A FREE BOUNDARY PROBLEM ARISING IN ELECTROPHOTOGRAPHY: SOLUTIONS WITH CONNECTED TONER REGION

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A FREE BOUNDARY PROBLEM ARISING IN ELECTROPHOTOGRAPHY: SOLUTIONS WITH CONNECTED TONER REGION

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Abstract. A free boundary problem which arises in the development of a photocopy is studied. The electric potential $-u$ satisfies the equation $\Delta u = 1$ in the toner region and $\Delta u = 0$ elsewhere. We show that the $C^{1+\alpha}$ of the free boundary would imply the $C^{2+\alpha}$ of the solution up to both side of the free boundary. Using this fact we prove the existence of a solution with connected toner region with $\partial u/\partial n = 0$ on the free boundary when the electrical charge length 2ε is "small".

Key Words. Free boundary problems, electrophotography, photocopy, elliptic estimates.

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1. Introduction. One of the steps in the photocopying process is the development of the electrical image into a visible image. A positively charged toner is brushed on to the electrical image and the visible dark image is therefore produced. This process is modeled as a non-standard free boundary problem. (See [1,2] for more details.)

We set

$$\begin{aligned}
\Omega^+ &= \{(x,y);\ |x| < a, 0 < y < b\},\\
\Omega^- &= \{(x,y);\ |x| < a, -h < y < 0\},\\
I &= \{(x,0);\ |x| < \varepsilon\},\\
J &= \{(x,0);\ |x| < a\},\\
\Omega &= \{(x,y);\ |x| < a, -h < y < b\} = \Omega^+ \cup J \cup \Omega^-.
\end{aligned}$$

We shall denote by A the toner region, then $A \subset \Omega^+$. Let $-u$ be the potential, then the problem becomes the following.

Find the pair (u, A) such that

$$\Delta u = 1 \qquad \text{in } A, \tag{1.1}$$

$$\Delta u = 0 \qquad \text{in } \Omega \setminus \overline{A}, \tag{1.2}$$

$$u \in C^1(\overline{\Omega} \setminus \overline{I}), \tag{1.3}$$

$$u_y(x, 0+) - u_y(x, 0-) = -\sigma \qquad \text{in } I \tag{1.4}$$

and u satisfies the free boundary condition

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma \tag{1.5}$$

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where $\Gamma = \partial A \cap \Omega^+$ and $\frac{\partial}{\partial n}$ is the outward normal to A, and also the boundary conditions:

$$u(x,-h) = 0 \qquad -a < x < a, \tag{1.6}$$

$$u(x,b) = M \qquad -a < x < a, \tag{1.7}$$

$$u_x(\pm a, y) = 0 \qquad -h < y < b. \tag{1.8}$$

Of course, a, b, h, M, σ are positive constants, and it is reasonable to assume that (see [2])

$$M < \sigma h, \qquad b \geq h. \tag{1.9}$$

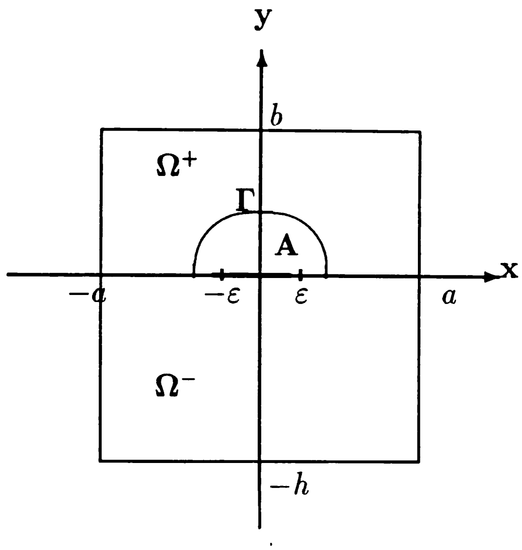

Figure 1.

When $a - \varepsilon$ is small, the problem reduces to a variational inequality; it is proved in [2] that in this case the problem has a unique solution.

When ε is small, it is proved in [2] that the problem is no longer a variational inequality and there are infinitely many solutions with two symmetric component, it is not clear however whether such solutions are physical. It it also shown that there exists a "ε^m-approximate" solution for which the toner set consists of a single component. The difficulty for finding a real solution with one connected toner region is that the corresponding u will have $\nabla u = 0$ at the point $\Gamma \cap \{x = 0\}$, which makes a saddle point for the corresponding dynamical system. In [2], $W^{2,p}$ estimate is employed for the PDE solution; however for this dynamical system coupled with the solution of the PDE, it is clear that more regularity is required for the solution of the PDE in order to use a fixed point theorem.

We shall prove in this paper that if ε is small, then the problem has a solution with one connected toner region. In §2, we prove an elliptic estimate which is of independent interest itself. Using the elliptic estimate we prove in §3 the existence result.

We want to express our thanks to IMA at the University of Minnesota for its hospitality where this work was completed.

2. Elliptic estimate. We shall first prove that the $C^{1+\alpha}$ of the free boundary would imply the $C^{2+\alpha}$ of the solution up to both sides of the free boundary. Suppose that Γ is given by $y = g(x)$ for $-2 < x < 2$ with $g(0) = 0$. We shall use $B(s)$ to denote a ball of radius s centered at (0,0). Suppose that

$$\Delta u = \theta \chi_E \qquad \text{for } (x,y) \cap B(2) \qquad (0 < \theta \leq 1) \tag{2.1}$$

where χ_E is the characteristic function of E and $E = \{y > g(x)\}$.

