

INEQUALITIES BETWEEN DIRICHLET AND NEUMANN EIGENVALUES

BY

HOWARD A. LEVINE

AND

HANS F. WEINBERGER

IMA Preprint Series # 195

November 1985

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

#	Author(s)	Title	Recent IMA Preprints #	Abstracts for the Workshop on Bayesian Analysis in Economics and Game Theory	Limited number of preprints are available on request.)	Author(s)	Title
40	William Reckle,	The Strong \mathbb{Q} Topology on Symmetric Sequence Spaces		78	Abstracts for the Workshop on Bayesian Analysis in Economics and Game Theory		
41	Charles R. Johnson,	A Characterization of Borda's Rule Via Optimization		79	G. Chirilus-Brukner, G.M. Heal, Existence of a Competitive Equilibrium in L_1 and Sobolev Spaces		
42	Hans Weisberger,	Kazuo Kishimoto, The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains		80	Thomas P. Sotomayor, Time-dependent Solutions of a Nonlinear System in Semiconductivity Theory, II: Boundedness and Periodicity		
43	K.A. Pericak-Spector, W.O. Williams,	On Work and Constraints in Mixtures		81	Yakar Kannan, Engaging in R&D and the Emergence of Expected Non-convex Technologies		
44	M. Rosenber,	E. Toubliana, Some Remarks on Deformations of Minimal Surfaces		82	Herve Moullia, Choice Functions over a Finite Set: A Summary		
45	Stephen Pollan,	The Duration of Transients		83	Herve Moullia, Choosing from a Tournament		
46	V. Capasso, K.L. Cooke, M. Witten,	Random Fluctuations of the Duration of Harvest		84	David Schmeidler, Subjective Probability and Expected Utility Without Additivity		
47	E. Fabes, D. Stroock,	The L^p -integrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations		85	I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Polikan, The Numerical Computation of Invariant Circles of Maps		
48	M. Brazis,	Semilinear Equations in \mathbb{R}^n without Conditions at Infinity		86	F. William Levere, State Categories, Closed Categories, and the Existence of Semi-Continuous Entropy Functions		
49	M. Slemrod,	Lax-Friedrichs and the Viscosity-Capillarity Criterion		87	F. William Levere, Functional Remarks on the General Concept of Chaos		
50	C. Johnson, M. Barrett,	Spanning Tree Extensions of the Hadamard-Fischer Inequalities		88	Steven R. Williams, Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process		
51	Andrew Postlewaite, David Schmeidler,	Revelation and Implementation under Differential Information		89	Steven R. Williams, Implementing a Generic Smooth Function		
52	Paul Blanchard,	Complex Analytic Dynamics on the Riemann Sphere		90	Dilip Abreu, Infinitely Repeated Games with Discounting: A General Theory		
53	G. Levitt, M. Rosenberg,	Topology and Differentiability of Labyrinths in the Disc and Annulus		91	J.S. Jordan, Instability in the Implementation of Walrasian Allocations		
54	G. Levitt, M. Rosenberg,	Symmetry of Constant Mean Curvature Hypersurfaces in Hyperbolic Space		92	Myra Holtz Woodard, William R. Zame, Large Games: Fair and Stable Outcomes		
55	Emanio Staechetti,	Analysis of a Dynamic, Decentralized Exchange Economy		93	J.L. Moskos, Critical Sets and Negative Bundles		
56	Henry Simpson, Scott Spector,	On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics		94	Graciela Chichilnisky, Von Neumann-Morgenstern Utilities and Cardinal Preferences		
57	Craig Tracy,	Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations		95	J.L. Erickson, Twinning of Crystals		
58	Tongren Ding,	Boundedness of Solutions of Duffing's Equation		96	Anna Nagurny, On Some Market Equilibrium Theory Paradoxes		
59	Abstracts for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles			97	Anna Nagurny, Sensitivity Analysis for Market Equilibrium		
60	Rafael Rob,	The Coase Theorem an Informational Perspective		98	Abstracts for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations		
61	Joseph Jerome,	Approximate Newton Methods and Homotopy for Stationary Operator Equations		99	Miller Beatty, A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability		
62	Rafael Rob,	A Note on Competitive Bidding with Asymmetric Information		100	Filomena Pacella, Central Configurations of the N-Body Problem via the Equivalent Morse Theory		
63	Rafael Rob,	Equilibrium Price Distributions		101	D. Carlson and A. Neger, The Derivative of a Tensor-valued Function of a Tensor		
64	William Reckle,	The Linearization Projection, Global Theories		102	Kenneth Mount, Privacy Preserving Correspondence		
65	Russell Johnson, Kenneth Palmer, George R. Sell,	Ergodic Properties of Linear Dynamical Systems		103	Miller Beatty, Finite Amplitude Vibrations of a Neo-Hookean Oscillator		
66	Stanley Reiter,	How a Network of Processors can Schedule its Work		104	D. Emson and M. Yannelis, On Perfectly Competitive Economies: Loeb Economies		
67	R.N. Goldman, D.C. Heath,	Linear Subdivision is Strictly a Polynomial Phenomenon		105	E. Mascolo and R. Schianchi, Existence Theorems in the Calculus of Variations		
68	R. Giachetti, R. Johnson,	The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients		106	D. Kinderlehrer, Twinning of Crystals (II)		
69	Steve Williams,	Realization and Nash Implementation: Two Aspects of Mechanism Design		107	R. Chen, Solutions of Minimax Problems Using Equivalent Differentiable Equations		
70	Steve Williams,	Sufficient Conditions for Nash Implementation		108	D. Abreu, D. Pearce, and E. Staechetti, Optimal Cartel Equilibria with Imperfect Monitoring		
71	Nicholas Yannelis, William R. Zame,	Equilibria in Banach Lattices Without Ordered Preferences		109	R. Lauterbach, Hopf Bifurcation from a Turning Point		
72	M. Morris, Y. Sibuya,	The Reciprocals of Solutions of Linear Ordinary Differential Equations		110	C. Kahn, An Equilibrium Model of Quits under Optimal Contracting		
73	Steve Pollan,	A Dynamical Meaning of Fractal Dimension		111	M. Kaneko and M. Woodard, The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results		
74	D. Heath, M. Suddarth,	Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach a Goal		112	Maria Brazis, Remarks on Sublinear Equations		
75	J.S. Jordan,	Information Flows Intrinsic to the Stability Economic Equilibrium		113	D. Carlson and A. Neger, On the Derivatives of the Principal Invariants of a Second-order Tensor		
76	J. Jerome,	An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition		114	Raymond Desoeters and Steve Pollan, Competitive Chaos		
77	David Schmeidler,	Integral Representation Without Additivity		115	Abstracts for the Workshop on Homogenization and Effective Moduli of Materials and Media		
				116	Abstracts for the Workshop on the Classifying Spaces of Groups		
				117	Roberto Mosco, Pointwise Potential Estimates for Elliptic Obstacle Problems		
				118	J. Rodrigues, An Evolutionary Continuous Casting Problem of Stefan Type		
				119	C. Mueller and F. Weisler, Single Point Blow-up for a General Semilinear Heat Equation		

