Symmetry Property and Construction of Wavelets With a General Dilation Matrix

Bin Han *

Abstract

In this note, we are interested in the symmetry property of a refinable function with a general dilation matrix. We investigate the symmetry group of a mask so that its associated refinable function with a general dilation matrix has certain kind of symmetry. Given two dilation matrices which produce the same lattice, we demonstrate that if a mask has certain kind of symmetry, then its associated refinable functions with respect to the two dilation matrices are the same; therefore, the two corresponding derived wavelet systems are essential the same. Finally, we illustrate that for any dilation matrix, orthogonal masks, as well as interpolatory masks having nonnegative symbols, can be easily constructed with any preassigned order of sum rules by employing a linear transform. Without solving any equation, the method in this note on constructing masks with certain desirable properties is simple, painless and general. Examples of quincunx wavelets are presented to illustrate the general theory.

Key words: Symmetry, wavelets, dilation matrix, sum rules, quincunx lattice.

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Correspondence address: Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1. Email: bhan@ualberta.ca

Current address: Institute for Mathematics and its Applications, University of Minnesota, 400 Lind Hall, 207 Church Street S.E., Minneapolis, MN 55455-0436. USA. Email: bhan@ima.umn.edu

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1 Introduction

Multidimensional wavelets are useful in dealing with high dimensional problems. It has been observed in the literature, for example, [2, 5, 9], that tensor product dyadic wavelets are not enough to deal with a variety of problems from a broad range of applications. Consequently, multidimensional wavelets with a general dilation matrix have been extensively studied in the literature [1, 2, 3, 6, 7, 9, 10, 11, 12] and references therein. For example, due to their special features, quincunx wavelets have been discussed in [1, 2, 3, 7, 9, 10, 12] and in many other papers. For detailed arguments about advantages of multidimensional wavelets such as quincunx wavelets, the reader is referred to [2, 5, 9, 10].

An $s \times s$ integer matrix M is called a **dilation matrix** if $\lim_{k\to\infty} M^{-k} = 0$. We say that a is a **mask** on $\mathbb{Z}^s$ if a is a finitely supported sequence on $\mathbb{Z}^s$ such that $\sum_{\beta\in\mathbb{Z}^s} a(\beta) = 1$. Wavelets are derived from refinable functions via a standard multiresolution technique. A **refinable function** ϕ is a solution to the following refinement equation

$$\phi = |\det M| \sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(M\cdot-\beta), \tag{1.1}$$

where a is a mask and M is a dilation matrix. For a mask a on $\mathbb{Z}^s$ and an $s\times s$ dilation matrix M, it is known that there exists a unique compactly supported distributional solution, denoted by ϕ_a^M throughout the note, to the refinement equation (1.1) such that $\widehat{\phi_a^M}(0) = 1$.

An orthogonal wavelet is derived from an **orthogonal** refinable function $\phi_a^M \in L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} \phi_a^M(x)\overline{\phi_a^M(x+\beta)}\,dx = \delta(\beta) \qquad \forall\, \beta\in\mathbb{Z}^s, \tag{1.2}$$

where $\delta(0) = 1$ and $\delta(\beta) = 0$ for all $\beta \in \mathbb{Z}^s\backslash\{0\}$.

In the tensor product dyadic wavelets, the dilation matrix $2I_s$ is used. When a general dilation matrix M is used in a wavelet system, it is necessary to use $|\det M| - 1$ wavelet functions with compact support to generate a wavelet system. Therefore, the case $|\det M| = 2$ is of particular interest in the literature since only a single wavelet function is needed to generate a wavelet system. In fact, when $|\det M| = 2$ and (1.2) holds, the associated wavelet function ψ can be easily obtained as follows:

$$\psi = |\det M| \sum_{\beta\in\mathbb{Z}^s} (-1)^{|\alpha-\beta|}\overline{a(\alpha-\beta)}\phi_a^M(M\cdot-\beta),$$

where $\alpha \in \mathbb{Z}^s\backslash M\mathbb{Z}^s$ and $|(\alpha_1,\cdots,\alpha_s)| = |\alpha_1| + \cdots + |\alpha_s|$.

In dimension two, the following two quincunx dilation matrices

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.3}$$

are of particular interest. Their associated lattice $Q\mathbb{Z}^2 = T\mathbb{Z}^2 = \{(\beta_1, \beta_2) \in \mathbb{Z}^2 \ : \ \beta_1 + \beta_2 \text{ is an even integer } \}$ is called the **quincunx lattice** in the literature. Quincunx wavelets using these dilation matrices have attracted a lot of interest and have been discussed in detail in [1, 2, 3, 7, 9, 10, 12] and many other references in the literature. For example, orthogonal quincunx wavelets are constructed in [1, 2, 9, 10] and biorthogonal quincunx wavelets are discussed in [2, 7].

However, the results in the literature [1, 2] reflect a sharp contrast between the dilation matrices Q and T. On one hand, arbitrarily smooth compactly supported quincunx orthogonal refinable functions with the dilation matrix T were reported in Cohen and Daubechies [2] for separable ones, and Belogay and Wang [1] for nonseparable ones. On the other hand, to our best knowledge, it is still an unsolved problem whether there exists a C^1 orthogonal compactly supported refinable function with respect to the dilation matrix Q (see [1, 2]). Due to these facts, it is believed explicitly or implicitly that it is much harder to deal with the dilation matrix Q than to deal with T. Sometimes in the literature, Q is referred as the "real" quincunx dilation matrix and T as a "separable" one since $T^2 = 2I_2$ and $Q^4 = -4I_2$.

In many applications, symmetry is a much desirable property of a wavelet system. However, it seems to us that the symmetry of quincunx wavelets is seldom discussed in the literature, needless to say symmetry of wavelets with a general dilation matrix. Let E be an integer matrix such that $|\det E| = 1$. If a is a mask satisfying $a(E\beta) = a(\beta)$ for all $\beta \in \mathbb{Z}^s$, then it is straightforward to see from the refinement equation (1.1) that $\phi_a^{E^{-1}ME} = \phi_a^M(E\cdot)$. When $M = 2I_s$, then $\phi_a^M = \phi_a^M(E\cdot)$ which means the symmetry of a mask carries over to its refinable function directly. But this is not necessarily true in general. In Section 2, we shall study the symmetry group of a mask such that its associated refinable function with a general dilation matrix possesses certain kind of symmetry.

