, \underline{y}_i) \underline{q}_x^j + \frac{N}{\alpha} I_3 \underline{q}_x^i = 0, \quad i = 1, 2, \dots, N.$$

Here $\alpha = 6\pi\mu\alpha'$ and I_K is the $K \times K$ identity matrix.

Construct now a vector $\underline{\tilde{q}}$ of length $3N$ which is composed of N chains of length 3, the k^{th} chain being $\underline{q}_x^k$, the matrix $\underline{\tilde{\psi}}$ as in Section 2, and the $3N \times 3$ matrix $\underline{\tilde{\Phi}}(\underline{x}, \{\underline{y}_j\})$

$$\underline{\tilde{\Phi}} = \begin{pmatrix} \underline{\psi}(\underline{x}, \underline{y}_1) \\ \vdots \\ \underline{\psi}(\underline{x}, \underline{y}_N) \end{pmatrix}.$$

Then, (3.2) can be written as

$$\tilde{\Phi} \underline{e} + (\tilde{\Psi} + \frac{N}{\alpha} I_{3N}) \tilde{\underline{q}}_x = 0, \text{ so}$$

$$(3.3) \quad \tilde{\underline{q}}_x = (\frac{\alpha}{N} \tilde{\Psi} + I_{3N})^{-1} (-\frac{\alpha}{N} \tilde{\Phi} \underline{e}).$$

From Lemma 2, $\|\frac{\alpha}{N} \tilde{\Psi}\| \leq c\sigma^{-\beta}$, so $\tilde{\underline{q}}_x$ is well defined for λ large enough. Returning to (3.1) we rewrite $\underline{z}^N$ as

$$(3.4) \quad \underline{z}_e^N = \underline{\Psi}(x, y) \underline{e} - \frac{\alpha}{N} \tilde{\Phi}^T(\{y_i\}, y) (\frac{\alpha}{N} \tilde{\Psi} + I_{3N})^{-1} \tilde{\Phi} \underline{e}.$$

Note that

$$(3.5) \quad \underline{u}^N(y) = \int \underline{w}^N(x, y) \underline{f}(x) dx$$

and similarly we set

$$(3.6) \quad \underline{n}^N(y) = \int [\underline{\Psi}(x, y) \underline{f}(x) dx - \frac{\alpha}{N} \tilde{\Phi}^T(y) (\frac{\alpha}{N} \tilde{\Psi} + I_{3N})^{-1} \tilde{\Phi}(x) \underline{f}(x)] dx.$$

Theorem 2

$$\|\underline{n}^N(y) - \underline{u}^N(y)\|_2 (D_1^N) \leq cN^{-\gamma} \text{ for any } \gamma < \frac{1}{6},$$

and $\{y_j\} \in \Omega_1^N$.

Proof

Let $\underline{\delta}^N = \underline{n}^N(y) - \underline{u}^N(y)$. $\underline{\delta}^N$ solves

$$(3.7) \quad \mu \Delta \underline{\delta}^N - \lambda \underline{\delta}^N = \tilde{\nabla} P^N \text{ in } D_1^N$$

$$\underline{\nabla} \cdot \underline{\delta}^N = 0 \text{ in } D_1^N$$

$$\underline{\delta}^N(y) = \underline{n}^N(y) \text{ for } y \in S_i \quad i = 1, 2, \dots, N.$$

We want to use the variational principle (Lemma 3). First we need an estimate for $\underline{n}^N(\underline{y})$ on the boundaries. Recalling (3.2) we write

$$(3.8) \quad \underline{n}^N(\underline{y}) = \int [\underline{\psi}(\underline{x}, \underline{y}) - \underline{\psi}(\underline{x}, \underline{y}_i)] \underline{f}(\underline{x}) d\underline{x} + \\ + \sum_j' [(\underline{\psi}(\underline{y}_j, \underline{y}_i) - \underline{\psi}(\underline{y}_j, \underline{y})) \underline{q}_f^j + \\ + [\underline{\psi}(\underline{y}_i, \underline{y}) - \frac{N}{\alpha} I_3] \underline{q}_f^i] \quad \underline{y} \in S_i,$$

where

$$(3.9) \quad \underline{\tilde{q}}_f = - \frac{\alpha}{N} \int (\frac{\alpha}{N} \underline{\tilde{\Psi}} + I_{3N})^{-1} \underline{\tilde{\Phi}}(\underline{x}, \{y_i\}) \underline{f}(\underline{x}) d\underline{x}.$$