INEQUALITIES BETWEEN DIRICHLET AND NEUMANN EIGENVALUES

by

Howard A. Levine & Hans F. Weinberger

To Jim Serrin on his sixtieth birthday.

1. Introduction

The purpose of this paper is to derive some inequalities of the form

$$(1.1) \quad \mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

between the eigenvalues $\lambda_1 < \lambda_2 < \dots$ of the Dirichlet problem

$$(1.2) \quad \begin{aligned} \Delta u + \lambda u &= 0 & \text{in } D \subset \mathbb{R}^N \\ u &= 0 & \text{on } \partial D \end{aligned}$$

and the eigenvalues $0 = \mu_1 < \mu_2 < \dots$ of the Neumann problem

$$(1.3) \quad \begin{aligned} \Delta v + \mu v &= 0 & \text{in } D \\ \frac{\partial v}{\partial \nu} &= 0 & \text{on } \partial D \end{aligned}$$

for some classes of N -dimensional domains D . Here $\partial/\partial \nu$ denotes the outward normal derivative.

It is an immediate consequence of the variational formulation of the eigenvalue problems that

$$\mu_k \leq \lambda_k \quad \text{for } k = 1, 2, \dots$$

POLYA [6] proved that

$$\mu_2 < \lambda_1.$$

PAYNE [5] showed that when $N = 2$ and D is a convex domain whose boundary ∂D is twice continuously differentiable,

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Convexity in two dimensions is characterized by the fact that the curvature, which is a scalar defined on the boundary ∂D , is nonnegative. In higher dimensions there are $N - 1$ principal curvatures at each point of ∂D , and hence there are a number of possible generalizations of PAYNE'S result.

In Section 2 we shall derive a generalization of the form (1.1) for each integer R on the interval $[1, N]$ under conditions which depend upon the principal curvatures of ∂D . The most important of these results is the fact that the inequality

$$\mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

is valid for all smooth bounded convex domains.

We also obtain the inequality

$$\mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

when ∂D has nonnegative mean curvature, which is a recent result of AVILES [1].

In Section 3 we show that the conditions on the principal curvatures can be stated in terms of their symmetric functions, which are the coefficients of the characteristic polynomial of the curvature matrix. A simpler but more restrictive set of such conditions is also presented.

The above results are derived under the hypothesis that the boundary ∂D has Hölder continuous second derivatives. In Section 4 we establish a semicontinuity result for the Neumann eigenvalues, which allows us to obtain the nonstrict version of the inequality (1.1) for some classes of domains whose boundary is not smooth.

In particular, we shall establish the inequality

$$\mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

for all bounded convex domains.

2. Inequalities for smooth domains

If D is a bounded domain in R^N with smooth boundary, the curvature matrix of a point P of the boundary ∂D is defined as follows: Let $v(x)$ be a continuously differentiable vector field defined in a neighborhood of P with the property that when x lies on ∂D , v is an outward unit normal vector to ∂D at x . The matrix

$$(2.1) \quad K_{ij} = v_{i,j} - \sum_{k=1}^N v_{i,k} v_k v_j$$

evaluated at x is called the curvature matrix of ∂D at x . It has the eigenvalue 0 with the eigenvector v . The $N - 1$ other eigenvalues of K_{ij} are called the principal curvatures

$$\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_{N-1}$$

of ∂D at x . (The symbol $_{,j}$ denotes differentiation with respect to x_j .)

It is easy to see that the matrix K_{ij} is symmetric.

Our results are based on the following proposition:

PROPOSITION 2.1 Let D be a bounded domain whose boundary ∂D is of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. If for some integer $R \in [1,N]$ the principal curvatures of ∂D at each of its points have the property that the sum of any $N - R + 1$ of the set of N numbers

$$(2.2) \quad \{\kappa_1, \kappa_2, \dots, \kappa_{N-1}, \sum_{j=1}^{N-1} \kappa_j\}$$

is nonnegative, then

$$(2.3) \quad \mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. We recall that [3, p. 399] if ϕ is continuously differentiable and

$$(2.4) \quad \int_D \phi v_i dx = 0 \quad \text{for } i = 1, 2, \dots, k + R - 1,$$

where v_1, v_2, \dots , is an orthonormal set of eigenfunctions of (1.3) corresponding to the Neumann eigenvalues $0 = \mu_1 < \mu_2 < \dots$, then

$$(2.5) \quad \int_D |\nabla \phi|^2 dx > \mu_{k+R} \int_D \phi^2 dx.$$

We consider the set of functions

$$(2.6) \quad \phi = \sum_{i=1}^k a_i u_i + \sum_{p=1}^N b_p u_{k,p}$$

where $\{u_1, u_2, \dots\}$ is an orthonormal set of eigenfunctions of (1.2) corresponding to the Dirichlet eigenvalues $0 = \lambda_1 < \lambda_2 < \dots$. Because the u_i are orthonormal and satisfy (1.2), integration by parts shows that

$$(2.7) \quad \int_D |\nabla \phi|^2 dx - \lambda_k \int_D \phi^2 dx = \sum_{i=1}^k a_i^2 (\lambda_i - \lambda_k) + \oint_{\partial D} w \frac{\partial w}{\partial \nu} dS,$$

where we have defined

$$w = \sum_{p=1}^N b_p u_{k,p}.$$

The sum on the right of (2.7) is clearly nonpositive. In order to see whether the integral is also nonpositive, we note that

$$(2.8) \quad \begin{aligned} \frac{\partial w}{\partial \nu} &= \sum_{q,r=1}^N b_q u_{k,qr} v_r \\ &= \sum b_q (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) u_{k,r} - \lambda_k u_k \sum b_q v_q. \end{aligned}$$