THEOREM 2.1. *Suppose that*

$$\sup_{B(2)} |u(x,y)| \leq L, \tag{2.2}$$

and

$$\|g\|_{C^{1+\alpha}(-2,2)} \leq K, \tag{2.3}$$

where $0 < \alpha < 1$. *Then*

$$\|u\|_{C^{2+\alpha}(\overline{E}\cap B(1))} \leq C \tag{2.4}$$

$$\|u\|_{C^{2+\alpha}((\overline{B(1)\setminus E}))} \leq C \tag{2.5}$$

where the constant C *depends only on* L, K *and* α. □

Remark: The proof below will actually show that the conclusion is also true for n dimensional situation.

We shall divide the proof into several lemmas.

LEMMA 2.2. *If*

$$\Delta u = f \qquad \text{in } B(s) \tag{2.6}$$

$$u = 0 \qquad \text{on } \partial B(s) \tag{2.7}$$

where $0 < s \leq 1$. *Then*

$$\int_{B(s)} |u| \leq \frac{1}{4} \int_{B(s)} |f|. \tag{2.8}$$

Proof.

$$u(x,y) = \int_{B(s)} G(x-\xi, y-\eta) f(\xi,\eta) d\xi d\eta \tag{2.9}$$

where G is the Green function on $B(s)$. Therefore,

$$\begin{aligned}
\int_{B(s)} |u(x,y)| dxdy &\leq \int_{B(s)} \left(\int_{B(s)} |G(x-\xi, y-\eta)| dxdy \right) |f(\xi,\eta)| d\xi d\eta \\
&= \int_{B(s)} \frac{1}{4}(s^2 - \xi^2 - \eta^2) |f(\xi,\eta)| d\xi d\eta \\
&\leq \frac{1}{4} \int_{B(s)} |f(\xi,\eta)| d\xi d\eta. \quad \square
\end{aligned}$$

LEMMA 2.3. *Suppose that for some $s \in [3/4, 1]$, we have*

$$\Delta v = 0 \qquad in\ B(s) \tag{2.10}$$

$$\int_{\partial B(s)} |v| d\sigma \leq L. \tag{2.11}$$

Then there exists a constant C_L depending only on L such that

$$|v(x,y) - P_2[v](x,y)| \leq C_L r^3 \qquad for\ r = \sqrt{x^2+y^2} \leq \frac{1}{2} \tag{2.12}$$

where $P_2[v](x,y)$ is the second order polynomial of the Taylor series of v at $(0,0)$.

Proof. Using (2.11) and Poisson formula we conclude immediately that

$$\|D^3 v\|_{L^\infty(B(1/2))} \leq C_L, \tag{2.13}$$

from which the Lemma follows immediately. □

We now fix L and α. Then we take C_L as in Lemma 2.3, fixed. Next we take λ such that

$$0 < \lambda \leq \frac{1}{2}, \quad C_L \lambda^{1-\alpha} \leq \frac{1}{2\pi}. \tag{2.14}$$

For such a fixed λ, we take ε_0 such that

$$\varepsilon_0 \leq \lambda^{4+\alpha}. \tag{2.15}$$

Using the appropriate scaling $\overline{u}(x,y) = u(\delta x, \delta y)$ ($\delta = (\varepsilon_0/K)^{1/\alpha}$) if necessary we may assume without loss of generality that

$$[g']_{C^\alpha(-2,2)} \leq \varepsilon_0. \tag{2.16}$$

LEMMA 2.4. *Under the assumptions of Theorem 2.1 (assuming also (2.16)) there exists a constant C such that for any $Q \in \Gamma \cap B(1)$, there exists P_Q*

$$\left\| u - \left(P_Q + \frac{\theta}{2}\left(\langle (x,y) - Q, \mathbf{n}_Q \rangle^+ \right)^2 \right) \right\|_{L^1(B_Q(r))} \leq C r^{4+\alpha} \qquad for\ 0 < r < \lambda \tag{2.17}$$

where P_Q is a harmonic polynomial of second degree, $\mathbf{n}_Q$ is the normal of Γ at Q in the direction of y-axis.

Proof. Let us assume without loss of generality that

$$Q = (0,0), \quad g(0) = 0, \quad g'(0) = 0.$$

It follows that

$$|g(x)| \leq \frac{\varepsilon_0}{1+\alpha} |x|^{1+\alpha} \leq \varepsilon_0 |x|^{1+\alpha}. \tag{2.18}$$

Set

$$w_1(x,y) = u(x,y) - \frac{\theta}{2}(y^+)^2, \tag{2.19}$$

and define v_1 by

$$\Delta v_1 = 0 \qquad \text{in } B(1) \tag{2.20}$$
$$v_1 = w_1 \qquad \text{on } \partial B(1). \tag{2.21}$$

Then

$$\Delta(w_1 - v_1) = 0 \qquad \text{in } B(1) \setminus \{|y| \geq \varepsilon_0 |x|\} \tag{2.22}$$
$$|\Delta(w_1 - v_1)| \leq 1 \qquad \text{in } B(1). \tag{2.23}$$

Therefore by Lemma 2.2 and (2.15),

$$\|w_1 - v_1\|_{L^1(B(1))} \leq \frac{1}{4}\text{meas}(B(1) \cap \{|y| \leq \varepsilon_0|x|\}) \leq \frac{1}{2}\varepsilon_0 \leq \frac{1}{2}\lambda^{4+\alpha}. \tag{2.24}$$

Clearly

$$\int_{\partial B(1)} |v_1| = \int_{\partial B(1)} |w_1| \leq L, \tag{2.25}$$

thus by Lemma 2.3

$$|v_1(x,y) - P_2[v_1](x,y)| \leq C_L r^3 \leq C_L \lambda^{1-\alpha}\lambda^{2+\alpha} \leq \frac{1}{2\pi}\lambda^{2+\alpha} \quad \text{for } r = \sqrt{x^2+y^2} \leq \lambda, \tag{2.26}$$

which implies

$$\|v_1 - P_2[v_1]\|_{L^1(B(\lambda))} \leq \frac{1}{2\pi}\lambda^{2+\alpha}\pi\lambda^2 \leq \frac{1}{2}\lambda^{4+\alpha}. \tag{2.27}$$

