Anisotropic inverse conductivity and scattering problems

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Abstract

Uniqueness in inverse conductivity and scattering problems is considered. In case the medium consists of two discontinuous constant anisotropic conductive parts, the measurements of potential and induced currents on the boundary of surrounding body are enough to guarantee uniqueness to determine conductivity and region of embedded unknown material under a very weak condition. The analogous uniqueness result is also obtained for an inverse scattering problem in the case that the medium is composed of two anisotropic and homogeneous materials.

Key words: anisotropic medium, inverse problem, conductivity, scattering

1 Introduction

An inverse conductivity problem, sometimes called an electrical impedance tomography problem, is to determine the interior conductivity profile or the location, shape, and size of an obstacle by measuring a pair or many pairs of current and voltage on the boundary or its part. To describe physically more rigorously, let u denote the electric potential (voltage) in a bounded medium Ω with a Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^2$ with the conductivity of the medium $\gamma = (\gamma_{jk})_{j,k=1,2}$. Then the potential f on $\partial\Omega$ and u satisfies

$$L_\gamma u := \nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega, \quad (1.1a)$$

$$u = f \text{ on } \partial\Omega. \quad (1.1b)$$

Then current flux on $\partial\Omega$ is given by $\nu \cdot \gamma \nabla u$, where ν denotes the unit outward normal to Ω .

A mathematical interpretation of an inverse conductivity problem proposed by Calderon [6] is to determine γ from the information on the Dirichlet-to-Neumann map $\Lambda_\gamma : H^{1/2}(\Omega) \rightarrow H^{-1/2}(\Omega)$. It is well-known that the knowledge of Λ_γ suffices to determine continuous, isotropic conductivity uniquely in $\mathbb{R}^n$ for $n = 2$ [5, 30], and for $n \geq 3$ [1, 29, 37, 38]. For the case of discontinuous (*e.g.*, piecewise constant) isotropic conductivity, the question of unique determination has been considered in [4, 12, 20, 21, 22]. However, most of such results cannot be extended directly to the corresponding problem for anisotropic media, although many interesting materials are in fact anisotropic. For instance, see [25, 26, 27, 28, 36, 39] and the bibliographies therein. For the continuous conductivity case, anisotropic inverse problems have been studied in [2, 3, 27, 30, 36, 37]

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For discontinuous anisotropic media, uniqueness has been known if the difference of discontinuities (or refractive indices) is positive-definite [17], and in the two dimensional case this condition is weakened so that the determinant of obstacle conductivity is different from that of background material if the conductivity matrix is small in a certain sense [16]. In this article, the smallness condition in [16] is removed in case the medium consists of two regions of homogeneous conductive materials with the determinants of conductivity of the obstacle and surrounding medium being different.

One of the popular tools in the analysis of uniqueness of inverse problems is the use of singular solutions. In what follows, we limit to anisotropic cases only. Denoting by $\Phi^{j,x}$, $j = 2, 3$, the fundamental solutions to the differential operator L_{γ_j} , with x being the singular point, Isakov [17] observed that $\Phi^{j,x}$ satisfies the equation (2.4) in place of u^j 's. He then proceeded to induce a contradiction by showing that the left hand side in (2.4) tends to infinity as x converges to a certain point, say, the origin $\mathbf{O}$. In two dimension under certain conditions, Ikehata [16] used complex functions of $O(\frac{1}{|x|})$ growth, instead. In our paper, real functions of $O(\log |x|)$ growth are used, which have the same growth as fundamental solutions.

Our analysis can be also applied to *inverse scattering problems* with discontinuous anisotropic homogeneous media. which is to determine material properties (including refractive indices) or the shape and size of an obstacle by measuring scattered fields at a large distance, or equivalently measuring far-field patterns. Let u and u^s denote the scattered and total waves, respectively, generated by the incident wave $e^{ikx \cdot d}$ of wave number k in the direction $d \in S^1$. Then the following equations hold:

$$L_{\gamma,k,q}u := \nabla \cdot (\gamma \nabla u) + k^2 q u = 0 \text{ in } \mathbb{R}^2, \quad (1.2a)$$

$$u(x) = e^{ikx \cdot d} + u^s(x), \quad (1.2b)$$

$$\lim_{r \rightarrow \infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad (1.2c)$$

where γ is the conductivity and q is the refractive index in the medium. Here, $q - 1$ is assumed to be an $L^\infty(\mathbb{R}^2)$ function with compact support. The far-field pattern u_∞ then satisfies

$$u^s(x) = |x|^{-\frac{1}{2}} e^{ik|x|} \left(u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right),$$

with $\hat{x} = \frac{x}{|x|}$. An inverse scattering problem is to find γ and q where the far field patterns $u_\infty(\cdot; d)$ are given for all incident direction $d \in S^1$ with a fixed frequency k [7, 8].

The theory of inverse scattering problems for isotropic media has been well-developed mathematically and numerically (see [7, 8, 9, 19, 23, 24, 32, 35] and the references therein). Corresponding theory for anisotropic media needs more attention [10, 11, 14, 33, 34].

For discontinuous anisotropic media, the far-field pattern determines the medium structure uniquely provided that the jump of discontinuity is positive-definite [18]. This condition can be weakened in two dimension such that the determinants of the background and obstacle are different as far as the anisotropy near the boundary of the obstacle is known [31]. In the present paper, uniqueness is obtained without knowing the anisotropy at the boundary for discontinuous anisotropic homogeneous medium in $\mathbb{R}^2$.