From the potential estimate

$$(3.10) \quad \sup_x \left| \int \frac{f(\underline{y})}{|\underline{x} - \underline{y}|^\alpha} d\underline{y} \right| \leq \|f\|_p, \quad p > \frac{1}{1 - \frac{\alpha}{3}}$$

we get

$$(3.11) \quad |\underline{n}^N(\underline{y})| \leq \frac{c}{N} \|f\|_{q_1} + \frac{c}{N} \sum_j' \|q_f^j\| |\underline{y}_j - \underline{y}_i|^{-2} + c \|\underline{q}_f^i\|, \\ \underline{y} \in S_i, \quad q_1 > 3.$$

Another useful estimate which is obtained from (3.9), (3.10) and Lemma 2 is

$$(3.12) \quad \|\underline{\tilde{q}}_f\| \leq cN^{-1/2} \|f\|_2$$

We turn now to construct a candidate field $\underline{\tau}$ to be used in the variational principle. Let $\{T_i\}$ $i = 1, 2, \dots, N$ be the following domains

$$T_i = \{\underline{x}; \frac{\alpha'}{N} \leq |\underline{x} - \underline{y}_i| \leq \frac{2\alpha'}{N}\}.$$

From (2.6) these are "security domains". In each T_i let $\underline{\tau}^i$ be the vector satisfying

$$(3.13) \quad \begin{aligned} \underline{\nabla} \cdot \underline{\tau}^i &= 0 \quad \text{in } T_i \\ \underline{\tau}^i(\underline{y}) &= \underline{n}^N(\underline{y}) \quad y \in S_i \\ \underline{\tau}^i(\underline{y}) &= 0 \quad |\underline{y} - \underline{y}^i| = \frac{2\alpha'}{N}. \end{aligned}$$

Such a vector exists since the compatibility condition $(\int_{S_i} \underline{n}^N \cdot \underline{n} \, ds = 0)$ is satisfied by the definition of $\underline{n}^N$.

For smooth domains, there are a priori estimates for problems like (3.13). In order to use them we rescale

$$\underline{x} = \frac{\underline{z}}{N}$$

and observe that

$$(3.14) \quad \|\underline{\tau}^i(\underline{x})\|_{2,1} \leq N^{-1/2} \|\underline{\tau}^i(\underline{z})\|_{2,1}.$$

But on the $\underline{z}$ scale, problem (3.13) is considered in a smooth domain, and we can use the estimate

$$(3.15) \quad \begin{aligned} \|\underline{\tau}^i\|_{2,1}(T_i) &\leq c \|\underline{n}^N\|_{2,1/2}(\partial T_i) \quad (\text{Ladyzenskaya [10]}). \\ \int_{\partial T_i} ds \quad \text{and} \quad \int_{\partial T_i} \int_{\partial T_i} \frac{d\underline{z} d\underline{z}'}{|\underline{z} - \underline{z}'|^3} &\text{ are bounded, hence it follows from (3.11) that} \\ \|\underline{n}^N\|_{2,1/2}(\partial T_i) &\leq \frac{c}{N} \|\underline{f}\|_{q_1} + \frac{c}{N} \sum_j' \|\underline{q}_f^j\| |\underline{y}_j - \underline{y}_i|^{-2} + c \|\underline{q}_f^i\|. \end{aligned}$$