The inequalities (2.24) and (2.27) imply

$$\|w_1 - P_2[v_1]\|_{L^1(B(\lambda))} \leq \lambda^{4+\alpha}. \tag{2.28}$$

Next, define

$$w_2(x,y) = \frac{(w_1 - P_2[v_1])(\lambda x, \lambda y)}{\lambda^{2+\alpha}}. \tag{2.29}$$

Then by (2.28)

$$\|w_2\|_{L^1(B(1))} = \frac{1}{\lambda^{4+\alpha}}\|w_1 - P_2[v_1]\|_{L^1(B(\lambda))} \leq 1; \tag{2.30}$$

it follows that there exists $s_0 \in [3/4, 1]$ such that

$$\int_{\partial B(s_0)} |w_2| \leq 4 \leq L. \tag{2.31}$$

Now, define v_2 by

$$\Delta v_2 = 0 \quad \text{in } B(s_0) \tag{2.32}$$
$$v_2 = w_2 \quad \text{on } \partial B(s_0). \tag{2.33}$$

Then

$$\Delta(w_1 - v_1) = 0 \quad \text{in } B(s_0) \setminus \{|y| \geq \varepsilon_0 \lambda^\alpha |x|\} \tag{2.34}$$
$$|\Delta(w_1 - v_1)| \leq \frac{1}{\lambda^\alpha} \quad \text{in } B(s_0). \tag{2.35}$$

Therefore by Lemma 2.2 and (2.15),

$$\|w_2 - v_2\|_{L^1(B(s_0))} \leq \frac{1}{4}\int_{(B(s_0))\cap\{|y|\leq\varepsilon_0\lambda^\alpha|x|\})} \frac{1}{\lambda^\alpha} \leq \frac{1}{2}\varepsilon_0 \leq \frac{1}{2}\lambda^{4+\alpha}. \tag{2.36}$$

Clearly by Lemma 2.3, using (2.31),

$$|v_2(x,y) - P_2[v_2](x,y)| \leq C_L r^3 \leq C_L \lambda^{1-\alpha}\lambda^{2+\alpha} \leq \frac{1}{2\pi}\lambda^{2+\alpha} \quad \text{for } r = \sqrt{x^2+y^2} \leq \lambda, \tag{2.37}$$

and therefore

$$\|v_2 - P_2[v_2]\|_{L^1(B(\lambda))} \leq \frac{1}{2\pi}\lambda^{2+\alpha}\pi\lambda^2 \leq \frac{1}{2}\lambda^{4+\alpha}. \tag{2.38}$$

The inequalities (2.36) and (2.38) imply

$$\|w_2 - P_2[v_2]\|_{L^1(B(\lambda))} \leq \lambda^{4+\alpha}. \tag{2.39}$$

Now we inductively define

$$w_n(x,y) = \frac{(w_{n-1} - P_2[v_{n-1}])(\lambda x, \lambda y)}{\lambda^{2+\alpha}}. \tag{2.40}$$

Notice that whenever we scale the domain by λ, we get one more $1/\lambda^\alpha$ factor on the right-hand side of the equation; but that is compensated by the fact that we get one more factor of λ^α for the domain at the same time. Hence we obtain

$$\|w_n - P_2[v_n]\|_{L^1(B(\lambda))} \leq \lambda^{4+\alpha}, \tag{2.41}$$

where $P_2[v_n]$ is a harmonic polynomial of the second degree. It follows from (2.40) and (2.41) that

$$\|w_1 - P_n\|_{L^1(B(\lambda^n))} \leq C(\lambda^n)^{4+\alpha}, \tag{2.42}$$

where $C = 1/\lambda^2$, and

$$P_n = \sum_{k=1}^{n} P_2[v_k]\left(\frac{x}{\lambda^{k-1}}, \frac{y}{\lambda^{k-1}}\right)(\lambda^{k-1})^{2+\alpha}. \tag{2.43}$$

It is obvious that all coefficients of the harmonic polynomials $P_2[v_k]$ are bounded with the bounds depending only on L. If we set

$$P = \lim_{n\to\infty} P_n = \sum_{k=1}^{\infty} P_2[v_k]\left(\frac{x}{\lambda^{k-1}}, \frac{y}{\lambda^{k-1}}\right)(\lambda^{k-1})^{2+\alpha}, \tag{2.44}$$

then (assuming that $\lambda^\alpha \leq 1/2$)

$$|\text{0th order coefficients of } (P-P_n)| \leq C\sum_{k=n}^{\infty}(\lambda^{2+\alpha})^k \leq 2C(\lambda^{2+\alpha})^n \tag{2.45}$$

$$|\text{1st order coefficients of } (P-P_n)| \leq C\sum_{k=n}^{\infty}(\lambda^{1+\alpha})^k \leq 2C(\lambda^{1+\alpha})^n \tag{2.46}$$

$$|\text{2nd order coefficients of } (P-P_n)| \leq C\sum_{k=n}^{\infty}(\lambda^{\alpha})^k \leq 2C(\lambda^{\alpha})^n. \tag{2.47}$$

So

$$\begin{aligned} &\|P-P_n\|_{L^1(B(\lambda^n))} \\ \leq\ & C\left[(\lambda^{2+\alpha})^n(\lambda^n)^2 + (\lambda^{1+\alpha})^n\lambda^n(\lambda^n)^2 + (\lambda^\alpha)^n(\lambda^n)^2(\lambda^n)^2\right] \\ \leq\ & C(\lambda^n)^{4+\alpha}. \end{aligned} \tag{2.48}$$

For each $0 < r < \lambda$, choose n so that $\lambda^{n+1} < r \leq \lambda^n$. Then by (2.48) and (2.41), we obtain,

$$\|w_1 - P\|_{L^1(B(r))} \leq Cr^{4+\alpha} \qquad \text{for } 0 < r < \lambda, \tag{2.49}$$

where P is a harmonic polynomial of second degree. □

Next, for any point $Q \in B(1)$, take $\pi(Q) \in \Gamma$ such that

$$d(Q, \pi(Q)) = \inf\{|Q-S|;\ S \in \Gamma\}. \tag{2.50}$$

Although Γ is not in C^2 and therefore $\pi(Q)$ is not uniquely determined by Q, the map $\pi : B(1) \to \Gamma$ is still well defined by axiom of choice.

