FINITE SPEED OF PROPAGATION AND ASYMPTOTIC RATES FOR SOME
NONLINEAR HIGHER ORDER PARABOLIC EQUATIONS WITH ABSORPTION

BY

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ABSTRACT

The "energy solutions" of the equation

$$(|u|^{q-1} \operatorname{sgn} u)_t + (-\Delta)^m u + |u|^{r-1} \operatorname{sgn} u = f$$

have finite speed of propagation if $1 < q < 2$ or $1 < r < 2$. If $1 < r < 2$ ($\forall q > 1$) support $u(\cdot, t)$ is uniformly bounded for $t > 0$ (localization property) and if $q < 2 \leq r$ sharp upper bounds of the interface (or free boundary) are obtained. We use a weighted energy method, the weights being powers of the distance to a variable half-space. We also study decay rates as $t \rightarrow \infty$ and extinction in finite time for bounded and unbounded domains (with null Dirichlet boundary conditions). Our equation includes the porous media equation with absorption. Analogous results hold if $(-\Delta)^m$ is replaced by an appropriate quasilinear elliptic operator.

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1. Introduction

Given the real numbers $p, q, r > 1$ and the integers $m, n \geq 1$, we consider the initial-boundary value problem

$$\left. \begin{aligned} \frac{\partial}{\partial t} (|u|^{q-1} \operatorname{sgn} u) + Au + |u|^{r-1} \operatorname{sgn} u &= f \quad \text{on } \Omega \times (0, \infty) \equiv 0 \\ u &\text{ satisfies null Dirichlet boundary conditions on } \partial\Omega \times (0, \infty) \\ u &= u_0 \quad \text{on } \Omega \quad \text{for } t = 0 \\ Au &= (-1)^m \sum_{|\alpha|=m} D^\alpha (|D^\alpha u|^{p-1} \operatorname{sgn} D^\alpha u) \end{aligned} \right\} \quad (1.1)$$

where $u = u(x, t)$, $(x, t) \in \Omega \times (0, \infty)$ and the sum extends to all x -derivatives of order m . (For $p = 2$ $A = (-\Delta)^m$ and for $m = 1$ $A = -\Delta_p$). Ω will be an arbitrary open set of R^n , unless otherwise stated.

We prove that if $1 < q < p$ or $1 < r < p$ the energy solutions (Definition 2.1) of (1.1) have finite speed of propagation (Theorems 3.1 to 3.4). Although these theorems also give useful information for bounded domains, we shall present now simplified statements for Ω unbounded. Set

$$\zeta(T) = \sup\{|x| : x \in \operatorname{support} u(\cdot, T)\} \quad (1.2)$$

Assume that the "energy" $E(\infty)$ is finite (see (1.16) and Remark 2.1) and that $\operatorname{support} u_0$ and the projection on Ω of $\operatorname{support} f$ are bounded. Then $\zeta(T) < \infty$ for all $T > 0$ and as $T \rightarrow \infty$

$$\zeta(T) = O(1) \quad \text{if } 1 < r < p \quad (\forall q > 1) \quad (1.3)$$

$$\zeta(T) = O(\ln T) \quad \text{if } 1 < q < p = r \quad (1.4)$$

$$\zeta(T) = O(T^\delta) \quad \text{if } 1 < q < p < r \leq p + mpq/n \quad (1.5)$$

$$\zeta(T) = O(T^{\beta_0}) \quad \text{if } 1 < q < p \quad \text{and} \quad r > p + mpq/n \quad (1.6)$$

$$\delta = (r-p)/(mp(r-q)), \quad \beta_0 = (mp + n(p-q)/q)^{-1} \quad (1.7)$$

((1.3) is known in the literature as a localization property).

The proofs of Theorems 3.1 to 3.4 are found in Sections 4 and 5. We use the weighted energy method of [B1] combined with the interpolation inequalities of Appendixes I and II. (In [B1], we studied (1.1) without the absorption term $|u|^{r-1} \text{sgn } u$). All upper bounds obtained in [B1] extend to (1.1) and many of them are improved. For example, (1.6) holds for the equation without absorption if $1 < q < p$. No hypothesis on the sign of u_0 is made. Theorems 3.1 to 3.4 extend to some nonhomogeneous Dirichlet data, as sketched in [B1].

We take $f = 0$ in Sections 6 to 8. Sections 6 and 7 give decay rates as $t \rightarrow \infty$ of the L^q norm of $u(\cdot, t)$ and results on extinction in finite time. In particular, support u is bounded in x and t (i.e., in Q) if $r < p$ and $r < q$, provided that $f = 0$, support u_0 bounded and $1/q \geq 1/p - m/n$. (This inequality plays no role if $n \leq mp$). This follows from Theorems 3.2 and 7.2. The property of extinction in finite time is found in Theorems 6.1, 7.2 and 8.3

For the porous media equation with absorption (see (1.11) below), it is known that there exists a solution of the initial-boundary value problem such that $v \in L^\infty(0, \infty; L^1(\Omega))$, i.e. $u \in L^\infty(0, \infty; L^{q-1}(\Omega))$ in our notation. Then, if $f = 0$ (1.5)-(1.6) are improved by the following bounds (see Section 8)

$$\zeta(T) = O(T^\delta) \quad \text{if } 1 < q < p < r \leq p + mp(q-1)/n \quad (1.8)$$

$$\zeta(T) = O(T^{\beta_1}) \quad \text{if } 1 < q < p \text{ and } r \geq p + mp(q-1)/n \quad (1.9)$$

$$\beta_1 = (mp + n(p-q)/(q-1))^{-1}. \quad (1.10)$$

The porous media equation with absorption

The case $m = 1$, $p = 2$, $f = 0$ of the equation of (1.1) is related to the porous media equation with absorption

$$\frac{\partial v}{\partial t} - \Delta(|v|^M \operatorname{sgn} v) + |v|^N \operatorname{sgn} v = 0 \quad (1.11)$$

by the change $v = |u|^{q-1} \operatorname{sgn} u$, $q - 1 = 1/M$. Thus, $N/M = r - 1$ and $N = (r - 1)/(q - 1)$. For the commodity of the reader, we give the following relations:

$$\begin{aligned} 1 < q < 2 &\Leftrightarrow M > 1, \quad 1 < r < 2 \Leftrightarrow 0 < N < M \\ 1 < r < q &\Leftrightarrow 0 < N < 1, \quad 1 < q < 2 \leq r \Leftrightarrow 1 < M \leq N \end{aligned}$$

Related work

As far as we know, the above results are new for order ≥ 4 , i.e. for $m \geq 2$. The following references deal with equation (1.11). The properties of localization and finite speed of propagation were obtained for $n = 1$ by Kalashnikov [K1,K2], Martinson and Pavlov [M] and Gurtin and MacCamy [G2]. In dimension n by Díaz and Herrero [D1] and Bertsch, Nambu and Peletier [B6]. The bounds (1.4), (1.8) and (1.9) were obtained for $n = 1$ by Knerr [K7], Kersner [K4,K5], Herrero [H1] (see also Herrero and Vazquez [H2]) and Bertsch, Kersner and Peletier [B4]. These papers also prove that the bounds are optimal. Notice that our results (1.4), (1.5) and (1.6) allow $f \neq 0$. Detailed results on large time behaviour are found in [B6]; see also Kamin and Peletier [K3] and references therein. Neumann boundary conditions are studied by Alikakos and Rostamian [A1]. Results on extinction in finite time were obtained in [K1]; see also Payne [P, pp. 58-59] and Knops and Straughan [K8]. The method of [P] and [K8] is surveyed in the book of Straughan [S].