Fundamental solutions of general elliptic partial differential and Lippmann-Schwinger-type equations are used in the analysis of inverse scattering problems [17, 31], respectively. We again make use of real-valued functions of $O(\log |x|)$ which have the same type of singularity as fundamental solutions. The relation between

the far-field operator and Dirichlet-to-Neumann map enables to apply to inverse scattering problem the analysis carried out for our inverse conductivity problem [17, 19].

We analyze in detail the inverse conductivity problem in discontinuous anisotropic media in the section to follow. Then in the subsequent section, we will apply the analysis to the inverse scattering problem of discontinuous anisotropic coefficient.

2 Anisotropic Inverse Conductivity Problem

2.1 Uniqueness Result

This section develops a uniqueness result related with the inverse problem of (1.1). Particularly, we restrict our attention to the case of the discontinuous anisotropic conductivity of the form

$$\gamma = A + (B - A)\chi_D,$$

where A is a known symmetric positive-definite matrix, B a symmetric positive-definite matrix to be determined, and D a Lipschitz domain, also to be determined, compactly contained in $\Omega \subset \mathbb{R}^2$ such that $\Omega \setminus \overline{D}$ is connected. Furthermore, assume that A and B are constant matrices.

We will then prove the following theorem.

Theorem 2.1. *Let $\gamma_j = A + (B_j - A)\chi_{D_j}$ for $j = 1, 2$, where D_1 and D_2 are Lipschitz domains compactly embedded in $\Omega \subset \mathbb{R}^2$ such that $\Omega \setminus \overline{D_j}, j = 1, 2$ are connected, and B_1 and B_2 are 2×2 constant symmetric positive-definite matrices. If $\det(B_j) \neq \det(A)$, then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies*

$$D_1 = D_2 \text{ and } B_1 = B_2.$$

2.2 The Proof of Theorem 2.1: $D_1 = D_2$

We begin by simplifying the types of $\gamma_j, j = 1, 2$. Since A is symmetric positive-definite, there exist an invertible matrix C such that $A = C^t C$. Identify C with the linear transformation $\tilde{x} \mapsto x = C\tilde{x}$, and denoting $\tilde{\Omega} = C^{-1}(\Omega)$, $\tilde{D} = C^{-1}(D)$, $\tilde{f}(\tilde{x}) = f(C\tilde{x})$, and $\tilde{\gamma}(\tilde{x}) = C^{-t}AC^{-1} + (C^{-t}BC^{-1} - C^{-t}AC^{-1})\chi_{\tilde{D}}(C\tilde{x})$. Then Problem (1.1) can be reduced to

$$\begin{aligned} \nabla_{\tilde{x}} \cdot (I + (\tilde{B} - I)\chi_{\tilde{D}})\nabla_{\tilde{x}}\tilde{u} &= 0 & \text{in } \tilde{\Omega}, \\ \tilde{u} &= \tilde{f} & \text{on } \partial\tilde{\Omega}, \end{aligned}$$

where $\tilde{B} = C^{-t}BC^{-1}$ is a symmetric positive-definite matrix. Therefore, without loss of generality, the matrix A may be assumed to be the identity matrix I in the original equations (1.1). The tildes will be dropped from now on.

Furthermore, consider an orthogonal matrix P such that $B_1 = P^t \Sigma P$, where Σ is a diagonal matrix whose diagonal elements consist of eigenvalues of B_1 . Again, by the identification of P^{-1} with the induced linear transformation $\tilde{x} \mapsto x = P^{-1}\tilde{x}$, Equations (2.3) can be reduced into a simpler setting in which B_1 is a diagonal matrix with positive diagonal entries while B_2 is a symmetric, positive-definite matrix.

Summing up, we henceforth assume that

$$A = I, \quad B_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1, \lambda_2 > 0,$$

and B_2 is a positive-definite matrix in Equations (2.3). Also, without loss of generality, we may assume

$$\begin{cases} \lambda_1 \leq \lambda_2 & \text{if } \lambda_1 \lambda_2 > 1 \\ \lambda_1 \geq \lambda_2 & \text{if } \lambda_1 \lambda_2 < 1. \end{cases} \quad (2.3)$$

Indeed, this can be done, if necessary, by using the transformation $\tilde{x} \mapsto x = Q\tilde{x}$, where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $Q^{-t}B_1Q^{-1}$ then switches the order of diagonal entries of B_1 . It is thus enough to prove Theorem 2.1 for the case where $\gamma_j = I + (B_j - I)\chi_{D_j}$, $j = 1, 2$ with B_1 being a diagonal matrix satisfying (2.3).

Next, assume that $D_1 \neq D_2$, say, D_1 is not contained in D_2 . Following [17], choose a point $p \in \partial D_1 \setminus \overline{D_2}$, which also lies in the intersection of the convex hull of $D_1 \cup D_2$ and the closure of the connected component of $\Omega \setminus (D_1 \cup D_2)$ that is in contact with $\partial\Omega$. Consider a disk $B_r(p)$ with center p of radius $r > 0$ to be specified in a moment. Set $D_3 = D_1 \cup B_r(p)$. Also, let D_4 be the union of $D_1 \cup D_2$ and all the components of $\Omega \setminus (D_1 \cup D_2)$ that do not touch $\partial\Omega$. Finally, let $D_5 = \Omega \setminus D_4$, the connected component of $\Omega \setminus (\overline{D_1 \cup D_2})$ whose boundary contains $\partial\Omega$.