LEMMA 2.5. *Under the assumptions of Theorem 2.1 (assuming also (2.16)) there exists a constant C such that for any $Q \in B(1)$, there exists P_Q*

$$\left\| u - \left(P_Q + \frac{\theta}{2}\left(\langle (x,y) - \pi(Q), \mathbf{n}_{\pi(Q)}\rangle^+\right)^2\right)\right\|_{L^1(B_Q(r))} \leq Cr^{4+\alpha} \quad \textit{for } 0 < r < \lambda \tag{2.51}$$

where P_Q is a harmonic polynomial of second degree, $\mathbf{n}_{\pi(Q)}$ is the normal of Γ at $\pi(Q)$ in the direction of y-axis.

Proof. Let us assume without loss of generality that

$$\pi(Q) = (0,0), \quad g(0) = 0, \quad g'(0) = 0.$$

Set

$$w(x,y) = u(x,y) - \frac{\theta}{2}(y^+)^2, \tag{2.52}$$

and

$$G = \{\Delta w \neq 0\}. \tag{2.53}$$

Then it is clear that

$$|G \cap B_Q(r)| \leq \varepsilon_0 r^{2+\alpha} \qquad \text{for } 0 < r < 1, \tag{2.54}$$

where $B_Q(r)$ is a ball of radius r centered at Q. Now the remaining of the proof is exactly the same as the proof of Lemma 2.4 except we shall use (2.54) when Lemma 2.2 is applied. □

Lemmas 2.4 and 2.5 immediately imply that

$$u(Q) = P_Q(Q) \tag{2.55}$$

$$Du(Q) = DP_Q(Q) \tag{2.56}$$

and if $Q \notin \Gamma$, then

$$D^2u(Q) = D^2P_Q(Q) + \frac{\theta}{2}D^2\left(\langle (x,y) - \pi(Q), \mathbf{n}_{\pi(Q)}\rangle^+\right)^2\Big|_{(x,y)=Q}, \tag{2.57}$$

which already implies that $\|u\|_{W^{2,\infty}(B(1))} \leq C$.

Lemma 2.6 *For any $Q_1, Q_2 \in B(1)$, we have*

$$\begin{aligned} \Big| P_{Q_1}(x,y) - P_{Q_2}(x,y) &+ \frac{\theta}{2}\left(\langle (x,y) - \pi(Q_1), \mathbf{n}_{\pi(Q_1)}\rangle^+\right)^2 \\ &- \frac{\theta}{2}\left(\langle (x,y) - \pi(Q_2), \mathbf{n}_{\pi(Q_2)}\rangle^+\right)^2 \Big| \leq Cr^{2+\alpha} \\ &\textit{for } \left|(x,y) - \frac{Q_1+Q_2}{2}\right| \leq \frac{r}{2} \end{aligned} \tag{2.58}$$

where $r = 3|Q_1 - Q_2|$.

Proof. Let $d_i = d(Q_i, \pi(Q_i))$ for $i = 1, 2$. Without loss of generality we assume that $d_1 \geq d_2$.

Case 1: $d_2 \geq 2r$,

In this case the balls $B_{Q_1}(r)$ and $B_{Q_2}(r)$ do not intersect with each other and $B_{\frac{1}{2}(Q_1+Q_2)}(r)$ does not intersect with Γ. It follows that

$$\Delta\left(P_{Q_1} - P_{Q_2} + q\right) = 0 \qquad \text{in } B_{\frac{1}{2}(Q_1+Q_2)}(r) \tag{2.59}$$

where

$$q(x,y) = \frac{\theta}{2}\left(\langle (x,y) - \pi(Q_1), \mathbf{n}_{\pi(Q_1)}\rangle^+\right)^2 - \frac{\theta}{2}\left(\langle (x,y) - \pi(Q_2), \mathbf{n}_{\pi(Q_2)}\rangle^+\right)^2.$$

It follows that (we shall use $B(r)$ to denote balls centered at $\frac{1}{2}(Q_1+Q_2)$)

$$\|P_{Q_1} - P_{Q_2} + q\|_{L^1(\partial B(s_r))} \le \frac{Cr^{4+\alpha}}{r/4} \le Cr^{3+\alpha} \tag{2.60}$$

for some $s_r \in (\frac{3}{4}r, r]$. Now the Poisson formula implies that

$$\|P_{Q_1} - P_{Q_2} + q\|_{L^\infty(B(r/2))} \le \frac{C}{r/4}\|P_{Q_1} - P_{Q_2} + q\|_{L^1(\partial B(s_r))} \le Cr^{2+\alpha}. \tag{2.61}$$

Case 2: $d_2 < 2r$,

In this case

$$d_1 = |Q_1 - \pi(Q_1)| \le |Q_1 - \pi(Q_2)| \le |Q_1 - Q_2| + d_2 \le 5r$$

and so

$$|\pi(Q_1) - \pi(Q_2)| \le d_1 + |Q_1 - Q_2| + d_2 \le 5r + 3r + 2r = 10r. \tag{2.62}$$