Since u_k vanishes on the boundary, the last term is zero and

$$(2.9) \quad u_{k,r} = v_r \frac{\partial u_k}{\partial v} \quad \text{for } \ell = 1, \dots, N.$$

Because the directional derivatives which appear in (2.8) are in directions tangent to ∂D , we can apply them to both sides of (2.9). Then (2.8) yields

$$w \frac{\partial w}{\partial v} = \sum_{p,q,r=1}^N b_p v_p \frac{\partial u_k}{\partial v} b_q (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) (v_r \frac{\partial u_k}{\partial v})$$

or

$$(2.10) \quad w \frac{\partial w}{\partial v} = \frac{1}{2} \sum b_p b_q (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) (v_p v_r |\nabla u_k|^2) \\ - \frac{1}{2} \sum b_p b_q |\nabla u_k|^2 [v_r (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) v_p \\ - v_p (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) v_r].$$

We see from the definition (2.1) of the curvature matrix K_{pq} that since $|v| = 1$ on ∂D ,

$$\sum_{r=1}^N [v_r (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) v_p - v_p (v_r \frac{\partial}{\partial x_q} - v_q \frac{\partial}{\partial x_r}) v_r] = K_{pq} + \sum K_{rr} v_p v_q.$$

The divergence theorem shows that the integral over ∂D of the first sum in (2.10) is zero. Thus

$$(2.11) \quad \int_{\partial D} w \frac{\partial w}{\partial v} dS = - \frac{1}{2} \sum_{p,q=1}^N b_p b_q \int_{\partial D} J_{pq}(x) dS$$

where

$$(2.12) \quad J_{pq} = \left(\frac{\partial u_k}{\partial v} \right)^2 [K_{pq} + (\sum K_{rr}) v_p v_q].$$

We recall that the vector v lies in the null space of the curvature matrix. It follows that J_{pq} has the eigenvalues

$$(2.13) \quad \left\{ \left(\frac{\partial u_k}{\partial v} \right)^2 \kappa_1, \dots, \left(\frac{\partial u_k}{\partial v} \right)^2 \kappa_{N-1}, \left(\frac{\partial u_k}{\partial v} \right)^2 \sum_{r=1}^{N-1} \kappa_r \right\}.$$

We rotate the x-coordinates so that the matrix

$$(2.14) \quad \int_{\partial D} J_{pq} dS$$

on the right of (2.8) is diagonal, with its diagonal elements arranged in non-decreasing order. By the first monotonicity principle [8, p. 58] the ordered eigenvalues of the $(N - R + 1) \times (N - R + 1)$ matrix $\bar{J}(x)$ defined by

$$\bar{J}_{\alpha\beta}(x) = J_{\alpha\beta}(x) \quad \text{for } \alpha, \beta = 1, \dots, N - R + 1$$

are upper bounds for the $N - R + 1$ lowest eigenvalues of the matrix J . We add these inequalities and use the hypothesis that the sum of any $N - R + 1$ of the numbers (2.2) is nonnegative to see that

$$(2.15) \quad \text{tr}(\bar{J}) = \sum_{p=1}^{N-R+1} J_{pp}(x) > 0.$$

Since this is true at each boundary point, we have

$$(2.16) \quad \sum_{p=1}^{N-R+1} \oint_{\partial D} J_{pp} dS > 0.$$

To show that this inequality is strict, we apply the Gauss-Bonnet formula

$$\oint_{\partial D^*} \kappa dS = \omega_{N-1}$$

to the convex hull D^* of D . Here

$$\kappa = \kappa_1 \cdots \kappa_{N-1}$$

is the Gaussian curvature of ∂D^* .

It is easily seen that $\kappa = 0$ on $\partial D^* \setminus \partial D$. Since the area ω_{N-1} of the unit sphere is positive, there must be an open subset S of $\partial D^* \cap \partial D$ on which $\kappa > 0$. Since D^* is convex, all the κ_j must be positive on S . We therefore

see from (2.12) and (2.16) that the inequality (2.16) is strict unless $\partial u_k / \partial \nu \equiv 0$ on S . Since also $u_k = 0$ on ∂D and $\Delta u_k + \lambda_k u_k = 0$, the generalized maximum principle [7, p. 73] would then imply that $u_k \equiv 0$ in a neighborhood in \bar{D} of S , and the analyticity of u_k in D would imply that $u_k \equiv 0$. Since this would contradict the definition of an eigenfunction, we conclude that equality cannot hold in (2.16). That is,

$$\sum_{p=1}^{N-R+1} \oint_{\partial D} J_{pp} dS > 0.$$

This clearly implies that the largest of the terms in this sum is positive. Because the elements of the diagonal matrix (2.14) appear in nondecreasing order, we conclude that

$$(2.17) \quad \int_{\partial D} J_{pp} dS > 0 \quad \text{for } p = N - R + 1, N - R + 2, \dots, N.$$

We now choose the $k + N$ numbers a_i and b_p so that they are not all zero and that they satisfy the $k + R - 1$ conditions (2.4) and the $N - R$ conditions $b_1 = \dots = b_{N-R} = 0$. We then see from (2.11) and (2.17) that the integral on the right of (2.7) is nonpositive. The sum on the right is also nonpositive. Thus we have

$$(2.18) \quad \int_D |\nabla \phi|^2 dx \leq \lambda_k \int_D \phi^2 dx.$$

If $\phi \equiv 0$, equality holds, so that the integral in (2.7) must vanish, which, in turn, implies that $b = 0$. Since the u_i are orthonormal, $\phi = \sum a_i u_i \equiv 0$ would imply that a as well as b is zero, contrary to our construction. Therefore $\phi \not\equiv 0$, and (2.5) and (2.18) imply that

$$(2.19) \quad \mu_{k+R} \leq \lambda_k.$$

Equality holds if and only if both (2.5) and (2.16) are equalities.