Note that there does not exist a 2×2 matrix E such that $EQE^{-1} = T$ since $T^2 = 2I_2 \neq Q^2$ for the two quincunx dilation matrices in (1.3). For a same mask a on $\mathbb{Z}^2$, it seems that the refinable functions ϕ_a^Q and ϕ_a^T are unrelated as demonstrated by the following discussion. Let b_r be the interpolatory mask in [4] which is supported on $[1-2r, 2r-1]$. Then we can obtain a mask $q_{2r-1,0}$ on $\mathbb{Z}^2$ by $q_{2r-1,0}(j,k) = b_r(j)\delta(k), j,k \in \mathbb{Z}$. Even though $q_{2r-1,0} (r \in \mathbb{N})$ are symmetric about the two coordinate axes, Cohen and Daubechies in [2] demonstrated that $\phi^T_{q_{2r-1,0}}$ can be made arbitrarily smooth by increasing r while $\phi^Q_{q_{2r-1,0}} \notin C^1(\mathbb{R}^2)$ for all $r \in \mathbb{N}$. A similar phenomenon happens for other quincunx masks in [7] which are also symmetric about the two coordinate axes.

On the other hand, we observed that for a family of masks with better symmetry in [7], the smoothness exponents of the quincunx refinable functions with both Q and T computed in Tables 1 and 2 of [7] are the same. This motivates us to think about whether there is a relation between ϕ_a^Q and ϕ_a^T when a has certain kind of symmetry.

Given two dilation matrices M and N, for a mask a, we shall investigate in Section 2 the symmetry group of the mask a under which we guarantee $\phi_a^M = \phi_a^N$. As a consequence, we can demonstrate that all the quincunx biorthogonal wavelets constructed in [2], as well as some other examples in [7], are essentially the same with respect to either Q or T.

Finally, in Section 3, we shall further investigate the relations between different dilation matrices. Using a linear transform which preserves cosets, we are able to demonstrate that for any dilation matrix, orthogonal masks, as well as interpolatory masks having nonnegative symbols, can be easily obtained with any preassigned order of sum rules. Comparing with other methods on constructing wavelets in the literature, the method proposed in this note is simple, painless and general without solving any equation.

2 Symmetry of a Mask and a Refinable Function

Let $U(\mathbb{Z}^s)$ denote all the $s \times s$ integer matrices E such that $|\det E| = 1$. That is, $E \in U(\mathbb{Z}^s)$ if and only if E is an isomorphism on $\mathbb{Z}^s$. Given a mask a on $\mathbb{Z}^s$ and $E \in U(\mathbb{Z}^s)$, we say that a is **invariant** under E if

$$a(E\beta) = a(\beta) \qquad \forall\, \beta \in \mathbb{Z}^s.$$

Moreover, we define

$$G_a := \{E \in U(\mathbb{Z}^s) \ : \ a \ \text{ is invariant under } E\}. \tag{2.1}$$

Then it is obvious that G_a is a group under the matrix multiplication; thus we say that G_a is the **symmetry group** associated with the mask a. By the following result, we see that G_a is a special subgroup of $U(\mathbb{Z}^s)$.

Proposition 2.1 *Let a be a finitely supported mask on $\mathbb{Z}^2$ such that the span of $\{\beta \in \mathbb{Z}^2 \ : \ a(\beta) \neq 0\}$ is $\mathbb{R}^2$ (that is, the support of a is not contained in a straight line through the origin). Then G_a is a subset of*

$$\{\pm I_2\} \cup \left\{ \begin{bmatrix} m & j \\ k & -m \end{bmatrix} \ : \ jk = \pm 1 - m^2 \right\} \cup \left\{ \begin{bmatrix} m & j \\ k & \pm 1 - m \end{bmatrix} \ : \ jk = m(\pm 1 - m) - 1 \right\}, \tag{2.2}$$

where $j, k, m \in \mathbb{Z}$.

Proof: Let $E \in G_a$ and $\operatorname{supp} a := \{\beta \in \mathbb{Z}^2 \ : \ a(\beta) \neq 0\}$. Since a is invariant under E, we deduce that $\{E^\ell \beta \ : \ \ell \in \mathbb{N}, \beta \in \operatorname{supp} a\} \subseteq \operatorname{supp} a$. Since $\operatorname{supp} a$ is a finite set, there exists an integer ℓ such that $E^\ell \beta = \beta$ for all $\beta \in \operatorname{supp} a$. Now $E^\ell = I_2$ follows directly from the fact that the span of the set $\operatorname{supp} a$ is $\mathbb{R}^2$. So the eigenvalues of E must have absolute value 1. Since E is an integer matrix, it yields that $\operatorname{trace}(E) = 0, \pm 1$ or ± 2.

Case 1: $\operatorname{trace}(E) = \pm 2$. Then the two eigenvalues of E must be equal and take value either 1 or -1. Using the Jordan matrix form of E, it follows from $E^\ell = I_2$ that $E = \pm I_2$.

Case 2: $\operatorname{trace}(E) = 0$. Since $\det E = \pm 1$, we can assume that $E = \begin{bmatrix} m & j \\ k & -m \end{bmatrix}$ with $j, k, m \in \mathbb{Z}$ and $jk = \pm 1 - m^2$. It is easy to check that $E \in U(\mathbb{Z}^2)$ and $E^2 = \pm I_2$.

Case 3: $\operatorname{trace}(E) = \pm 1$. In this case, the eigenvalues of E must not be real numbers and therefore, we must have $\det E = 1$. Assume $E = \begin{bmatrix} m & j \\ k & \pm 1 - m \end{bmatrix}$ with $j, k, m \in \mathbb{Z}$ and $jk = m(\pm 1 - m) - 1$. It is easy to check that $E \in U(\mathbb{Z}^2)$ and $E^3 = \mp I_2$. ■

For any element E of the set in (2.2), we can easily construct a finitely supported mask on $\mathbb{Z}^2$ such that it is invariant under E. Though the discussion will be more complicated, the arguments in the proof of Proposition 2.1 can apply to higher dimensions.

For any complex-valued matrix A, by A^* we denote the complex conjugate of the transpose of the matrix A. Given a mask a, its **symbol** is defined to be

$$\widetilde{a}(\xi) := \sum_{\beta \in \mathbb{Z}^s} a(\beta) e^{-i\beta \cdot \xi}, \qquad \xi \in \mathbb{R}^s. \tag{2.3}$$

In terms of symbol, a is invariant under E if and only if $\widetilde{a}(E^*\xi) = \widetilde{a}(\xi)$ for all $\xi \in \mathbb{R}^s$. Using the symbol of a mask, we can rewrite the refinement equation in (1.1) as follows

$$\widehat{\phi_a^M}(\xi) = \widetilde{a}((M^*)^{-1}\xi)\widehat{\phi_a^M}((M^*)^{-1}\xi), \qquad \xi \in \mathbb{R}^s. \tag{2.4}$$

Theorem 2.2 *Let a be a mask on $\mathbb{Z}^s$ and M be an $s \times s$ dilation matrix. Define*

$$G_a^M := \{E \in U(\mathbb{Z}^s) \ : \ M^j E M^{-j} \in G_a \quad \forall\, j \in \mathbb{N}\} = U(\mathbb{Z}^s) \cap \cap_{j=1}^{\infty} M^{-j} G_a M^j, \tag{2.5}$$

where G_a is the symmetry group of a in (2.1). Then G_a^M is a group under the matrix multiplication and $\phi_a^M(E\cdot) = \phi_a^M$ for all $E \in G_a^M$. That is, ϕ_a^M is invariant under all the elements in G_a^M.