Thus

$$\left|\langle \pi(Q_1) - \pi(Q_2), \mathbf{n}_{\pi(Q_2)}\rangle\right| \le Cr^{1+\alpha}$$

$$|\mathbf{n}_{\pi(Q_1)} - \mathbf{n}_{\pi(Q_2)}| = \left|\left[\frac{(-g',1)}{\sqrt{1+(g')^2}}\right]_{\pi(Q_1)}^{\pi(Q_2)}\right| \le Cr^\alpha, \tag{2.63}$$

which implies that

$$\begin{aligned} |q(x,y)| &\le Cr\left|\langle (x,y) - \pi(Q_1), \mathbf{n}_{\pi(Q_1)}\rangle^+ - \langle (x,y) - \pi(Q_2), \mathbf{n}_{\pi(Q_2)}\rangle^+\right| \\ &\le Cr\left|\langle (x,y) - \pi(Q_1), \mathbf{n}_{\pi(Q_1)} - \mathbf{n}_{\pi(Q_2)}\rangle\right| \\ &\quad + Cr\left|\langle \pi(Q_1) - \pi(Q_2), \mathbf{n}_{\pi(Q_2)}\rangle\right| \\ &\le Cr^{2+\alpha} \end{aligned} \tag{2.64}$$

for $(x,y) \in B(r)$. Now repeat the proof of case 1 for $P_{Q_1} - P_{Q_2}$ (which is harmonic), and the error term q is controlled by (2.64) which does not cause any problems. □

Proof of Theorem 2.1. Take $Q_1, Q_2 \in B(1) \setminus E$ (in which $\Delta u = 0$). Notice that in this case the second term of (2.57) is zero for both $Q = Q_1$ and $Q = Q_2$; therefore by Lemma 2.6 and (2.55)-(2.57) we obtain

$$\begin{aligned} &\Big|\Big(u(Q_1) + Du(Q_1)\cdot(X-Q_1) + \frac{1}{2}(X-Q_1)^T D^2u(Q_1)(X-Q_1)\Big) \\ &- \Big(u(Q_2) + Du(Q_2)\cdot(X-Q_2) + \frac{1}{2}(X-Q_2)^T D^2u(Q_2)(X-Q_2)\Big)\Big| \\ &\le C|Q_1-Q_2|^{2+\alpha} \end{aligned} \tag{2.65}$$

for all X with $\left|X - \frac{1}{2}(Q_1+Q_2)\right| \le 3r$. This immediately implies that

$$\|u\|_{C^{2+\alpha}((\overline{B(1)\setminus E}))} \le C.$$

The estimates on $\overline{E} \cap B(1)$ is obtained by considering $-\left(u - \frac{\theta}{4}(x^2+y^2)\right)$. □

3. Existence of a solution with connected toner region. We shall always assume (1.9). We fix $0 < \alpha < 1$. Our theorem is

THEOREM 3.1 *The free boundary problem* (1.1)–(1.8) *has a solution* (u, A) *with a connected toner region* A *such that*

$$\Gamma = \partial A \cap \Omega^+ \in C^{1+\alpha} \tag{3.1}$$

provided ε *is small enough.* □

Suppose that u_ε satisfy (1.1)–(1.4) and (1.6)–(1.8). Set

$$\tilde{u}_\varepsilon(x,y) = \frac{1}{\varepsilon}\left[u_\varepsilon(\varepsilon x, \varepsilon y) - u_\varepsilon(0,0)\right]. \tag{3.2}$$

Then

$$\Delta \tilde{u}_\varepsilon = \begin{cases} \varepsilon & \text{for } (x,y) \in \frac{1}{\varepsilon} A_\varepsilon \equiv \tilde{A}_\varepsilon \\ 0 & \text{for } (x,y) \notin \tilde{A}_\varepsilon \cup \{(x,0); |x| \le 1\} \end{cases}$$

where A_ε is the region in which $\Delta u_\varepsilon = 1$.

It has been proved in [2, section 8] that for any compact $F \subset R^2$

$$\|\tilde{u}_\varepsilon - \tilde{v}\|_{W^{2,p}(F)} \le C_{F,p} \left(\text{meas}(A_\varepsilon) + \varepsilon^p\right)^{1/p} \tag{3.3}$$

for any $p > 2$, where

$$\tilde{v}(x,y) = -\frac{\sigma}{2\pi}\int_{-1}^{1} \log\sqrt{(x-\xi)^2 + y^2}\, d\xi + \frac{My}{b+h} - \frac{\sigma}{\pi}. \tag{3.4}$$

The equation $\frac{\partial \tilde{v}}{\partial n} = 0$ gives the level curves of the harmonic conjugate $z(x,y)$ of $\tilde{v}$, where

$$\begin{aligned} z(x,y) = \frac{\sigma}{4\pi}\Big\{ & y \log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} - 2(1-x)\arctan\frac{1-x}{y} \\ & +2(1+x)\arctan\frac{1+x}{y}\Big\} - \frac{M}{b+h}x. \end{aligned} \tag{3.5}$$

There is only one curve: $y = \varphi_0(x)$ that passes through y axis. $\varphi_0(x)$ is analytic and it is shown in [2] that

$$\begin{aligned} &\varphi_0(0) = y_0, && \varphi_0'(0) = 0 \\ &\varphi_0'(x) < 0 && \text{for } 0 < x \le \overline{x}_0 \\ &\varphi_0''(x) < 0 && \text{for } 0 \le x \le \overline{x}_0 \end{aligned} \tag{3.6}$$

where

$$y_0 = \cot\left(\frac{\pi}{\sigma}\frac{M}{b+h}\right), \qquad \overline{x}_0 = \frac{b+h}{M}\frac{\sigma}{2} > 1. \tag{3.7}$$

It is clear that

$$\frac{\partial \tilde{v}}{\partial n} = 0 \qquad \text{on } y = \varphi_0(x) \tag{3.8}$$

We shall use Schauder's fixed point theorem to prove the existence. Now fix $0 < \alpha < 1$ and set

$$\begin{aligned} X \;=\; \Big\{\gamma(x);\quad & \gamma \in C^{1+\alpha}[0, \overline{x}_0 + l], \quad \gamma'(0) = 0, \quad \gamma(\overline{x}_0 + l) \le 0, \\ & \gamma(x) \ge \sqrt{\mu^2 - (x-1)^2}, \quad \frac{1}{2} y_0 \le \gamma(x) \le \frac{3}{2} y_0 \quad \text{for } 0 \le x \le \mu, \\ & \gamma(x) \ge c_0 > 0 \quad \text{for } \mu \le x \le 1, \qquad \|\gamma\|_{C^{1+\alpha}[0,\overline{x}_0+l]} \le K \Big\}, \end{aligned} \tag{3.9}$$

where l, μ and c_0 are fixed small positive constants, K is to be determined later on.