Equality in (2.5) is valid only if ϕ is an eigenfunction of the Neumann problem, so that $\Delta\phi + \mu_{k+R}\phi = 0$ in D and $\partial\phi/\partial\nu = 0$ on ∂D . As we have already seen, equality in (2.18) implies that $b = 0$ so that $\phi = 0$ on ∂D . As we showed in the derivation of (2.17), these conditions imply that $\phi \equiv 0$, which contradicts what we proved above. Therefore equality cannot hold in (2.19), and the statement (2.3) of the Proposition is proved.

There are two important cases in which the conditions of Proposition 2.1 are easily verified.

THEOREM 2.1. Let D be a convex domain in R^N whose boundary has Hölder continuous second derivatives. Then the Neumann and Dirichlet eigenvalues satisfy the inequalities

$$(2.20) \quad \mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. Since D is convex, all the principal curvatures κ_j are non-negative at each boundary point. Hence each of the numbers in the set (2.2) is nonnegative, so that the hypotheses of Proposition 2.1 hold with $R = N$. Thus Proposition 2.1 implies (2.20).

The sum of the N numbers in (2.2) is

$$2 \sum_{j=1}^{N-1} \kappa_j = 2(N-1)H,$$

where H is called the mean curvature. Hence for $R = 1$ Proposition 2.1 yields a result which is contained in a recent paper of AVILES [1].

THEOREM 2.2 (AVILES). Let the boundary ∂D of the domain D be of class $C^{2,\alpha}$ and let its mean curvature be nonnegative at all its points. Then

$$(2.21) \quad \mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

The conditions of Proposition 2.1 can also be simplified for other values of R .

THEOREM 2.3. If the inequalities

$$(2.22) \quad 2 \sum_{i=1}^{N-1} \kappa_i - \kappa_j > 0 \quad \text{for } j = 1, \dots, N-1$$

are satisfied and $\partial D \in C^{2,\alpha}$, then

$$(2.23) \quad \mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. By adding the inequalities (2.22) we find that

$$(2.24) \quad \sum_{i=1}^{N-1} \kappa_i > 0.$$

The left-hand side of the latter inequality is the sum of the first $N - 1$ elements of the set (2.2). The left-hand side of (2.22) is the sum of all the elements of (2.2) except for κ_j . Thus (2.22) implies the hypotheses of Proposition 2.1 with $R = 2$, and the Theorem is proved.

THEOREM 2.4. If $R > 2$, if every sum of $N - R + 1$ members of the set

$$(2.25) \quad \{\kappa_1, \kappa_2, \dots, \kappa_{N-1}\}$$

is nonnegative, and if $\partial D \in C^{2,\alpha}$, then

$$(2.26) \quad \mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. Since $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_{N-1}$, the above condition is equivalent to

$$(2.27) \quad \sum_{i=1}^{N-R+1} \kappa_i > 0.$$

At least one of the terms in the sum must be nonnegative, so that

$$0 \leq \kappa_{N-R+1} \leq \kappa_{N-R+2} \leq \dots \leq \kappa_{N-1}.$$

Therefore

$$\sum_{i=1}^{N-1} \kappa_i > \kappa_{N-1}$$

and the set (2.2) is ordered. Thus the hypotheses of Proposition 2.1 follow from (2.27). This proves the Theorem.

REMARK: It is easily seen that each of the Theorems 2.1 through 2.4 implies Proposition 2.1 for the corresponding values of R .

3. Conditions involving curvature invariants

The eigenvalues of the curvature matrix K_{ij} are the zeros of the characteristic polynomial

$$(3.1) \quad \det(\lambda\delta_{ij} - K_{ij}) = \sum_{k=0}^{N-1} (-1)^k S_k(\kappa_1, \dots, \kappa_{N-1}) \lambda^{N-k}.$$

$S_k(\kappa_1, \dots, \kappa_{N-1})$ is the k th elementary symmetric function, which is defined as the sum of all products of k distinct elements of the set $\{\kappa_1, \dots, \kappa_{N-1}\}$. By definition, $S_0(\kappa_1, \dots, \kappa_{N-1}) = 1$.

At least in principle, these symmetric functions can be found from the equation 3.1 without computing the eigenvalues κ_j . The hypotheses of Theorem 2.1 are easily seen to be equivalent to

$$(3.2) \quad S_\ell(\kappa_1, \dots, \kappa_{N-1}) \geq 0 \quad \text{for } \ell = 1, \dots, N-1.$$

The condition of Theorem 2.2 is, of course, just

$$(3.3) \quad S_1(\kappa_1, \dots, \kappa_{N-1}) > 0.$$

We may restate Theorem 2.3 in a similar fashion.

THEOREM 3.1. Let $\partial D \in C^{2,\alpha}$. If (3.2) is valid, then

$$(3.4) \quad \mu_{k+N} < \lambda_k \quad \text{for } k = 1, 2, \dots .$$

If (3.3) is valid, then

$$(3.5) \quad \mu_{k+1} < \lambda_k \quad \text{for } k = 1, 2, \dots .$$

If

$$(3.6) \quad \sum_{\ell=0}^{N-1-j} (-1)^{N-1-\ell} \binom{N-1-\ell}{j} S_\ell(\kappa_1, \dots, \kappa_{N-1}) [2S_1(\kappa_1, \dots, \kappa_{N-1})]^{N-1-\ell-j} > 0.$$

for $j = 0, \dots, N-2,$

then

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \dots .$$

Proof. We have already shown that (3.2) and (3.3) imply (3.4) and (3.5), respectively.