Choose the radius r such that $B_r(p) \subset \Omega$, $B_r(p) \cap \overline{D_2} = \emptyset$ and $\partial D_4 \cap B_r(p)$ is the image of a Lipschitz function $\psi : [0, 1] \rightarrow \mathbb{R}^2$. Without loss of generality, we may assume p to be the origin $\mathbf{O}$. For the notational simplicity write $B_3 = B_1$, and define γ_3 by

$$\gamma_3 = I + (B_3 - I)\chi_{D_3} (= I + (B_1 - I)\chi_{D_3}).$$

Recall the following lemma due to Isakov [17].

Lemma 2.2. *For $j = 2, 3$, let w^j satisfy $L_{\gamma_j} w^j = 0$ in a neighborhood of D_4 for $j = 2, 3$. Then*

$$\int_{D_1} (B_1 - I) \nabla u^2 \cdot \nabla u^3 dy = \int_{D_2} (B_2 - I) \nabla u^2 \cdot \nabla u^3 dy. \quad (2.4)$$

Notice that the restrictions of γ_j to $B_r(\mathbf{O})$ are the constant matrices. Indeed, they are the identity matrix and the matrix B_1 for $j = 2$ and $j = 3$, respectively, which we will denote by γ'_2 and γ'_3 . Then the following lemma is a suitable modification of a result in [15].

Lemma 2.3. *Suppose that a function $\tilde{U}^j \in L^1_{loc}(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2 \setminus \{\mathbf{O}\})$, for $j = 2, 3$, satisfies*

$$\nabla \cdot (\gamma'_j \nabla \tilde{U}^j) = 0 \text{ in } \mathbb{R}^2 \setminus \mathbf{O}. \quad (2.5)$$

Then there exists a family of functions $\{\tilde{w}^{j,\xi}\}$, $\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D_4}$, such that

$$\sup_{\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D_4}} \|\tilde{w}^{j,\xi}(\cdot) - \tilde{U}^j(\cdot - \xi)\|_{H^1(D_4)} < \infty, \quad (2.6a)$$

$$\tilde{w}^{j,\xi}|_{\partial\Omega} = 0, \quad (2.6b)$$

$$\nabla \cdot (\gamma_j \nabla \tilde{w}^{j,\xi}) = 0 \text{ in } D_{4,\xi} = \{y \mid d(y, D_4) < d(\xi, D_4)/2\}. \quad (2.6c)$$

Proof. Let $\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D_4}$ be arbitrary. First, a mollification argument will provide a sequence of $C^\infty(\overline{\Omega})$ -functions $\{\phi_m^{j,\xi}\}_{m \geq 1}$ such that

$$\phi_m^{j,\xi}(\cdot) \rightarrow \tilde{U}^j(\cdot - \xi) \text{ in } H_{loc}^1(\Omega \setminus \{\xi\}) \quad \text{as } m \rightarrow \infty.$$

For each m , let $w_m^{j,\xi}$ and $z_m^{j,\xi} \in H^1(\Omega)$ be the unique solutions of the following problems:

$$\nabla \cdot (\gamma_j \nabla w_m^{j,\xi}) = \nabla \cdot (\gamma_j' \nabla \phi_m^{j,\xi}) \quad \text{in } \Omega, \quad (2.7a)$$

$$w_m^{j,\xi} = 0 \quad \text{on } \partial\Omega, \quad (2.7b)$$

and

$$\begin{aligned} \nabla \cdot (\gamma_j' \nabla z_m^{j,\xi}) &= 0 \quad \text{in } \Omega, \\ z_m^{j,\xi} &= -\phi_m^{j,\xi} \quad \text{on } \partial\Omega. \end{aligned}$$

By elliptic regularity and the trace theorem, we get

$$\|z_m^{j,\xi} - z_{m'}^{j,\xi}\|_{H^1(\Omega)} \leq C \|\phi_m^{j,\xi} - \phi_{m'}^{j,\xi}\|_{H^{1/2}(\partial\Omega)} \leq C \|\phi_m^{j,\xi} - \phi_{m'}^{j,\xi}\|_{H^1(\Omega \setminus B_r(\mathbf{O}))}, \quad (2.8)$$

from which we have $z^{j,\xi} \in H^1(\Omega)$ such that

$$\begin{aligned} z_m^{j,\xi} &\rightarrow z^{j,\xi} \text{ as } m \rightarrow \infty \text{ in } H^1(\Omega), \\ \sup_{\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D_4}} \|z^{j,\xi}\|_{H^1(\Omega)} &< \infty. \end{aligned}$$

Setting $v_m^{j,\xi} = w_m^{j,\xi} - \phi_m^{j,\xi} - z_m^{j,\xi}$, we see that

$$\nabla \cdot (\gamma_j \nabla v_m^{j,\xi}) = \nabla \cdot ((\gamma_j' - \gamma_j) \nabla (\phi_m^{j,\xi} + z_m^{j,\xi})) \quad \text{in } \Omega, \quad (2.9a)$$

$$v_m^{j,\xi} = 0 \quad \text{on } \partial\Omega. \quad (2.9b)$$

Since $\gamma_j' = \gamma_j$ in $B_r(\mathbf{O})$, the estimate (2.8) leads to

$$\begin{aligned} \|v_m^{j,\xi} - v_{m'}^{j,\xi}\|_{H^1(\Omega)} &\leq C \{\|\phi_m^{j,\xi} - \phi_{m'}^{j,\xi}\|_{H^1(\Omega \setminus B_r(\mathbf{O}))} + \|z_m^{j,\xi} - z_{m'}^{j,\xi}\|_{H^1(\Omega \setminus B_r(\mathbf{O}))}\} \\ &\leq C' \|\phi_m^{j,\xi} - \phi_{m'}^{j,\xi}\|_{H^1(\Omega \setminus B_r(\mathbf{O}))}. \end{aligned}$$

