# LOCAL BEHAVIOR OF SOLUTIONS OF SOME ELLIPTIC EQUATIONS

By

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52 **Paul Blanchard**, Complex Analytic Dynamics on the Riemann Sphere
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56 **Henry Simpson, Scott Spector**, On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics
57 **Craig Tracy**, Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations
58 **Tongren Ding**, Boundedness of Solutions of Duffing's Equation
59 **Abstracts** for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles
60 **Rafael Rob**, The Coase Theorem an Informational Perspective
61 **Joseph Jerome**, Approximate Newton Methods and Homotopy for Stationary Operator Equations
62 **Rafael Rob**, A Note on Competitive Bidding with Asymmetric Information
63 **Rafael Rob**, Equilibrium Price Distributions
64 **William Ruckle**, The Linearization Projection, Global Theories
65 **Russell Johnson, Kenneth Palmer, George R. Sell**, Ergodic Properties of Linear Dynamical Systems
66 **Stanley Reiter**, How a Network of Processors can Schedule its Work
67 **R.N. Goldman, D.C. Heath**, Linear Subdivision is Strictly a Polynomial Phenomenon
68 **R. Glachett, R. Johnson**, The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients
69 **Steve Williams**, Realization and Nash Implementation: Two Aspects of Mechanism Design
70 **Steve Williams**, Sufficient Conditions for Nash Implementation
71 **Nicholas Yannelis, William R. Zame**, Equilibria in Banach Lattices Without Ordered Preferences
72 **W. Harris, Y. Sibuya**, The Reciprocals of Solutions of Linear Ordinary Differential Equations
73 **Steve Pelikan**, A Dynamical Meaning of Fractal Dimension
74 **D. Heath, W. Sudderth**, Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach a Goal
75 **J.S. Jordan**, Information Flows Intrinsic to the Stability Economic Equilibrium
76 **J. Jerome**, An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition
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79 **G. Chichilnisky, G.M. Heal**, Existence of a Competitive Equilibrium in $L^p$ and Sobolev Spaces
80 **Thomas Seidman**, Time-dependent Solutions of a Nonlinear System in Semiconductory Theory, II: Boundedness and Periodicity
81 **Yakar Kannai**, Engaging in R&D and the Emergence of Expected Non-convex Technologies
82 **Herve Moulin**, Choice Functions over a Finite Set: A Summary
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84 **David Schmeidler**, Subjective Probability and Expected Utility Without Additivity
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88 **Steven R. Williams**, Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process
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96 **Anna Nagurney**, On Some Market Equilibrium Theory Paradoxes
97 **Anna Nagurney**, Sensitivity Analysis for Market Equilibrium
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100 **Filomena Pacella**, Central Configurations of the N-Body Problem via the Equivariant Morse Theory
101 **D. Carlson and A. Hoger**, The Derivative of a Tensor-valued Function of a Tensor
102 **Kenneth Mount**, Privacy Preserving Correspondence
103 **Millard Beatty**, Finite Amplitude Vibrations of a Neo-hookean Oscillator
104 **D. Emmons and N. Yannelis**, On Perfectly Competitive Economies: Loeb Economies
105 **E. Mascolo and R. Schianchi**, Existence Theorems in the Calculus of Variations
106 **D. Kinderlehrer**, Twinning of Crystals (II)
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108 **D. Abreu, D. Pearce, and E. Stacchetti**, Optimal Cartel Equilibria with Imperfect Monitoring
109 **R. Lauterbach**, Hopf Bifurcation from a Turning Point
110 **C. Kahn**, An Equilibrium Model of Quits under Optimal Contracting
111 **M. Kaneko and M. Wooders**, The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results
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113 **D. Carlson and A. Hoger**, On the Derivatives of the Principal Invariants of a Second-order Tensor
114 **Raymond Deneckere and Steve Pelikan**, Competitive Chaos
115 **Abstacts** for the Workshop on Homogenization and Effective Moduli of Materials and Media
116 **Abstracts** for the Workshop on the Classifying Spaces of Groups
117 **Umberto Mosco**, Pointwise Potential Estimates for Elliptic Obstacle Problems
118 **J. Rodrigues**, An Evolutionary Continuous Casting Problem of Stefan Type
119 **C. Mueller and F. Weissler**, Single Point Blow-up for a General Semilinear Heat Equation

# LOCAL BEHAVIOR OF SOLUTIONS OF SOME ELLIPTIC EQUATIONS

by

Patricio Aviles

## § 1. INTRODUCTION

Here we shall describe the local behavior of singular positive solutions of certain elliptic equations. Theorem A generalizes in an important manner one of our main results and indeed answers an open problem posed in [A]. There, corresponding upper and lower bounds for the singularity of the solution were given. To obtain Theorem A of this article considerably more arguments are needed. We point out that when n=3 the equation (1.1) below seems to be relevant in Yang-Mills-Higgs theory. See L. Sibner and R. Sibner [S-S].

Also we remark that equations of type (1.1) seem to be relevant to Astrophysics, a fact pointed out to the author by J. Serrin, (see [C], [F], [H]).

Our result reads as follows. Let $B = \{x \in R^n : |x| < 1 , n > 3\}$ . Then

THEOREM A . *Let* $u \in C^2(B|\{0\})$ *be a non negative solution of*

$$\Delta u + |x|^{\sigma} u^{(n+\sigma)/(n-2)} = 0 \text{ in } B|\{0\} \tag{1.1}$$

*where* $-2 < \sigma < 2$ . *Then* $u$ *has either a removable singularity at* $\{0\}$ *or*

$$\lim_{|x|\to 0} |x|^{n-2} (-\ell n |x|)^{(n-2)/(\sigma+2)} u(x) = \left[ \frac{(n-2)}{(\sigma+2)}^{(\sigma+2)} \right]^{(n-2)/(\sigma+2)} .$$

(The existence of singular solutions was shown in [A]).

Next, we would like to say a few words about the proof of Theorem A. In [G-S] the strategy of the proof of the corresponding statement for singular solution of $\Delta u + u^q = 0$ , $\frac{n}{n-2} < q < \frac{n+2}{n-2}$ was based on the fact that there is only one non-trivial solution of $\Delta u + u^q = 0$ in $R^n | \{0\}$ . However, there are no non-trivial solutions of $\Delta u + u^{n/(n-2)} = 0$ in $R^n | \{0\}$ . The strategy of our proof is to compute the Laplacian of $v(x) = |x|^{n-2} (-\ell n|x|)^{(n-2)/2} u$ . Then by means of the change of variable $t = -\ell n |x|$ we transform that equation in a time dependent equation. By using energy methods we prove then that

$$(1.2) \qquad \lim_{|x|\to 0} |x|^{n-2}(-\ell n\ |x|)^{(n-2)/2}\ u(x) \quad \text{exists.}$$

(Lemmas 3 to 17). Then, by completely different means we compute the limit (Lemmas 1,2,18). The method of using time dependent equations in the spirit used here has also been used by L. Simon [S]. For parabolic singularities related ideas have been applied by Y. Giga and R. Kohn [G-K].

(1.2) is a difficult point in our proof. The reason roughly being the following. After making the time transformation $t = -\ell n|x|$ we obtain the equation (2.8), that is

$$v_{tt} + (n-2)[1-t^{-1}]v_t + \Delta_\theta v = - t^{-1}\ v^{n/(n-2)} + \frac{1}{2}(n-2)^2\ t^{-1}\ v$$
$$- \frac{(n-2)n}{4}\ t^{-2}\ v.$$

In Lemma 4 we shall prove that

$$v_{tt}\ ,\ v_t \to 0 \ \text{ as } \ t \to \infty .$$

Since by Lemma 2 $v$ is bounded, we have from standard elliptic estimates that for each sequence $\{t_k\}$, $t_k \to \infty$ , there exists a subsequence $\{t_k'\}$ such that

$$v(t_k' ,\theta) \to v(\theta) \ \text{ as } t_k' \to \infty \ \text{ and } \ \Delta_\theta\ v = 0 .$$

So $v(\theta) = c(t_k) = \text{constant}$.

To reason that $c(t_k)$ is independent of the choice of the sequence $\{t_k\}$ is a serious point, because in principle all constants are admissible. This is contrary to what happens for singular solutions of $\Delta u + u^q = 0$ , $\frac{n}{(n-2)} < q < \frac{(n+2)}{(n-2)}$ in where only a discrete set of constants are admissible, see appendix.

In Lemma 2 we find the best possible upper bound for the set of admissible constants. Using many energy type arguments we have shown in Lemmas 4-17 that there is only one possible admissible constant. As we shall see the essential point is to control the angular derivatives $|\nabla_\theta v|$ and $|\nabla_\theta v_t|$. In Lemma 18, we prove that the upper bound of Lemma 2 is that constant.

We also mention that radial singular solutions of $\Delta u + u^q = 0$ have been studied by Fowler [F] and Hopf [H] .

Ni and Serrin [N-S] have informed us of work in preparation in which they study singular radial solutions for some general classes of equations.

Finally, in the appendix we shall give a new proof of Theorem 3.3 of Gidas-Spruck [G-S] in which there is an assertion which seems to require further explanation. The precise statement is given in the appendix. An analogous statement which is equally in need of clarification was made later in Lemma 2, (2.1.22) of [A]. This statement is an immediate consequence of the much more delicate result in Theorem A of this paper.

## §2. PROOF OF THEOREM A

We begin by considering the average

$$\overline{u}(r) = \frac{1}{w_{n-1}} \int_{S^{n-1}} u(r,\theta)\, d\omega \quad , \quad 0 < r < 1$$

where $w_{n-1}$ is the "volume" of the sphere $S^{n-1}$ . Taking average in (1.1) and assuming that $\sigma = 0$ we obtain

$$\overline{u}'' + \left(\frac{n-1}{r}\right) \overline{u}' + \overline{u}(r)^{n/(n-2)} \leq 0 , \qquad 0 < r < 1 .$$

We show the following.

LEMMA 1.

$$\limsup_{r \to 0} \; (-\ln r)^{(n-2)/2}\, r^{n-2}\, u(r) \leq \left(\frac{n-2}{2^{1/2}}\right)^{n-2} .$$

PROOF. Define $A(s) = \overline{u}(s^{-1/(n-2)})$ . Then

$$\ddot{A}(s) + \frac{1}{(n-2)^2} s^{-2(n-1)/(n-2)} A(s)^{n/(n-2)} < 0 ,$$

Let $B(r) = rA(r^{-1})$ with $r$ closes to zero. Then $B$ satisfies

$$\ddot{B}(r) + \frac{1}{(n-2)^2} \frac{1}{r^2} B(r)^{n/(n-2)} < 0 .$$

It follows from [A] p. 778 that $B$ is non-decreasing and $B(0) = 0$ . These facts imply that

$$\dot{B}(\rho) > \frac{1}{(n-2)^2} \frac{B(\rho)^{n/(n-2)}}{\rho} , \quad \rho \text{ near } 0 .$$

(see proof of Lemma 1 of [A]). By considering the functions $C(\rho) = B(\lambda\rho)$ $0 < \lambda < 1$ we may suppose that the above relation holds for $0 < \rho < 1$ . Integrating from $\rho$ to 1 we obtain

$$-\frac{(n-2)}{2} [- B(\rho)^{-2/(n-2)} + B(1)^{-2/(n-2)} ] > - \frac{1}{(n-2)^2} \ell n \ \rho .$$

Hence

$$\left( \frac{n-2}{2} \right) B(\rho)^{-2/(n-2)} > \frac{(n-2)}{2} B(1)^{-2/(n-2)} + \frac{1}{(n-2)^2} (-\ell n \ \rho) .$$

Since $B(1) > 0$ we get

$$\frac{2}{(n-2)} B(\rho)^{2/(n-2)} < (n-2)^2 (-\ell n \ \rho)^{-1} .$$

So

$$\left( \frac{1}{(n-2)} \right)^{(n-2)/2} B(\rho) < \frac{(n-2)^{(n-2)}}{2^{(n-2)/2}} (-\ell n \ \rho)^{-(n-2)/2} .$$

The definition of $B(\rho)$ yields

$$\left(\frac{1}{(n-2)}\right)^{(n-2)/2} (-\ell n\rho)^{(n-2)/2} \quad \rho\, \bar{u}(\rho^{1/(n-2)}) < \left(\frac{n-2}{2^{1/2}}\right)^{n-2} .$$

Setting $\rho = r^{(n-2)}$ we obtain Lemma 1.