Generalizations to other second order equations are found in [K1, K7, K6, D2, B5, B7]. In these papers the powers of equation (1.11) are replaced by more general functions. [D2] includes also the case $p \neq 2$ and further generalizations.

In the case $r = q$ ($N = 1$ in (1.11)), the change of variables introduced by [M] or [G2] (see also [A3])

$$u(x, t) = w(x, \tau) e^{-t/(q-1)}, \quad \tau = (1 - e^{-(p-q)t/(q-1)})(q-1)/(p-q)$$

($\tau = t$ if $p = q$) transforms the equation with absorption on $\Omega \times (0, \infty)$ into the equation without absorption on $\Omega \times (0, T_0)$. This is true for $p, q > 1$ and $m, n \geq 1$. T_0 is finite if $q < p$ and infinite if $q \geq p$. Thus, the results of [B1] imply corresponding results for the case $r = q$. For example, take $q = r < p$. Then T_0 is finite and (1.1) has the property of localization of the support.

Notations

C : positive constant which may be different in different occurrences

β_0 and β_1 : interface exponents, see (1.7) and (1.10)

δ : interface exponent, see (1.7)

$S(T)$, $S_n(T)$: bounds of the supports of the data. See (3.6) and (3.2)

$\zeta(T)$, $\zeta_n(T)$: bounds of $\partial \text{supp } u(\cdot, T)$ (interface or free boundary).

See (1.2) and (3.1)

p' : conjugate exponent of p : $1/p + 1/p' = 1$

$$W = W_0^{m,p}(\Omega) \cap L^q(\Omega) \quad \text{if } 1 < q \leq p \text{ or } 1 < r \leq p \text{ or } \Omega \text{ bounded.} \quad (1.12)$$

See Section 7 (Remark 7.2) for the meaning of W if $q > p$, $r > p$ and Ω unbounded. Definition (1.12) always applies in Sections 3 to 6. We always consider real spaces.

$$x = (x_1, \dots, x_n), \quad z = x_n$$

$D^j u$: vector of all x -derivatives of order j of u ($j \geq 0$)

$$|D^j u| = \left(\sum_{|\alpha|=j} |D^\alpha u|^p \right)^{1/p} \quad (1.13)$$

$$\|D^j u\|_p = \left(\int_{\Omega} |D^j u|^p dx \right)^{1/p} \quad (1.14)$$

$$E(T_1, T_2) = \int_{T_1}^{T_2} \int_{\Omega} |D^m u|^p dx dt = \int_{T_1}^{T_2} \|D^m u(t)\|_p^p dt \quad (1.15)$$

$$E(T) = E(0, T) \quad (1.16)$$

2. Concept of Solution

We assume always that

$$u_0 \in L^q(\Omega), \quad f \in L^{p'}(0, T; W') + L^{r'}(\Omega \times (0, T)) \quad \text{for all finite } T > 0 \quad (2.1)$$

Definition 2.1 Given (2.1), we say that u is an energy solution of (1.1) if

$$u \in C([0, \infty); L^q(\Omega)) \cap L^p(0, T; W) \cap L^r(\Omega \times (0, T)) \quad \text{for all finite } T > 0, \quad (2.2)$$

u satisfies the equation of (1.1) in the sense of distributions of $\mathcal{D}'(Q)$ and $u(0) = u_0$. (See (1.12) for the meaning of W ; W' is the dual of W).

In [B2] we prove that there exists at least one solution satisfying Definition 2.1. (Related existence results are found in Grange and Mignot [G1] and Alt and Luckhaus [A2]). Furthermore, the formula of integration by parts

$$\left. \begin{aligned} & \frac{1}{q'} \int_{\Omega} |u(x,T)|^q dx + \int_0^T \int_{\Omega} (|D^m u|^p + |u|^r) dx dt = \\ & = \frac{1}{q'} \int_{\Omega} |u_0|^q dx + \int_0^T (f(t), u(t)) dt \quad \text{for all finite } T \geq 0 \end{aligned} \right\} (2.3)$$

holds for the energy solutions (see [B2]). In (2.3), $(\cdot, \cdot)$ stands for the duality between $W' + L^{r'}(\Omega)$ and $W \cap L^r(\Omega)$.

Remark 2.1 In order that $E(\infty)$, defined by (1.15)-(1.16), be finite, it is enough to assume the following additional hypothesis: f has the form

$$f = f_1 + f_2 + \sum_{|\alpha|=m} D^\alpha f_\alpha, \quad f_1 \in L^1(0, \infty; L^{q'}(\Omega)), \quad f_2 \in L^{r'}(Q), \quad f_\alpha \in L^{p'}(Q)$$

This follows from (2.3). At the same time, $E(\infty)$ is estimated in terms of

$$\|u_0\|_q; \int_0^\infty \|f_1(t)\|_{q'} dt, \int_0^\infty \|f_2(t)\|_{r'} dt, \sum_{|\alpha|=m} \int_0^\infty \|f_\alpha(t)\|_{p'}^{p'} dt$$

In the particular case $f = 0$, we obtain simply

$$q' E(\infty) \leq \|u_0\|_q^q$$

Remark 2.2 (Dimensional analysis). The differential equation of (1.1) is invariant on $R^n \times (0, \infty)$ under the one-parameter group defined by

$$\hat{u}(x, t) = Ku(x/L, t/M), \quad K = L^{mp/(p-r)}, \quad M = L^{1/\delta}, \quad L > 0,$$

where δ is given by (1.7). This gives the following operational rule for dimensional analysis: we associated to x, u, t the dimensionality exponents $1, mp/(p-r)$ and $1/\delta$, respectively. Both sides of the estimates of Theorems 3.1 to 3.4, 6.1, 6.2 and 7.2 have the same dimensionality exponent, as expected. (The case $r = p$ requires a small modification: take exponents $0, 1$ and $q-p$, respectively.)