From this it follows that, for all j, ξ , there is a $v^{j,\xi} \in H^1(\Omega)$ such that

$$\begin{aligned} v_m^{j,\xi} &\rightarrow v^{j,\xi} \text{ in } H^1(\Omega) \quad \text{as } m \rightarrow \infty, \\ \sup_{\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D_4}} \|v^{j,\xi}\|_{H^1(\Omega)} &< \infty. \end{aligned}$$

For each j , let $\tilde{u}^{j,\xi} = \tilde{U}^j(\cdot - \xi) + v^{j,\xi} + z^{j,\xi}$. It is then immediate to see that $\{\tilde{u}^{j,\xi}\}$ satisfies the properties of (2.6). This completes the proof. $\square$

Since $\mathbf{O}$ lies in the convex hull of $D_1 \cup D_2$, there is a straight line passing through $\mathbf{O}$ such that $D_1 \cap B_r(\mathbf{O})$ is contained in one side of this straight line. We may assume this straight line to be the y_1 axis and the y_2 axis to be a vertical line to this line. Assume further that the y_2 axis passes through $D_1 \cap B_r(\mathbf{O})$, if necessary, after

rotating the y_1 axis. Choose the positive directions of the y_2 axis toward $D_1 \cap B_r(\mathbf{O})$. Since D_1 is Lipschitz, there are two cones K^i and K^e with angles $0 \leq \theta^i < \frac{\pi}{2}$ at $\mathbf{O}$ satisfying [13]

$$B_r(\mathbf{O}) \cap \overline{K}^i \subset \overline{D}_1, \quad B_r(\mathbf{O}) \cap \overline{K}^e \cap \overline{D}_1 = \{\mathbf{O}\}.$$

More explicitly, let us denote such cones by

$$K^i = \{(y_1, y_2) \in \mathbb{R}^2 \mid \frac{\pi}{2} - \theta^i < \arg(y_1 + iy_2) < \frac{\pi}{2} + \theta^i\},$$

with K^e being the reflection of K^i to the y_1 -axis. Here and in what follows, the local branch of the argument function $\arg$ is chosen as the negative y_2 axis. For $j = 2, 3$, let

$$\tilde{u}^j = \frac{1}{2} \log(y_1^2 + \tau_j^2 y_2^2),$$

where $\tau_2 = 1, \tau_3 = \sqrt{\frac{\lambda_1}{\lambda_2}}$. Then one can verify that the function $\tilde{u}^j \in L_{loc}^1(\mathbb{R}^2) \cap H_{loc}^1(\mathbb{R}^2 \setminus \{\mathbf{O}\})$ satisfies (2.5). Then let $\{\tilde{u}^{j,\xi}\}_{\xi \in B_{r/2}(\mathbf{O}) \setminus \overline{D}_4}$, $j = 2, 3$, be given such that (2.6) is fulfilled with $\tilde{u}^j$ in place of $\tilde{U}^j$, respectively, as in Lemma 2.3. Recalling from (2.6c) that $D_{4,\xi}$ is a neighborhood of D_4 , we see that the orthogonality relation (2.4) applies to $\tilde{u}^{j,\xi}$, $j = 2, 3$, with $\xi \in K^e \cap B_{r/2}(\mathbf{O})$, which yields

$$\begin{aligned} & \int_{D_1 \cap B_r(\mathbf{O})} (B_1 - I) \nabla \tilde{u}^{2,\xi} \cdot \nabla \tilde{u}^{3,\xi} dy \\ &= \int_{D_1} (B_1 - I) \nabla \tilde{u}^{2,\xi} \cdot \nabla \tilde{u}^{3,\xi} dy - \int_{D_1 \setminus B_r(\mathbf{O})} (B_1 - I) \nabla \tilde{u}^{2,\xi} \cdot \nabla \tilde{u}^{3,\xi} dy \\ &= \int_{D_2} (B_2 - I) \nabla \tilde{u}^{2,\xi} \cdot \nabla \tilde{u}^{3,\xi} dy - \int_{D_1 \setminus B_r(\mathbf{O})} (B_1 - I) \nabla \tilde{u}^{2,\xi} \cdot \nabla \tilde{u}^{3,\xi} dy. \end{aligned} \tag{2.10}$$

Notice that the last two integrals are bounded regardless of $\xi \in K^e \cap B_{r/2}(\mathbf{O})$ by (2.6a). Consequently, the integral (2.10) is bounded.