Next we show, that if in (1.1) $\sigma=0$, then

LEMMA 2.

$$\limsup_{|x|\to 0} (-\ell n\ |x|)^{(n-2)/2}\ |x|^{(n-2)}\ u(x) < \left(\frac{n-2}{2^{1/2}}\right)^{n-2} .$$

PROOF. Consider

$$v(x) = (-\ell n\ |x|)^{(n-2)/2}\ u(x) .$$

We have

$$\Delta v = -u^{n/(n-2)} (-\ell n|x|)^{(n-2)/2} - (n-2) \frac{(-\ell n|x|)^{(n-4)/2}}{|x|^2} \nabla_x u\cdot x \tag{2.1}$$

$$+\ u \frac{(n-2)}{2|x|^2} (-\ell n|x|)^{(n-4)/2}[(2-n) + \frac{(n-4)}{2} (-\ell n|x|)^{-1}] .$$

We now consider

$$v_\lambda(x) = \lambda^{n-2}\ v(\lambda x) \quad , \quad 0 < \lambda < 1 .$$

Using Lemma 1 and the Harnack inequality, cf. [G-S] Theorem 3.1,* we get

$$u(x) < C \frac{(-\ell n|x|)^{-(n-2)/2}}{|x|^{n-2}} \tag{2.2}$$

where $C > 0$ is a constant independent of $u$. Hence

$$0 < v_\lambda(x) < \frac{C}{|x|^{n-2}} \quad \text{if} \quad 0 < |x| < \lambda^{-1} .$$

* See Appendix II for a simple proof.

We next prove that for each sequence $\lambda_i \to 0$ there exists a subsequence $\lambda_i'$ so that $v_{\lambda_i'}$ converges uniformly on compact subsets of $R^n | \{0\}$ to $w$, where $w$ is a harmonic function in $R^n | \{0\}$. We first show that if

$$F(x) = \Delta v_{\lambda_i}(x)$$

and if $K$ is a compact set in $R^n | \{0\}$, then

$$\sup_{x \in K} |F(x)| \to 0 \text{ as } \lambda_i \to 0 .$$

Indeed we notice that

$$|\nabla u(x)| < C \frac{(-\ln|x|)^{-(n-2)/2}}{|x|^{n-1}}, \quad 0 < |x| < \frac{1}{2} \tag{2.3}$$

where $C > 0$ is a constant independent of $u$. This follows by writing the equation for $u$ in the form

$$\Delta u(x) = - u(x)^{n/(n-2)} ,$$

by using the well-known gradient bound

$$|\nabla u(x)| < \frac{C}{|x|} \sup_{x \in B(x,|x|/2)} u(x) + C\,|x| \sup_{x \in B(x,|x|/2)} |f(x)|, \quad 0 < |x| < \frac{1}{2}$$

with $f(x) = -u(x)^{n/(n-2)}$, $B(x, |x|/2) = \{y \in R^n : |y-x| < \frac{|x|}{2}\}$, $C>0$ a constant independent of $u$, and by using the estimate (2.2).

Now, since

$$F(x) = - \lambda_i^n (u(\lambda_i x))^{n/(n-2)}(-\ln|\lambda_i x|)^{(n-2)/2} - (n-2)\lambda_i^{n+1} \frac{(-\ln|\lambda_i x|)^{(n-4)/2}}{|\lambda_i x|^2} \nabla_{\lambda_i x} u(\lambda_i x) \cdot x$$

$$+ \lambda_i^n\, u(\lambda_i x) \frac{(n-2)}{2\lambda_i^2 |x|^2} (-\ln|\lambda_i x|)^{(n-4)/2}((2-n) + (\frac{n-4}{2})(-\ln|\lambda_i x|)^{-1})$$

we obtain

$$|F(x)| < C(-\ln|\lambda_i x|)^{-1}/|x|^n$$

where $C>0$ is a constant. Hence $\sup_{x \in K} |F(x)| \to 0$ as $\lambda_i \to 0$.

Next, consider $K_i$, a sequence of compact sets, such that $\bigcup K_i = R^n | \{0\}$, $K_i \subset K_{i+1}$. Let $\lambda_j$ be sufficiently small so that $K_i \subset \{x: 0 < |x| < \lambda_j^{-1}\}$. Since $\{v^\lambda\}$, $\lambda < \lambda_j$ are uniformly bounded on $K_i$, standard elliptic estimates imply

$$|v^\lambda|_{C^{2,\alpha}(K_i)} < M(K_i) \quad , \quad \lambda < \lambda_j \quad , \quad 0 < \alpha < 1 ,$$

where $M(K_i) > 0$ is a constant. Hence $\{v^\lambda , Dv^\lambda , D^2 v^\lambda\}$ form equicontinuous families. By the Arzela-Ascoli theorem there exists a subsequence $\lambda_{k_j}$ such that $v_{\lambda_{k_j}} \to w$ in the $C^2$ topology of $K_i$. Clearly $\Delta w = 0$ on $K_i$. Let $K_\ell \supset K_i$. Repeating the above argument with $K_\ell$ and the sequence $v_{\lambda_{k_j}}$ we conclude that there exists a subsequence $\lambda_{k_j'}$ so that $v_{\lambda_{k_j'}} \to w'$ on $K_\ell$ and by analytic continuation $w' = w$ on $K_i$. We therefore conclude by using a standard diagonalization argument that there exists a subsequence which we call $\lambda_j'$ so that $v_{\lambda_j'} \to w$ in the $C^2$ topology of $R^n | \{0\}$ and moreover $w$ satisfies $\Delta w = 0$ in $R^n | \{0\}$. Since $0 < v_\lambda < \frac{C}{|x|^{n-2}}$, we conclude that

$$w = \frac{C(\lambda_j')}{|x|^{n-2}}$$

where $C(\lambda_j') > 0$ is a constant which depends on the sequence $\lambda_j'$. Therefore, given a sequence $\lambda_j \to 0$, there exists a subsequence $\lambda_j' \to 0$, a constant $C(\lambda_j') > 0$ such that

$$|\lambda_j' x|^{n-2} (-\ell n |\lambda_j' x|)^{(n-2)/2} \quad u(\lambda_j' x) \to C(\lambda_j')$$

as $\lambda_j' \to 0$, uniformly on compact subsets of $R^n | \{0\}$.

To this end, we consider a sequence $y_j$, $y_j \to 0$ so that

$$|y_j|^{n-2} (-\ell n |y_j|)^{(n-2)/2} u(y_j) \to \hat{C}$$

where $\limsup_{y\to 0} (-\ln|y|)^{(n-2)/2}|y|^{n-2}u(y) = \hat{C}$ .

Let $\lambda_j = |y_j|$ . By the above argument, there exists a subsequence $\lambda_j' = |y_j'|$ so that

$$(2.4) \qquad \sup_{|x|=1}\left| |y_j'|^{n-2}(-\ln|y_j'|)^{(n-2)/2}u(|y_j'|x) - C(\lambda_j')\right| \to 0$$

as $\lambda_j' \to 0$ .

Since

$$\left||y_j'|^{n-2}(-\ln|y_j'|)^{(n-2)/2}u(y_j') - C(\lambda_j')\right| < \sup_{|x|=1}\left||y_j'|^{n-2}(-\ln|y_j'|)^{(n-2)/2}u(|y_j'|x) - C(\lambda_j')\right)$$

we conclude that

$$\tilde{C} = C(|y_j'|) .$$

On the other hand from (2.4) we have

$$\frac{1}{w_{n-1}}\int_{S^{n-1}}|y_j'|^{n-2}(-\ln|y_j'|)^{(n-2)/2}u(|y_j'|x)\,d\omega \to C(|y_j'|) .$$

Hence

$$|y_j'|^{n-2}(-\ln|y_j'|)^{(n-2)/2}\overline{u}(|y_j'|x) \to C(|y_j'|) .$$

By Lemma 1,

$$\hat{C} = C(|y_j'|) < \left(\frac{n-2}{2^{1/2}}\right)^{n-2} ,$$

which proves the assertion of the lemma.

We shall now prove that if in (1.1) $\sigma = 0$ ,then

<u>LEMMA</u> 3.

$$\lim_{|x|\to 0} |x|^{n-2} (-\ell n\ |x|)^{(n-2)/2} u(x) \ \underline{\text{exists}} .$$

<u>Proof</u>. We compute the Laplacian of

$$v(x) = |x|^{n-2} ( - \ell n|x|)^{(n-2)/2} u(x) .$$

We show that $v$ satisfies

$$\Delta v - k(|x|)\nabla v\cdot x = - u\ |x|^{n-2}(-\ell n|x|)^{(n-2)/2} \tag{2.5}$$

$$\{-\frac{1}{2}(n-2)^2 \frac{(-\ \ell n|x|)^{-1}}{|x|^2} + (\frac{n-2}{4})n \frac{(-\ \ell n|x|)^{-2}}{|x|^2} + u^{2/(n-2)}\}$$

where $k(|x|) = \frac{(n-2)}{|x|^2}(2-(-\ \ell n|x|)^{-1})$ .

Let $f(|x|) = |x|^{n-2} (-\ell n|x|)^{(n-2)/2}$. Then

$$\nabla f(|x|) = \dot{f}(|x|)\frac{x}{|x|} , \quad \Delta f(|x|) = \ddot{f}(|x|) + \frac{(n-1)}{|x|}\dot{f}(|x|) ,$$

$$\dot{f}(|x|) = (n-2)\ |x|^{n-3}(-\ell n|x|)^{(n-2)/2} - \frac{(n-2)}{2} |x|^{n-3} (-\ell n|x|)^{(n-4)/2} ,$$

$$\ddot{f}(|x|) = (n-2)(n-3)\ |x|^{n-4} (-\ \ell n|x|)^{(n-2)/2} - \frac{(n-2)^2}{2} |x|^{n-4} (-\ell n|x|)^{(n-4)/2} -$$

$$(n-3)\frac{(n-2)}{2} |x|^{n-4} ( - \ell n|x|)^{(n-4)/2} + \frac{(n-2)(n-4)}{4} |x|^{n-4} (-\ell n\ |x|)^{(n-6)/2} ;$$

$$\Delta f = (n-2)|x|^{(n-4)} (-\ell n|x|)^{(n-2)/2} [2(n-2) - \tfrac{3}{2}(n-2)(-\ell n|x|)^{-1} + \tfrac{(n-4)}{4}(-\ell n|x|)^{-2}]$$

$$= |x|^{(n-4)}(-\ell n\ |x|)^{(n-2)/2}[2(n-2)^2 + \delta(|x|)] ,$$

where $\delta(|x|) = -\frac{3}{2}(n-2)^2 (-\ell n|x|)^{-1} + (n-2)\frac{(n-4)}{4}(-\ell n\ |x|)^{-2}$ .