3. Theorems on finite speed of propagation and localization

We recall that $x = (x_1, \dots, x_n)$ and $z = x_n$. We set the following definitions and hypotheses:

$$\zeta_n(T) = \sup\{z: x \in \text{support } u(\cdot, T)\} \quad (3.1)$$

$$\left. \begin{aligned} S_{nf}(T) &= \sup\{z: (x, t) \in \text{support } f, 0 \leq t \leq T\} \\ S_n(T) &= \sup\{S_{nf}(T), \zeta_n(0)\} \end{aligned} \right\} \quad (3.2)$$

$$\left. \begin{aligned} p, q, r &\in \mathbb{R}, p, q, r > 1; m, n \in \mathbb{N}, m, n \geq 1 \\ u &\text{ is an energy solution of (1.1) (Definition 2.1)} \end{aligned} \right\} \quad (3.3)$$

$$S_n(T) \text{ is finite} \quad (3.4)$$

In the next theorems the constants $C, \hat{C}$ depend only on the indicated parameters. Other results are found in Remarks 5.2-5.3.

Theorem 3.1 Under the hypotheses (3.3), (3.4)

$$\zeta_n(T) - S_n(T) \leq C_{mpqn} T^{\beta_0} E(T)^{\lambda_0} \quad \text{if } q < p \quad (\forall r)$$

where $\lambda_0 = (p - q)\beta_0/q$, β_0 is given by (1.7) and $E(T)$ by (1.15)-(1.16).

Theorem 3.2 (Localization). Under the hypotheses (3.3), (3.4)

$$\zeta_n(T) - S_n(T) \leq C_{mpqrn} E(T)^\mu \quad \text{if } r < p \quad (\forall q)$$

$$\mu = (p - r)/(mpq + np - nr) \quad (3.5)$$

Theorem 3.3 Under the hypotheses (3.3), (3.4)

$$\zeta_n(T) - S_n(T) \leq C_{mpqn} \lambda_n (1 + \hat{C}_{mpqn} T^{\beta_0} E(T)^{\lambda_0}) \quad \text{if } q < p = r$$

where λ_0 and β_0 are as in Theorem 3.1.

We recall $\zeta(T)$ defined by (1.2) and set

$$\begin{aligned} S_f(T) &= \supremum\{|x|: (x,t) \in \text{support } f, 0 \leq t \leq T\} \\ S(T) &= \supremum\{S_f(T), \zeta(0)\} \end{aligned} \quad (3.6)$$

Theorem 3.4 Assume (3.3) and $S(T)$ finite. Then

$$\zeta(T) \leq C_{mpqrn}(S(T) + T^\delta) \quad \text{if } q < p < r$$

where δ is given by (1.7).

The proofs of these theorems are given in Sections 4 and 5. It is remarkable that the constant of Theorem 3.4 does not depend on u_0 and f . (This was known for equation (1.11) and $n = 1$: see [H1]). Theorem 3.1 also holds for problem (1.1) without absorption: see [B1].

Theorems 3.1 to 3.4 imply easily (1.3) to (1.6) of the introduction. In order to obtain (1.5) and (1.6), we only need to compare β_0 and δ by using the identity

$$\delta - \beta_0 = (nr - np - mpq)(p - q)\beta_0/(mpq(r - q))$$

If $f = 0$, $S_n(T) = \zeta_n(0)$ and $S(T) = \zeta(0)$. Since we can apply the theorems on $[T_1, T_2]$ instead of $[0, T]$, we obtain the following corollaries.

Corollary 3.1 Assume (3.3), $f = 0$ and $\zeta_n(0)$ finite. Then for all $T_2 > T_1 > 0$ we can replace in Theorems 3.1 to 3.3 $\zeta_n(T)$, $S_n(T)$, T and $E(T)$ by $\zeta_n(T_2)$, $\zeta_n(T_1)$, $T_2 - T_1$ and $E(T_1, T_2)$, respectively.

Corollary 3.2 Assume (3.3), $f = 0$ and $\zeta(0)$ finite. Then for all $T_2 > T_1 > 0$ we can replace in Theorem 3.4 $\zeta(T)$, $S(T)$ and T by $\zeta(T_2)$, $\zeta(T_1)$ and $T_2 - T_1$, respectively.

Remark 3.1 Corollary 3.1 implies (for $q < p$) that $\zeta_n(T)$ is Hölder continuous if it is nondecreasing. For equation (1.11) and dimension 1, [K7] proves that $\zeta_1(T)$ is nondecreasing and Lipschitz continuous if $M > 1$ and $N \geq 1$ ($q < 2$ and $q \leq r$ in our notation). Furthermore, if $N \geq M > 1$ (i.e. $q < 2 \leq r$), $\zeta_1(T)$ is C^1 and strictly increasing after a possibly positive time: [H2]. On the other hand, notice that from (2.3) with $f = 0$

$$q' E(T_1, T_2) \leq \|u(\cdot, T_2)\|_q^q - \|u(\cdot, T_1)\|_q^q \leq \|u_0\|_q^q.$$

4. Proofs of Theorems 3.1 to 3.3

In this section we always assume (3.3). We take a fixed T . By translation, we can suppose that $S_n(T) = 0$. We assume that the half-space $\{z > 0\}$ intersects Ω (if not, we have trivially $\zeta_n(T) \leq S_n(T)$). We define for $z_0 \geq 0$ and $s \geq 0$

$$E_s(z_0) = \int_0^T \int_{\substack{\Omega \\ z > z_0}} (z - z_0)^s |D^m u|^p dx dt \quad (4.1)$$

Notice that $E_s(z_0)$ is nonincreasing in z_0 , $E'_s = -sE_{s-1}$ and

$$E_0(0) \leq E(T) \quad (4.2)$$

Theorem 3.1 is proved exactly as in [B1] for the case without absorption, because the absorption term $|u|^{r-1} \text{sgn } u$ generates nonnegative terms in the left-hand side of the estimates. We also take from [B1, Lemma 4.3] the following:

Lemma 4.1 Assume $q \leq p$ or $r \leq p$. If $s \geq mp$, then for all $z_0 \geq 0$

$$\frac{1}{q^r} \sup_{0 \leq t \leq T} \int_{\substack{\Omega \\ z > z_0}} (z - z_0)^s |u(x, t)|^q dx \leq C_{mps} E_s(z_0) \quad (4.3)$$

$$\int_0^T \int_{\substack{\Omega \\ z > z_0}} (z - z_0)^s |u|^r dx dt \leq C_{mps} E_s(z_0) \quad (4.4)$$

$$E_s(z_0) \leq C_{mps} \int_0^T \int_{\substack{\Omega \\ z > z_0}} (z - z_0)^{s-p} |D^{m-1} u|^p dx dt < +\infty \quad (4.5)$$

(The hypothesis $q \leq p$ or $r \leq p$ is needed for unbounded domains in order to perform the proof of [B1, Appendix II]).

Remark 4.1 We are going to apply to our solution u the inequalities of Lemmas I.1 and II.1 (see Appendixes I and II). Since the zero extension operator maps $W_0^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$ and commutes with the derivatives of order $\leq m$, we can actually apply these inequalities on half-spaces of $\mathbb{R}^n$ and thus the constants are independent of z_0 and Ω .

Remark 4.2 We are going to obtain that $D^m u$ is zero a.e. for $z > z_1(T)$ and $0 < t < T$. But this implies (e.g. by (4.3)) that u is zero a.e. on the same set.