In the meanwhile, we are going to show in the following that the integral (2.10) is unbounded as $\xi \rightarrow \mathbf{O}$, which leads to a contradiction. Decompose (2.10) into the sum of I_1, I_2, I_3 and I_4 , where

$$\begin{aligned} I_1 &= \int_{D_1 \cap B_r(\mathbf{O})} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy, \\ I_2 &= \int_{D_1 \cap B_r(\mathbf{O})} (B_1 - I) \nabla \cdot (\tilde{u}^{2,\xi}(y) - \tilde{u}^2(y - \xi)) \nabla \tilde{u}^3(y - \xi) dy, \\ I_3 &= \int_{D_1 \cap B_r(\mathbf{O})} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla (\tilde{u}^{3,\xi}(y) - \tilde{u}^3(y - \xi)) dy, \\ I_4 &= \int_{D_1 \cap B_r(\mathbf{O})} (B_1 - I) \nabla (\tilde{u}^{2,\xi}(y) - \tilde{u}^2(y - \xi)) \cdot \nabla (\tilde{u}^{3,\xi}(y) - \tilde{u}^3(y - \xi)) dy. \end{aligned}$$

Due to (2.6a), an upper bound of I_4 is easily obtained, say, $|I_4| < C_0$. Next, again by (2.6a) and Young's inequality, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$|I_2| + |I_3| \leq C_1 \epsilon \int_{D_1 \cap B_r(\mathbf{O})} \frac{dy}{|y - \xi|^2} + \frac{C_2}{\epsilon}$$

for all $\xi \in K^i \cap B_{r/2}(\mathbf{O})$ and $\epsilon > 0$. Using the following inequality,

$$\begin{aligned} |I_1 + I_2 + I_3 + I_4| &\geq |I_1| - |I_2| - |I_3| - |I_4| \\ &\geq |I_1| - C_1 \epsilon \int_{D_1 \cap B_r(\mathbf{O})} \frac{dy}{|y - \xi|^2} - \frac{C_2}{\epsilon} - C_0, \end{aligned} \tag{2.11a}$$

we will show that (2.11a) is unbounded as $\xi \rightarrow \mathbf{O}$.

For $s < r/2$, choose $\xi_s \in K^e \cap B_{r/2}(\mathbf{O})$ such that $\xi_s = (0, -s)$. Depending on s , denote by (ρ_s, θ_s) the polar coordinate system with origin at ξ_s such that $y - \xi_s = (\rho_s \cos \theta_s, \rho_s \sin \theta_s)$, where $\rho_s(y) = |y - \xi_s|$ and $\theta_s(y) = \arg(y - \xi_s)$. Let $M_s = \max_{y \in S} |y - \xi_s|$, where $S = \partial D_1 \cap \partial B_r(\mathbf{O})$ consists of a set of two points for sufficiently small r . Notice that M_s converges to r as s converges to 0. Take the two points $P_1, P_2 \in \partial K^i$ such that $\arg(P_1) = \frac{\pi}{2} - \theta^i$, $\arg(P_2) = \frac{\pi}{2} + \theta^i$ and $|P_1| = |P_2| = s$. Then we have $\arg(P_1 - \xi_s) = \frac{\pi}{2} - \frac{\theta^i}{2}$, $\arg(P_2 - \xi_s) = \frac{\pi}{2} + \frac{\theta^i}{2}$ and $|P_1 - \xi_s| = |P_2 - \xi_s| = \sqrt{2}s\sqrt{1 + \cos \theta^i}$. Let us decompose $D_1 \cap B_r(\mathbf{O})$ into the three regions:

$$\begin{aligned} E_s &= \{y \in D_1 \cap B_r(\mathbf{O}) \mid \sqrt{2}s\sqrt{1 + \cos \theta^i} \leq \rho_s(y) \leq M_s\}, \\ F_s &= \{y \in D_1 \cap B_r(\mathbf{O}) \mid s \leq \rho_s(y) < \sqrt{2}s\sqrt{1 + \cos \theta^i}\}, \\ G_s &= \{y \in D_1 \cap B_r(\mathbf{O}) \mid \rho_s(y) > M_s\}. \end{aligned} \quad (2.12)$$

By (2.11) and (2.12), $|I_1 + I_2 + I_3 + I_4|$ is bounded below as follows:

$$\begin{aligned} |I_1 + I_2 + I_3 + I_4| &\geq \left| \int_{E_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| - C_1 \epsilon \int_{E_s} \frac{dy}{|y - \xi|^2} \\ &\quad - \left| \int_{F_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| - C_1 \epsilon \int_{F_s} \frac{dy}{|y - \xi|^2} \\ &\quad - \left| \int_{G_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| - C_1 \epsilon \int_{G_s} \frac{dy}{|y - \xi|^2} \\ &\quad - \frac{C_2}{\epsilon} - C_0. \end{aligned} \quad (2.13)$$

We will show that the integrals over F_s and G_s are uniformly bounded, while that over E_s is unbounded. Since $G_s \subset G_r$ for all $s \leq r \leq 1$, the integral over G_s is bounded uniformly in s as follows:

$$\begin{aligned} &\left| \int_{G_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| + C_1 \epsilon \int_{G_s} \frac{dy}{|y - \xi|^2} \\ &\leq \max(|\lambda_1 - 1|, |\lambda_2 - 1|) \int_{G_r} |\nabla \tilde{u}^2(y - \xi)| \cdot |\nabla \tilde{u}^3(y - \xi)| dy + C_1 \epsilon \int_{G_r} \frac{dy}{|y - \xi|^2} \\ &\leq (C_3 + C_1 \epsilon) \frac{|G_r|}{r^2} \leq (C_3 + C_1 \epsilon) \pi. \end{aligned} \quad (2.14)$$

Turn to estimate the integrals over E_s and F_s .