Hence

$$\Delta v = -u^{n/(n-2)} |x|^{n-2} (-\ell n|x|)^{(n-2)/2} +$$

$$+ 2(n-2)|x|^{n-4} (-\ell n|x|)^{(n-2)/2} \nabla u\cdot x - (n-2)|x|^{n-4} (-\ell n|x|)^{(n-4)/2} \nabla u\cdot x$$

$$+ u(x)\ |x|^{n-4} (-\ell n\ |x|)^{(n-2)/2} (2(n-2)^2 + \delta(|x|)) .$$

$$= -u^{n/(n-2)} |x|^{n-2} (-\ell n\ |x|)^{(n-2)/2} + \frac{(n-2)}{|x|^2} f(|x|)\nabla u\cdot x \ (2-(-\ell n\ |x|)^{-1})$$

$$+ u(x)\ |x|^{n-4}(-\ell n\ |x|)^{(n-2)/2} (2(n-2)^2 + \delta(|x|) .$$

Let $k(|x|) = \frac{(n-2)}{|x|^2} (2-(-\ell n\ |x|)^{-1})$ . Since

$$k(|x|)\nabla v\cdot x = k(|x|)f(|x|)\nabla u\cdot x + k(|x|)u\ \nabla f(|x|)\cdot x , \text{ and}$$

$$k(|x|)\nabla f\cdot x = (n-2)k(|x|)|x|^{n-2}(-\ell n|x|)^{(n-2)/2} - \frac{(n-2)}{2} |x|^{n-2}k(|x|)(-\ell n|x|)^{(n-4)/2}$$

$$= 2(n-2)^2 |x|^{n-4} (-\ell n|x|)^{(n-2)/2} - 2(n-2)^2 |x|^{n-4} (-\ell n\ |x|)^{(n-4)/2}$$

$$+ \frac{(n-2)^2}{2} |x|^{(n-4)} (-\ell n\ |x|)^{(n-6)/2} ,$$

we have

$$\Delta v - k(|x|)\ \nabla v\cdot x = -u^{n/(n-2)} |x|^{n-2}(-\ell n\ |x|)^{(n-2)/2}$$

$$-2(n-2)^2 |x|^{(n-4)} (-\ell n|x|)^{(n-2)/2} u + 2(n-2)^2 |x|^{n-4} (-\ell n|x|)^{(n-4)/2}u$$

$$- \frac{(n-2)^2}{2} |x|^{n-4}(-\ell n\ |x|)^{(n-6)/2} u$$

$$+ 2(n-2)^2|x|^{n-4}(-\ell n|x|)^{(n-2)/2} u - \frac{3}{2}(n-2)^2 |x|^{(n-4)} (-\ell n|x|)^{(n-4)/2} u$$

$$+ (n-2)(\frac{n-4}{4}) \quad |x|^{(n-4)} \quad (-\ln|x|)^{(n-6)/2} \quad u \;;$$

Since this equation can be written as

$$\Delta v - k(|x|) \; \nabla v \cdot x = - u|x|^{n-2}(-\ln|x|)^{(n-2)/2} \; \{u^{2/(n-2)}$$

$$- \frac{1}{2}(n-2)^2 \; \frac{(-\ln|x|)^{-1}}{|x|^2} + \frac{(n-2)n}{4} \; \frac{(-\ln|x|)^{-2}}{|x|^2} \; \} \;,$$

we obtain (2.5).

In terms of $v$ this equation reads as follows.

$$\Delta v - k(r) \; \nabla v \cdot x = - \; |x|^{-2} \, (-\ln|x|)^{-1} \; v^{n/(n-2)} + \frac{1}{2} \, (n-2)^2(-\ln|x|)^{-1} \; |x|^{-2} v$$

(2.5)'

$$- (n-2) \, \frac{n}{4} \, v \; (-\ln|x|)^{-2} \; |x|^{-2} \;.$$

Next, we transform the above equation in a time dependent equation. Let

$$t = - \ln \, |x| = - \ln \, r$$

$$v_t = - v_r e^{-t} \;, \; v_{tt} = v_{rr} \; e^{-2t} + v_r \; e^{-t}$$

So

(2.6) $$v_{tt} - (n-2) \; v_t = e^{-2t} \; (v_{rr} + \frac{(n-1)}{r} \; v_r).$$

Also

$$\nabla v \cdot x = v_r r = -e^t v_t e^{-t} = - \; v_t \;,$$

hence

(2.7) $$-k(r) \; \nabla v \cdot x = (n-2)e^{2t} \; (2-t^{-1}) \; v_t \;.$$

On the other hand the right hand side of (2.5)', say A, reads as:

$$A = - \; e^{2t} t^{-1} \; v^{n/(n-2)} + \frac{1}{2} \, (n-2)^2 e^{2t} t^{-1} \; v - \frac{(n-2)n}{4} \, e^{2t} \; t^{-2} v \;.$$

So from (2.6), (2.7) we have that (2.5)' can be written as

$$e^{2t}(v_{tt} - (n-2)v_t) + (n-2)e^{2t}(2-t^{-1})v_t + e^{2t}\Delta_\theta v$$

$$= A$$

Therefore we obtain

$$v_{tt} + (n-2)[1-t^{-1}]v_t + \Delta_\theta v = -t^{-1}\, v^{n/(n-2)} + \tfrac{1}{2}(n-2)^2\, t^{-1}\, v$$

(2.8)

$$- (n-2)\tfrac{n}{4}\, t^{-2}\, v\ .$$

Lemma 3 then follows from Lemmas 4 and 5.

LEMMA 4. For each sequence $t_k \to \infty$, there exists a subsequence $\{t_k'\}$ such that

$$v(t_k', \theta) \to c(t_k) \quad \text{as} \quad t'_k \to \infty$$

where $c(t_k) > 0$ is a constant.

PROOF. The ideas of this lemma are very similar to the one in [G-S] p. 558-559. See also [S].

Multiplying (2.8) by $v_t$ we get

(2.8)'
$$\tfrac{1}{2}(v_t^2)_t + (n-2)(1-t^{-1})\, v_t^2 + \tfrac{1}{2}\ \Delta_\theta\, (v^2)_t$$

$$= -\frac{(n-2)}{2(n-1)}\, t^{-1}\, (v^{2(n-1)/(n-2)})_t + \frac{(n-2)^2}{4}\, t^{-1}(v^2)_t - \frac{(n-2)\ n}{8}\, t^{-2}\, (v^2)_t\ .$$

We now assert the following:

$|v|$, $|v_t|$, $|v_\theta|$, $|v_{tt}|$, $|v_{t\theta}|$, $|v_{ttt}|$ $|v_{tt\theta}|$ are uniformly bounded for $t > t_0$

where $t_0 > 0$ is a constant. It follows from the definition and (2.2) that $v$ is bounded. From (2.3) and the facts that $|v_t| < r\,|\nabla v|$ and $|\nabla_\theta v| < r|\nabla v|$ we get that $|v_t|$ and $|v_\theta|$ are uniformly bounded. We next differentiate the equation $\Delta u + u^{n/(n-2)} = 0$ to obtain $\Delta u_i + \frac{n}{(n-2)} u_i\, u^{2/(n-2)} = 0$. Hence for $0 < |x| < \frac{1}{2}$ and $f = -n/(n-2)\, u_i u^{2/(n-2)}$ we have

$$|\nabla u_i(x)| < C[\frac{1}{|x|} \sup_{x \in B(x,|x|/2)} |u_i| + |x| \sup_{x \in B(x,|x|/2)} |f|\,]$$

where $C>0$ is a constant indpendent of $u_i$. This implies that for $0 < |x| < \frac{1}{2}$,

$$|v_{ij}| < C\,\frac{(-\ln|x|)^{-(n-2)/2}}{|x|^n}\,, \quad \text{for all } i,j = 1,\ldots n$$

with $C > 0$ a constant independent of $u$. Hence from (2.6) we get that $|v_{tt}|$ is uniformly bounded. Since $|\nabla v_r|^2 = |v_{rr}|^2 + r^{-2}\,|\nabla_\theta v_r|^2$, we have $|\nabla_\theta v_r|^2 < r^2\,|\nabla v_r|^2$. This implies that $v_{\theta t}$ is uniformly bounded. The bound on $|v_{ttt}|$ is obtained as follows. We have $\Delta u_i + \frac{n}{(n-2)} u^{2/(n-2)}\, u_i = 0$. Hence if $h(x) = \lambda^{n-1} u_i(\lambda x)$, $0 < \lambda < 1$, then

$$\Delta h + \frac{n}{(n-2)} \lambda^2\, u^{2/(n-2)}(\lambda x)\, h = 0\,.$$

Let $c(x) = \frac{n}{(n-2)}\lambda^2\, u^{2/(n-2)}\,(\lambda x) = \frac{n}{n-2}\,[\lambda^{(n-2)} u(\lambda x)]^{2/(n-2)}$. For $\frac{1}{4} < |x| < \frac{1}{2}$, we get from (2.2) and from the fact that $f(t) = t^{2/(n-2)}$ is Hölder continuous that

$$|c(x)|_{C^{0,\alpha}(\frac{1}{4} < |x| < \frac{1}{2})} < M \qquad , \ \alpha = \frac{2}{n-2}$$

where $M > 0$ is a constant independent of $\lambda$, (if $n=3$ or $n=4$ we take the $C^1$ norm). Hence, standard elliptic estimates imply that

$$|D^2 h|_{C(\frac{1}{4} < |x| < \frac{1}{2})} < M_1$$

where $M_1 > 0$ is a constant. Setting $\lambda x = y$ we obtain that near the origin

$$|u_{ijk}(y)| < \frac{M_2}{|y|^{n+1}}$$

where $M_2 > 0$ is a constant. This estimate easily implies that $v_{ttt}$ is uniformly bounded if $t$ is sufficiently large. Same for $|v_{tt\theta}|$.

Let $t_0 > 0$ be sufficiently large so that $t_0^{-1} < \frac{1}{2}$. Integrating (2.8)' from $t_0$ to $T$, $t_0 < T$, and using integration by parts we obtain

$$\frac{1}{2}(n-2)\int_{t_0}^{T} v_t^2\,dt < \frac{1}{2}[v_t^2(t_0) - v_t^2(T)] - \int_{t_0}^{T} v_t\,\Delta_\theta v\,dt$$

$$-\frac{(n-2)}{2(n-1)}\Big[t^{-1}v^{2(n-1)/(n-2)}\Big|_{t_0}^{T} + \int_{t_0}^{T} t^{-2}\,v^{2(n-1)/(n-2)}dt\Big]$$

$$\frac{(n-2)^2}{4}\Big[t^{-1}\,v^2\Big|_{t_0}^{T} + \int_{t_0}^{T} t^{-2}v^2dt\Big] - \frac{(n-2)}{8}\,n\,\Big[t^{-2}\,v^2\Big|_{t_0}^{T}$$

$$+\quad 2\int_{t_0}^{T} t^{-3}\,v^2\,dt\Big].$$

Integrating over $S^{n-1}$, using the uniform bounds on $v$, $v_t$ and the fact that

$$\int_{t_0}^{\infty}\int_{S^{n-1}} v_t\,\Delta_\theta\,v\,d\omega\,dt = -\frac{1}{2}\int_{S^{n-1}}|\nabla_\theta v|^2\Big|_{t_0}^{T}d\omega$$

we get

(2.9)
$$\int_{t_0}^{\infty}\int_{S^{n-1}} v_t^2\,d\omega\,dt < C < \infty$$

We prove that $v_t(t,\theta) \to 0$, uniformly on $\theta \in S^{n-1}$. For we define

$$g(t) = \int_{S^{n-1}} v_t^2\,d\omega\,.$$

Since $v_t\,v_{tt}$ is uniformly bounded we get that

$$\dot g(t) = 2\int_{S^{n-1}} v_t\,v_{tt}\,d\omega$$

is uniformly bounded. It follows that $g(t) \to 0$ as $t \to \infty$. Indeed if not given $\varepsilon > 0$ there exists a sequence $t_i \to \infty$ so that $g(t_i) > 2\varepsilon$. Let $M$ be chosen such that $|\dot{g}(t)| < M$. Therefore, if $|t-t_j| < \varepsilon/M$ then

$$g(t) > g(t_j) - |\int_t^{t_j} \dot{g}(s)ds| > \varepsilon .$$

Let now $\{t'_j\}$ be a subsequence of $\{t_j\}$ satisfying $t'_{j+1} > t'_j + \varepsilon/M$, $t'_0 > t_0$. Since if

$$t'_{j-1} < t < t'_j, \text{ then } -\varepsilon/M < t - t'_j < 0 \text{ , i.e. } |t-t'_j| < \varepsilon/M$$

we obtain

$$\sum_{j=1}^{N} \int_{t'_{j-1}}^{t'_j} g(t)\, dt > \frac{\varepsilon^2}{M} N \to \infty \text{ as } N \to \infty ,$$

contradicting (2.9). Thus

$$g(t) = \int_{S^{n-1}} v_t^2 (t,\theta)d\omega \to 0 \text{ as } t \to \infty . \tag{2.10}$$

Since $v_t(t,\theta)$ and $v_{t\theta}(t,\theta)$ are uniformly bounded, we can invoke Arzela-Ascoli's theorem to assert that given a sequence $\{t_k\}$, $t_k \to \infty$ there exists a subsequence $\{t_{k_j}\}$ such that as $t_{k_j} \to \infty$

$$v_t (t_{k_j}, \theta) \to X(\theta) \text{ , uniformly on } \theta .$$

By using the fact that $v_t$ is uniformly bounded and the dominated convergence theorem we can assert then that

$$\int_{S^{n-1}} (X(\theta))^2 \, d\omega = 0$$

Thus $X(\theta) = 0$ on $S^{n-1}$. Therefore $v_t(t_{k_j}, \theta) \to 0$.