The conditions of Theorem 2.3 are that the $(N - 1) \times (N - 1)$ matrix whose eigenvalues are the left-hand sides of $2S_1(\kappa_1, \dots, \kappa_{N-1}) - \kappa_j$ has nonnegative eigenvalues. This is again equivalent to the condition that the coefficients of its characteristic polynomial have alternating signs. It is easily seen that this characteristic polynomial can be obtained from the characteristic polynomial of the $(N - 1) \times (N - 1)$ matrix with eigenvalues κ_j by replacing the variable λ by $2S_1(\kappa_1, \dots, \kappa_{N-1}) - \lambda$. The characteristic polynomial of the latter matrix is just (3.1) divided by λ . In this way we obtain the conditions (3.6) and the Theorem is proved.

For other values of R we can use similar reasoning. The conditions of Theorem 2.4 are equivalent to the nonnegativity of the symmetric functions of the $\binom{N-1}{R-2}$ sums of $N - R + 1$ distinct elements of $\{\kappa_1, \dots, \kappa_{N-1}\}$.

THEOREM 3.2. If $R > 3$, $\partial D \in C^{2, \alpha}$, and

$$(3.7) \quad S_j \left(\left\{ \sum_{i=1}^{N-R+1} \kappa_{\ell_i} : \ell_1 < \ell_2 < \dots < \ell_{N-R+1} \right\} \right) > 0 \quad \text{for } j = 1, \dots, \binom{N-1}{R-2},$$

then

$$\mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \dots .$$

We remark that because the left hand sides of (3.7) are symmetric in the κ_i , they can be written as polynomials in the elementary symmetric functions $S_j(\kappa_1, \dots, \kappa_{N-1})$. When $R = 3$, the sums are all of the form

$S_1(\kappa_1, \dots, \kappa_{N-1}) = \kappa_j$ and there are $N - 1$ of them. By a derivation like that of (3.6) we find the following result:

THEOREM 3.3 If

$$(8.8) \quad \sum_{\ell=0}^{N-1-j} (-1)^{N-1-\ell} \binom{N-1-\ell}{j} S_{\ell}(\kappa_1, \dots, \kappa_{N-1}) [S_1(\kappa_1, \dots, \kappa_{N-1})]^{N-1-\ell-j} > 0$$

for $j = 1, 2, \dots, N-2,$

then

$$\mu_{k+3} < \lambda_k \quad \text{for } k = 1, 2, \dots .$$

We can now write down the conditions for the important case of 3 dimensions in terms of the mean curvature $H = (\kappa_1 + \kappa_2)/2$ and the Gaussian curvature $\kappa = \kappa_1 \kappa_2$. We see that

$$(3.9) \quad H > 0$$

implies

$$(3.10) \quad \mu_{k+1} > \lambda_k.$$

The conditions

$$(3.11) \quad H > 0, \quad \kappa > 0$$

imply that

$$(3.12) \quad \mu_{k+3} > \lambda_k.$$

Finally, the intermediate conditions

$$(3.13) \quad H > 0, \quad 8H^2 + \kappa > 0$$

imply that

$$(3.14) \quad \mu_{k+2} < \lambda_k.$$

It is easily seen from the proofs that for a fixed R , Theorem 3.1 or 3.2 is equivalent to Proposition 2.1. While it is therefore clear that the conditions of Theorem 3.2 for a larger R imply those for a smaller R , the number

$$\binom{N-1}{R-2}$$

of these conditions is not monotone in R .

We shall provide a partial remedy for this situation by establishing a slightly weaker result in which the number of conditions increases with R . We begin with the following algebraic lemma.

LEMMA 3.1 Suppose that for some (not necessarily ordered) set of numbers μ_1, \dots, μ_ν and some positive integer $\ell < \nu$

$$(3.15) \quad S_j(\mu_1, \dots, \mu_\nu) > 0 \quad \text{for } j = 1, \dots, \ell.$$

Then

$$(3.16) \quad S_j(\mu_1, \dots, \mu_{\nu-1}) > 0 \quad \text{for } j = 1, \dots, \ell-1.$$

Proof. It is easily seen that

$$(3.17) \quad S_j(\mu_1, \dots, \mu_\nu) = S_j(\mu_1, \dots, \mu_{\nu-1}) + \mu_\nu S_{j-1}(\mu_1, \dots, \mu_{\nu-1})$$

where, as always we define $S_0(\mu_1, \dots, \mu_{\nu-1}) = 1$. By using this identity, we find that

$$\begin{aligned}
 & S_j(\mu_1, \dots, \mu_v) S_j(\mu_1, \dots, \mu_{v-1}) - S_{j+1}(\mu_1, \dots, \mu_v) S_{j-1}(\mu_1, \dots, \mu_{v-1}) \\
 &= S_j(\mu_1, \dots, \mu_{v-1})^2 - S_{j+1}(\mu_1, \dots, \mu_{v-1}) S_{j-1}(\mu_1, \dots, \mu_{v-1}) \\
 &= \frac{v}{(j+1)(v-j)} S_j(\mu_1, \dots, \mu_{v-1})^2 \\
 &\quad + \left\{ \frac{j(v-j-1)}{(j+1)(v-j)} S_j(\mu_1, \dots, \mu_{v-1})^2 \right. \\
 &\quad \left. - S_{j+1}(\mu_1, \dots, \mu_{v-1}) S_{j-1}(\mu_1, \dots, \mu_{v-1}) \right\}.
 \end{aligned}$$

Newton's inequality [4, p. 104, Theo. 144] states that the term in braces is nonnegative. Therefore

$$\begin{aligned}
 S_j(\mu_1, \dots, \mu_v) S_j(\mu_1, \dots, \mu_{v-1}) &> S_{j+1}(\mu_1, \dots, \mu_v) S_{j-1}(\mu_1, \dots, \mu_{v-1}) \\
 &\quad + \frac{v}{(j+1)(v-j)} S_j(\mu_1, \dots, \mu_{v-1})^2.
 \end{aligned}$$

We now see from the hypothesis (3.15) that if $j \leq \ell - 1$ and if $S_{j-1}(\mu_1, \dots, \mu_{v-1}) > 0$, then $S_j(\mu_1, \dots, \mu_{v-1}) > 0$. (Note that $S_j(\mu_1, \dots, \mu_v) = 0$ implies that $S_j(\mu_1, \dots, \mu_{v-1}) = 0$.) Since $S_0(\mu_1, \dots, \mu_{v-1}) = 1 > 0$, the Lemma is proved by induction.