Proof: When $E, F \in G_a^M$, then $EF^{-1} \in G_a^M$ by $M^j EF^{-1} M^{-j} = (M^j E M^{-j})(M^j F M^{-j})^{-1} \in G_a$ for all $j \in \mathbb{N}$. So G_a^M is a group.

For any $E \in G_a^M$, we have $M^j E M^{-j} \in G_a$ for all $j \in \mathbb{N}$. Hence,

$$\widehat{\phi_a^M}(E^*\xi) = \prod_{j=1}^{\infty} \widetilde{a}((M^*)^{-j}E^*\xi) = \prod_{j=1}^{\infty} \widetilde{a}((M^j E M^{-j})^*(M^*)^{-j}\xi) = \prod_{j=1}^{\infty} \widetilde{a}((M^*)^{-j}\xi) = \widehat{\phi_a^M}(\xi).$$

Hence, $\phi_a^M(E\cdot) = \phi_a^M$ for all $E \in G_a^M$. ■

Let a be a mask on $\mathbb{Z}^2$ such that $G_a = \{\pm I_2\}$ or $G_a = \{\pm I_2, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$. Then $G_a^T = G_a^Q = G_a^{Q^*} = \{\pm I_2\}$, where the quincunx dilation matrices Q and T are defined in (1.3). Therefore, when the dilation matrix is not a multiple of the identity matrix, better symmetry of a mask may not be able to guarantee better symmetry of its refinable function.

Theorem 2.3 *Let M and N be two $s \times s$ dilation matrices such that $N^j M^{-j}$ are integer matrices for all $j \in \mathbb{N}$. Define a set $S_{M,N}$ of integer matrices as follows:*

$$S_{M,N} := \{N^j M^{-j} \quad : \quad j \in \mathbb{N}\}. \tag{2.6}$$

Let a be a mask on $\mathbb{Z}^s$. If a is invariant under all the elements in $S_{M,N}$; that is, $S_{M,N} \subseteq G_a$, then $\phi_a^M = \phi_a^N$.

Proof: Since $N^j M^{-j} \in G_a$ for all $j \in \mathbb{N}$, we have

$$\widehat{\phi}_a^M(\xi) = \prod_{j=1}^{\infty} \widetilde{a}((M^*)^{-j}\xi) = \prod_{j=1}^{\infty} \widetilde{a}((N^j M^{-j})^*(N^*)^{-j}\xi) = \prod_{j=1}^{\infty} \widetilde{a}((N^*)^{-j}\xi) = \widehat{\phi}_a^N(\xi).$$

Therefore, $\phi_a^M = \phi_a^N$. ■

Example 2.4 Let Q and T be the quincunx dilation matrices in (1.3). By computation, we have

$$S_{Q,T} = S_{T,Q} = S_{Q^*,T} = S_{T,Q^*} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \tag{2.7}$$

and

$$S_{Q,Q^*} = S_{Q^*,Q} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

which are groups under the matrix multiplication. If a mask a is invariant under all the elements in $S_{Q,T}$, then $S_{Q,T} \subseteq G_a^Q$ and by Theorems 2.2 and 2.3, $\phi_a^T = \phi_a^Q = \phi_a^{Q^*}$ which is symmetric under all the elements in the group $S_{Q,T}$. All the masks for the quincunx biorthogonal wavelets constructed in [2] are invariant under $S_{Q,T}$ and therefore, their associated refinable functions with respect to T, Q and Q^* are the same. Theorem 2.3 also explains why the smoothness exponents of the refinable functions $\phi_{g_r}^Q$ and $\phi_{g_r}^T$ computed in Tables 1 and 2 of [7] are the same for a family of masks $g_r (r \in \mathbb{N})$ in [7] since all $g_r (r \in \mathbb{N})$ are invariant under $S_{Q,T}$.

That a mask is invariant under $S_{Q,T}$ is equivalent to saying that it is symmetric about the two coordinate axes $x_1 = 0, x_2 = 0$ and the lines $x_1 = x_2$ and $x_1 = -x_2$. In fact, the group $S_{Q,T}$ is quite natural and maximal due to the following fact.

Proposition 2.5 *Let a be a mask on $\mathbb{Z}^2$ such that the support of a is not contained in a straight line through the origin. If a is invariant under either $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then G_a must be a subgroup of $S_{Q,T}$ in (2.7).*

Proof: Let $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If $F \in G_a$ comes from Case 2 in the proof of Proposition 2.1, then $F = \begin{bmatrix} m & j \\ k & -m \end{bmatrix}$ with $j, k, m \in \mathbb{Z}$ and $jk = \pm 1 - m^2$. Since $E, F \in G_a$, we must have $EF = \begin{bmatrix} m & j \\ -k & m \end{bmatrix} \in G_a$. By Proposition 2.1, we have $\text{trace}(EF) = 2m = \pm 2$ or 0; that is, $m = 0, \pm 1$. When $m = 0$, then $jk = \pm 1 - m^2 = \pm 1$ implies $F \in S_{Q,T}$. When $m = \pm 1$, then $\text{trace}(EF) = \pm 2$ and therefore, by Proposition 2.1, $EF = \pm I_2$ which implies $F = \pm E \in S_{Q,T}$.

If $F \in G_a$ comes from Case 3 in the proof of Proposition 2.1, then $\det F = 1$. However, by a simple computation, we deduce that $\text{trace}(EF) = \pm 1$ which implies that EF must also come from Case 3. But $\det EF = \det E \det F = -1$ which is impossible. ∎

3 Constructing Wavelets via a Linear Transform

Let a be a mask and M be a dilation matrix. When the refinable function ϕ_a^M satisfies (1.2), then it is necessary that a is an **orthogonal mask** (with respect to the lattice $M\mathbb{Z}^s$), that is,

$$\sum_{\beta \in \mathbb{Z}^s} a(\alpha + \beta)\overline{a(\beta)} = \delta(\alpha)/|\det M| \qquad \forall\, \alpha \in M\mathbb{Z}^s. \tag{3.1}$$

It is known that ϕ_a^M is an orthogonal refinable function if and only if a is an orthogonal mask with respect to the lattice $M\mathbb{Z}^s$ and the subdivision scheme associated with mask a and the dilation matrix M converges in the L_2 norm. For simplicity, in this section we only deal with a mask a rather than the more subtle issue of the stability of the refinable function ϕ_a^M, though such stability property of ϕ_a^M can be established when all the zeros of $\widetilde{a}$ are known.