It follows that $v_t(t,\theta) \to 0$ as $t \to \infty$. Indeed, if not there exists a sequence $t_k \to \infty$, $\theta_0 \in S^{n-1}$ so that $v_t(t_k,\theta_0) \to A \neq 0$. The above argument shows that we can find a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ such that $v_t(t_{k_j},\theta) \to 0$, uniformly on $\theta$. Since $v_t(t_{k_j},\theta_0) \to A$, we have that $A = 0$, a contradiction.

We now assert that $v_{tt}(t,\theta) \to 0$ as $t \to \infty$. Proceeding as above we conclude that it suffices to show

$$\int_{t_0}^{\infty} \int_{S^{n-1}} v_{tt}^2 \, d\omega \, dt < M < \infty \tag{2.11}$$

and that

$$\int_{S^{n-1}} v_{tt} \, v_{ttt} \, d\omega \tag{2.11'}$$

is uniformly bounded. (2.11)' follows at once from the uniform bound on $v_{tt}$ and $v_{ttt}$.

Let now $w(t,\theta) = v_t(t,\theta)$. We differentiate the equation for $v$ to obtain

$$\begin{aligned} & v_{ttt} + (n-2)\,(1-t^{-1})v_{tt} + (n-2)t^{-2}\, v_t + \Delta_\theta\, v_t \\ & = t^{-2}\, v^{n/(n-2)} - \frac{n}{(n-2)} t^{-1}\, v^{2/(n-2)}\, v_t \\ & + \frac{1}{2}(n-2)^2\, t^{-1}\, v_t - \frac{1}{2}(n-2)^2\, t^{-2}\, v + (n-2)\frac{n}{2}\, t^{-3}\, v - (n-2)\frac{n}{4}\, t^{-2}\, v_t\,. \end{aligned}$$

Writing the equation in terms of $w$ we get

$$\begin{aligned} & w_{tt} + (n-2)(1-t^{-1})\, w_t + (n-2)t^{-2}\, w + \Delta_\theta\, w \\ & = t^{-2}\, v^{n/(n-2)} - \frac{n}{(n-2)} t^{-1}\, v^{2/(n-2)}\, w \\ & + \frac{1}{2}(n-2)^2\, t^{-1} w - \frac{1}{2}(n-2)^2 t^{-2}\, v + (n-2)\frac{n}{2}\, t^{-3}\, v - (n-2)\frac{n}{4}\, t^{-2}\, w\,. \end{aligned} \tag{2.12}$$

Then we multiply (2.12) by $w_t$ and we integrate from $t_0$ to $T$ and over $S^{n-1}$ to obtain

$$\frac{1}{2}\int_{t_0}^{T}\int_{S^{n-1}}(w_t^2)_t\,d\omega dt+\int_{t_0}^{T}\int_{S^{n-1}}(n-2)(1-t^{-1})\,w_t^2\,d\omega dt+\int_{t_0}^{T}\int_{S^{n-1}}\frac{(n-2)}{2}\,t^{-2}\,(w^2)_t\,d\omega\,dt$$

$$\int_{t_0}^{T}\int_{S^{n-1}}w_t\,\Delta_\theta w\,d\omega=\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,v^{n/(n-2)}\,w_t\,d\omega dt-\frac{n}{2(n-2)}\int_{t_0}^{T}\int_{S^{n-1}}t^{-1}v^{2/(n-2)}(w^2)_t d\omega dt$$

$$+\frac{1}{4}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}}t^{-1}\,(w^2)_t\,d\omega dt-\frac{1}{2}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,vw_t\,d\omega dt$$

$$+(n-2)\frac{n}{2}\int_{t_0}^{T}\int_{S^{n-1}}t^{-3}\,v\,w_t d\omega dt-\frac{(n-2)n}{8}\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,(w^2)_t\,d\omega\,dt$$

We now observe the following facts. We recall that $w$, $w_t$, and $v$ are uniformly bounded.

I. $$\frac{1}{2}\int_{t_0}^{T}\int_{S^{n-1}}({w_t}^2)_t\,d\omega\,dt=\frac{1}{2}\int_{S^{n-1}}({w_t}^2(T)-w_t^2\,(t_0))\,d\omega$$

$$< M \quad ;$$

II. $$\left|\frac{(n-2)}{2}\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}(w^2)_t\,d\omega dt\right|< C\int_{t_0}^{T}t^{-2}\,dt< M$$

where $M > 0$ is a constant independent of $t$;

The same is true for

$$\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,v^{n/(n-2)}\,w_t\,d\omega\,dt\,,\;-\frac{1}{2}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,vw_t\,d\omega\,dt\,,$$

$$(n-2)\,\frac{n}{2}\int_{t_0}^{T}\int_{S^{n-1}}t^{-3}\,vw_t\,d\omega\,dt\quad\text{and}\quad-\frac{(n-2)n}{8}\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,(w^2)_t\,d\omega\,dt$$

III. $$\int_{t_0}^{T}\int_{S^{n-1}}w_t\,\Delta_\theta\,w d\omega dt=-\frac{1}{2}\int_{S^{n-1}}|\nabla_\theta w|^2\,\Big|_{t_0}^{T}\,d\omega\quad;$$

IV. By Holder inequality and (2.9) we have

$$-\frac{n}{(n-2)}\int_{t_0}^{T}\int_{S^{n-1}}t^{-1}\,v^{2/(n-2)}\,w_t w\,d\omega dt<$$

$$<\tilde{M}\Big[\int_{t_0}^{T}\int_{S^{n-1}}t^{-2}\,d\omega dt\Big]^{1/2}\,\Big[\int_{t_0}^{T}\int_{S^{n-1}}v^2{}_t d\omega\,dt\Big]^{1/2}< M$$

where $\tilde{M}$, $M > 0$ are constants independent of $T$ (here one used the uniform bound on $v_{tt}$);

V. Finally, by using integration by parts we can easily show that

$$\left|\frac{1}{4}(n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} (w^2)_t \, d\omega \, dt\right| < M$$

with $M > 0$ a constant independent of $T$. Now using I-V, (2.11) follows at once.

So we have shown that

(2.13) $$v_{tt} \to 0 \, , \; v_t \to 0 \quad \text{as} \quad t \to \infty \, , \quad \text{uniformly on} \quad \theta \quad S^{n-1} \, .$$

Since $v_\theta(t,\theta)$ is uniformly bounded for each sequence $t_k \to \infty$ there exists a subsequence $\{t_k'\}$, $t_k' \to \infty$ such that

$$v(t_k', \theta) \to v(\theta) \quad \text{uniformly on} \quad \theta \quad \text{as} \quad t_k' \to \infty \, .$$

Therefore, because of (2.8) we have that

$$\Delta_\theta \, v(\theta) = 0$$

Hence $v(\theta) = c(t_k) = \text{constant}$.

<u>LEMMA</u> 5. <u>The limits $c(t_k)$ are independent of the choice of the sequence</u> $\{t_k\}$.

<u>REMARK</u>. If in (1.1) $u$ were radially symmetric, then Lemma 5 is very simple. Indeed because of the proof of Lemma 1

$$v(t) = r^{n-2}(-\ln r)^{(n-2)/2} u(r) < (n-2)^{(n-2)}/2^{(n-2)/2} \, .$$

So $-v^{n/(n-2)} + \frac{1}{2}(n-2)^2 v > 0$. Hence from (2.8) we get

$$v_{tt} + (n-2)(1-t^{-1})v_t > -(n-2)\frac{n}{4} t^{-2} v.$$

Let now $t_k$, $s_k$ be sequences such that $v(t_k) \to c(t_k)$ as $t_k \to \infty$, $v(s_k) \to c(s_k)$ as $s_k \to \infty$ and $c(t_k) > c(s_k)$. We next observe that by taking subsequences we may suppose that $t_k < s_k$. Now, since by (2.13) $v_t \to 0$ as $t \to \infty$, since $t^{-2} v$ is integrable and since by (2.9), Holder inequality and the inequality $ab < \frac{a^2}{2} + \frac{b^2}{2}$ we also have that $t^{-1} v_t$ is integrable, we conclude by integrating the above relation from $t_k$ to $s_k$ that

$$c(s_k) > c(t_k).$$

So $c(t_k) = c(s_k)$.

We shall divide the proof in several steps (Lemmas 8-17). The essential idea of the proof is to show that all limits $c(t_k)$ are the same by using the energy defined in Lemma 7. This is finally accomplished in Lemma 17. To show that the energy $E(t)$ of Lemma 7 has a limit we shall need

<u>Lemma</u> 6 $|\nabla_\theta v|^2$ <u>is an integrable function</u> i.e.

$$\int_{t_0}^{\infty} \int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega \, dt < \infty .$$

<u>PROOF</u>. Let $\overline{v} = \frac{1}{w_{n-1}} \int_{S^{n-1}} v \, d\omega$ .

We multiply (2.8) by $(v-\overline{v})$ and integrate from $t_0$ to $T$ and over $S^{n-1}$ to obtain

$$\int_{t_0}^{T} \int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega dt = \int_{t_0}^{T} \int_{S^{n-1}} v_{tt} (v-\overline{v}) \, d\omega \, dt$$

$$+ (n-2) \int_{t_0}^{T} \int_{S^{n-1}} (1-t^{-1}) \; v_t (v-\overline{v}) \, d\omega \, dt + \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} v^p (v-\overline{v}) \, d\omega \, dt$$

$$- \frac{1}{2} (n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} v(v-\overline{v}) \, d\omega \, dt + (n-2) \frac{n}{4} \int_{t_0}^{T} \int_{S^{n-1}} t^{-2} v(v-\overline{v}) d\omega \, dt$$

where $p = n/(n-2)$.