THEOREM 3.4. If $\partial D \in C^{2,\alpha}$, $R > 3$, and at each point of ∂D

$$(3.18) \quad S_j(\kappa_1, \dots, \kappa_{N-1}) > 0 \quad \text{for } j = 1, \dots, R - 1,$$

then

$$(3.19) \quad \mu_{k+R} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. Apply Lemma 3.1 $R - 2$ times to find that

$$S_1(\kappa_1, \dots, \kappa_{N-R+1}) > 0.$$

Since $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_{N-1}$, this give the conditions of Theorem 2.4, and the Theorem is proved.

When $R = 2$ we find the following simpler result.

THEOREM 3.5. If $\partial D \in C^{2,\alpha}$ and if

$$(3.20) \quad \begin{aligned} S_1(\kappa_1, \dots, \kappa_{N-1}) &> 0 \\ S_2(\kappa_1, \dots, \kappa_{N-1}) + S_1(\kappa_1, \dots, \kappa_{N-1})^2 &> 0, \end{aligned}$$

then

$$\mu_{k+2} < \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. Clearly

$$S_1(\kappa_1, \dots, \kappa_{N-2}, \sum_{i=1}^{N-1} \kappa_i, \kappa_{N-1}) = 2S_1(\kappa_1, \dots, \kappa_{N-1}) > 0$$

while by (3.17) and (3.20)

$$S_2(\kappa_1, \dots, \kappa_{N-2}, \sum_{i=1}^{N-1} \kappa_i, \kappa_{N-1}) = S_1(\kappa_1, \dots, \kappa_{N-1}) + S_1(\kappa_1, \dots, \kappa_{N-1})^2 > 0.$$

Thus Lemma 3.1 shows that

$$S_1(\kappa_1, \dots, \kappa_{N-2}, \sum_{i=1}^{N-1} \kappa_i) > 0,$$

and the result follows from Theorem 2.3.

We note that when $\kappa_1 = -2, \kappa_2 = \dots = \kappa_{N-2} = 0, \kappa_{N-1} = 5$, the conditions (3.6) are satisfied while the conditions (3.20) are not, which shows that Theorem 3.4 is strictly weaker than Theorem 3.1. Similarly, the example $\kappa_1 = -2, \kappa_2 = \dots = \kappa_{N-R} = 0, \kappa_{N-R+1} = \dots = \kappa_{N-1} = 3$ shows that when $3 \leq R \leq N - 1$, Theorem 3.3 is strictly weaker than Theorem 3.2.

4. Extension to nonsmooth domains

We shall extend the inequalities (3.2) to more general domains by means of a limiting process. We will have to give up the strictness of such inequalities in the process.

We begin with an elementary lemma.

Lemma 4.1. If v is a C^1 function on $[0,b]$ and $0 < a < b$, then

$$(4.1) \quad \int_a^{a+b} v^2 dt \leq 2b(\coth a + 2\pi^{-2}b) \int_0^{a+b} (v^2 + v'^2) dt.$$

Proof. A standard variational argument shows that the maximum μ of the ratio of the integral on the left to the integral on the right is the largest root of the equation

$$(4.2) \quad (\mu^{-1} - 1)^{1/2} \tan(\mu^{-1} - 1)^{1/2} b = \tanh a.$$

The well known estimates

$$2/\pi \leq \sin \rho / \rho \leq 1 \quad \text{for } 0 < \rho < \pi/2$$

show that

$$\tan \rho \leq \frac{2\rho}{[1 - (2\rho/\pi)^2]}.$$

By applying this inequality to (4.2) and solving for μ , we find that

$$\mu \leq \frac{2b(1 + 2\pi^{-2}b \tanh a)}{(1 + 4\pi^{-2}b^2)\tanh a + 2b} \leq 2b(\coth a + 2\pi^{-2}b),$$

which yields the statement (4.1) of the Lemma.

As usual, we define the Minkowski distance $d(A,B)$ between the two point sets A and B to be the infimum of numbers δ such that every point of A

is within distance δ of B and every point of B is within distance δ of A .

We shall impose the following conditions on the domain D :

CONDITION η : The boundary ∂D can be covered by a finite collection of open sets Z_ν such that

a) For some $\theta > 0$

$$U(Z_\nu) \supset \{x: d(x, \partial D) < \theta\}$$

b) Each Z_ν is the image of the cylinder $B_{N-1} \times (0,1)$, where B_{N-1} is the open unit ball in R^{N-1} , under an invertible differentiable mapping $x = \gamma^\nu(\eta, t)$ whose Jacobian is bounded above and below by positive numbers. (Here $\eta \in B_{N-1}$, $t \in (0,1)$)

c) There is a continuous function g^ν on B_{N-1} such that

$$(4.3) \quad D \cap Z_\nu = \{x = \gamma^\nu(\eta, t): \eta \in B_{N-1}, 0 < t < g^\nu(\eta)\},$$

$$\partial D \cap Z_\nu = \{x = \gamma^\nu(\eta, t): \eta \in B_{N-1}, t = g^\nu(\eta)\},$$

$$\alpha < g^\nu(\eta) < \beta,$$

$$0 < \alpha < \beta < 1.$$

and

d) There is a positive constant s such that if $x \in Z_\nu \setminus \bar{D}$, $d(x, \bar{D})$ is the Euclidean distance from x to \bar{D} , and $x = \gamma^\nu(\eta, t)$, then

$$t - g^\nu(\eta) < s d(x, \bar{D}).$$

Our convergence results will be based on the following semicontinuity lemma.