We define now $\underline{\tau}$ to be $\underline{\tau}^i$ in T_i and zero elsewhere.

$$\begin{aligned} \|\underline{\tau}\|_{2,1}(D_1^N) &= \sum_i \|\underline{\tau}^i(\underline{x})\|_{2,1}(T_i) \leq \\ &\leq N^{-1/2} \sum_i \frac{c}{N} \|\underline{f}\|_{q_1} + c \cdot N^{-3/2} \sum_{i,j}' \|\underline{q}_f^j\| |\underline{y}_j - \underline{y}_i|^{-2} + c N^{-1/2} \sum_i \|\underline{q}_f^i\|, \end{aligned}$$

where we use (3.14) and (3.15)

Let now $\underline{Q}$ be the matrix

$$(\underline{Q})_{ij} = \begin{cases} |\underline{y}_j - \underline{y}_i|^{-2} & i \neq j \\ 0 & i = j \end{cases}$$

$$(3.16) \quad \|\underline{Q}\| = \left(\sum_{i,j} |\underline{y}_i - \underline{y}_j|^{-4} \right)^{1/2} \leq cN^1 + ((1-\nu)(1+\xi)/2)$$

for $\nu < \frac{1}{3}$, $\xi > 0$, and we used (2.6) and (2.7).

Hence (Applying Cauchy Schwartz inequality)

$$\begin{aligned} \|\underline{\tau}\|_{2,1}(D^n) &\leq cN^{-1/2} \|\underline{f}\|_{q_1} + cN^{-3/2} \|\underline{q}_f\|_Q \cdot N^{1/2} + c\|\underline{q}_f\| \\ &\leq cN^{-1/2} \|\underline{f}\|_{q_1} + cN^{-(1/6)+\varepsilon} \|\underline{f}\|_2. \end{aligned}$$

(In the last step we used (3.12) and (3.16))

Recalling Lemma 3 and the remark on the equivalence of $\|\cdot\|_{2,1}$ and $F(\cdot)$, we obtain

$$\|\underline{\delta}^N\|_{2,1} \leq cN^{-\gamma} \quad \gamma < \frac{1}{6}$$

which is in fact stronger than the statement of the theorem.

Q.E.D.

Remarks

- 1) It is obvious from the proof that we could have used weaker conditions on $\underline{f}$. For example we can take $\underline{f} \in L_q(D)$, $q > 3$, with fast decay for large $|\underline{x}|$.
- 2) $\underline{n}^N(\underline{y})$ can be extended to all D , and the difference between the full operator and the reduced one can be shown to be less than $cN^{-1/2}$ (in L_2 norm). The proof is exactly the same as the one given by Figari et. al [5]. Notice also that in our scaling

$$\text{Vol}(D - D_1^N) = O(N^{-2}) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

4. Homogenization of the Point Sources Approximation

The work we have put into the point sources approximation pays off now, since we do not have to deal anymore with the complicated geometry. Our next step is to prove that $\underline{u}^N$ converges in the appropriate sense to $\underline{v}$. We shall follow here the homogenization procedure suggested by Figari et. al [5]. Eventhough their work dealt with the Laplace equations, a careful examination of their homogenization theorem shows that it applies to a very general family of operators, among them is the Stokes equation. The idea is to compare the resolvent expansion of $\underline{u}^N(\underline{y})$ and $\underline{v}(\underline{y})$. Let

$$(4.1) \quad \underline{r}_f \equiv \int \underline{r}(\underline{x}, \underline{y}) \underline{f}(\underline{x}) d\underline{x}, \quad \underline{f}_r \equiv (\underline{r}_f)^T$$