We next observe that

$$\int_{t_0}^{T} v_{tt}\ (v-\bar{v})\ dt = v_t(v-\bar{v})\ \Big|_{t_0}^{T}$$

$$- \int_{t_0}^{T} v_t^2\ dt + \frac{1}{w_{n-1}} \int_{t_0}^{T} v_t\ (\int_{S^{n-1}} v_t\ d\omega)dt\ .$$

Hence by Fubini's theorem and Hölder inequality we obtain

(i) $$\Big|\int_{t_0}^{T} \int_{S^{n-1}} v_{tt}(v-\bar{v})\ d\omega\ dt\ \Big| < C\ [\ \int_{S^{n-1}} |v_t|\ d\omega + \int_{t_0}^{T} \int_{S^{n-1}} v_t^2\ d\omega\ dt]$$

where the Hölder inequality was used to guarantee that

$$\frac{1}{w_{n-1}} \int_{t_0}^{T} (\int_{S^{n-1}} v_t\ d\omega)^2 dt < C \int_{t_0}^{T} \int_{S^{n-1}} v_t^2\ d\omega\ dt\ ,$$

with $C>0$ a constant independent of $T$.

On the other hand

$$\int_{t_0}^{T} v_t(v-\bar{v})\ dt = v\ (v-\bar{v})\ \Big|_{t_0}^{T} - \int_{t_0}^{T} v(v-\bar{v})_t\ dt.$$

But

$$I = \int_{t_0}^{T} v_t(v-\bar{v})dt - \int_{t_0}^{T} v(v-\bar{v})_t dt = - \int_{t_0}^{T} v_t\ \bar{v}\ dt$$

$$+ \int_{t_0}^{T} v\ \bar{v}_t\ dt\ .$$

So by integration by parts we obtain

$$I = v\ \bar{v}\ \Big|_{t_0}^{T} - 2 \int_{t_0}^{T} v_t\ \bar{v}\ dt\ .$$

Hence

$$\int_{t_0}^{T} \int_{S^{n-1}} v_t\ (v-\bar{v})\ d\omega\ dt - \int_{t_0}^{T} \int_{S^{n-1}} v(v-\bar{v})_t\ d\omega\ dt$$

$$= \frac{1}{w_{n-1}} [(\int_{S^{n-1}} v d\omega)^2\ \Big|_{t_0}^{T} - \int_{t_0}^{T} (\int_{S^{n-1}} v d\omega)_t^2\ dt]$$

$$= 0\ .$$

So we have shown that

(ii) $$\left| \int_{t_0}^{T} \int_{S^{n-1}} v_t\, (v-\bar{v})\, d\omega\, dt \right| = \frac{1}{2} \left| \int_{S^{n-1}} v(v-\bar{v})\, d\omega \Big|_{t_0}^{T} \right| < C$$

where $C>0$ is a constant independent of $T$:

We also have

$$\int_{t_0}^{T} t^{-1}\, v^{p}(v-\bar{v})\, dt < \bar{C} \int_{t_0}^{T} t^{-2}\, dt + \varepsilon \int_{t_0}^{T} |v-\bar{v}|^2 dt$$

where $\bar{C}>0$ is a constant independent of $T$ and $\varepsilon>0$ is also independent of $T$ and small enough so that when we integrate over $S^{n-1}$ and we use the Poincaré inequality on $S^{n-1}$ we obtain

(iii) $$\left| \int_{t_0}^{T} \int_{S^{n-1}} t^{-1}\, v^{p}\, (v-\bar{v})\, d\omega\, dt \right| < C + \frac{1}{4} \int_{t_0}^{T} \int_{S^{n-1}} |\nabla_\theta v|^2\, d\omega\, dt$$

where $C>0$ is a positive constant independent of $T$.

Working similarly we get that

(iv) $$\left| \int_{t_0}^{T} \int_{S^{n-1}} t^{-1}\, v(v-\bar{v})\, d\omega\, dt \right| < C + \frac{1}{4} \int_{t_0}^{T} \int_{S^{n-1}} |\nabla_\theta v|^2\, d\omega\, dt$$

We also have that

(v) $$\left| \int_{t_0}^{T} \int_{S^{n-1}} t^{-2}\, v(v-\bar{v})\, d\omega\, dt \right| < C$$

where in (iv), (v) $C>0$ is a constant independent of $T$. Now from (i) - (v) the assertion of the lemma is evident

<u>LEMMA</u> 7. <u>The energy associated to the equation</u> (2.8)

$$E(t) = \frac{1}{2}\, t \int_{S^{n-1}} v_t^2\, d\omega - \frac{1}{2}\, t \int_{S^{n-1}} |\nabla_\theta v|^2\, d\omega + \int_{S^{n-1}} \frac{v^{p+1}}{p+1}\, d\omega$$

$$- \frac{(n-2)^2}{4} \int_{S^{n-1}} v^2\, d\omega + (n-2)\, \frac{n}{8}\, t^{-1} \int_{S^{n-1}} v^2\, d\omega\, ,$$

*where* $p = n/(n-2)$ *has the following properties:*

(a) $\frac{dE(t)}{dt} < 0$ , *for* $t > t_0$ .

(b) $\lim_{t\to\infty} E(t)$ *exists and it is less than* $-\infty$ .

PROOF. To see the first part we multiply (2.8) by $tv_t$ and we integrate over $S^{n-1}$ to obtain that

$$\frac{dE}{dt} = \frac{1}{2} \int_{S^{n-1}} v_t^2 \, d\omega - (n-2) \int_{S^{n-1}} (t-1) \; v_t^2 \, d\omega - \frac{1}{2} \int_{S^{n-1}} |\nabla_\theta v|^2 d\omega$$

(2.14)

$$- (n-2)\frac{n}{8} \int_{S^{n-1}} t^{-2} \, v^2 \, d\omega .$$

Hence $\frac{dE}{dt} < 0$ and therefore $\lim_{t\to\infty} E(t)$ exits.

To see that

(2.15) $$\lim_{t\to\infty} E(t) < -\infty ,$$

we notice the following facts.

(i) By Lemma 2, $0 \lesssim c(t_k) < [\frac{(n-2)}{2^{1/2}}]^{(n-2)}$ for all sequence $t_k \to \infty$ ;

and

(ii) By Lemma 6 $\liminf_{t\to\infty} \; t \int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega < \infty$ .

Indeed if (ii) does not hold, then $\int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega$ is not integrable, contradicting Lemma 6. Now from (i) and (ii) (2.15) follows. Indeed let $\{t_k\}$, $t_k \to \infty$ be a sequence such that

$$\liminf_{t\to\infty} \; t \int_{S^{n-1}} |\nabla_\theta \, v|^2 \, d\omega = \lim_{t_k\to\infty} t_k \int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega < \infty$$

Then

$$\lim_{t\to\infty} E(t) = \lim_{t_k\to\infty} E(t_k) < -\infty .$$

The next step is to show that

$$t \int_{S^{n-1}} v_t^2 \, d\omega \quad \text{and} \quad t \int_{S^{n-1}} |\nabla_\theta v|^2 \, d\omega \to 0 \quad \text{as } t \to \infty ,$$

which are proved in Lemma 16. The proof of this lemma needs several facts which are grouped together in Lemmas 8-15. The reader might wish to proceed directly to the proof of Lemma 16 and then return to the proofs of Lemmas 8-15 after the necessity of these lemmas has become clear.

LEMMA 8 . $tv_t^2$ *is an integrable function* i.e.

$$\int_{t_0}^{\infty} \int_{S^{n-1}} tv_t^2 \, d\omega \, dt < \infty .$$

PROOF. By integrating (2.14) from $t_0$ to $T$ we get that

$$(n-2) \int_{t_0}^{T} \int_{S^{n-1}} tv_t^2 \, d\omega \, dt \quad < - E(T) + E(t_0) + (n-3/2) \int_{t_0}^{T} \int_{S^{n-1}} v_t^2 \, d\omega dt.$$

But $\lim_{t \to \infty} E(t) = c < - \infty$ . Hence we obtain Lemma 8.

In Lemma 16 we shall use the fact that the function $t\, v_{tt}^2$ is integrable. This is proved in Lemma 11 for which we need Lemmas 9 and 10.

LEMMA 9 . $|\nabla_\theta w|^2$ *is integrable, where* $w = v_t$ .

PROOF. We multiply (2.12) by $w$ and we then integrate from $t_0$ to $T$ and over $S^{n-1}$ to obtain

$$\int_{t_0}^{T} \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega \, dt = \int_{t_0}^{T} \int_{S^{n-1}} w_{tt} \, w \, d\omega \, dt$$

$$+ (n-2) \int_{t_0}^{T} \int_{S^{n-1}} (1-t^{-1}) \, w_t \, w \, d\omega \, dt + (n-2) \int_{t_0}^{T} \int_{S^{n-1}} t^{-2} \, w^2 \, d\omega \, dt$$

$$- \int_{t_0}^{T} \int_{S^{n-1}} t^{-2} \, v^p \, w \, d\omega \, dt + p \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} \, v^{p-1} \, w^2 d\omega \, dt$$

$$- \frac{1}{2}\ (n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} t^{-1}\, w^2\, d\omega\, dt + \frac{1}{2}(n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} t^{-2}\, vw\, d\omega\, dt$$

$$- (n-2)\frac{n}{2} \int_{t_0}^{T} \int_{S^{n-1}} t^{-3}\, vw\, d\omega\, dt$$

$$+ (n-2)\frac{n}{4} \int_{t_0}^{T} \int_{S^{n-1}} t^{-2}\, w^2\, d\omega\, dt$$

We next observe the following facts

(i) $\displaystyle\int_{t_0}^{T} w_{tt}\, w\, dt = w_t w \Big|_{t_0}^{T} - \int_{t_0}^{T} w_t^2\, dt$ .

Hence

$$\left| \int_{t_0}^{T} \int_{S^{n-1}} w_{tt}\, w\, d\omega\, dt \right| < C\Big[ \int_{S^{n-1}} w_t^2\, d\omega + \int_{S^{n-1}} w^2\, d\omega\Big]$$

$$+ \int_{t_0}^{T} \int_{S^{n-1}} w_t^2\, d\omega\, dt ,$$

where $C>0$ is constant independent of $T$. Since by the proof of Lemma 4, statement (2.13), we have

$$|w_t| , |w| \to 0 \quad \text{as} \quad t \to \infty ,$$

and since also by Lemma 4, (2.11), we have that $w_t^2$ is integrable we get that

$$\left| \int_{t_0}^{\infty} \int_{S^{n-1}} w_{tt}\, w\, d\omega\, dt \right| < \infty ,$$

Also

(ii) $\displaystyle\left|(n-2)(1-t^{-1}) \int_{t_0}^{\infty} \int_{S^{n-1}} w_t w\, d\omega\, dt \right| < C \Big[ \int_{t_0}^{\infty} \int_{S^{n-1}} w_t^2\, d\omega\, dt$

$$+ \int_{t_0}^{\infty} \int_{S^{n-1}} w^2\, d\omega\, dt\Big]$$

which by Lemma 4 is bounded.

Since all the other terms involved are easily seen to be integrable we conclude the lemma.

LEMMA 10. The energy associated to the equation (2.12)

$$J(t) = \frac{1}{2}\, t \int_{S^{n-1}} w_t^2 \, d\omega - \frac{1}{2}\, t \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega$$

$$+ \int_{S^{n-1}} \int_t^\infty t^{-1} v^p \, w_t \, dt\, d\omega - p \int_{S^{n-1}} \int_t^\infty v^{p-1} \, w\, w_t \, dt\, d\omega$$

$$- \frac{(n-2)^2}{4} \int_{S^{n-1}} w^2 \, d\omega - \frac{1}{2}(n-2)^2 \int_{S^{n-1}} \int_t^\infty t^{-1} v w_t \, dt\, d\omega$$

$$+ (n-2)\frac{n}{2} \int_{S^{n-1}} \int_t^\infty t^{-2} v w_t \, dt\, d\omega - (n-2)\frac{(n+4)}{4} \int_{S^{n-1}} \int_t^\infty t^{-1} w w_t \, dt\, d\omega$$

where $p = n/(n-2)$ has the following properties

(i) $\frac{dJ(t)}{dt} < 0$ for $t > t_0$ ;

(ii) $\lim_{t\to\infty} J(t) = \lim_{t\to\infty} J^0(t) = c < -\infty$

where $J^0(t) = \frac{1}{2}\, t \int_{S^{n-1}} w_t^2 \, d\omega - \frac{1}{2}\, t \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega$ .