LEMMA 4.2. Let the domain D satisfy the conditions η . Let D_n $n = 1, 2, \dots$ be a sequence of domains, each of which satisfies a cone condition, and such that

$$(4.4) \quad D_n \supset D$$

and

$$(4.5) \quad \lim_{n \rightarrow \infty} d(D_n, D) = 0.$$

If $\mu_j(D_n)$ is the j th Neumann eigenvalue of D_n and $\mu_j(D)$ is the corresponding eigenvalue of D , then

$$(4.6) \quad \liminf_{n \rightarrow \infty} \mu_\ell(D_n) \geq \mu_\ell(D).$$

Proof. We see from (4.3a) and (4.5) that when n is sufficiently large, D_n lies in the union of D and the Z_ν . For such an n we shall derive a bound for the difference in the integrals of ψ^2 over D and over D_n for any smooth function defined on D_n in terms of the integral of $\psi^2 + |\nabla\psi|^2$ over D_n .

In order to do this we note that

$$(4.7) \quad \int_{(D_n - D) \cap Z_\nu} \psi^2 dx = \int_{\gamma^{-1}[(D_n - D) \cap Z_\nu]} [\psi(\gamma^\nu(t, n))]^2 J dt dn$$

$$\leq \int_{B_{N-1}} \int_{g^\nu(n)}^{g^\nu(n)+r(n)} \psi(\gamma^\nu(t, n))^2 J dt dn.$$

Since J is bounded above and $r(n) \leq sd(D_n, D)$ by (4.3d), Lemma 4.1 shows that there is a constant c_1 such that

$$\int_{g^\nu(n)}^{g^\nu(n)+r(n)} \psi^2 J dt \leq c_1 d(D_n, D) \int_0^{g^\nu+r} (\psi^2 + \psi_t^2) dt.$$

We substitute this in (4.7), transform back to the x-coordinates, and use the fact that J has a positive lower bound to see that there is a constant c_2 such that

$$\int_{(D_n - D) \cap Z_\nu} \psi^2 dx \leq c_2 d(D_n, D) \int_{D_n \cap Z_\nu} (\psi^2 + |\nabla \psi|^2) dx.$$

We add these inequalities to find that

$$(4.8) \quad \int_{D_n} \psi^2 dx - \int_D \psi^2 dx \leq cd(D_n, D) \int_{D_n} (\psi^2 + |\nabla \psi|^2) dx,$$

where $c = Lc_2$ with L the number of sets Z_ν .

Let w_1, \dots, w_ℓ be a set of infinitely differentiable functions which are orthogonal on D_n and which approximate the first ℓ Neumann eigenfunctions of D_n so well that for any linear combination

$$(4.9) \quad \psi(x) = \sum_i^{\ell} c_i w_i(x)$$

satisfies

$$(4.10) \quad \int_{D_n} |\nabla \psi|^2 dx < (\mu_\ell(D_n) + \epsilon) \int_{D_n} \psi^2 dx.$$

Select the constants in (4.9) so that they are not all zero and

$$\int_D \psi v_i dx = 0, \quad i = 1, \dots, \ell-1.$$

Then

$$(4.11) \quad \mu_\ell(D) \int_D \psi^2 dx \leq \int_D |\nabla \psi|^2 dx.$$

We observe that

$$(4.12) \quad \int_D |\nabla \psi|^2 dx \leq \int_{D_n} |\nabla \psi|^2 dx \leq (\mu_\ell(D_n) + \epsilon) \int_{D_n} \psi^2 dx,$$

while by (4.8) and (4.10)

$$(4.13) \quad \int_D \psi^2 dx > (1 - cd(D_n, D)[1 + \mu_\ell(D_n) + \epsilon]) \int_{D_n} \psi^2 dx.$$

Since the Dirichlet eigenvalue $\lambda_\ell(D)$ decreases as D grows [3, p. 409, Theo. 3], we see that

$$\mu_\ell(D_n) < \lambda_\ell(D_n) < \lambda_\ell(D).$$

Thus, if n is so large that $cd(D_n, D)[1 + \lambda_\ell(D) + \epsilon] < 1$, we see from (4.11), (4.12), and (4.13) that

$$\mu_\ell(D) < \frac{\mu_\ell(D_n) + \epsilon}{1 - cd(D_n, D)[1 + \lambda_\ell(D) + \epsilon]}.$$

Since this is true for any positive ϵ , we have

$$\mu_\ell(D) < \frac{\mu_\ell(D_n)}{1 - cd(D_n, D)[1 + \lambda_\ell(D)]}.$$

We now let $n \rightarrow \infty$ through a sequence to obtain the statement (4.6) of the Lemma.

REMARKS 1. If D is bounded and convex and 0 is any of its points, then a closed ball of some radius R_1 centered at 0 lies in D and an open ball of radius R_2 centered at 0 contains \bar{D} . If (r, ω) are polar coordinates centered at 0 , we can divide the annular region between the two balls into finitely many "cylinders" with the coordinates $t = (r - R_1)/(R_2 - R_1)$ and $n = n(\omega)$. Thus any bounded convex domain satisfies condition η .

2. If D and D_n are any convex domains with $d(D_n, D)$ small, there are dilations of D with constants near 1, one of which takes \bar{D} inside D_n while the other makes \bar{D} contain D_n . The above proof then shows that for bounded

convex domains $d(D_n, D) \rightarrow 0$ implies that

$$\mu_\ell(D_n) \rightarrow \mu_\ell(D).$$

3. We recall [3, p. 423, Theo. 11] that the Dirichlet eigenvalues are continuous in the sense that $d(D_n, D) \rightarrow 0$ implies that $\lambda_\ell(D_n) \rightarrow \lambda_\ell(D)$.