A much desirable property of a wavelet system is its order of vanishing moments since it guarantees that the wavelet representation of a piecewise smooth function is sparse. The order of vanishing moments of a wavelet system is closely related to the order of sum rules satisfied by a mask. We say that a satisfies the **sum rules** of order ℓ (with respect to the lattice $M\mathbb{Z}^s$) if

$$\sum_{\beta \in M\mathbb{Z}^s} a(\alpha + \beta) q(\alpha + \beta) = \sum_{\beta \in M\mathbb{Z}^s} a(\beta) q(\beta) \qquad \forall\, q \in \Pi_{\ell - 1}, \tag{3.2}$$

where $\Pi_{\ell-1}$ denotes the set of all polynomials of (total) degree less than ℓ.

A closely related concept to an orthogonal mask is an interpolatory mask. We say that a mask b is an **interpolatory mask** (with respect to the lattice $M\mathbb{Z}^s$) if

$$b(\beta) = \delta(\beta)/|\det M| \qquad \forall\, \beta \in M\mathbb{Z}^s. \tag{3.3}$$

Given a mask a, we can define another mask b by $\widetilde{b}(\xi) = |\widetilde{a}(\xi)|^2$. Then a is an orthogonal mask with respect to the lattice $M\mathbb{Z}^s$ if and only if b is an interpolatory mask with respect to the lattice $M\mathbb{Z}^s$, or equivalently,

$$\sum_{j=1}^{|\det M|} \widetilde{b}(\xi + 2\pi\varepsilon_j) = 1 \qquad \forall\, \xi \in \mathbb{R}^s,$$

where $\{\varepsilon_j \;:\; j = 1, \cdots, |\det M|\}$ is a complete set of representatives of the distinct cosets of the quotient group $(M^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$. For discussion on interpolatory masks, the reader is referred to [3, 4, 6, 7, 11].

A simple observation is that the definitions of an orthogonal mask in (3.1), an interpolatory mask in (3.3) and the order of sum rules in (3.2) depend only on the lattice $M\mathbb{Z}^s$ rather than the dilation matrix M since $|\det M|$ equals the number of distinct cosets in the quotient group $\mathbb{Z}^s/M\mathbb{Z}^s$. On the other hand, let a be a mask and E be an integer matrix with $|\det E| = 1$. Consider a new sequence b given by $b(\beta) = a(E\beta), \beta \in \mathbb{Z}^s$. Suppose that E maps the coset $M\mathbb{Z}^s$ into the coset $M\mathbb{Z}^s$. Then we observe that E must map any coset $\alpha + M\mathbb{Z}^s$ one-to-one and onto another coset $E\alpha + M\mathbb{Z}^s$. Hence, it is easy to verify that if a is an orthogonal mask, or is an interpolatory mask, or satisfies the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^s$, then so does the mask b.

So a more accurate definition of a symmetry group associated with a mask in (2.1) of Section 2 should consider only elements $E \in U(\mathbb{Z}^s)$ such that $EM\mathbb{Z}^s \subseteq M\mathbb{Z}^s$. Fortunately, all the elements in the group $S_{Q,T}$ in (2.7) maps the quincunx lattice $Q\mathbb{Z}^2$ into itself.

In the following, let us explore the idea of using a linear transform to obtain new masks from known ones in a more complicated setting. Given two lattices $M\mathbb{Z}^r$ and $N\mathbb{Z}^s$, if we have a linear transform $P : \mathbb{R}^s \mapsto \mathbb{R}^r$ such that it maps a coset $\alpha + N\mathbb{Z}^s (\alpha \in \mathbb{Z}^s)$ into a coset $P\alpha + M\mathbb{Z}^r$ if $P\alpha \in \mathbb{Z}^r$, then such a linear mapping will transform a mask a on $\mathbb{Z}^s$ into a mask b on $\mathbb{Z}^r$ by

$$b(\alpha) = c \sum_{\{\beta\in\mathbb{Z}^s \,:\, P\beta=\alpha\}} a(\beta), \qquad \alpha \in \mathbb{Z}^r,$$

where c is a constant so that $\sum_{\alpha\in\mathbb{Z}^r} b(\alpha) = 1$. If P is carefully chosen, then the order of sum rules of mask a with respect to $N\mathbb{Z}^s$ can be preserved so that the new mask b on $\mathbb{Z}^r$ can also satisfy the same order of sum rules with respect to $M\mathbb{Z}^r$ since the equations in the definition of the sum rules in (3.2) still hold for the new mask b and lattice $M\mathbb{Z}^r$ under certain conditions on the linear transform P.

Before we state the result, let us introduce some notation. Let $D_s = [\partial_1, \cdots, \partial_s]^*$ where ∂_j denotes the partial derivative with respect to the j-th coordinate axis. The Kronecker product of two matrices $A = (a_{ij})_{1\le i\le l, 1\le j\le n}$ and B, written as $A \otimes B$, is defined to be the following matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1}B & a_{l2}B & \cdots & a_{ln}B \end{bmatrix}.$$

It is easy to see ([8]) that a satisfies the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^s$ if and only if

$$[\underbrace{D_s \otimes \cdots \otimes D_s}_{k\ \ times} \otimes \widetilde{a}](2\pi\varepsilon) = 0 \qquad \forall\, \varepsilon \in (M^*)^{-1}\mathbb{Z}^s \backslash \mathbb{Z}^s, 0 \le k < \ell.$$

The following lemma will be needed later.

Lemma 3.1 *Let P be an $s \times r$ real-valued matrix and F be an $m \times n$ matrix of $C^\infty(\mathbb{R}^s)$ functions. Then $D_r \otimes [F(P\cdot)] = [(P^* D_s) \otimes F](P\cdot) = (P^* \otimes I_m)(D_s \otimes F)(P\cdot)$ and consequently, for any positive integer k,*

$$\begin{aligned}\underbrace{D_r \otimes \cdots \otimes D_r}_{k\ times} \otimes [F(P\cdot)] &= \Big[\underbrace{(P^*D_s) \otimes \cdots \otimes (P^*D_s)}_{k\ times} \otimes F\Big](P\cdot) \\ &= \Big(\underbrace{P^* \otimes \cdots \otimes P^*}_{k\ times} \otimes I_m\Big)\Big(\underbrace{D_s \otimes \cdots \otimes D_s}_{k\ times} \otimes F\Big)(P\cdot).\end{aligned}$$

Proof: By a simple computation, we have

$$[D_r \otimes [F(P\cdot)]]_{i1,jk} = \partial_i [F_{jk}(P\cdot)] = \sum_{t=1}^{s} \partial_t F_{jk} P_{ti} = \sum_{t=1}^{s} P^*_{it} \partial_t F_{jk} = [(P^*D_s) \otimes F]_{i1,jk},$$

which completes the proof. ∎

Now we have the following result on constructing new masks from known ones via a linear transform.