Denote by $\tilde{A}_\gamma$ the connected component of the area enclosed by γ and x axis which contains $\{(0, y),\ 0 < y < \gamma(0)\}$. For each $\gamma \in X$, we always extend it by letting $\gamma(-x) = \gamma(x)$. Next, define w by

$$\Delta w = \chi_{\tilde{A}_\gamma} \qquad \text{in } R_{2N} \tag{3.10}$$

$$w = 0 \qquad \text{on } \partial R_{2N} \tag{3.11}$$

where $R_{2N} = \{|x| < 2N,\ |y| < 2N\}$ $(N/2 > 3y_0/2)$ and $\chi_{\tilde{A}_\gamma}$ is the characteristic function of $\tilde{A}_\gamma$.

By $W^{2,p}$ estimates and Sobolev's embedding theorem,

$$\|w\|_{C^{1+\alpha}(R_{2N})} \le C. \tag{3.12}$$

Here and below we shall use C to denote constants independent of K and C_K to denote the constants depending on K.

Let $V = ([0,1] \times [c_0, N]) \cup (([1, \overline{x}_0 + l] \times [0, N]) \setminus \{(x-1)^2 + y^2 < \mu^2\})$, then by Theorem 2.1

$$\|w\|_{W^{2,\infty}(R_N \cap V)} \le C_K. \tag{3.13}$$

For this $\gamma \in X$, let u_γ be the solution of (1.1)–(1.4) and (1.6)–(1.8) with $A = \varepsilon \tilde{A}_\gamma$, and let

$$\tilde{u}_{\gamma,\varepsilon}(x, y) = \frac{1}{\varepsilon}\left[u_\gamma(\varepsilon x, \varepsilon y) - u_\varepsilon(0,0)\right];$$

then

$$\Delta\left(\tilde{u}_{\gamma,\varepsilon} - \tilde{v} - \varepsilon w\right) = 0 \qquad \text{in } R_{2N}. \tag{3.14}$$

Therefore by (3.12) and (3.3) (with $\tilde{u}_\varepsilon = \tilde{u}_{\gamma,\varepsilon}$ in (3.3)), we can apply the Schauder's interior estimates to obtain

$$\|\tilde{u}_{\gamma,\varepsilon} - \tilde{v} - \varepsilon w\|_{C^3(R_N)} \le C\varepsilon + C\varepsilon^{2/p} \tag{3.15}$$

where the constant is independent of K.

By symmetry

$$\frac{\partial}{\partial x}(\widetilde{u}_{\gamma,\varepsilon} - \widetilde{v} - \varepsilon w) = 0 \qquad \text{on } x = 0, \tag{3.16}$$

and so by (3.15),

$$\frac{\partial}{\partial x}(\widetilde{u}_{\gamma,\varepsilon} - \widetilde{v} - \varepsilon w) \le (C\varepsilon + C\varepsilon^{2/p})x \qquad \text{for } (x,y) \in R_N. \tag{3.17}$$

Using (3.13) and $\frac{\partial w}{\partial x}(0,y) = 0$ (by symmetry), we get

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}(x,y) - \frac{\partial w}{\partial x}(0,y) \le C_K x \qquad \text{for } (x,y) \in V. \tag{3.18}$$

Clearly

$$\widetilde{v}_x = \frac{\sigma}{4\pi}\log\frac{(1-x)^2+y^2}{(1+x)^2+y^2} \le -c_1 x < 0 \qquad \text{for } (x,y) \in V. \tag{3.19}$$

It follows that

$$\frac{\partial}{\partial x}\widetilde{u}_{\gamma,\varepsilon} \le \left(C\varepsilon + C\varepsilon^{2/p} + C_K\varepsilon - c_1\right)x \le -\frac{1}{2}c_1 x \qquad \text{in } V \tag{3.20}$$

provided $0 < \varepsilon \le \varepsilon_K$ and ε_K is small enough (depending on K).

Write

$$\widetilde{u}_{\gamma,\varepsilon} \equiv \widetilde{v} + \varepsilon w + \varepsilon^{2/p}\widetilde{w} \tag{3.21}$$

where $\|\widetilde{w}\|_{C^3(R_N)} \le C$ by (3.15), also, $w_x(x,\gamma(x))$ and $w_y(x,\gamma(x))$ are $C^{1+\alpha}$ functions with their $C^{1+\alpha}$ norms bounded by C_K, by Theorem 2.1.

Let $w^* = \varepsilon^{1-1/p}w + \varepsilon^{1/p}\widetilde{w}$, then

$$\widetilde{u}_{\gamma,\varepsilon} \equiv \widetilde{v} + \varepsilon^{1/p}w^*. \tag{3.22}$$

If $0 < \varepsilon < \varepsilon_K$ and ε_K is small enough, then

$$\|\zeta_1\|_{C^{1+\alpha}} \le 1, \qquad \|\zeta_2\|_{C^{1+\alpha}} \le 1, \tag{3.23}$$

where $\zeta_1(x) = w_x^*(x,\gamma(x))$ and $\zeta_2(x) = w_y^*(x,\gamma(x))$.