PROOF. We multiply (2.12) by $tw_t$ and we integrate over $S^{n-1}$ to get that

$$\frac{dJ}{dt} = (-(n-2)t + (n-3/2)) \int_{S^{n-1}} w_t^2 \, d\omega$$

(2.16)
$$- \frac{1}{2} \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega \, .$$

From this relation (i) follows.

Since by (2.13) $w(t) \to 0$ as $t \to \infty$ and since the functions $t^{-1} v^p w_t$ , $v^{p-1} \; ww_t$ , $t^{-1} \; vw_t$ , $t^{-2} vw_t$ and $t^{-1} \; ww_t$ are integrable we conclude that

$$c = \lim_{t\to\infty} J(t) = \lim_{t\to\infty} J^0(t) \; < -\infty \, .$$

But since $|\nabla_\theta w|^2$ is integrable we have

$$L = \liminf_{t\to\infty} t \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega < \infty .$$

Hence there exists a sequence, say $\{t_k\}$, $t_k \to \infty$ such that $L = \lim_{t_k\to\infty} t_k \int_{S^{n-1}} |\nabla_\theta w|^2 dw$ and therefore

$$\lim_{t\to\infty} J^0(t) = \lim_{t_k\to\infty} J^0(t_k) + c < -\infty .$$

This concludes this proof.

LEMMA 11. $t\, v_{tt}^2$ is integrable. That is,

$$\int_{t_0}^{\infty} \int_{S^{n-1}} t v_{tt}^2 \, d\omega \, dt < \infty$$

PROOF. We integrate (2.16) from $t_0$ to $\infty$ to get that

$$\int_{t_0}^{\infty} \int_{S^{n-1}} t v_{tt}^2 \, d\omega \, dt < C[-J(\infty) + J(t_0)] + C$$

where $C > 0$ is a constant. Since $-J(\infty) < \infty$, the lemma follows.

In the proof of Lemma 16 we shall also use the fact that the function $t^2 v_{tt}^2$ is integrable. We show this in Lemma 14. As in the case of the proof of the integrability of $tv_t^2$ and $tv_{tt}^2$ we shall need some previous facts which we group in Lemmas 12 and 13.

LEMMA 12. $t \, |\nabla_\theta \, w|^2$ is integrable where $w = v_t$ .

PROOF. We multiply (2.12) by $tw$ and integrate from $t_0$ to $T = t_k$ and over $S^{n-1}$ to get

$$\int_{t_0}^{T} \int_{S^{n-1}} t |\nabla_\theta w|^2 \, d\omega \, dt = \int_{t_0}^{T} \int_{S^{n-1}} t \, w_{tt} \, w \, d\omega \, dt$$

$$+ (n-2) \int_{t_0}^{T} \int_{S^{n-1}} (t-1) \, w_t w \, d\omega \, dt + (n-2) \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} w^2 d\omega \, dt$$

$$- \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} v^p \, w \, d\omega \, dt + p \int_{t_0}^{T} \int_{S^{n-1}} v^{p-1} w^2 \, d\omega \, dt$$

$$-\frac{1}{2}(n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} w^2 \, d\omega \, dt + \frac{1}{2}(n-2)^2 \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} v w \, d\omega \, dt$$

$$-(n-2)\frac{n}{2} \int_{t_0}^{T} \int_{S^{n-1}} t^{-2} vw \, d\omega \, dt + (n-2)\frac{n}{4} \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} w^2 \, d\omega \, dt$$

where $p = n/(n-2)$ and where $t_k \to \infty$ is a sequence to be chosen below.

Using Hölder inequality, the inequality $ab < \frac{1}{2}(a^2 + b^2)$, and the integrability of $tw_t^2$, $tw^2$, $w^2$, all the terms but the one below are easily seen to be integrable,

$$I = \int_{t_0}^{T} \int_{S^{n-1}} tw_{tt} \, w \, d\omega \, dt.$$

We prove that $|I| < M$, where $M > 0$ is a constant. Indeed,

$$I = \int_{t_0}^{T} \int_{S^{n-1}} tw_{tt} \, w \, d\omega \, dt = t \int_{S^{n-1}} w_t w \, d\omega \Big|_{t_0}^{T} - \int_{t_0}^{T} \int_{S^{n-1}} tw_t^2 \, d\omega \, dt$$

$$- \int_{t_0}^{T} \int_{S^{n-1}} w_t w \, d\omega \, dt$$

So letting $T = t_k$ we get

$$|I| < t_k \int_{S^{n-1}} w_t^2 \, d\omega + t_k \int_{S^{n-1}} w^2 \, d\omega + \int_{t_0}^{t_k} \int_{S^{n-1}} tw_t^2 \, d\omega \, dt$$

$$+ \int_{t_0}^{t_k} \int_{S^{n-1}} w_t^2 \, d\omega \, dt + \int_{t_0}^{t_k} \int_{S^{n-1}} w^2 \, d\omega \, dt + C$$

where $C>0$ is a constant independent of $\{t_k\}$.

We shall now select the sequence $t_k$, $t_k \to \infty$. Let $t_k$ be such that $t_k \to \infty$ and

$$t_k \int_{S^{n-1}} w_t^2 \, d\omega \to 0 \quad \text{as} \quad t_k \to \infty .$$

This sequence always exists because otherwise we could find $C>0$ a constant independent of $t$, such that

$$\int_{S^{n-1}} w_t^2 \, d\omega > \frac{C}{t} ,$$

which contradicts (2.11).

We next observe that $t \int_{S^{n-1}} w^2 \, d\omega$ is bounded.

Indeed let $t_0 > 0$ be a constant and observe that

$$t^{1/2} w(t,\theta) = t_0^{1/2} w(t_0,\theta) + \int_{t_0}^{t} (t^{1/2} w)_t dt .$$

So the Holder inequality and the inequality $ab < \frac{a^2}{2} + \frac{b^2}{2}$ imply

$$tw^2 < t_0 \, w^2(t_0,\theta) + \int_{t_0}^{t} t^{-1} w^2 dt + \int_{t_0}^{t} tw_t^2 \, dt .$$

Integrating over $S^{n-1}$, using (2.9) and Lemma 11 we easily conclude that

$$t \int_{S^{n-1}} w^2 \, d\omega < M < \infty$$

where $M>0$ is a constant independent of $t$.

It is now evident in view of (2.9), (2.11) and Lemma 11 to conclude that I is bounded. This proves the lemma.

LEMMA 13. *The energy associated to the equation* (2.12)

$$\begin{aligned} F(t) = {} & \frac{1}{2} t^2 \int_{S^{n-1}} w_t^2 \, d\omega + \frac{(n-2)}{2} \int_{S^{n-1}} w^2 \, d\omega \\ & - \frac{1}{2} t^2 \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega - \int_{S^{n-1}} w v^p \, d\omega \\ & - p \int_{S^{n-1}} \left( \int_t^\infty t v^{p-1} w \, w_t dt \right) d\omega + \frac{1}{2} (n-2)^2 \int_{S^{n-1}} \left( \int_t^\infty tw \, w_t \, dt \right) d\omega \\ & + \frac{1}{2} (n-2)^2 \int_{S^{n-1}} \int_t^\infty v \, w_t \, dt \, d\omega + \; + (n-2) \frac{n}{2} \int_{S^{n-1}} \int_t^\infty t^{-1} v w_t \, dt \, d\omega \\ & + (n-2) \frac{n}{8} \int_{S^{n-1}} w^2 \, d\omega \end{aligned}$$

*where* $p = n/(n-2)$ *has the following properties:*

(i) $\frac{dF}{dt}(t) < 0$ *if* $t$ *is sufficiently large;*

(ii) $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} F^0(t) = c < -\infty$

where $F^0(t) = \frac{1}{2} t^2 \int_{S^{n-1}} (w_t^2 - |\nabla_\theta w|^2) d\omega$ .

PROOF. By multiplying (2.12) by $t^2 w_t$ and then by integrating over $S^{n-1}$ it is easy to see that

$$\frac{dF}{dt} = -(n-2)t^2 \int_{S^{n-1}} w_t^2 \, d\omega + (n-1)t \int_{S^{n-1}} w^2 \, d\omega$$

(2.17)

$$- t \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega - p \int_{S^{n-1}} w^2 v^{p-1} \, d\omega \quad .$$

From this formula (i) is clear.

To show (ii) we recall that $w = v_t \to 0$ as $t \to \infty$ and that $w^2$ , $w_t^2$ are integrable. Since we can easily see that $tv^{p-1} w w_t$ , $t w w_t$ are also integrable and since

$$\int_{S^{n-1}} \int_t^\infty v w_t \, dt \, d\omega = - \int_{S^{n-1}} vw \, d\omega - \int_{S^{n-1}} \int_t^\infty w^2 dt \, d\omega$$

we conclude that

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} F^0(t) = c.$$

To see that $c < -\infty$ , we observe that by Lemma 12

$$L = \liminf_{t\to\infty} t^2 \int_{S^{n-1}} |\nabla_\theta w|^2 \, d\omega < \infty$$

Hence there exists a sequence $t_k \to \infty$ such that $L = \lim_{t_k\to\infty} t_k^2 \int_{S^{n-1}} |\nabla_\theta w|^2 d\omega$ and therefore

$$c = \lim_{t_k\to\infty} F(t_k) < -\infty.$$

LEMMA 14.

$$\int_{t_0}^\infty \int_{S^{n-1}} t^2 v_{tt}^2 \, d\omega \, dt < \infty$$

PROOF. Consider $t_0$ sufficiently large and integrate (2.17) from $t_0$ to $\infty$ to obtain

$$(n-2)\int_{t_0}^{\infty}\int_{S^{n-1}} t^2 w_t^2 \, d\omega \, dt < [-F(\infty) + F(t_0)] + C$$

where $C>0$ is a constant. Since by Lemma 13, $-F(\infty) < \infty$, the assertion follows.

LEMMA 15.

$$\int_{t_0}^{\infty}\int_{S^{n-1}} v_{ttt}^2 \, d\omega \, dt < \infty .$$

PROOF. We multiply (2.12) by $w_{tt} = v_{ttt}$ and we integrate from $t_0$ to $T$ and over $S^{n-1}$ to obtain that

$$\begin{aligned}(2.18)\qquad &\int_{t_0}^{T}\int_{S^{n-1}} w_{tt}^2 d\omega \, dt + (n-2)\int_{t_0}^{T}\int_{S^{n-1}}(1-t^{-1})\, w_{tt}\, w_t \, d\omega \, dt \\ &+ \int_{t_0}^{T}\int_{S^{n-1}} t^{-2}\, w\, w_{tt}\, d\omega \, dt + \int_{t_0}^{T}\int_{S^{n-1}} w_{tt}\, \Delta_\theta w \, d\omega \, dt \\ &= I\end{aligned}$$

where

$$\begin{aligned} I = &\int_{t_0}^{T}\int_{S^{n-1}} t^{-2} v^p w_{tt} \, d\omega \, dt - p\int_{t_0}^{T}\int_{S^{n-1}} t^{-1} v^{p-1} w w_{tt} \, d\omega \, dt \\ &+ \frac{1}{2}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}} t^{-1}\, w\, w_{tt} \, d\omega \, dt - \frac{1}{2}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}} t^{-2}\, v w_{tt} \, d\omega \, dt \\ &+ (n-2)\frac{n}{2}\int_{t_0}^{T}\int_{S^{n-1}} t^{-3}\, v\, w_{tt} \, d\omega \, dt - (n-2)\frac{n}{4}\int_{t_0}^{T}\int_{S^{n-1}} t^{-2}\, w w_{tt} \, d\omega \, dt \end{aligned}$$

with $p = n/(n-2)$.