We can now obtain limiting forms of Proposition 2.1

THEOREM 4.1 Let the domain D satisfy the condition and suppose that
there is a sequence of domains $D_n \subset D$ such that each D_n satisfies the
hypotheses of Proposition 2.1, and let

$$\lim_{n \rightarrow \infty} d(D_n, D) = 0.$$

Then the eigenvalues of D satisfy the inequalities

$$\mu_{k+R} \leq \lambda_k \quad \text{for } k = 1, 2, \dots$$

Proof. Apply lemma 4.2 to the inequalities

$$\mu_{k+R}(D_n) < \lambda_k(D_n) \leq \lambda_k(D).$$

A theorem of MINKOWSKI [2, p. 35ff.] states that a convex domain D can be approximated in the sense of set distance by smooth convex domains $D_n \subset D$.

This and Remark 1 after Lemma 4.2 yield the following result:

THEOREM 4.2: If D is any nonempty bounded convex domain, then

$$(4.14) \quad \mu_{k+N} \leq \lambda_k \quad \text{for } k = 1, 2, \dots$$

It is not known whether there is a convex domain in $N > 1$ dimensions for which equality holds in (4.14).

It is easy to find nonconvex domains for which the condition η can be

verified, but it is difficult to determine whether a nonconvex domain is the limit of a sequence of larger domains which satisfy the conditions of Proposition 2.1 for some $R < N$. We present one class of domains where this can be done.

Let $N = 3$ and let D be obtained by rotating a two-dimensional domain D_0 about a coplanar line which does not intersect the closure of D_0 . Suppose that the boundary of D_0 is of class $C^{2,\alpha}$ with the exception of a subset of the interior of $\partial D \cap \partial D^*$ where D^* is the convex hull of D . By the Minkowski construction we construct a smooth convex domain $D_0' \supset D_0$ which is arbitrarily close to the convex hull D_0^* of D_0 . We then construct a convex domain D_0'' with smooth boundary such that $\partial D_0''$ coincides with ∂D_0 on the part of $\partial D_0 \cap \partial D^*$ near the closure of $\partial D_0 \setminus \partial D^*$, and with $\partial D_0'$ near the set where ∂D_0 is not smooth. Let D_0''' be the subdomain of D_0'' whose boundary consists of $\partial D_0 \setminus \partial D_0^*$ and a part of $\partial D_0''$. Let D''' be obtained by rotating D_0''' . Then D''' contains and is arbitrarily close to D , and it has nonnegative mean curvature if this is true of $\partial D \setminus \partial D^*$. It is easily verified that D satisfies Condition η .

The same reasoning also works when D is obtained by rotating D_0 about a line of symmetry.

We conclude from Theorem 4.1 that if D is a domain of revolution and if $\partial D \cap \partial D^*$ is smooth and has nonnegative mean curvature, then

$$\mu_{k+1} \leq \lambda_k \quad \text{for } k = 1, 2, \dots .$$

The same result follows for an N -dimensional domain which is obtained by rotating a two-dimensional domain about an $N-2$ -dimensional hyperplane.

It would be interesting to find a larger class of domains to which Theorem 4.1 can be applied.

It was observed by Joseph Hersch that if a domain D' is obtained from a domain D by removing a set of measure zero, then $\lambda_k(D') \geq \lambda_k(D)$ while

$\mu_k(D') \leq \mu_k(D)$. Therefore if D satisfies the conditions of Proposition 2.1, then the eigenvalues of D' still satisfy the inequalities (2.3). Since the boundary of D' may not even have nonnegative mean curvature, this observation makes one wonder whether the inequalities (2.3) are not, in fact true for all domains.

However, numerical computation shows that $\mu_3 < \lambda_1$ for the two-dimensional annular sector

$$D = \{(r, \theta): 1 < r < 2, 0 < \theta < 3\pi/2\}.$$

By Lemma 4.1 one can find smooth domains containing this D for which the same inequality is valid. Thus the inequality (2.3) is not true for all smooth two-dimensional domains.

Computation shows that $\mu_3 < \lambda_1 < \mu_4$ for a disc and that $\mu_4 < \lambda_1 < \mu_5$ for a three-dimensional ball. Thus μ_{N+k} cannot be replaced by μ_{N+k+1} in the inequality (2.20). On the other hand, one can show that there are constants $a > 0$ and $c > 1$ such that for the N -ball

$$\mu_{ac}^N < \lambda_1.$$

This suggests that perhaps (2.20) can be replaced by a better inequality of the form

$$\mu_\phi(N, k) < \lambda_k$$

for convex N -dimensional domains.

ACKNOWLEDGEMENT. LEVINE was supported in part by the two NSF Institutes, by the C.N.R. (Italy), and by AFOSR Grant 84-0252. WEINBERGER was supported by grants from the NSF and the AFOSR.

REFERENCES

1. P. AVILES, Symmetry theorems related to Pompeiu's problem. American J. of Math. (in press)
2. T. BONNESSEN & W. FENCHEL, Theorie der Konvexen Korper. Springer, Berlin, 1934, Chelsea, New York, 1948.
3. R. COURANT & D. HILBERT, Methods of Mathematical Physics, vol. I, Interscience, New York (1962).
4. G.H. HARDY, J.E. LITTLEWOOD, & G. POLYA, Inequalities, 2nd ed., Cambridge U. Press, London, 1952.
5. L.E. PAYNE, Inequalities for eigenvalues of plates and membranes, J. Rat. Mech. Anal. 4(1955), 517-529.
6. G. POLYA, Remarks on the foregoing paper, J. of Math. and Phys. 31 (1952), 55-57.
7. M.H. PROTTER & H.F. WEINBERGER, Maximum Principles in Differential Equations, Prentice Hall, Englewood Cliffs, 1967, Springer, New York, 1984.
8. H.F. WEINBERGER, Variational Methods for Eigenvalue Approximation, C.B.M.S. Regional Conf. Ser. #15, S.I.A.M., Philadelphia, 1974.

Iowa State University
Ames

and

Institute for Mathematics
and its Applications
University of Minnesota
Minneapolis