$$\begin{aligned} &\left| \int_{E_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| - C_1 \epsilon \int_{E_s} \frac{dy}{|y - \xi|^2} \\ &= \left| \int_{\sqrt{2}s\sqrt{1 + \cos \theta^i}}^{M_s} \frac{1}{\rho_s} \int_{\alpha(s)}^{\beta(s)} \left[\frac{\lambda_1 \lambda_2}{\lambda_2 \cos^2 \theta_s + \lambda_1 \sin^2 \theta_s} - 1 \right] d\theta_s d\rho_s \right| \\ &\quad - C_1 \epsilon \int_{\sqrt{2}s\sqrt{1 + \cos \theta^i}}^{M_s} \frac{1}{\rho_s} \int_{\alpha(s)}^{\beta(s)} d\theta_s d\rho_s, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned}
& \left| \int_{F_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| + C_1 \epsilon \int_{F_s} \frac{dy}{|y - \xi|^2} \\
&= \left| \int_s^{\sqrt{2}s\sqrt{1+\cos\theta^i}} \frac{1}{\rho_s} \int_{\alpha(s)}^{\beta(s)} \left[\frac{\lambda_1 \lambda_2}{\lambda_2 \cos^2 \theta_s + \lambda_1 \sin^2 \theta_s} - 1 \right] d\theta_s d\rho_s \right| \\
&\quad + C_1 \epsilon \int_s^{\sqrt{2}s\sqrt{1+\cos\theta^i}} \frac{1}{\rho_s} \int_{\alpha(s)}^{\beta(s)} d\theta_s d\rho_s,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
\alpha(s) &= \min_{y \in D_1 \cap B_r(\mathbf{O})} \arg(y - \xi_s), & \beta(s) &= \max_{y \in D_1 \cap B_r(\mathbf{O})} \arg(y - \xi_s). \\
|y - \xi_s| &= \rho_s & |y - \xi_s| &= \rho_s
\end{aligned}$$

Notice that

$$\left| \int_{\alpha}^{\beta} \left[\frac{\lambda_1 \lambda_2}{\lambda_2 \cos^2 \theta + \lambda_1 \sin^2 \theta} - 1 \right] d\theta \right| \leq (\beta - \alpha) \left[1 + \frac{\lambda_1 \lambda_2}{\min(\lambda_1, \lambda_2)} \right] \leq \pi \left[1 + \frac{\lambda_1 \lambda_2}{\min(\lambda_1, \lambda_2)} \right] := C_4. \tag{2.17}$$

More explicitly, a change of variables implies, for $0 \leq \alpha < \frac{\pi}{2} < \beta \leq \pi$,

$$\begin{aligned}
& \int_{\alpha}^{\beta} \frac{(\lambda_1 - 1)\lambda_2 \cos^2 \theta + \lambda_1(\lambda_2 - 1)\sin^2 \theta}{\lambda_2 \cos^2 \theta + \lambda_1 \sin^2 \theta} d\theta \\
&= \sqrt{\lambda_1 \lambda_2} \left(\pi - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan(\pi - \beta)\right) - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan \alpha\right) \right) - (\beta - \alpha),
\end{aligned}$$

where the range of $\arctan$ is contained in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Define the continuous functions $\tilde{\delta} : [0, \frac{\pi}{2}] \times [\frac{\pi}{2}, \pi] \rightarrow \mathbb{R}$, and $\delta : [0, r] \rightarrow \mathbb{R}$ by

$$\tilde{\delta}(\alpha, \beta) = \begin{cases} \sqrt{\lambda_1 \lambda_2} \left(\pi - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan(\pi - \beta)\right) - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan \alpha\right) \right) - (\beta - \alpha), & 0 \leq \alpha < \frac{\pi}{2} < \beta \leq \pi, \\ \sqrt{\lambda_1 \lambda_2} \left(\pi - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan(\pi - \beta)\right) - \frac{\pi}{2} \right) - (\beta - \frac{\pi}{2}), & \alpha = \frac{\pi}{2}, \frac{\pi}{2} < \beta \leq \pi, \\ \sqrt{\lambda_1 \lambda_2} \left(\pi - \frac{\pi}{2} - \arctan\left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan \alpha\right) \right) - (\frac{\pi}{2} - \alpha), & 0 \leq \alpha < \frac{\pi}{2}, \beta = \frac{\pi}{2}, \end{cases}$$

and

$$\delta(s) = \tilde{\delta}(\alpha(s), \beta(s)).$$

Using the constant C_4 in (2.17), an upper bound of the integral (2.16) is obtained:

$$\begin{aligned}
& \left| \int_s^{\sqrt{2}s\sqrt{1+\cos\theta^i}} \frac{1}{\rho_s} \delta(\rho_s) d\rho_s \right| + C_1 \epsilon \int_s^{\sqrt{2}s\sqrt{1+\cos\theta^i}} \frac{1}{\rho_s} d\rho_s \\
&\leq (C_4 + C_1 \epsilon) \log \left(\sqrt{2} \sqrt{1 + \cos \theta^i} \right).
\end{aligned} \tag{2.18}$$