Proof of Theorem 3.2

Lemma 4.2 Assume $r < p$ ($\forall q$). If $s - p$ is integer and $s - p \geq mp$, then for all $z_0 \geq 0$

$$E_s(z_0) \leq C_{mpqrns} E_{s-p}(z_0)^{1+p/(s-p+1/\mu)}$$

Proof of Lemma 4.2. We apply Lemma I.1 (in Appendix I) to (4.5) with $k = s - p$, $j = m - 1$. Then we use (4.3) and (4.4) with s replaced by $s - p$ and the lemma follows.

Now Theorem 3.2 follows from Lemma 4.2, Lemma III.2 (in Appendix III) and (4.2). Explicit computations show that the exponent μ of Theorem 3.2 does not depend on s . Then, we choose a value of s ; for example, the smallest s such that $s - p$ is integer $\geq mp$. (From the point of view of localization property, the key point of the proof was that T does not appear as a separate factor in the inequality of Lemma I.1).

For the next proofs we shall need the following lemmas.

Lemma 4.3 Assume $q < p$ ($\forall r$). If $s - p$ is an integer and $s - p \geq mp$, then for all $z_0 > 0$

$$E_s(z_0) \leq C_{mpqns} T^{q(\theta-1)/(p-q)} E_{s-p}(z_0)^\theta$$

$$\theta - 1 = p/(1/\lambda_0 + s - p)$$

The proof is exactly as that of [B1, Lemma 4.4]. Lemma 4.3 and Lemma III.1 imply the following lemma. (Again, the exponents β_0 and $1 + \lambda_0$ do not depend on s ; β_0 and λ_0 are defined in Theorem 3.1).

Lemma 4.4 Assume $q < p$ ($\forall r$). Then, for all $z_0 > 0$

$$E_1(z_0) \leq C_{mpqn} T^{\beta_0} E_0(z_0)^{1+\lambda_0}$$

Proof of Theorem 3.3

Part of this proof is a particular case of the proof of Theorem 3.4 (in Section 5), but we consider more transparent to give it separately.

Lemma 4.5 Assume $p = r$ ($\forall q$). Then, for all $z_0 \geq 0$

$$E_1(z_0) \leq C_{mpn} E_0(z_0)$$

Proof of Lemma 4.5. We take the smallest s such that $s - p$ is integer $\geq mp$. We apply Lemma II.1 to (4.5) with $p = r$, $k = s - p$ and $j = m - 1$. Then, we use (4.4) with s replaced by $s - p$, obtaining

$$E_s(z_0) \leq C_{mpn} E_{s-p}(z_0)$$

This inequality and Lemma III.1 (case $\theta = 1$) imply Lemma 4.5.

Let $q < p = r$ as in Theorem 3.3. From Lemmas 4.4 and 4.5

$$T^{-\beta_0/(1+\lambda_0)} E_1(z_0)^{1/(1+\lambda_0)} + E_1(z_0) \leq C E_0(z_0)$$

for $z_0 \geq 0$. We obtain Theorem 3.3 by applying Lemma III.3 with $b = 1/(1 + \lambda_0)$, $a = T^{\beta_0/(1+\lambda_0)}$ and $A = 1$. Notice that $E'_1 = -E_0$.

5. Proof of Theorem 3.4

In this section we always assume (3.3), $S(T)$ finite and $q < p < r$. Lemmas 5.1 to 5.3 deal with $\zeta_n(T)$ (rather than with $\zeta(T)$) and we adopt the conventions of the former section. In particular, $S_n(T) = 0$. Remarks 4.1 and 4.2 will be used again. Since $S(T)$ is finite and $q < p$, by Theorem 3.1 there exists a finite R such that

$$\text{support } u(\cdot, t) \subset \overline{B_R} \text{ for all } t \in [0, T] \quad (5.1)$$

where B_R is a ball of radius R .

Lemma 5.1 Let s be the smallest real number such that $s - p$ is integer $\geq mp$. Then for all $z_0 \geq 0$

$$E_s(z_0) \leq C_{mprn} (T R^{n+s-p})^{(r-p)/mr} E_{s-p}(z_0)^{1-(r-p)/mr}$$

Proof. We apply Lemma II.1 to (4.5) with $k = s - p$, $j = m - 1$. Then we use (4.4) with s replaced by $s - p$. The lemma follows.

Lemma 5.2 Let s be as in Lemma 5.1. Then for all $z_0 \geq 0$

$$E_1(z_0) \leq C_{mprn} (T R^{n+s-p})^{1/(s-w)} E_0(z_0)^{1-1/(s-w)}$$

$$w = p - mpr/(r - p) < 0$$

This lemma follows from Lemma 5.1 and Lemma III.1

Lemma 5.3 Take s as in Lemma 5.1. Then

$$\zeta_n(T) - S_n(T) \leq C_{mpqrn} R^d T^e$$

$$1 - d = e/\delta, \quad e = p/((p - q)(1/\lambda_0 + s - w))$$

where δ , λ_0 and w are defined in Theorem 3.4, Theorem 3.1 and Lemma 5.2, respectively.

Remark 5.1 For the proof of Theorem 3.4, we do not need to know the explicit values of e and d . It is enough to know that $d > 0$, $e > 0$ and $1 - d = e/\delta$. The relation $1 - d = e/\delta$ agrees with dimensional analysis (see Remark 2.2).

Proof of Lemma 5.3. Recall that $E'_1 = -E_0$. From Lemmas 4.4 and 5.2, we obtain a differential inequality as that of Lemma III.4 with

$$\begin{aligned} b &= 1/(1 + \lambda_0) , & a &= T^{\beta_0/(1 + \lambda_0)} \\ B &= (s - w)/(s - w - 1), & A &= (T R^{n+s-p})^{1/(s-w-1)} \end{aligned}$$

Now Lemma 5.3 follows from Lemma III.4

Proof of Theorem 3.4 from Lemma 5.3. Since $S(T)$ is finite, the bound of Lemma 5.3 holds "along all coordinate semi-axes". Thus

$$\zeta(T) \leq \hat{C}_{mpqrn} (S(T) + R^d T^e) = h(T) \quad (5.2)$$

where the equality defines $h(T)$. Since h is an increasing function, we can take $R = h(T)$, see (5.1). On the other hand, by Young's inequality

$$R^d T^e \leq \varepsilon R + C_\varepsilon T^{e/(1-d)} = \varepsilon R + C_\varepsilon T^\delta \quad (5.3)$$

where we have used $1 - d = e/\delta$. Choosing ε small enough (depending on $\hat{C}_{mpqrn}$), Theorem 3.4 follows from $R = h(T)$, (5.2) and (5.3).

Remark 5.2 The following results can be proved by the same method. (I) has the interest of giving an estimate independent of the data. (II) and (III) may have interest for the local behaviour of $\zeta_n(T)$ (recall Remark 3.1).