Theorem 3.2 *Let M be an $r \times r$ integer matrix and N be an $s \times s$ integer matrix such that both M and N have nonzero determinants. Let P be an $r \times s$ real-valued matrix such that*

$$PN\mathbb{Z}^s \subseteq M\mathbb{Z}^r \quad and \quad (\alpha + M\mathbb{Z}^r) \cap P\mathbb{Z}^s \ne \emptyset \qquad \forall\, \alpha \in \mathbb{Z}^r. \tag{3.4}$$

Let $\{\varepsilon_j\}_{j=1}^t$ be a complete set of representatives of the distinct cosets of the quotient group $P^*\mathbb{Z}^r/\mathbb{Z}^s$. For any mask a on $\mathbb{Z}^s$ such that a satisfies the sum rules of order 1 with respect to the lattice $N\mathbb{Z}^s$, define a sequence Pa on $\mathbb{Z}^r$ as follows:

$$\widetilde{Pa}(\xi) = \sum_{j=1}^{t} \widetilde{a}(P^*\xi + 2\pi\varepsilon_j), \qquad \xi \in \mathbb{R}^r, \tag{3.5}$$

or equivalently,

$$Pa(\alpha) := t \sum_{\{\beta\in\mathbb{Z}^s\,:\,P\beta=\alpha\}} a(\beta), \qquad \alpha \in \mathbb{Z}^r, \tag{3.6}$$

where by convention $Pa(\alpha) := 0$ when $\{\beta \in \mathbb{Z}^s \ : \ P\beta = \alpha\}$ is the empty set. Then Pa is a well-defined finitely supported mask on $\mathbb{Z}^r$ such that

(1) *If a satisfies the sum rules of order ℓ with respect to the lattice $N\mathbb{Z}^s$, then Pa also satisfies the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^r$;*

(2) *If $\widetilde{a}(\omega) \geq 0$ for all $\omega \in \mathbb{R}^s$, then $\widetilde{Pa}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^r$;*

(3) *If the following extra condition*

$$\{\beta \in \mathbb{Z}^s \ : \ P\beta \in M\mathbb{Z}^r\} \subseteq N\mathbb{Z}^s \tag{3.7}$$

holds, then for any interpolatory mask a with respect to the lattice $N\mathbb{Z}^s$, Pa is an interpolatory mask with respect to the lattice $M\mathbb{Z}^r$.

Proof: Let $S = M^{-1}PN$. Then by assumption in (3.4) S is an integer matrix and $P^* = (N^*)^{-1}S^*M^*$. Hence, $P^*\mathbb{Z}^r/\mathbb{Z}^s$ is a subgroup of $(N^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$ and t is a finite integer.

Since $P^*\mathbb{Z}^r/\mathbb{Z}^s$ is a subgroup of $(N^*)^{-1}\mathbb{Z}^s/\mathbb{Z}^s$ and a satisfies the sum rules of order 1, we have $\widetilde{Pa}(0) = \widetilde{a}(0) = 1$. To demonstrate that Pa is finitely supported, it suffices to show that $\widetilde{Pa}$ is a $2\pi\mathbb{Z}^r$-periodic trigonometric polynomial. For any $\alpha \in \mathbb{Z}^r$,

$$\widetilde{Pa}(\xi + 2\pi\alpha) = \sum_{j=1}^{t} \widetilde{a}(P^*\xi + 2\pi(P^*\alpha + \varepsilon_j)) = \sum_{j=1}^{t} \widetilde{a}(P^*\xi + 2\pi\varepsilon_j) = \widetilde{Pa}(\xi)$$

since $\{P^*\alpha + \varepsilon_j \ : \ j = 1, \cdots, t\}$ is also a complete set of representatives of the distinct cosets of $P^*\mathbb{Z}^r/\mathbb{Z}^s$. Therefore, Pa is a well-defined mask on $\mathbb{Z}^r$.

Note that for any $\beta \in \mathbb{Z}^s$, the set $\{e^{-i2\pi\beta\cdot\varepsilon_j}\}_{j=1}^{t}$ is a group under multiplication. Moreover, if $\langle\beta, \varepsilon_j\rangle$ is an integer for all $j = 1, \cdots, t$, since $\{\varepsilon_j\}_{j=1}^{t}$ is a complete set of representatives of the distinct cosets of $P^*\mathbb{Z}^r/\mathbb{Z}^s$, then $\langle\beta, P^*\alpha\rangle = \langle P\beta, \alpha\rangle$ is an integer for all $\alpha \in \mathbb{Z}^r$. Therefore, $P\beta$ must be an integer whenever $\beta \in \mathbb{Z}^s$ and $\langle\beta, \varepsilon_j\rangle$ is an integer for all $j = 1, \cdots, t$. By a simple argument, we observe that

$$\sum_{j=1}^{t} e^{-i2\pi\beta\cdot\varepsilon_j} = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \text{ such that } P\beta \notin \mathbb{Z}^r.$$

It follows from the above identity that

$$\widetilde{Pa}(\xi) = \sum_{j=1}^{t} \widetilde{a}(P^*\xi + 2\pi\varepsilon_j) = \sum_{j=1}^{t}\sum_{\beta\in\mathbb{Z}^s} a(\beta)e^{-i\beta\cdot(P^*\xi+2\pi\varepsilon_j)} = \sum_{j=1}^{t}\sum_{\beta\in\mathbb{Z}^s} a(\beta)e^{-iP\beta\cdot\xi}\sum_{j=1}^{t} e^{-i2\pi\beta\cdot\varepsilon_j}$$

$$= \sum_{\alpha\in P\mathbb{Z}^s} e^{-i\alpha\cdot\xi} \sum_{\{\beta\in\mathbb{Z}^s\,:\,P\beta=\alpha\}} a(\beta)\sum_{j=1}^{t} e^{-i2\pi\beta\cdot\varepsilon_j} = t\sum_{\alpha\in\mathbb{Z}^r} e^{-i\alpha\cdot\xi} \sum_{\{\beta\in\mathbb{Z}^s\,:\,P\beta=\alpha\}} a(\beta).$$

Therefore, (3.5) and (3.6) are equivalent.