Using Hölder inequality, the inequality $ab < \frac{1}{2}a^2 + \frac{1}{2}b^2$, the fact that $|w_{tt}| < M$ and the integrability of $w$ we can easily conclude that

$$|I| < \infty .$$

We also have

$$\int_{t_0}^{\infty} \int_{S^{n-1}} t^{-2} \; |w| \; |w_{tt}| \; d\omega \, dt < \infty .$$

Hence to conclude the integrability of $v_{ttt}^2$ it suffices to study the terms

$$I_1 = \int_{t_0}^{T} \int_{S^{n-1}} (1-t^{-1}) \; w_{tt} \; w_t \; d\omega \, dt \quad \text{and} \quad I_2 = \int_{t_0}^{T} \int_{S^{n-1}} w_{tt} \Delta_\theta \; w \; d\omega \, dt$$

By integration by parts we conclude that

$$I_2 = - \int_{S^{n-1}} \nabla_\theta \; w_t \cdot \nabla_\theta w \; d\omega \Bigg|_{t_0}^{T} + \int_{t_0}^{T} \int_{S^{n-1}} |\nabla_\theta w_t|^2 \, d\omega \, dt$$

and

$$I_1 = \frac{1}{2} \int_{S^{n-1}} w_t^2 \, d\omega \Bigg|_{t_0}^{T} - \int_{t_0}^{T} \int_{S^{n-1}} t^{-1} \; w_{tt} \; w_t \; d\omega \, dt.$$

Using Lemma 4, (2.13), Hölder inequality, the inequality $ab < \frac{1}{2} a^2 + \frac{1}{2} b^2$ and the integrability of $w_t^2$ and $|\nabla_\theta \, w_t|^2$, and the fact that $w_{tt}$ is uniformly bounded we conclude that

$$|I_1| \; , \; |I_2| < \infty .$$

Therefore from (2.18) we conclude that

$$\int_{t_0}^{T} \int_{S^{n-1}} w_{tt}^2 d\omega \, dt < M < \infty$$

where $M>0$ is constant independent of $T$.

<u>LEMMA</u> 16. <u>The following holds,</u>

(a) $t \int_{S^{n-1}} v_t^2 \; d\omega \to 0$, <u>as</u> $t \to \infty$,

(b) $t \int_{S^{n-1}} v_{tt}^2 \; d\omega \to 0$ <u>as</u> $t \to \infty$

(c) $t \int_{S^{n-1}} |\nabla_\theta \, v|^2 \, d\omega \to 0$ <u>as</u> $t \to \infty$.

PROOF. Consider

$$\frac{d}{dt}\, t \int_{S^{n-1}} v_t^2 \, d\omega = \int_{S^{n-1}} v_t^2 \, d\omega + 2t \int_{S^{n-1}} v_t \, v_{tt} \, d\omega .$$

Let $\{t_k\}$, $\{s_k\}$ be two sequences such that $t_k, s_k \to \infty$. Integrating from $t_k$ to $s_k$, using Hölder inequality and the inequality $ab < \frac{1}{2} a^2 + \frac{1}{2} b^2$ we get

$$|t_k \int_{S^{n-1}} v_t^2 \, d\omega - s_k \int_{S^{n-1}} v_t^2 d\omega| <$$

$$\int_{\ell_k}^{\infty} \int_{S^{n-1}} v_t^2 \, d\omega \, dt + \int_{\ell_k}^{\infty} \int_{S^{n-1}} t v_t^2 \, d\omega \, dt + \int_{\ell_k}^{\infty} \int_{S^{n-1}} t v_{tt}^2 \, d\omega \, dt$$

where $\ell_k = \min(t_k, s_k)$. Since by Lemmas 4,8 and 11 the functions on the right are integrable we conclude that

$$\lim_{t \to \infty} t \int_{S^{n-1}} v_t^2 \, d\omega \ \text{ exists}$$

But since $v_t^2$ is integrable this limit must be zero.

(b) follows from the fact that

$$|t_k \int_{S^{n-1}} v_{tt}^2 (t_k, \theta) \, d\omega - s_k \int_{S^{n-1}} v_{tt}^2 (s_k, \theta) d\omega \, | <$$

$$\int_{\ell_k}^{\infty} \int_{S^{n-1}} v_{tt}^2 (t,\theta) \, d\omega \, dt + \int_{\ell_k}^{\infty} \int_{S^{n-1}} t^2 \, v_{tt}^2 \, (t,\theta) \, d\omega \, dt$$

$$+ \int_{\ell_k}^{\infty} \int_{S^{n-1}} v_{ttt}^2 \, (t,\theta) \, d\omega \, dt ,$$

and the facts that the functions $v_{tt}^2$, $t^2 \, v_{tt}^2$ and $v_{ttt}^2$ are integrable, by Lemmas 4, 14 and 15 respectively.

We prove (c). We multiply (2.8) by $t(v-\bar{v})$ and we integrate over $S^{n-1}$ to get

$$(2.19)\qquad t\int_{S^{n-1}}|\nabla_\theta v|^2\,d\omega = t\int_{S^{n-1}}(v-\bar{v})\,v_{tt}\,d\omega + (n-2)(t-1)\int_{S^{n-1}}(v-\bar{v})v_t\,d\omega$$

$$+\int_{S^{n-1}}v^p\,(v-\bar{v})\,d\omega - \frac{1}{2}(n-2)^2\int_{S^{n-1}}v(v-\bar{v})\,d\omega$$

$$+(n-2)\frac{n}{4}\,t^{-1}\int_{S^{n-1}}v(v-\bar{v})d\omega\ .$$

Now, using the Hölder inequality and the inequality $ab < \frac{1}{\varepsilon}a^2 + \varepsilon b^2$, where $\varepsilon>0$ is a small number chosen such that when we applied the Poincaré inequality on $S^{n-1}$ we have

$$(n-2)\varepsilon\, t\int_{S^{n-1}}|v-\bar{v}|^2 d\omega < \frac{1}{4}\,t\int_{S^{n-1}}|\nabla_\theta v|^2 d\omega$$

we get that

$$t\int_{S^{n-1}}|v-\bar{v}|v_{tt}\,d\omega < \frac{1}{\varepsilon}\,t\int_{S^{n-1}}v_{tt}^2\,d\omega + \frac{1}{4}\,t\int_{S^{n-1}}|\nabla_\theta v|^2 d\omega\ ,$$

and

$$(n-2)(t-1)\int_{S^{n-1}}(v-\bar{v})v_t d\omega < \frac{(n-2)}{\varepsilon}\,t\int_{S^{n-1}}v^2\,d\omega + \frac{1}{4}\,t\int_{S^{n-1}}|\nabla_\theta v|^2 d\omega.$$

Hence from (2.19) we obtain that

$$(2.20)\qquad t\int_{S^{n-1}}|\nabla_\theta v|^2\,d\omega < C[\,t\int_{S^{n-1}}v_{tt}^2 d\omega + t\int_{S^{n-1}}v_t^2\,d\omega + \int_{S^{n-1}}|v-\bar{v}|d\omega]$$

where $C>0$ is a constant independent of $t$.

Let now $t_k \to \infty$ be a sequence such that

$$\lim_{t_k\to\infty} t_k\int_{S^{n-1}}|\nabla_\theta v|^2 d\omega = \limsup_{t\to\infty} t\int_{S^{n-1}}|\nabla_\theta v|^2\,d\omega\ .$$

Since $v(t_k) \to c(t_k)$ (we take a subsequence if necessary) we have that

$$|v-\bar{v}| \to 0 \quad\text{as}\quad t_k \to \infty\ .$$

Hence since from (a) and (b) we know that

$$t\int_{S^{n-1}} v_{tt}^2\, d\omega\ ,\quad t\int_{S^{n-1}} v_t^2\, d\omega \to 0$$

as $t \to \infty$, we have from (2.20) that

$$\lim_{t\to\infty}\sup\ t\int_{S^{n-1}} |\nabla_\theta v|^2\, d\omega \to 0.$$

This proves (c).

<u>LEMMA</u> 17. <u>Lemma</u> 5 <u>holds</u>.

<u>PROOF</u>. From Lemmas 7 and 16 we have that as $t \to \infty$

$$\int_{S^{n-1}} (v^{\,p+1}/(p+1) - \frac{(n-2)^2}{4} v^2)\, d\omega \to c < -\infty$$

where $c$ is a constant and $p = n/(n-2)$. Since the function

$$f(t) = \frac{t^{\,p+1}}{p+1} - \frac{(n-2)^2}{4} t^2\ ,\ 0 < t < \frac{(n-2)^{(n-2)}}{2^{(n-2)/2}} = t^*$$

is strictly decreasing, and by Lemma 2 all possible constants satisfies the relation $c(t_k) < t^*$, we have that

$$c(t_k) = f^{-1}(c/w_{n-1})$$

where $f^{-1}$ is the inverse of $f$ and $w_{n-1}$ is the volume of $S^{n-1}$. Hence Lemma 5 holds.

<u>LEMMA</u> 18. <u>Theorem</u> A <u>holds</u>.

<u>PROOF</u>. In view of our previous lemmas it suffices to show that if

$$\lim_{|x|\to 0} |x|^{n-2} (-\ln |x|)^{(n-2)/2} u(x) = C \tag{2.21}$$

with $C < \frac{(n-2)^{(n-2)}}{2^{(n-2)/2}}$, then the singularity must be removable. Indeed by Lemma 3 we have

$$\lim_{|x|\to 0} |x|^{n-2} \ (-\ln |x|)^{(n-2)/2} \ u(x) = \hat{C}$$

for some $\hat{C} > 0$. Now by Lemma 2 we have that

$$\hat{C} < \frac{(n-2)^{(n-2)}}{2^{(n-2)/2}}$$

Therefore it suffices to prove (2.21).

For we consider the auxiliary function

$$\ell(x) = (-\ln |x|^2)^{-s} \ u(x) \ , \ s > 0 \ .$$

Then as Lemma 2 of [A] we have that

$$\Delta \ \ell + \sum_{i=1}^{n} b_i(x) \ \ell_{x_i} = \ell[4s(s-1) \frac{(-\ln|x|^2)^{-2}}{|x|^2} + 2s(n-2) \frac{(-\ln|x|^2)^{-1}}{|x|^2}$$

$$- u^{2/(n-2)}] \ ,$$

where

$$b_i(x) = - 4s \ (-\ln |x|^2)^{-1} \frac{x_i}{|x|^2} \ .$$

We then consider the function

$$\psi(r) = \int_r^{1/2} \frac{(-\ln t)^{-2s}}{t^{n-1}} \ dt \quad , \ 0 < r < \frac{1}{2} \ .$$

This function satisfies

$$\Delta\psi + \sum_{i=1}^{n} b_i(x) \ \psi_{x_i} = 0 \quad ,$$

and it has the property that

$$\psi(x) > M \ \frac{(-\ln|x|)^{-2s}}{|x|^{n-2}} \quad ,$$

where $M > 0$ is a constant.

Since

$$\lim_{|x|\to 0} |x|^{n-2} \ (- \ \ln \ |x|)^{(n-2)/2} \ \ u(x) = C < [\frac{(n-2)^2}{2} \ ]^{(n-2)/2} \ ,$$

by taking

$$\frac{C^{2/(n-2)}}{2(n-2)} < s < (n-2)/2 \tag{2.22}$$

we get (I) - (II) :

(I) $$\Delta \ell + \sum_{i=1}^{n} b_i(x)\, \ell_{x_i} \geq 0 \text{ near } 0 ,$$

(II) $$\ell(x) = 0\, ((-\ell n\, |x|)^{-(n-2)/2 - s}\, |x|^{-(n-2)})$$

Because of (2.22) we also have, $(\frac{(n-2)}{2} + s) > 2s$ and hence $\ell(x) < \psi(|x|)$ near 0 .