- (I) Theorem 3.4 also holds if $r < p < q$
- (II) $\zeta_n(T) - S_n(T) \leq C T^\delta \ln(1 + \hat{C} T^{\beta_0 - \delta} E(T)^{\lambda_0})$ if $q < p \leq r$
- (III) $\zeta_n(T) - S_n(T) \leq C T^\delta \ln(1 + \hat{C} T^{-\delta} E(T)^\mu)$ if $r < p \leq q$

(β_0, λ_0 are defined in Th. 3.1, μ in Th. 3.2, δ in Th. 3.4).

Remark 5.3 If $q = p \leq r$ or $r = p \leq q$, $E_0(z_0)$ decays exponentially as $z_0 \rightarrow \infty$ (for fixed T).

6. Decay rates as $t \rightarrow \infty$ and extinction in finite time for bounded domains
($f = 0$)

If $f = 0$ and Ω bounded, formula (2.3) of integration by parts implies sharp decay rates of $\|u(\cdot, t)\|_q$ as $t \rightarrow \infty$. Let B_R be a ball of radius R . We set:

$$g(t) = \|u(\cdot, t)\|_q^q = \int_{\Omega} |u(x, t)|^q dx \quad (6.1)$$

$$\Omega \subset B_R \quad (6.2)$$

Theorem 6.1 Assume (3.3), $f = 0$ and (6.2). Then (for all $r > 1$)

(I) If $q < p$, for all $t > 0$

$$g(t) \leq (g(0))^{-(p-q)/q} + C_{mpqn} R^{-1/\beta_0} t^{-q/(p-q)} \leq \left(\frac{R^{1/\beta_0}}{C_{mpqn} t} \right)^{q/(p-q)}$$

(II) $g(t) \leq g(0) \exp(-C_{mpn} t/R^{mp})$, $\forall t > 0$, if $q = p$

(III) If $p < q$ and $1/q > 1/p - m/n$, $u(\cdot, t) = 0$ for all $t > T_0$
(extinction in finite time) and

$$T_0 \leq C_{mpqn} R^{1/\beta_0} g(0)^{(q-p)/q}$$

where we interpret $1/\beta_0 = 0$ if $1/q = 1/p - m/n$.

Remark 6.1 Additional results on extinction in finite time are found in Theorems 7.2 and 8.3; Theorem 8.3 has interest only for unbounded domains.

Theorem 6.2 Assume (3.3), $f = 0$ and (6.2). Then for all $t > 0$ (and all $p > 1$)

(I) If $q < r$

$$\begin{aligned} g(t) &\leq (g(0))^{-(r-q)/q} + C_{qrn} R^{-n(r-q)/q} t^{-q/(r-q)} \leq \\ &\leq R^n (C_{qrn} t)^{-q/(r-q)} \end{aligned}$$

(II) $g(t) \leq g(0)\exp(-q't)$ if $q = r$

Remark 6.2 In the case $p = 2$ and $m = 1$ ($\forall n$), the results of Bertsch, Nambu and Peletier [B6] imply that as $t \rightarrow \infty$ the exponent $q/(2 - q)$ of Theorem 6.1(I) is the best possible if $q < 2 \leq r$, while $q/(r - q)$ of Theorem 6.2 (I) is the best possible if $q < r \leq 2$. Furthermore, the exponential bound of Theorem 6.2 (II) is optimal if $r = q < 2$: [A3].

Proof of Theorem 6.1. From (2.3) and $f = 0$, g is absolutely continuous, nonincreasing and

$$-g'(t)/q' = \int_{\Omega} |D^m u(x, t)|^p dx + \int_{\Omega} |u(x, t)|^r dx \quad (6.3)$$

for almost all $t > 0$. From Poincaré-Friedrichs inequality, Sobolev imbedding (which requires $1/q \geq 1/p - m/n$) and/or Hölder inequality

$$\int_{\Omega} |u|^q dx \leq C R^{q/(p\beta_0)} \left(\int_{\Omega} |D^m u|^p dx \right)^{q/p} \quad (6.4)$$

(We extend $u(\cdot, t)$ by zero to B_R : recall Remark 4.1. The dependence on R is obtained taking $R = 1$ and scaling). If $1/q = 1/p - m/n$, R does not appear in (6.4). From (6.3) -(6.4)

$$g(t) \leq C R^{q/(p\beta_0)} |g'(t)|^{q/p} \quad (6.5)$$

Explicit integration of this differential inequality gives Theorem 6.1.

Proof of Theorem 6.2. By Hölder inequality

$$\int_{\Omega} |u|^q dx \leq C R^{n(r-q)/r} \left(\int_{\Omega} |u|^r dx \right)^{q/r}$$

This inequality and (6.3) imply

$$g(t) \leq C R^{n(r-q)/r} |g'(t)|^{q/r}$$

Integration of this inequality gives Theorem 6.2. For $q = r$ we check directly the value of the constants.

7. Decay rates as $t \rightarrow \infty$ and extinction in finite time for arbitrary domains
($f = 0$)

Theorem 7.1 gives decay rates as $t \rightarrow \infty$ of $\|u(\cdot, t)\|_q$, while Theorem 7.2 deals with the property of extinction in finite time. Theorem 7.1 is obtained by combining the properties of localization and finite speed of propagation with the results of Section 6. Now Ω is again an arbitrary open set of R^n .

Theorem 7.1 Assume (3.3), $f = 0$ and support u_0 bounded. Recall that $g(t)$ is defined by (6.1). Then as $t \rightarrow \infty$

$$g(t) = O(t^{-q/(r-q)}) \quad \text{if } q < r < p \quad (7.1)$$

$$g(t) = O((\ln t)^{1/\lambda_0} t^{-q/(p-q)}), \lambda_0 \text{ given by (7.5), if } q < p = r \quad (7.2)$$

$$g(t) = O(t^{-(mpq+np-nr)/(mp(r-q))}) \quad \text{if } q < p < r < p+mpq/n \quad (7.3)$$

$$g(t) = O((\ln t)^{-q/(p-q)}) \quad \text{if } q < p < r \text{ and } r \geq p + mpq/n \quad (7.4)$$

Remark 7.1 (a) If $r = q$ ($\forall p > 1$), then $g(t) \leq g(0) \exp(-q't)$, $\forall t \geq 0$, without assuming support u_0 bounded (see the proof of Theorem 6.2). (b) (7.2) to (7.4) have interest only for unbounded domains, since Theorem 6.1 (I) gives a better result for bounded domains. (c) (7.3)-(7.4) can be improved if $u \in L^\infty(0, \infty; L^{q-1}(\Omega))$, see Section 8.