A calculation shows

$$\widetilde{v}_{yy} = -\widetilde{v}_{xx} = \frac{\sigma}{4\pi}\left(\frac{2(1-x)}{(1-x)^2+y^2} + \frac{2(1+x)}{(1+x)^2+y^2}\right). \tag{3.24}$$

So

$$\widetilde{v}_{yy}(0,y) = \frac{\sigma}{\pi}\frac{1}{y^2}. \tag{3.25}$$

It follows that (using also (3.22) and (3.13)),

$$\frac{\partial^2 \tilde{u}_{\gamma,\varepsilon}}{\partial y^2}(0,y) \geq \frac{1}{2}\frac{\sigma}{\pi}\frac{1}{y^2} > 0 \qquad \text{in } V \tag{3.26}$$

provided ε_K is small enough.

Therefore the equation

$$\frac{\partial \tilde{u}_{\gamma,\varepsilon}}{\partial y}(0,y) = 0 \qquad \frac{1}{2}y_0 \leq y \leq \frac{3}{2}y_0 \tag{3.27}$$

has a unique solution $y = y_\gamma$ provided ε_K is small enough. It is obvious that y_γ depends continuously on the $C^{1+\alpha}$ norm of γ.

Now we take $\tau > 0$ and define $\eta = T\gamma$ to be the solution of the ordinary differential equation:

$$\eta(0) = y_\gamma, \tag{3.28}$$

$$\eta'(x) = \frac{\dfrac{\tilde{v}_y(x,\eta(x)) - \tilde{v}_y(x,y_\gamma)}{x+\tau} + \varepsilon^{1/p}\dfrac{\zeta_2(x)-\zeta_2(0)}{x+\tau}}{\dfrac{\tilde{v}_x(x,\eta(x))}{x} + \varepsilon^{1/p}\left(\dfrac{\zeta_1(x)}{x}\right) * \rho_\tau(x)}, \tag{3.29}$$

where $\rho_\tau(x)$ is a C^∞ mollifier (see [4, chapter 7]) and the convolution is defined after extending the definition of $\zeta_1(x)/x$ to be $\zeta_1(\overline{x}_0 + l)/(\overline{x}_0 + l)$ for $x > \overline{x}_0 + l$ and $\lim_{x\to 0+}(\zeta_1(x)/x)$ for $x < 0$.

We write the right-hand-side of (3.29) as $\dfrac{f_1(x,y)}{f_2(x,y)}$ where $y = \eta(x)$. Then by analyticity of $\tilde{v}$, noticing also that $\tilde{v}_{xy}(0,y) \equiv 0$, we get

$$\left\|\frac{\partial f_2}{\partial y}\right\|_{L^\infty(V)} = \left\|\frac{1}{x}\tilde{v}_{xy}\right\|_{L^\infty(V)} \leq C. \tag{3.30}$$

LEMMA 3.2

$$\left[\frac{\zeta_1(x)}{x}\right]_{C^\alpha[0,\overline{x}_0+l]} \leq 3. \tag{3.31}$$

Proof. Suppose that $0 < x_1 < x_2$. If $x_2 - x_1 \geq x_1$, then $x_2 = x_2 - x_1 + x_1 \leq 2(x_2 - x_1)$. So by (3.23),

$$\begin{aligned}\left|\frac{\zeta_1(x_1)}{x_1} - \frac{\zeta_1(x_2)}{x_2}\right| &\leq \left|\frac{\zeta_1(x_1)}{x_1} - \zeta_1'(0)\right| + \left|\frac{\zeta_1(x_2)}{x_2} - \zeta_1'(0)\right| \\ &\leq x_1{}^\alpha + x_2{}^\alpha \leq (1+2^\alpha)|x_1 - x_2|^\alpha.\end{aligned}$$

If $x_2 - x_1 < x_1$, then we use

$$\zeta_1(x_2) = \zeta_1(x_1) + \zeta_1'(x_1)(x_2 - x_1) + r, \qquad |r| \leq |x_2 - x_1|^{1+\alpha}$$

to obtain

$$\begin{aligned}
\left|\frac{\zeta_1(x_1)}{x_1} - \frac{\zeta_1(x_2)}{x_2}\right| &\le \left|\frac{(x_1 - x_2)\zeta_1(x_1) + x_1\zeta_1'(x_1)(x_2 - x_1)}{x_1x_2}\right| + \frac{|x_2 - x_1|^{1+\alpha}}{x_2} \\
&\le \frac{|x_1 - x_2|}{x_1x_2}|\zeta_1(x_1) - x_1\zeta_1'(x_1)| + \frac{|x_2 - x_1|^{1+\alpha}}{x_2} \\
&\le \frac{|x_1 - x_2|}{x_1x_2}|x_1|^{1+\alpha} + \frac{|x_2 - x_1|^{1+\alpha}}{x_2} \quad (\text{since } \zeta_1(0) = 0) \\
&\le 2|x_1 - x_2|^{\alpha};
\end{aligned}$$

now the lemma follows. □

Since $\tilde{v}_x(0, y) = 0$, Lemma 3.2 and the analyticity of $\tilde{v}$ imply that

$$\sup_V |f_2(x, y)| + \sup_{x_1 \ne x_2;\, (x_1,y),(x_2,y)\in V} \frac{|f_2(x_1, y) - f_2(x_2, y)|}{|x_1 - x_2|^{\alpha}} \le C. \tag{3.32}$$

By (3.19), we have

$$f_2(x, y) \le -\frac{1}{2}c_1 < 0 \tag{3.33}$$

provided ε_K is small enough.