If a satisfies the sum rules of order ℓ with respect to the lattice $\mathbb{N}\mathbb{Z}^s$, then

$$[\underbrace{D_s\otimes\cdots\otimes D_s}_{k\ times}\otimes\widetilde{a}](2\pi\beta) = 0 \qquad \forall\,\beta\in(N^*)^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s, 0\le k<\ell.$$

From Lemma 3.1, for all $0\le k<\ell$ and $\beta\in(M^*)^{-1}\mathbb{Z}^r\backslash\mathbb{Z}^r$, we have

$$\left[\underbrace{D_r\otimes\cdots\otimes D_r}_{k\ times}\otimes\widetilde{Pa}\right](2\pi\beta) = \sum_{j=1}^{t}\left[\underbrace{D_r\otimes\cdots\otimes D_r}_{k\ times}\otimes[\widetilde{a}(P^*\cdot+2\pi\varepsilon_j)]\right](2\pi\beta)$$

$$= \sum_{j=1}^{t}\left(\underbrace{P^*\otimes\cdots\otimes P^*}_{k\ times}\otimes I_1\right)\left(\underbrace{D_s\otimes\cdots\otimes D_s}_{k\ times}\otimes\widetilde{a}\right)(2\pi(P^*\beta+\varepsilon_j)) = 0$$

provided that $P^*\beta+\varepsilon_j\in(N^*)^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$ for all $j=1,\cdots,t$.

Since $\varepsilon_j\in P^*\mathbb{Z}^r$ and $P^*=(N^*)^{-1}S^*M^*$, obviously $P^*\beta+\varepsilon_j\in(N^*)^{-1}\mathbb{Z}^s$. To prove that $P^*\beta+\varepsilon_j\in(N^*)^{-1}\mathbb{Z}^s\backslash\mathbb{Z}^s$ for all $j=1,\cdots,t$ and $\beta\in(M^*)^{-1}\mathbb{Z}^r\backslash\mathbb{Z}^r$, it suffices to prove that if $P^*\beta$ is an integer for some $\beta\in(M^*)^{-1}\mathbb{Z}^r$, then β must be an integer. Suppose that $P^*\beta$ is an integer. Then $\langle\beta,P\alpha\rangle=\langle P^*\beta,\alpha\rangle$ must be an integer for all $\alpha\in\mathbb{Z}^s$. By assumption in (3.4), there exist $\alpha_j\in\mathbb{Z}^r, j=1,\cdots,|\det M|$ such that $\{\alpha_j\ :\ j=1,\cdots,|\det M|\}$ is a complete set of representatives of the distinct cosets of $\mathbb{Z}^r/M\mathbb{Z}^r$ and $\alpha_j\in P\mathbb{Z}^s$. Therefore, $\langle\beta,\alpha_j\rangle$ are integers for all $j=1,\cdots,|\det M|$. By $\beta\in(M^*)^{-1}\mathbb{Z}^r$, we have $\langle\beta,\alpha_j+M\alpha\rangle$ must be an integer for all $j=1,\cdots,|\det M|$ and $\alpha\in\mathbb{Z}^r$. Since $\{\alpha_j\ :\ j=1,\cdots,|\det M|\}$ is a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^r/M\mathbb{Z}^r$, it yields that $\langle\beta,\alpha\rangle$ must be an integer for all $\alpha\in\mathbb{Z}^r$. So β must be an integer.

When $\{\beta\in\mathbb{Z}^s\ :\ P\beta\in M\mathbb{Z}^r\}\subseteq N\mathbb{Z}^s$, from (3.6) it is obvious that Pa is an interpolatory mask. ∎

The requirement in (3.4) is equivalent to saying that the linear transform P maps the coset $N\mathbb{Z}^s$ into the coset $M\mathbb{Z}^r$, and $P\mathbb{Z}^s$ should intersect every coset of $\mathbb{Z}^r/M\mathbb{Z}^r$. In other words, $P=MSN^{-1}$ such that S is an $r\times s$ integer matrix and $M^{-1}\mathbb{Z}^r/\mathbb{Z}^r\subseteq S(N^{-1}\mathbb{Z}^s/\mathbb{Z}^s)$. On the other hand, we can obtain a converse to Theorem 3.2 in a certain sense so that we can demonstrate that for any mask b on $\mathbb{Z}^r$, then it is necessary that $b = Pa$ for some mask a, some linear transform P and some dilation matrix $N = dI_s$ for some positive integer d.

Without assuming the extra condition on P in (3.7), if a is an interpolatory mask and satisfies the sum rules of order ℓ with respect to $N\mathbb{Z}^s$, then we can modify Pa as follows: $b(\beta) = Pa(\beta)$ for all $\beta \in \mathbb{Z}^r \backslash M\mathbb{Z}^r$, and $b(\beta) = \delta(\beta)/|\det M|$ for all $\beta \in M\mathbb{Z}^r$. Then b is an interpolatory mask and satisfies the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^r$. The key point here is that $\sum_{\beta \in N\mathbb{Z}^s} a(\beta)q(\beta) = q(0)$ for all $q \in \Pi_{\ell-1}$ which means that the sums on the left side of the definition of sum rules in (3.2) are independent of the mask a.

Example 3.3 The checkerboard lattice on $\mathbb{Z}^s$ is given by

$$CB_s := \{(\beta_1, \cdots, \beta_s) \in \mathbb{Z}^s \;:\; \beta_1 + \cdots + \beta_s \text{ is an even integer}\}.$$

In particular, when $s = 2$, $CB_2 = Q\mathbb{Z}^2 = T\mathbb{Z}^2$ is the well known quincunx lattice. Let $M = EM_sE^{-1}$ where the $s \times s$ integer matrices E and M_s are given by

$$E := \begin{bmatrix} 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad M_s := \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then it is easy to check that M is a dilation matrix, $M\mathbb{Z}^s = CB_s$ and $M^s = 2I_s$. Let $P = M/2$ and $N = 2I_s$. Then it is easy to check that all the conditions in (3.4) and (3.7) are satisfied.

Let b_r be the interpolatory mask given in [4] such that b_r is supported on $[1 - 2r, 2r - 1]$. We have the tensor product interpolatory masks t_r given by $\widetilde{t}_r(\xi_1, \cdots, \xi_s) = \widetilde{b}_r(\xi_1) \cdots \widetilde{b}_r(\xi_s)$. Then t_r is an interpolatory mask satisfying the sum rules of order $2r$ with respect to the lattice $2\mathbb{Z}^s$. Moreover, $\widetilde{t}_r(\xi) \geq 0$ for all $\xi \in \mathbb{R}^s$. Applying Theorem 3.2 to the tensor product interpolatory masks t_r, we obtain interpolatory masks $Pt_r (r \in \mathbb{N})$ on the checkerboard lattice which can satisfy any preassigned order of sum rules with respect to the lattice CB_s and the symbols of these masks are nonnegative. In dimension $s = 2$, $Pt_r (r \in \mathbb{N})$ are also obtained by Dahlke, Gröchnig and Maass in [3] using a different approach. For this particular construction in dimension two, our method here is simpler and more general than the method used in [3].