Then

$$(4.2) \quad \underline{v}(\underline{y}) = \sum_{n=0}^{\infty} \underline{\psi} \cdot (-\alpha)^n (\underline{\psi} \rho)_f^n$$

or

$$(4.3) \quad \underline{v}_{\beta}(\underline{y}) = \sum_{n=0}^{\infty} (-\alpha)^n \int \underline{\psi}_{\beta\beta_1}(\underline{y}, \underline{z}_1) \rho(\underline{z}_1) \underline{\psi}_{\beta_1\beta_2}(\underline{z}_1, \underline{z}_2) \rho(\underline{z}_2) \dots \\ \underline{\psi}_{\beta_{n-1}\beta_n}(\underline{z}_{n-1}, \underline{z}_n) \rho(\underline{z}_n) \underline{\psi}_{\beta_n\beta_{n+1}}(\underline{z}_n, \underline{z}_{n+1}) \underline{f}_{\beta_{n+1}}(\underline{z}_{n+1}) \\ \cdot d\underline{z}_1 \dots d\underline{z}_{n+1}$$

where $\beta_i = 1, 2, 3$. Also

$$(4.4) \quad \underline{u}^N(\underline{y}) = \underline{\psi}_f + \sum_{n=1}^{\infty} \left(-\frac{\alpha}{N} \right) \tilde{\Phi} \left(-\frac{\alpha}{N} \right)^{n-1} (\tilde{\psi})^{n-1} \tilde{\Phi}_f.$$

A general term in this expansion is of the form

$$(4.5) \quad \left(\frac{\alpha}{N} \right)^S \underline{\psi}_{\beta\beta_1}(\underline{y}, \underline{y}_{j_1}) \underline{\psi}_{\beta_1\beta_2}(\underline{y}_{j_1}, \underline{y}_{j_2}) \dots \underline{\psi}_{\beta_{s-1}\beta_s}(\underline{y}_{j_{s-1}}, \underline{y}_{j_s}) (\underline{\psi}_f)_{\beta_s}(\underline{y}_{j_s}).$$

We would like to dispose of terms which contain repeating points $\underline{y}_k$. (This

will make our averaging procedure much easier).

We decompose $\underline{n}^N$ into

$$(4.6) \quad \underline{n}^N(\underline{y}) = \underline{v}^N(\underline{y}) + \underline{\zeta}^N(\underline{y}),$$

where $\underline{\zeta}^N(\underline{y})$ contains all the terms with "self intersecting graphs", and $\underline{v}^N$ contains the rest, i.e. all the terms in (4.4) in which each point appears at most once.

The following Lemma holds:

Lemma 4

There is λ_0 , such that for every $\lambda > \lambda_0$, and for every $\epsilon > 0$, there is N_0 , such that for every $N > N_0$, and for $\{y_j\} \in \Omega_1^N$,

$$\|N^{1/2} \underline{\zeta}^N\|_2 \leq \epsilon \|f\|_2 \quad \forall f \in C_0(D).$$

The proof is given in the Appendix.

In the sequel, we emphasize the dependence of $\underline{n}^N$ on $\underline{f}$ by writing it as $\underline{n}^N(\underline{f})$. Let us define for every $\underline{f} \in L_2(D)$ and $\underline{g}$ which is continuous and has a compact support in D the random variable

$$(4.7) \quad \theta^N(\underline{f}, \underline{g}) = \sum_{s=0}^N (-\alpha)^s \cdot [N^{-s+1/2} \underline{f} \Phi(\tilde{\Psi}^{s-1})_* \tilde{\Phi}_{\underline{g}} - N^{1/2}(\underline{f}, \Psi(\Psi^s) \underline{g})]$$

where $(\underline{u}, \underline{v})$ is the scalar product in $L_2(D)$ of vector fields, and $(\tilde{\Psi}^n)_*$ means that we count only the terms which belong to $\underline{v}^N(\underline{g})$.