Suppose next that (I) and (II) hold for $0 < r < r_0$ . Let $M = \max_{|x|=r_0} \ell(x)$ . It follows from the maximum principle that for every $\varepsilon>0$ there exists $r(\varepsilon) < r < r_0$, $(r(\varepsilon) \to 0$ as $\varepsilon \to 0)$ such that

$$\ell(x) \leq \varepsilon\, \psi(|x|) + M \text{ if } r(\varepsilon) < |x| < r_0 .$$

Therefore, $\ell(x)$ is bounded. As in Lemma 2 of [A] this implies that the singularity must be removable.

The case where $- 2 < \sigma < 2$ is obtained with straightforward changes.

## § 3. APPENDIX.

In this appendix we shall give a new proof of a theorem of Gidas and Spruck.

In [G-S], Gidas and Spruck studied positive singular solutions of

$$\Delta u + u^q = 0 \text{ in } B|\{0\} ,\ \frac{n}{(n-2)} < q < \frac{(n+2)}{(n-2)} ,$$

where $B$ is the unit ball in $R^n$ , $n\geq 3$.

In Theorem 3.3 of [G-S] in where they claimed the estimate

$$u(x) \geq C\, |x|^{-2/(q-1)} , \tag{3.0}$$

where $C>0$ is a positive constant, the statement:

"If $\liminf_{x \to 0} |x|^{2/(q-1)} u(x) = 0$ then the Harnack inequality implies that

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = 0"$$

seems to need more explanation. The same statement was made later in [A].

However, as we shall see below, one can modify their proof of the theorem stated below, in such a way that one only need to use

$$u(x) < C/|x|^{2/(q-1)} . \tag{3.1}$$

Using only (3.1) we prove

<u>THEOREM</u> B (Gidas-Spruck). *Let u be a* $C^2(B\backslash\{0\})$ *nonnegative solution of*

$$\Delta u + |x|^{\sigma} u^q = 0 \quad \text{in } B\backslash\{0\} \tag{3.2}$$

*where*

$$1 < \frac{(n+\sigma)}{(n-2)} < q < \frac{(n+2)}{(n-2)} \quad , \quad -2 < \sigma < 2$$

*and* $q \neq (n-2 + 2\sigma)/(n-2)$.

*Then,* $u$ *has either a removable singularity at* $\{0\}$ *or*

$$\lim_{|x| \to 0} |x|^{(2+\sigma)/(q-1)} u(x) = C_0 , \tag{3.3}$$

*where* $C_0 = \left[ \frac{(2+\sigma)(n-2)}{(q-1)^2} \left(q - \frac{(n+\sigma)}{(n-2)}\right)\right]^{1/(q-1)}$

Clearly (3.3) is a stronger statement than (3.0)

<u>PROOF</u>. If $u$ is a solution of (3.2), then it follows from the work of Gidas and Spruck that

$$u(x) < C/ |x|^{(2+\sigma)/(q-1)} \tag{3.4}$$

where $C>0$ is a constant and $x$ is close to the origin.

Then we consider

$$t = -\ell n\ |x|$$

and

$$v(t,\theta) = |x|^{(2+\sigma)/(q-1)}\ u(r,\theta)\ ,$$

$$r = |x|\ ,\ \theta \in S^{n-1}\ ,\ t \in R\ .$$

Because of (3.4) $v$ is bounded. Now as in [G-S] (Theorem 1.4) we get the equation

$$v_{tt} + av_t + \Delta_\theta v - C_0^{q-1}\ v + v^q = 0$$

with

$$a = \frac{(n-2)}{(q-1)}\ \left(\frac{n+2+2\sigma}{n-2} - q\right)$$

$$C_0 = \left( \frac{(2+\sigma)(n-2)}{(q-1)^2}\ \left(q - \frac{(n+\sigma)}{(n-2)}\right)\right)^{1/(q-1)}$$

$$q \neq (n-2+2\sigma)/(n-2)\ .$$

Repeating the proof of their Theorem 1.4 (it should be observed that here one only need the fact that $v$ is bounded, see Lemma 4 of this article) we conclude that for each sequence $\{t_k\}$, $t_k \to \infty$, there exists a subsequence $t_k'$, such that

$$v(t_k'\ ,\ \theta) \to v(\theta)\ \text{ as }\ t_k' \to \infty$$

and where $v(\theta)$ satisfies

$$\text{(3.5)} \qquad \Delta_\theta\ v - C_0^{(q-1)}\ v + v^q = 0\ \text{ on }\ S^{n-1}\ .$$

It is shown in Appendix B of Gidas-Spruck [G-S] that the only solutions of (3.5) are

$$v=0 \quad \text{or} \quad v = C_0\ .$$

Hence because the limit set of a smooth function is a connected set we have

$$v(t,\theta) \to C_0 \quad \text{or} \quad v(t,\theta) \to 0 \quad \text{as} \quad t \to \infty .$$

(Observe that when $q = n/(n-2)$, then we do not know a-priori that the limit set of $v$ is a discrete set as it occurs in this case).

If the later occurs, it follows from the definition of $v$ that

$$\lim_{x \to 0} |x|^{(2+\sigma)/(q-1)} u(x) = 0 . \tag{3.6}$$

Then we define the auxiliary function

$$v(x) = |x|^s \; u(x) \; , \; s>0 .$$

By computing the Laplacian of this function and then by using the maximum principle, exactly as in the proof of Theorem 2 of [A] p. 785-786, (see also Lemma 18 of this article) we conclude that if (3.6) occurs, then the singularity must be removable. This concludes the proof of Theorem B.

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## § 4 APPENDIX II

In this appendix we shall give a simple proof of the Harnack inequality for positive solutions of (1.1).

In Lemma 2 we used the Harnack inequality for positive solutions of (1.1). This inequality is a consequence of Theorem 3.1 in [G-S]. However, their proof is rather complicated.

We begin by recalling the well known fact.

Lemma 4.1. *There are no non-negative* $C^\infty$ *solutions of* (1.1) *in* $R^n - K$, *where* $K \subset R^n$ *is a compact set of* $R^n$.

Proof. We assume $0 \in K$ and $K \subseteq \{x \in R^n : |x| < 1\}$, We then consider the average of $u$, $\bar{u}$, center at the origin. By averaging (1.1) we get (assuming $\sigma = 0$)

$$\bar{u}_{rr} + (n-1)\frac{\bar{u}_r}{r} + \bar{u}^{n/(n-2)} \leq 0, \quad r > 1.$$

(The case $\sigma \neq 0$, $-2<\sigma<2$ is treated in the same manner).

Next, we make the following change of variables,

$$v(r) = \bar{u}(r^{-1/(n-2)}) \quad , \quad r \leq 1$$

$$f(r) = r\, v(r^{-1}) \quad , \quad r \geq 1.$$

We obtain the differential inequality

$$f_{rr} + \frac{1}{(n-2)^2}\,\frac{f^{n/(n-2)}}{r^2} < 0 \quad , \ r > 1.$$

Therefore, $f_{rr} \leq 0$ and because $f \geq 0$ we obtain that $f_r \geq 0$. Hence

$$(4.1) \qquad f_r(r) = f_r(r_0) + \int_{r_0}^{r} f_{rr}(s)\, ds$$

$$\leq f_r(r_0) - \frac{1}{(n-2)^2}\,\frac{f^{n/(n-2)}(r_o)}{r_o} + \frac{1}{(n-2)^2}\,\frac{f^{n/(n-2)}(r_o)}{r}$$

If there is $r_0 \geq 1$ such that

$$(4.2) \qquad f_r(r_0) - \frac{1}{(n-2)^2}\,\frac{f^{n/(n-2)}(r_o)}{r_o} < 0,$$

then by letting $r \to \infty$ in (4.1) we conclude that

$$f_r(r) < 0., \text{ for } r \text{ sufficiently large}$$

This is a contradiction.

So, since (4.2) never holds we have

$$f_r(r) \geq \frac{1}{(n-2)^2}\,\frac{f^{n/(n-2)(r)}}{r} \quad , \; r \geq 1.$$

Hence, by integrating from 1 to $r$ we obtain

$$\frac{-(n-2)}{2} f^{-2/(n-2)}(r) + \frac{(n-2)}{2} f^{-2/(n-2)}(1) \geq \frac{1}{(n-2)^2} \ln r.$$

Since as $r \to \infty$, $f(r) \to c \lesssim \infty$ we get a contradiction.

Lemma 4.2 If $u \in C^\infty(\Omega)$ is a non-negative solution of (1.1) in $\Omega$, then

$$\sup_{x \in \tilde{\Omega}} u(x) \lesssim C(\tilde{\Omega}, n),$$

where $\tilde{\Omega} \subset\subset \Omega$ and $C(\tilde{\Omega}, n) > 0$ is a constant depending only on $\tilde{\Omega}$ and $n$ but independent of $u$.

Proof. This lemma follows at once from Lemma 4.1. (See [G - S, II] p.887 - 890). Indeed suppose there are a sequence of $C^\infty$-solutions of (1.1) in $\Omega$, say $u_i$, and a sequence of points $p_i \to p$, $p_i$, $p \in \bar{\tilde{\Omega}}$ such that

$$M_i = u_i(p_i) \to \infty \quad \text{as } i \to \infty.$$

We consider

$$v_i(x) = \lambda_i^{n-2} u_i(\lambda_i x + p_i), \qquad |x| \lesssim \lambda_i^{-1}$$

where we have asumed that $B_1(p_i) \subset \Omega$ and where $\lambda_i \to 0$ is defined by

$$\lambda_i^{n-2} u_i(p_i) = 1.$$

Since $p_i \to p \in \Omega$, and $B_{1/2}(p) \subset \Omega$, standard elliptic estimates and a diagonalization procedure imply we can find $v$ and a subsequence $i' \to \infty$ such that

$$v_{i'} \to v \text{ in the } C^2 \text{ topology of } R^n,$$

$$\Delta v + v^{n/(n-2)} = 0 \text{ in } R^n, \quad v(0) = 1$$

But this contradicts Lemma 4.2.

Lemma 4.3 Let $u \geq 0$, $u \in C^\infty(B|\{0\})$ be a solution of (1.1) in $B|\{0\}$, where $B = \{x \in R^n : |x| < 1\}$. Then

(4.3) $u(x) \leq C/|x|^{n-2}$ , $|x| \leq \frac{1}{2}$;

(4.4) $$\sup_{x \in B(x, \frac{|x|}{2})} u(x) < C \inf_{x \in B(x, \frac{|x|}{2})} u(x)$$

where $|x| \leq \frac{1}{2}$ , and $C > 0$ is a constant independent of $u$ and $x$;

(4.5) $$\sup_{\epsilon_0 \leq |x| \leq (1+\theta)\epsilon_0} u(x) \leq C \inf_{\epsilon_0 \leq |x| \leq (1+\theta)\epsilon_0} u(x)$$

where $C > 0$, $0 < \theta < \frac{1}{2}$, $\epsilon_0 > 0$ and small, are constants independent of $u$

Proof. We prove (4.3). Let $x_0 \neq 0$ , $|x_0| \leq \frac{1}{2}$. We consider

$w(x) = |x_0|^{n-2} u(|x_0|x + x_0)$ , $|x| < 1$.

By Lemma 4.2

$$w(x) \leq C \quad \text{if} \quad |x| \leq \frac{1}{2}.$$

In particular,

$$|x_0|^{n-2} u(x_0) = w(0) \leq C.$$

This proves (4.3).

(4.4) and (4.5) follow from (4.3) by using standard arguments. Indeed we write (1.1) in the form

$$\Delta u + u^{2/(n-2)} u = 0$$

(4.3) implies that we can apply standard results for linear equation to conclude (4.4) and (4.5). We refer to the proof of Theorem 3.1 of Gidas and Spruck [G-S] for further details.

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