A family of interpolatory masks $g_r (r \in \mathbb{N})$ with respect to the lattice $2\mathbb{Z}^2$ was reported in Han and Jia [6]. Applying Theorem 3.2, we obtain interpolatory masks $Pg_r (r \in \mathbb{N})$ with respect to the quincunx lattice. Such masks Pg_r were also obtained in [7] using a different approach. Both Pt_r and Pg_r are quincunx interpolatory masks, satisfy the sum rules of order $2r$ and have nonnegative symbols. Since Pt_r and Pg_r are invariant under all the elements of the group $S_{Q,T}$ in (2.7), we have $\phi^Q_{Pt_r} = \phi^T_{Pt_r}$ and $\phi^Q_{Pg_r} = \phi^T_{Pg_r}$. Moreover, the size of the support of the

Table 1: The Hölder smoothness exponents of the refinable functions $\phi^Q_{Pt_r}$ and $\phi^Q_{Pg_r}$.

r	1	2	3	4	5	6	7	8
$\nu_\infty(\phi^Q_{Pt_r})$	0.61152	1.51556	2.30354	3.0030	3.64031	4.23278	4.79257	5.32836
$\nu_\infty(\phi^Q_{Pg_r})$	0.61152	1.45934	2.21896	2.90350	3.53133	4.11667	4.67061	5.20149
r	9	10	11	12	13	14	15	16
$\nu_\infty(\phi^Q_{Pt_r})$	5.84611	6.34960	6.84111	7.32180	7.79197	8.25127	8.69891	9.13362
$\nu_\infty(\phi^Q_{Pg_r})$	5.71514	6.21534	6.70431	7.18321	7.65242	8.11171	8.56039	8.99752

mask Pg_r is roughly half of that of Pt_r. The Hölder smoothness exponents of these refinable functions are given in Table 1.

where $\nu_\infty(\phi) := \sup\{\nu : \phi \in C^\nu(\mathbb{R}^2)\}$ and the Hölder smoothness exponents $\nu_\infty(\phi^Q_{Pg_r}), r = 1, \cdots, 8$ were given in [7].

The following result is a direct consequence of Theorem 3.2.

Corollary 3.4 *Let L be a proper sublattice of $\mathbb{Z}^s$. Then there is a dilation matrix M such that $L = M\mathbb{Z}^s$ and $M^s = |\det M| I_s$. Let $P = M/|\det M|$ and $N = |\det M| I_s$. Then P satisfies* (3.4) *and* (3.7). *Moreover, with respect to the lattice L, we can easily construct an orthogonal mask satisfying any preassigned order of sum rules.*

Proof: Since L is a sublattice of $\mathbb{Z}^s$, $L = K\mathbb{Z}^s$ for some integer matrix K. Hence, there exist two integer matrices E and F such that $|\det E| = |\det F| = 1$ and

$$EKF = \operatorname{diag}(d_1, \cdots, d_s), \qquad d_1, \cdots, d_s \in \mathbb{N}, d_1 \ge d_2 \ge \cdots \ge d_s \ge 1.$$

Define

$$M := E^{-1} \begin{bmatrix} 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & d_{s-1} \\ d_s & 0 & \cdots & 0 & 0 \end{bmatrix} E.$$

Then it is easy to verify that $M^s = d_1 \cdots d_s I_s = |\det M| I_s$ and $M\mathbb{Z}^s = L$. Now it is easy to check the conditions in (3.4) and (3.7) since $\mathbb{Z}^s = PM^{s-1}\mathbb{Z}^s \subseteq P\mathbb{Z}^s$.

Let r be the largest integer such that $d_r > 1$. For any preassigned positive integer ℓ, let a be a tensor product orthogonal mask on $\mathbb{Z}^r$ such that a satisfies the sum rules of order ℓ with respect to the lattice $\operatorname{diag}(d_1, \cdots, d_r)\mathbb{Z}^r$. Define a mask b on $\mathbb{Z}^s$ as follows:

$$b(\beta_1, \cdots, \beta_r, \beta_{r+1}, \cdots, \beta_s) = a(\beta_1, \cdots, \beta_r)\delta(\beta_{r+1}) \cdots \delta(\beta_s), \qquad \beta_1, \cdots, \beta_s \in \mathbb{Z}.$$

Obviously, b is an orthogonal mask and satisfies the sum rules of order ℓ with respect to the lattice $\operatorname{diag}(d_1, \cdots, d_s)\mathbb{Z}^s$. Define a sequence $c(\beta) := b(E^{-1}\beta), \beta \in \mathbb{Z}^s$ (This is a special case of Theorem 3.2 by taking $M\mathbb{Z}^s = L, N\mathbb{Z}^s = \operatorname{diag}(d_1, \cdots, d_s)\mathbb{Z}^s$ and $P = E$). Then c is an orthogonal mask and satisfies the sum rules of order ℓ with respect to the lattice L. ■

A shortcoming of the above construction is that the resulting orthogonal mask may be separable. In the case that nonseparable orthogonal masks are preferred, we can modify the orthogonal masks in Corollary 3.4 into nonseparable ones using linear combinations of orthogonal masks as discussed in [10] for the dilation matrix Q.

Let a be a mask on $\mathbb{Z}^s$. Let M be a dilation matrix and $m := |\det M|$. Let $\{\varepsilon_j : j = 1, \cdots, m\}$ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^*\mathbb{Z}^s$. By convention, we assume that $\varepsilon_1 = 0$. For a sequence a on $\mathbb{Z}^s$, we can define a vector as follows:

$$\vec{\widetilde{a}}(\xi) := \big(\widetilde{a}(\xi + 2\pi\varepsilon_1), \cdots, \widetilde{a}(\xi + 2\pi\varepsilon_m)\big) \in \mathbb{C}^m, \qquad \xi \in \mathbb{R}^s. \tag{3.8}$$

The inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^m$ is defined to be

$$\langle x, y \rangle := \sum_{j=1}^{m} x_j \overline{y_j}, \qquad x = (x_1, \cdots, x_s), y = (y_1, \cdots, y_s) \in \mathbb{C}^m.$$

Proposition 3.5 *Let M be a dilation matrix and $m := |\det M|$. Suppose that $a_1, \cdots, a_k$ are finitely supported sequences on $\mathbb{Z}^s$ such that $\langle \vec{\widetilde{a}}_i, \vec{\widetilde{a}}_j \rangle = \delta(i-j)$ for all $i, j = 1, \cdots, k$ where $\vec{\widetilde{a}}_j$ is defined in* (3.8)*. Let $c_j, j = 1, \cdots, k$ be finitely supported sequences on $\mathbb{Z}^s$ such that all $\widetilde{c}_j$ are $2\pi(M^*)^{-1}\mathbb{Z}^s$-periodic and $\sum_{j=1}^{k} |\widetilde{c}_j(\xi)|^2 = 1$ for all $\xi \in \mathbb{R}^s$. Define a sequence b on $\mathbb{Z}^s$ as follows*

$$\widetilde{b}(\xi) = \sum_{j=1}^{k} \widetilde{c}_j(\xi)\widetilde{a}_j(\xi), \qquad \xi \in \mathbb{R}^s. \tag{3.9}$$

Then $\langle \vec{\widetilde{b}}, \vec{\widetilde{b}} \rangle = \sum_{j=1}^{m} |\widetilde{b}(\xi + 2\pi\varepsilon_j)|^2 = 1$ for all $\xi \in \mathbb{R}^s$. Moreover, when $k = m$, any finitely supported sequence b with $\langle \vec{\widetilde{b}}, \vec{\widetilde{b}} \rangle = 1$ must take the form in (3.9).