Using the resolvent expansions, and the fact that $(\tilde{\Psi}^n)_*$ does not contain repeating points we see that

$$(4.8) \quad \lim_{N \rightarrow \infty} \theta^N(\underline{f}, \underline{g}) - N^{1/2}(\underline{f}, \underline{v}^N(\underline{g}) - \underline{v}(\underline{g})) = 0.$$

Taking expectation on $\theta^N(\underline{f}, \underline{g})$ means taking expectations on terms like (4.5), only that now the $\underline{y}_j$ are all distinct. Hence, the expectation of such

a term is precisely the corresponding term in the resolvent expansion of $\underline{v}_\beta$ (4.3). We only have to count how many such terms (like (4.5) with distinct $\underline{y}_j$) appear, and one finds:

$$(4.9) \quad E[\theta^N(\underline{f}, \underline{g})] = N^{1/2} \sum_{s=1}^N (\gamma(N, s) - 1) (-\alpha)^s (\underline{f}, \underline{\psi}(\underline{\psi}_\rho) \underline{g})$$

$$\text{where } \gamma(N, s) = \frac{N!}{(N-s)! N^s}$$

Lemma 5

$$\lim_{N \rightarrow \infty} E[\theta^N(\underline{f}, \underline{g})] = 0$$

Proof

Notice first (from the proof of Lemma 2, for example) that

$$|(\underline{f}, \underline{\psi}(\underline{\psi}_\rho) \underline{g})^s| \leq c \lambda^{-s}.$$

Consider now $s: 1, 2, \dots, \lg N$ for N large. Then $\gamma(N, s)$ can be expanded as

$$\gamma(N, s) \approx 1 - \frac{s^2}{2N} + \dots$$

and for any s , $\gamma(N, s) \leq 1$.

Combining these observations we estimate

$$|E[\theta^N(\underline{f}, \underline{g})]| \leq \frac{\lg^2 N}{N^{1/2}} \sum_{s=1}^{\lg N} (c \lambda)^{-s} + c N^{1/2} \sum_{s=\lg N+1}^{\infty} (c \lambda)^{-s} \rightarrow 0$$

as $N \rightarrow \infty$ for λ large enough.

Q.E.D.

Next we compute the covariance for $\underline{f}, \underline{f}'$:

$$(4.10) \quad E[\theta^N(\underline{f}, \underline{g}) \theta^N(\underline{f}', \underline{g})] =$$

$$= \sum_{s, s'=1}^N (-\alpha)^{s+s'} E[N^{-s-s'+1} \tilde{f}_{\underline{f}} (\tilde{\psi})_{\star}^{s-1} \tilde{g}_{\underline{f}} (\tilde{\psi})_{\star}^{s'-1} \tilde{g}_{\underline{g}}] +$$

$$+ N[1 - \gamma(N, s) - \gamma(N, s')] (\underline{f}, \underline{\psi}(\underline{\psi}_\rho) \underline{g})^s (\underline{f}', \underline{\psi}(\underline{\psi}_\rho) \underline{g})^{s'}.$$

A tedious computation which is basically similar to our proof of Lemma 5 yields

Lemma 6

$$E[\theta^N(\underline{f}, \underline{g}) \theta^N(\underline{f}', \underline{g})] = G(\underline{f}, \underline{f}', \underline{g})$$

where G is a bounded functional on $L_2(D) \times L_2(D) \times L_2(D)$. (Explicit formula for G can be given, but we do not find it very illuminating, especially when Theorem 2 gives a convergence rate which is less than $\frac{1}{6}$).

Finally, we extend $\underline{u}^N(y)$ into D by defining it to be identically zero in $\bigcup_{j=1}^N B_j$. Lemma 6, Lemma 5, Lemma 4, (4.8), remark 3 in Section 3, Theorem 2 and Lemma 1 altogether complete the proof of Theorem 1.

5. Some Remarks

1) Brinkman's equation is often written in the form

$$(5.1) \quad \mu' \Delta \underline{v} - \frac{\mu}{K} \underline{v} = \nabla p$$

where K is the permeability of the porous medium and μ' is some effective viscosity (Lundgren [11]). It follows from our derivation that

$$(5.2) \quad \mu' = \mu, \quad K = \frac{1}{6\pi\alpha^*\rho}, \text{ or } K = \frac{2}{9} R^2 \cdot \beta^{-1}$$

where β is the volume fraction occupied by the spheres. (5.2) is the expression for K which was derived by Brinkman for dilute suspensions.