Finally, let us consider the integral (2.15). If $\lambda_1 \lambda_2 > 1$, by (2.3),

$$\arctan \left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan(\pi - \beta) \right) < \pi - \beta \text{ and } \arctan \left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan \alpha \right) < \alpha,$$

resulting in

$$\delta(s) \geq (\lambda_1 \lambda_2 - 1)(\beta(s) - \alpha(s)) > 0.$$

On the other hand if $\lambda_1 \lambda_2 < 1$, again by (2.3),

$$\arctan \left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan(\pi - \beta) \right) > \pi - \beta \text{ and } \arctan \left(\sqrt{\frac{\lambda_1}{\lambda_2}} \tan \alpha \right) > \alpha,$$

which leads to

$$\delta(s) \leq (\lambda_1 \lambda_2 - 1)(\beta(s) - \alpha(s)) < 0.$$

Therefore, as s varies, $\delta(s)$ does not change its sign for fixed λ_1, λ_2 . In E_s , it is obvious that $0 \leq \alpha(s) \leq \frac{\pi}{2} - \frac{\theta^i}{2}$ and $\frac{\pi}{2} + \frac{\theta^i}{2} \leq \beta(s) \leq \pi$, a lower bound of $|\delta(s)|$ being obtained:

$$|\delta(s)| \geq |\lambda_1 \lambda_2 - 1| \theta^i > 0.$$

The choice of $\epsilon = \frac{|\lambda_1 \lambda_2 - 1|}{2C_1}$ gives a lower bound of the integral (2.15) over E_s :

$$\begin{aligned} & \left| \int_{E_s} (B_1 - I) \nabla \tilde{u}^2(y - \xi) \cdot \nabla \tilde{u}^3(y - \xi) dy \right| - C_1 \epsilon \int_{E_s} \frac{dy}{|y - \xi|^2} \\ & \geq \int_{\sqrt{2}s\sqrt{1+\cos\theta^i}}^{M_s} \frac{1}{\rho_s} [|\lambda_1 \lambda_2 - 1| - C_1 \epsilon] d\rho_s \\ & \geq \frac{|\lambda_1 \lambda_2 - 1|}{2} \int_{\sqrt{2}s\sqrt{1+\cos\theta^i}}^{M_s} \frac{1}{\rho_s} d\rho_s, \end{aligned} \quad (2.19)$$

Summing up (2.13), (2.14), (2.18) and (2.19), we get

$$\begin{aligned} |I_1 + I_2 + I_3 + I_4| & \geq \frac{|\lambda_1 \lambda_2 - 1|}{2} \int_{\sqrt{2}s\sqrt{1+\cos\theta^i}}^{M_s} \frac{1}{\rho} d\rho - \left(C_4 + \frac{|\lambda_1 \lambda_2 - 1|}{2} \right) \log(\sqrt{2(1+\cos\theta^i)}) \\ & \quad - \left(C_3 + \frac{|\lambda_1 \lambda_2 - 1|}{2} \right) \pi - \frac{2C_2}{|\lambda_1 \lambda_2 - 1|} - C_0. \end{aligned} \quad (2.20)$$

All terms in the right side of (2.20) are constant except the first one that tends to $\frac{|\lambda_1 \lambda_2 - 1|}{2} \int_0^r \frac{1}{\rho} d\rho$ to blow up. Hence $I_1 + I_2 + I_3 + I_4$ becomes unbounded as ξ_s approaches to $\mathbf{O}$, which is a contradiction originated from the assumption that $D_1 \neq D_2$. This proves that $D_1 = D_2$.

2.3 The Proof of Theorem 2.1: $B_1 = B_2$

It remains to prove $B_1 = B_2$. Let $D = D_1 = D_2$ and take $\hat{f}^\eta(\xi) = \eta \cdot \xi$, for a fixed vector $\eta \in \mathbb{R}^2$. Then by the uniqueness of the Dirichlet problem, $\eta \cdot \xi$ is the unique solution of

$$\begin{aligned} \nabla \cdot (B_1 \nabla \hat{v}) &= 0 & \text{in } D, \\ \hat{v} &= \hat{f}^\eta & \text{on } \partial D. \end{aligned}$$

Next take a harmonic function v in $\Omega \setminus \overline{D}$ such that

$$\begin{aligned} v^\eta &:= \hat{v}^\eta = \hat{f}^\eta & \text{on } \partial D, \\ \frac{\partial v^\eta}{\partial \nu} &:= \nu \cdot B_1 \nabla \hat{v}^\eta & \text{on } \partial D. \end{aligned}$$

Let f^η be the trace on $\partial\Omega$ of $v^\eta \in H^1(\Omega \setminus \overline{D})$ and u_j^η be the solution of (1.1) replacing f by f^η for $j = 1, 2$, respectively. By uniqueness, $u_1^\eta(\xi) = v^\eta(\xi)\chi_{\Omega \setminus \overline{D}}(\xi) + \eta \cdot \xi\chi_D(\xi)$. Suppose that the Dirichlet-to-Neumann maps Λ_γ are identical corresponding to $\gamma = I + (B_1 - I)\chi_D$ and $\gamma = I + (B_2 - I)\chi_D$. Then $u_1^\eta = u_2^\eta$ in $\Omega \setminus \overline{D}$ by unique continuation. Thus the transmission condition implies that u_2^η satisfies

$$\begin{aligned} \nabla \cdot (B_2 \nabla u_2^\eta) &= 0 & \text{in } D, \\ u_2^\eta &= \hat{f}^\eta & \text{on } \partial D, \end{aligned}$$

and hence $u_2^\eta(\xi) = \eta \cdot \xi$ in D . Moreover,

$$\nu(B_1 - B_2)\eta = \frac{\partial u_1^\eta}{\partial \nu^+} - \frac{\partial u_2^\eta}{\partial \nu^+} = 0$$

on ∂D for all $\eta \in \mathbb{R}^2$. Consequently, get $B_1 = B_2$. Therefore we prove Theorem 2.1.