Proof of Theorem 7.1 (7.1) follows from the localization property (1.3) and Theorem 6.2(I). (7.2) follows from (1.4) and Theorem 6.1 (I). In fact, we can take in Theorem 6.1 $R \leq C \ln t$ for t large enough. Recall the definition of λ_0 (Theorem 3.1):

$$\lambda_0 = (p - q)\beta_0/q = (n + mpq/(p - q))^{-1} \quad (7.5)$$

Analogously, (7.3) follows from (1.5) and Theorem 6.1 (I), choosing $R \leq Ct^\delta$ for t large enough and taking into account the identity

$$(1 - \delta/\beta_0)q/(p - q) = (mpq + np - nr)/(mp(r - q))$$

Finally, we prove (7.4). From (1.6), we can take in (6.5) $R \leq Ct^{\beta_0}$ for t large enough. Thus, we obtain

$$g(t) \leq C t^{q/p} |g'(t)|^{q/p}$$

Integration of this differential inequality gives (7.4).

Remark 7.2 In the next theorem we allow by first time $q > p$, $r > p$ and Ω unbounded. Then we define the space W as the closure of $C_0^\infty(\Omega)$ in the Banach space

$$\{u \in L^q(\Omega), D^m u \in L^p(\Omega)\}, \|u\| = \|u\|_q + \|D^m u\|_p$$

The existence theorem for energy solutions of [B2] also holds in this case, as well as formula (2.3). Thus, (6.3) holds if $f = 0$. As in Remark 4.1, for $u \in W$ the operator of zero extension commutes with the derivatives of order $\leq m$.

Theorem 7.2 Assume (3.3) and $f = 0$ (support u_0 bounded is not required). Let (1) $r < q$ and $1/q \geq 1/p - m/n$ or (2) $q < r$ and $1/q \leq 1/p - m/n$. Then $u(\cdot, t) = 0$ for all $t \geq T_0$ and

$$T_0 \leq C_{mpqrn} g(0)^{mp(q-r)/(mpq+np-nr)}$$

Proof of Theorem 7.2. Under the hypotheses (1) or (2) the following Gagliardo-Nirenberg inequality holds (see Nirenberg [N])

$$\|u\|_q \leq C_{mpqrn} \|D^m u\|_p^a \|u\|_r^{1-a}$$

where a is given by $1/q = a(1/p - m/n) + (1 - a)/r$. This inequality applies e.g. to R^n . We extend by zero $u(\cdot, t)$ (recall Remark 4.1). Using the inequality $A^\lambda B^\mu \leq C(A + B)^{\lambda + \mu}$ we obtain:

$$\int |u|^q dx \leq C \left(\int |D^m u|^p dx + \int |u|^r dx \right)^{q(a/p + (1-a)/r)}$$

and from (6.3)

$$g(t) \leq C |g'(t)|^{q(a/p + (1-a)/r)}$$

Integration of this inequality gives Theorem 7.2. Notice that $q(a/p + (1-a)/r) = 1 + amq/n > 1$.

8. Improvements on the bounds if $u \in L^\infty(0, \infty; L^{q-1}(\Omega))$, $f = 0$

This section has interest only for unbounded domains. For $m = 1$, $p = 2$ it is well known that L^1 semigroup theory gives solutions of (1.11) satisfying $v \in L^\infty(0, \infty; L^1(\Omega))$ if $v(0) \in L^1(\Omega)$. In our notation this reads

$$u \in L^\infty(0, \infty; L^{q-1}(\Omega)) \tag{8.1}$$

For $f = 0$, $m = 1$, $q = 2$, L^1 theory of semigroups also gives a solution of (1.1) satisfying (8.1) if $u_0 \in L^1(\Omega)$. For $m \geq 2$ we do not know general results on the validity of (8.1), but the method of [B1, Remark 11.1] shows that (8.1) holds for energy solutions at least in some special cases.

Lemma 8.1 Assume (3.3), $f = 0$ and support u_0 bounded. Let β_1 be defined by (1.10) and $g(t)$ by (6.1). Then if $q < p$ ($\forall r$) the following three relations are equivalent:

- (I) $u \in L^\infty(0, \infty; L^{q-1}(\Omega))$
- (II) $g(t) = O(t^{-\gamma_1})$ as $t \rightarrow \infty$, $\gamma_1 = n\beta_1/(q-1)$
- (III) $\zeta(t) = O(t^{\beta_1})$ as $t \rightarrow \infty$

This is proved exactly as [B1, Corollary 6.1]. Notice that $\beta_1 < \beta_0$ if $q < p$.

Theorem 8.1 Assume the hypotheses of Theorem 7.1 and (8.1). Then as $T \rightarrow \infty$ (1.8) and (1.9) hold.

Proof. From Theorem 3.4, $\zeta(T) = O(T^\delta)$ holds. And $\zeta(T) = O(T^{\beta_1})$ from Lemma 8.1. Thus, we only need to compare δ and β_1 by using the identity

$$\delta - \beta_1 = (nr - np - mp(q-1))(p-q)\beta_1 / (mp(q-1)(r-q))$$

Theorem 8.2 Under the hypotheses of Theorem 8.1 with γ_1 as in lemma 8.1

$$g(t) = O(t^{-(mpq+np-nr)/(mp(r-q))}) \quad \text{if } q < p < r \leq p + mp(q-1)/n$$

$$g(t) = O(t^{-\gamma_1}) \quad \text{if } q < p < r \text{ and } r \geq p + mp(q-1)/n$$

Proof. The first bound holds because of (7.3) and the second one because of Lemma 8.1 Comparing the exponents by using the identity

$$(mpq + np - nr)/(mp(r-q)) - \gamma_1 = -(nr - np - mp(q-1))q\beta_1/(mp(q-1)\beta_0(r-q))$$

the theorem follows.

Theorem 8.3 Assume (3.3), $f = 0$ and (8.1), (support u_0 bounded is not required). Let γ_1 be defined in Lemma 8.1 (II). Then for all $r > 1$

$$I. \quad g(t) = O(t^{-\gamma_1}) \quad \text{as } t \rightarrow \infty \quad \text{if } \beta_1 > 0$$

II. If $\beta_1 < 0$ and $1/q > 1/p - m/n$, there exists a finite T_0 such that $u(\cdot, t) = 0$ for all $t > T_0$.

The proof is as that of [B1, Theorem 10.2]. Notice that $\beta_1 < 0$ if and only if $n > mp$ and $q > (np - mp)/(n - mp)$. If Ω is bounded, Theorem 6.1 gives better results.

Remark 8.1 As noted in the introduction, for $m = 1$, $p = 2$, $n = 1$, $\Omega = \mathbb{R}$, the bounds (1.8)-(1.9) are optimal: see [B4] and references therein. Then, from [B1, Theorem 6.1] follows that the bounds of Theorem 8.2 are also optimal.

APPENDIX I. Weighted Interpolation Inequalities with three factors in the right-hand side

We set

$$H = \{x \in \mathbb{R}^n : x_n > 0\}, \quad z = x_n \tag{I.1}$$

$$p, q, r \in \mathbb{R}, \quad p, q, r \geq 1, \quad m, j, k \in \mathbb{N}, \quad m \geq 1, \quad 0 \leq j < m, \quad k \geq 0 \tag{I.2}$$

$$u \in L^p(0, T; W_{loc}^{m, p}(H)), \quad T > 0 \tag{I.3}$$

where $w \in W_{loc}^{m, p}(H)$ means $w \in W^{m, p}(G)$ for any bounded open set G such that $\bar{G} \subset H$. ($\bar{G}$ is the closure of G in $\mathbb{R}^n$). We recall the notation $|D^j u|$

defined by (1.13). The following lemma is used in the proof of the localization property.