2) The scaling we used here corresponds to very low volume fraction of obstacles (high porosity). What happens in higher volume fractions? This is still an open question, though there have been some attempts to find corrections to (2.5) (see for example Childress [4] and Hinch [17]). We believe that a possible approach will be to improve Theorem 2 by going to higher multipoles.

3) It is not hard to check that the proof holds for a distribution of nonidentical spheres with $R_j = \frac{\alpha'_j}{N}$, provided that $0 < \alpha_- \leq \alpha'_j \leq \alpha_+ < \infty$, and that

there exists a function $\bar{\alpha}'(\underline{x})$ such that

$$\frac{1}{N} \sum_{j=1}^N \alpha'_j \phi(\underline{y}_j) \rightarrow \int \rho(\underline{y}) \bar{\alpha}'(\underline{y}) \phi(\underline{y}) d\underline{y} \quad \text{as } N \rightarrow \infty$$

for every continuous ϕ .

Appendix

The proof of Lemma 4 is similar to the one given in [5] but we give it here explicitly because of its importance.

We notice first that for $\underline{f} \in C_0(D)$, $\underline{u}^N(\underline{f}) \in L_2(D)$. Consider now (for $\sigma > 0$)

$$N^{1-\sigma}(\underline{h}, \underline{u}^N(\underline{f})) \quad \text{for } \underline{h} \in L_2, \quad \underline{f} \in C_0(D).$$

Let the first point which repeats itself (see (4.4), (4.5)) be at the n_1 place in the expansion, and assume that it appears again at the $n_1 + n_2$ place. The contribution of such a term is less than

$$\begin{aligned} T_{n_1 n_2} &= \frac{\alpha^{n+1}}{N^{n-1+\sigma}} \sum_{\substack{i,j,p,k \\ r,q,\ell=1}}^{3N} (\underline{h}\underline{\phi})_i (\underline{\psi}^{n_1-1})_{ip} (\underline{\psi})_{pk} (\underline{\psi})_{k\ell} (\underline{\psi}^{n_2-1})_{\ell q} (\underline{\psi})_{qk} (\underline{\psi})_{k'r} \cdot \\ &\quad \cdot (\underline{\psi}^{n-n_1-n_2-1})_{rj} (\underline{\phi}_f)_j \end{aligned}$$

where $|k - k'| < 3$.

Using the potential estimate (3.8) (recall that $(\underline{\psi})_{ij} \leq \frac{c}{|\underline{y}_i - \underline{y}_j|}$), and applying Cauchy Schwartz inequality repeatedly we find

$$T_{n_1 n_2} \leq c \alpha^{n+1} \|\underline{h}\|_2 \|\underline{f}\|_2 \frac{1}{N} \|\underline{\psi}\|^{n-4} \cdot N^{-1/2} \cdot L(N)$$

where

$$L(N) \leq N^{-3} \sum_{\substack{i,j,k \\ i \neq j \neq k}} |\underline{y}_i - \underline{y}_j|^{-2} |\underline{y}_j - \underline{y}_k|^{-2} + N^{-3} \sum_{i,j} |\underline{y}_i - \underline{y}_j|^{-4}.$$

From (2.7), (2.8) $L(N) \leq c$, for every n there are at most n^2 terms like $T_{n_1 n_2}$, and recalling further Lemma 2 we arrive at

$$N^{1-\sigma}(\underline{h}, \underline{u}^N(\underline{f})) \leq \|\underline{f}\|_2 \|\underline{h}\|_2 N^{-\sigma} \sum_{n=2}^{\infty} (c\lambda)^{-n\beta/2} n^2,$$

where the sum on the right converges for λ big enough.

Q.E.D.

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