3 Anisotropic Inverse Scattering Problem

In this section we apply the analysis carried out for the inverse conductivity problem (1.1) to the inverse scattering problem (1.2). As the argument of the proof is identical to that in Section 2, it will be briefly sketched.

3.1 Uniqueness Result

Analogously to the conductivity case, we consider the case

$$\gamma = A + (B - A)\chi_D \quad \text{and} \quad q = 1 + (\rho - 1)\chi_D,$$

where A is a known symmetric positive-definite matrix, B a positive-definite matrix, and D a Lipschitz domain compactly embedded in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus D$ is connected. Moreover, we assume that the matrices A and B are constant.

The uniqueness of our *inverse scattering problem* is stated in the following theorem.

Theorem 3.1. *Let $\gamma_j = A + (B_j - A)\chi_{D_j}$, $q_j = 1 + (\rho_j - 1)\chi_{D_j}$ and u_j be the solution of (1.2) for $\gamma = \gamma_j$ and $q = q_j$ for $j = 1, 2$. Assume $\det(B_j) \neq \det(A)$, $j = 1, 2$. Then*

$$u_\infty^1(\cdot; d) = u_\infty^2(\cdot; d) \text{ for all direction } d \in S^1 \quad (3.21)$$

implies

$$D_1 = D_2, B_1 = B_2 \text{ and } \rho_1 = \rho_2.$$

3.2 Proof of Theorem 3.1: a sketch

First of all, inverse conductivity and scattering problems are closely related with the following approximation property Lemmale:consca, which can be found in [18].

Lemma 3.2. *Let Ω be an open set where $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected, $\overline{D} \in \Omega$ be a Lipschitz domain such that $\Omega \setminus \overline{D}$ is connected and v be a solution to the equation*

$$\nabla \cdot (\gamma \nabla v) + k^2 q v = 0 \quad \text{in } \Omega.$$

Then v is the $H^1(D)$ -limit of linear combination of solutions $u(\cdot : d)$, $d \in S^1$ satisfying (1.2).

With the aid of this lemma we can apply the analysis carried out for our inverse conductivity problem in the previous section to analyze our inverse scattering problem.

We then follow the arguments in Section 2 to prove Theorem 3.1. The same changes of variables convert A, B_1, B_2 into I , a diagonal matrix, and a symmetric positive-definite matrix, respectively. Let D_3, D_4, D_5 and p be identical to those appeared in the proof of Theorem 2.1. Analogously to Lemma 2.2, the following lemma holds: (See also [18])

Lemma 3.3. *If (3.21) holds, then*

$$\int_{D_1} (-(B_1 - I) \nabla u^2 \cdot \nabla u^3 + k^2 q_1 u^3 u^2) \, dx = \int_{D_2} (-(B_2 - I) \nabla u^2 \cdot \nabla u^3 + k^2 q_2 u^3 u^2) \, dx,$$

for all $u_j, j = 2, 3$, such that

$$\nabla \cdot (\gamma_j \nabla u^j) + k^2 q_j u^j = 0 \quad j = 2, 3,$$

in some neighborhood of D_4 .

Similarly to Lemma 2.3, we also have

Lemma 3.4. *Suppose that $u^j, j = 2, 3$ satisfy*

$$u^j \in L^1_{loc}(\Omega) \cap H^1_{loc}(\Omega \setminus \{a\}) \quad \text{and} \quad \nabla \cdot (\gamma'_j \nabla u^j) = 0 \quad \text{in } D_4.$$

Then there exist $\{u^{j,x}\}_{x \in B_{r/2}}$ such that

$$\sup_{x \in B(r/2)} \|u^{j,x}(\cdot) - u^j(\cdot - x)\|_{H^1(D_4)} < \infty; \quad u^{j,x}|_{\partial\Omega} = 0$$

and

$$\nabla \cdot \gamma_j \nabla u^{j,x} + k^2 q_j u^{j,x} = 0 \quad \text{in } D_{4,x} = \{y \mid d(y, \overline{D}_4) < d(x, \overline{D}_4)/2\}.$$

Proof. The proof is identical to that of Lemma 2.3 except (2.7) and (2.9). Take $\phi_m^{j,x}, z_m^{j,x}$ in the same manner as in Lemma 2.3. Let $w_m^{j,x} \in H^1(\Omega)$ be such that

$$\begin{aligned} \nabla \cdot (\gamma_j \nabla w_m^{j,x}) + k^2 q w_m^{j,x} &= \nabla \cdot (\gamma'_j \nabla u_m^{j,x}) \quad \text{in } \Omega, \\ w_m^{j,x} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and $v_m^{j,x} = w_m^{j,x} - \phi_m^{j,x} - z_m^{j,x}$. Then $v_m^{j,x}$ satisfies

$$\begin{aligned} \nabla \cdot (\gamma_j \nabla v_m^{j,x}) + k^2 q v_m^{j,x} &= \nabla \cdot ((B'_j - B_j \chi_{D_j}) \nabla (\phi_m^{j,x} + z_m^{j,x})) \quad \text{in } \Omega, \\ v_m^{j,x} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The rest of the proof follows by the same argument as in Lemma 2.3. □

Choose u_j in the same fashion as in the proof of Theorem 2.1, and follow the arguments of Section 2. Then the uniqueness $D_1 = D_2$ and $B_1 = B_2$ is obtained. The uniqueness of ρ follows from the argument given in [5, 18, 30].

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