Proof: It is easy to see that $\langle \vec{\widetilde{b}}, \vec{\widetilde{b}} \rangle = 1$. Conversely, when $k = m$, then $\widetilde{b} = \sum_{j=1}^{m} \langle \vec{\widetilde{b}}, \vec{\widetilde{a}}_j \rangle \widetilde{a}_j$. Since $\langle \vec{\widetilde{b}}, \vec{\widetilde{a}}_j \rangle$ are $2\pi(M^*)^{-1}\mathbb{Z}^s$-periodic trigonometric polynomials and $\sum_{j=1}^{m} |\langle \vec{\widetilde{b}}, \vec{\widetilde{a}}_j \rangle|^2 = \langle \vec{\widetilde{b}}, \vec{\widetilde{b}} \rangle = 1$, we are done. ■

Let a be an orthogonal mask satisfying the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^s$. In Proposition 3.5 we can choose a particular one-dimensional finitely supported sequence d_1 such that $1 - |\widetilde{d}_1(\xi)|^2 \geq 0$ and $1 - |\widetilde{d}_1(\xi)|^2 = O(|\xi|^{2\ell})$, ξ near 0. For example, $1 - |\widetilde{d}_1(\xi)|^2 = c\sin^{2\ell}(\xi/2), 0 \leq c \leq 1$. Then we can find one-dimensional finitely supported sequences $d_j, j =$

$2, \cdots, k$ such that $\sum_{j=1}^{k} |\widetilde{d}_j|^2 = 1$. Let $\beta \in \mathbb{Z}^s$ and $\widetilde{c}_j(\xi) = \widetilde{d}_j(\beta^* M^* \xi), j = 1, \cdots, k$. Then the mask b, given in (3.9), is also an orthogonal mask satisfying the sum rules of order ℓ with respect to the lattice $M\mathbb{Z}^s$. Moreover, b can be made nonseparable by carefully choosing the sequences $d_j, j = 1, \cdots, k$.

Given a mask a on $\mathbb{Z}^2$, by a^* we denote another mask given by $a^*(j,k) := a(k,j), j, k \in \mathbb{Z}^2$. Let us illustrate the above procedure by a simple example as follows.

Example 3.6 Let

$$\widetilde{D}_4(\xi) = \frac{1+\sqrt{3}}{8} + \frac{3+\sqrt{3}}{8} e^{-i\xi} + \frac{3-\sqrt{3}}{8} e^{-i2\xi} + \frac{1-\sqrt{3}}{8} e^{-i3\xi}$$

be the one-dimensional Daubechies orthogonal mask ([5]) which satisfies the sum rules of order 2 with respect to the lattice $2\mathbb{Z}$. Let

$$\widetilde{a}_1(\xi_1, \xi_2) := \widetilde{D}_4(\xi_1), \quad \widetilde{a}_2(\xi_1, \xi_2) := e^{-i3\xi_1} \overline{\widetilde{D}_4(\xi_1 + \pi)}.$$

Let $M := Q$ be the quincunx dilation matrix. Then it is easy to check that $\langle \vec{\widetilde{a}}_j, \vec{\widetilde{a}}_k \rangle = \delta(j-k)$ for $j, k = 1, 2$. By Proposition 3.4, a_1 is an orthogonal mask satisfying the sum rules of order 2 with respect to the quincunx lattice. Let

$$\widetilde{d}_1(\xi) := 1 + (1-t_1)(e^{-i\xi} - 1) + t_2(e^{-i2\xi} - 1)^2, \qquad \widetilde{d}_2(\xi) = t(1 - e^{-i\xi})^2, \qquad \xi \in \mathbb{R},$$

where

$$t_1 := (1 + \sqrt{1 - 16\, t^2})/2, \quad t_2 = (t_1 + \sqrt{t_1^2 + 4\, t^2})/2, \quad \text{and} \quad |t| \le 1/4.$$

It is easy to check that $|\widetilde{d}_1|^2 + |\widetilde{d}_2|^2 = 1$ and $1 - |\widetilde{d}_1|^2 = O(|\xi|^4)$. Then by Proposition 3.5 the mask a given by

$$\widetilde{a}(\xi_1, \xi_2) := \widetilde{d}_1(\xi_1 + \xi_2) \widetilde{a}_1(\xi_1, \xi_2) + \widetilde{d}_2(\xi_1 + \xi_2) \widetilde{a}_2(\xi_1, \xi_2)$$

must be an orthogonal mask satisfying the sum rules of order 2 with respect to the quincunx lattice. When $t = 0$, a is the shifted version of a_1. By computation, $\nu_2(\phi_{a_1}^Q) = \nu_2(\phi_{a_1^*}^Q) \approx 0.324298$, where

$$\nu_2(\phi) := \sup\{\nu : \int_{\mathbb{R}^2} |\widehat{\phi}(\xi)|^2 (1 + |\xi|^\nu)^2 d\xi < \infty\}$$

is the Sobolev smoothness exponent of ϕ. When $t = \sqrt{3}/8$, the mask a is supported on $[0, 5] \times [0, 2]$ and is given by

$$\begin{bmatrix} 0 & 0 & \frac{3+\sqrt{3}}{16} & \frac{3+3\sqrt{3}}{16} & 0 & 0 \\ 0 & \frac{\sqrt{3}-1}{16} & \frac{3-\sqrt{3}}{16} & \frac{3+\sqrt{3}}{16} & \frac{-1-\sqrt{3}}{16} & 0 \\ 0 & 0 & \frac{3-3\sqrt{3}}{16} & \frac{3-\sqrt{3}}{16} & 0 & 0 \end{bmatrix}$$

with the bottom-left corner as the origin. This mask was given by Kovačević and Vetterli in [9]. It was demonstrated by Villemoes in [12] that ϕ_a^Q is a continuous orthogonal refinable function. By computation, we have $\nu_2(\phi_a^Q) \approx 0.588316$ and $\nu_2(\phi_{a^*}^Q) \approx 0.611459$. On the other hand, we found that when $t = 23/128$, we have better L_2 smoothness $\nu_2(\phi_a^Q) \approx 0.666682$ and $\nu_2(\phi_{a^*}^Q) \approx 0.674469$.

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