Lemma I.1 Assume (I.1) to (I.3) and $r \leq p$ ($\forall q \geq 1$). Then

$$\begin{aligned} \int_0^T \int_H z^k |D^j u|^p dx dt &\leq \\ &\leq C \left(\int_0^T \int_H z^k |D^m u|^p dx dt \right)^b \left(\int_0^T \int_H z^k |u|^r dx dt \right)^{1-b} \times \\ &\quad \sup_{0 \leq t \leq T} \left(\int_H z^k |u(x,t)|^q dx \right)^{(p-r)(1-b)/q} \end{aligned}$$

$$1 - b = pq(m - j)/(mpq + (n + k)(p - r))$$

where the constant C depends only on n, m, j, p, q, r, k . In particular, C is independent of T .

We take from [B3, Lemma 8] the following:

Lemma I.2 Assume (I.1), (I.2), $u \in W_{1,0C}^{m,p}(H)$ and $r \leq p$. Then

$$\left(\int_H z^k |D^j u|^p dx \right)^{1/p} \leq C \left(\int_H z^k |D^m u|^p dx \right)^{a_1/p} \left(\int_H z^k |u|^r dx \right)^{(1-a_1)/r} \quad (I.4)$$

where a_1 satisfies $j/m \leq a_1 < 1$ and is defined by

$$1/p = j/(n + k) + a_1(1/p - m/(n + k)) + (1 - a_1)/r \quad (I.5)$$

and the constant C depends only on n, m, j, p, r, k .

In order to prove Lemma I.1 we shall need another lemma, which deals with the following inequality. (This inequality has independent interest for other applications to partial differential equations).

$$\left(\int_H z^k |D^j u|^p dx\right)^{1/p} \leq C \left(\int_H z^k |D^m u|^p dx\right)^{b_1/p} \left(\int_H z^k |u|^r dx\right)^{b_2/r} \left(\int_H z^k |u|^q dx\right)^{b_3/q} \quad (I.6)$$

$$C \text{ depends only on } n, m, j, p, q, r, k \quad (I.7)$$

$$\left. \begin{aligned} b_1 + b_2 + b_3 &= 1 \\ 1/p &= j/(n+k) + b_1(1/p - m/(n+k)) + b_2/r + b_3/q \end{aligned} \right\} \quad (I.8)$$

We define a_2 by

$$1/p = j/(n+k) + a_2(1/p - m/(n+k)) + (1 - a_2)/q \quad (I.9)$$

((I.9) is obtained from (I.5) replacing r by q).

Given b_1 , (I.8) determines b_2 and b_3 , except if $r = q$. For $r = q$, (I.8) determines $b_2 + b_3$ and b_1 if $r \leq p$.

Lemma I.3. Assume (I.1), (I.2) and $u \in W_{loc}^{m,p}(H)$. Let $q \leq p$ or $r \leq p$. Then (I.6) to (I.8) hold for all b_1 in the following range:

Case I. $a_1 \leq b_1 \leq a_2$ or $a_2 \leq b_1 \leq a_1$ if $r \leq p$ and $q \leq p$

Case II. $j/m \leq b_1 \leq a_1$ if $r \leq p \leq q$

Case III. $j/m \leq b_1 \leq a_2$ if $q \leq p \leq r$,

where a_1 is defined by (I.5) and a_2 by (I.9).

Proof of Lemma I.3. If (I.6) and (I.7) hold, (I.8) follows by dimensional analysis (i.e., by scaling). Thus, we only need to establish the range of b_1 .

Case I. It is enough to prove the extreme cases $b_1 = a_1$ and $b_1 = a_2$. (The intermediate cases follow by raising the extreme cases to θ and $1 - \theta$, $0 < \theta < 1$, and multiplying them. The constants of the intermediate cases can be easily bounded in terms of the constants of the extreme cases). For $b_1 = a_1$,

we obtain $b_3 = 0$ and $b_2 = 1 - a_1$ from (I.8) and (I.5). Thus, Lemma I.2 applies. For $b_1 = a_2$, from (I.8) and (I.9) $b_2 = 0$ and $b_3 = 1 - a_2$. Thus, Lemma I.2 applies (with r replaced by q).

Case II The extreme case $b_1 = a_1$ holds as before. The extreme case $b_1 = j/m$ follows from Lemma I.2 with $r = p$ (then a_1 of (I.5) is equal to j/m) and the inequality

$$\left(\int_H z^k |u|^p dx \right)^{1/p} \leq \left(\int_H z^k |u|^r dx \right)^{d/r} \left(\int_H z^k |u|^q dx \right)^{(1-d)/q} \quad (\text{I.10})$$

where d is defined by

$$1/p = d/r + (1 - d)/q \quad (\text{I.11})$$

(I.10) is obtained by writing

$$z^k |u|^p = z^{kpd/r} |u|^{dp} \cdot z^{kp(1-d)/q} |u|^{(1-d)p}$$

and applying Hölder inequality with exponents $r/(pd)$ and $q/(p(1 - d))$. Notice that

$$b_2 = d(1 - j/m), \quad b_3 = (1 - d)(1 - j/m) \quad \text{if } b_1 = j/m. \quad (\text{I.12})$$

Case III. It is obtained from Case II interchanging the roles of r and q .

Proof of Lemma I.1 from Lemma I.3 Lemma I.1 follows from (I.6) (raising to the power p , integrating in t and applying Hölder inequality with exponents $1/b_1$ and $r/(pb_2)$) if b_1 is so chosen that

$$b_1/p + b_2/r = 1/p \quad (\text{I.13})$$

In fact, the linear system in b_1, b_2, b_3 (I.8), (I.13) has a unique solution given by

$$b_1 = b, b_2 = (1 - b)r/p, b_3 = (p - r)(1 - b)/p$$

where b is defined in Lemma I.1. Thus, to prove Lemma I.1 we only need to show that the choice (I.13) is possible. We consider cases I and II of Lemma I.3 and study the sign of $1/p - b_1/p - b_2/r$ for the extreme values of b_1 :

$$1/p - b_1/p - b_2/r = \begin{cases} (1 - a_1)(r - p)/pr \leq 0 & \text{if } b_1 = a_1 & \text{(I.14)} \\ (1 - a_2)/p > 0 & \text{if } b_1 = a_2 & \text{(I.15)} \\ (1 - j/m)(1 - d)/q \geq 0 & \text{if } b_1 = j/m & \text{(I.16)} \end{cases}$$

where (I.16) follows from (I.11), (I.12). Recall that $b_2 = 1 - a_1$ for $b_1 = a_1$, while $b_2 = 0$ for $b_1 = a_2$. Now, (I.14) and (I.15) show that the choice (I.13) is possible in case I. Furthermore, (I.14) and (I.16) show it in case II. Q.E.D.

APPENDIX II. An Interpolation inequality for $p \leq r$

Lemma II.1 Assume (I.1) to (I.3) and $p \leq r$ (q does not appear in this lemma). Assume, in addition, that

$$\text{support } u \subset \overline{B_R} \times [0, T] \quad \text{(II.1)}$$

where B_R is a ball of radius R . Then

$$\begin{aligned} & \int_0^T \int_H z^k |D^j u|^p dx dt \leq \\ & \leq C (T R^{n+k})^{(1-p/r)(1-j/m)} \left(\int_0^T \int_H z^k |D^m u|^p dx dt \right)^{j/m} \left(\int_0^T \int_H z^k |u|^r dx dt \right)^{(1-j/m)p/r} \end{aligned}$$

where the constant C depends only on n, m, j, p, r, k . If $p = r$, the hypothesis (II.1) can be dropped.

Proof. Consider (I.4) with r replaced by p (then $a_1 = j/m$, see (I.5)). We raise to the power p , integrate in t and apply Hölder inequality with exponents m/j and $m/(m-j)$. Then Lemma II.1 follows from

$$\int_0^T \int_H z^k |u|^p dx dt \leq C \left(\int_0^T \int_H z^k |u|^r dx dt \right)^{p/r} (T R^{n+k})^{1-p/r}$$

This inequality is obtained by writing

$$z^k |u|^p = z^{kp/r} |u|^p \cdot z^{k(1-p/r)},$$

applying Hölder inequality with exponents r/p , $r/(r-p)$ and using (II.1).

APPENDIX III Some ordinary and fractional differential inequalities

We set

$$E_s(z) = \int_z^\infty (x-z)^s g(x) dx, \quad g \in L^1(R_+), \quad g \geq 0 \text{ a.e on } R_+ \quad (\text{III.1})$$

$E_s/\Gamma(s+1)$ is the Weyl fractional integral of order $s+1$ of g . Since $g \geq 0$, E_s is nonincreasing on $\overline{R_+}$ if $s \geq 0$. Notice that $E'_s = -sE_{s-1}$ if $s \geq 1$. We consider the fractional differential inequality

$$E_s(z) \leq K E_{s-p}(z)^\theta \quad \text{for all } z \geq 0 \quad (\text{III.2})$$

$$K, \theta, s, p \in \mathbb{R}; \quad K > 0, \quad \theta > 0, \quad s \geq 1, \quad 0 < p \leq s \quad (\text{III.3})$$

The following two lemmas are proved in [B1, Appendix III]. Notice that, since $E'_1 = -E_0$, Lemma III.1 implies Lemma III.2 by explicit integration of the differential inequality.

Lemma III.1 Assume (III.1) to (III.3) and $E_{s-p}(0)$ finite. Define w by (III.4). If $\theta > 1$, assume, in addition, that $s < w$. Then

$$\begin{aligned} E_1(z) &\leq K^{1/((\theta-1)(w-s))} E_0(z)^{1+1/(w-s)} \quad \text{for all } z \geq 0, \quad \text{if } \theta \neq 1 \\ E_1(z) &\leq K^{1/p} E_0(z) \quad \text{for all } z \geq 0, \quad \text{if } \theta = 1 \\ w &= \theta p / (\theta - 1) \end{aligned} \quad (\text{III.4})$$

(Notice that $w < 0$ if $0 < \theta < 1$).

Lemma III.2 Assume the hypotheses of Lemma III.1 and $\theta > 1$. Then the support of E_0 is a bounded interval $[0, z_1]$ and

$$z_1 \leq (w - s + 1) K^{1/((\theta-1)(w-s))} E_0(0)^{1/(w-s)}$$

The next two lemmas deal with first order ordinary differential inequalities.

Lemma III.3 Let the function $y \in C^1(\overline{R}_+)$ be nonnegative and nonincreasing. Assume that

$$y(z)^b/a + y(z)/A \leq K|y'(z)| \quad \text{for all } z \geq 0$$

$$0 < b < 1; \quad a, A, K > 0$$

Then the support of y is a bounded interval $[0, z_1]$ and

$$z_1 \leq \frac{KA}{1-b} \ln(1 + a^{1/b} A^{-1} |Ky'(0)|^{1/b-1})$$

Proof. Notice that $|y'(z)| = -y'(z)$. If $y(z) \neq 0$ we obtain from the differential inequality

$$z \leq K F(y(0)) - K F(y(z)) \leq K F(y(0))$$

$$F(t) = \frac{A}{1-b} \ln(1 + \frac{a}{A} t^{1-b})$$

Notice that $F(0) = 0$, $F(t) > 0$ for $t > 0$ and

$$F'(t) = (t^b/a + t/A)^{-1}$$

The lemma follows taking into account that (from the differential inequality for $z = 0$)

$$y(0)^b \leq Ka |y'(0)|$$

Remark III.1 For the purposes of the present paper, we are interested in obtaining $|y'(0)|$ (rather than $y(0)$) in the right-hand side of the bound. ($|y'|$ corresponds to E_0 and y to E_1).

Lemma III.4 Let the function $y: \overline{R}_+ \rightarrow \overline{R}_+$ be nonincreasing and absolutely continuous on bounded intervals. Assume that

$$y(z)^b/a + y(z)^B/A \leq K |y'(z)| \quad \text{a.e. on } R_+$$

$$0 < b < 1 < B; \quad a, A, K > 0$$

Then the support of y is a bounded interval $[0, z_1]$ and

$$z_1 \leq K K_M (1-b)^{-1} a^{1-M} A^M$$

$$M = (1-b)/(B-b), \quad K_M = \pi M \operatorname{cosec}(\pi M)$$

Remark III.2 The bound of z_1 is independent of $y(0)$ and $|y'(0)|$. This is related to the fact that the constant of Theorem 3.4 does not depend on u_0 and f . On the other hand, the dependence on a and A is also used in this paper.

Proof of Lemma III.4. Notice that $0 < M < 1$. If $y(z) \neq 0$ we obtain from the differential inequality

$$z \leq K G(y(0)) - K G(y(z)) \leq K G(\infty)$$

$$G(t) = \int_0^t (s^{b/a} + s^{B/A})^{-1} ds$$

$G(t)$ is finite for finite t because $b < 1$. $G(\infty)$ is finite because $B > 1$. Performing the change of variable $x = (a/A)^M s^{1-b}$, we obtain

$$G(\infty) = (1 - b)^{-1} a^{1-M} A^M \int_0^\infty (1 + x^{1/M})^{-1} dx$$

and it is well-known (see any book of integral tables) that

$$\int_0^\infty (1 + x^{1/M})^{-1} dx = \pi M \operatorname{cosec}(\pi M)$$

The lemma is proved.

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