Using the analyticity of $\tilde{v}$, $\tilde{v}_{yx}(0, y) \equiv 0$ and a similar argument as in Lemma 3.2 for $\dfrac{\zeta_2(x) - \zeta_2(0)}{x + \tau}$, (using also $\zeta_2'(0) = 0$) we conclude

$$\sup_{x\ne y;\, (x_1,y),(x_2,y)\in V} \frac{|f_1(x_1, y) - f_1(x_2, y)|}{|x_1 - x_2|^{\alpha}} \le C, \tag{3.34}$$

and

$$\sup_{0\le t\le x} |f_1(t, \eta(t))| \le C \sup_{0<t\le x} \frac{|\eta(t) - \eta(0)|}{t} \tag{3.35}$$

where the constant C is independent of τ. And also

$$\left|\frac{\partial f_1}{\partial y}\right| = \left|\frac{\tilde{v}_{yy}(x, y)}{x + \tau}\right| \le \frac{C}{x + \tau}. \tag{3.36}$$

Therefore for each $\tau > 0$, the ODE (3.28) (3.29) is uniquely solvable. It is obvious that for each $\tau > 0$, $\eta \in C^{1+\alpha}$. Next, we shall derive $C^{1+\alpha}$ estimates independent of τ. From now on we shall assume that

(3.37) For the solution $\eta(x)$, we have $(x, \eta(x)) \in V$.

What we need to prove is actually the estimates near $x = 0$.

LEMMA 3.3.

$$\lim_{x\to 0} \frac{-x\tilde{v}_{yy}}{\tilde{v}_x} = 1. \tag{3.38}$$

Proof.

$$\lim_{x\to 0}\frac{-x\widetilde{v}_{yy}}{\widetilde{v}_x}=\lim_{x\to 0}\frac{-\widetilde{v}_{yy}}{\frac{\widetilde{v}_x}{x}}=\lim_{x\to 0}\frac{-\widetilde{v}_{yy}}{\widetilde{v}_{xx}}=1. \qquad \square$$

The convergence is uniform by the analyticity of $\widetilde{v}$. Thus there exists some small $\mu > 0$ such that

$$\frac{-x\widetilde{v}_{yy}}{\widetilde{v}_x}>0 \qquad \text{for } 0\le x\le \mu,\ \frac{1}{2}y_0\le y\le \frac{3}{2}y_0,$$

which implies

$$(3.39)\qquad \frac{(f_1)_y(x,y)}{f_2(x,\eta(x))}=\frac{\frac{\widetilde{v}_{yy}(x,y)}{x+\tau}}{f_2(x,\eta(x))}<0 \qquad \text{for } 0\le x\le\mu,\ \frac{1}{2}y_0\le y\le\frac{3}{2}y_0.$$

Let $\lambda(x)=\eta(x)-\eta(0)$, then

$$\begin{aligned}\lambda'(x) &= \frac{f_1(x,\eta(x))}{f_2(x,\eta(x))}\\ (3.40)\qquad &= \frac{f_1(x,\eta(x))-f_1(x,\eta(0))+\varepsilon^{1/p}\frac{\zeta_2(x)-\zeta_2(0)}{x+\tau}}{f_2(x,\eta(x))}\\ &\equiv q_1(x)\lambda(x)+q_2(x)\end{aligned}$$

where $q_1(x)=\dfrac{(f_1)_y(x,y)}{f_2(x,\eta(x))}<0$ by (3.39), and it is also clear that $q_1(x)\ge -\frac{C}{x+\tau}$.

By the definition of ζ_2 we have $\zeta_2'(0)=0$, therefore

$$(3.41)\qquad |q_2(x)|\le C\varepsilon^{1/p}\left|\frac{\zeta_2(x)-\zeta_2(0)}{x+\tau}\right|\le C^*x^\alpha$$

where the constant is independent of K, τ. It follows by comparison (we use $q_1(x)<0$ here) that

$$(3.42)\qquad |\lambda(x)|\le\int_0^x|q_2(\xi)|d\xi\le C^*x^{1+\alpha}.$$

This inequality together with (3.35) imply that

$$(3.43)\qquad |\eta'(x)|=|\lambda'(x)|\le C^*x^\alpha$$

It follows immediately that

$$(3.44)\qquad \|\eta'\|_{L^\infty[0,\bar{x}_0+l]}\le C^*.$$

Therefore we obtained the $W^{1,\infty}$ estimates for η.

Next, take $a > 0$ and consider $\tilde{\lambda}(x) = \eta(x+a) - \eta(x)$ for $x > 0$. Clearly

$$\begin{aligned} \tilde{\lambda}'(x) &= \frac{f_1(x+a,\eta(x+a))}{f_2(x+a,\eta(x+a))} - \frac{f_1(x,\eta(x))}{f_2(x,\eta(x))} \\ &= \frac{f_1(x+a,\eta(x+a)) - f_1(x+a,\eta(x)) + f_1(x+a,\eta(x)) - f_1(x,\eta(x))}{f_2(x+a,\eta(x+a))} \\ &\quad + \frac{f_1(x,\eta(x))[f_2(x,\eta(x)) - f_2(x+a,\eta(x+a))]}{f_2(x,\eta(x))f_2(x+a,\eta(x+a))} \\ &\equiv I_1 + I_2 \end{aligned} \tag{3.45}$$

where by (3.32) (3.33) (3.35) and (3.44),

$$|I_2| \le C^* a^\alpha, \tag{3.46}$$

and if we let

$$I_1 \equiv J_1 + J_2 \tag{3.47}$$

then by (3.34) (3.33),

$$|J_2| = \left| \frac{f_1(x+a,\eta(x)) - f_1(x,\eta(x))}{f_2(x+a,\eta(x+a))} \right| \le C^* a^\alpha, \tag{3.48}$$

and

$$J_1 = \frac{f_1(x+a,\eta(x+a)) - f_1(x+a,\eta(x))}{f_2(x+a,\eta(x+a))} \equiv q(x+a)(\eta(x+a) - \eta(x)) \tag{3.49}$$

where by (3.33) (3.36)

$$|q(x+a)| \le \frac{C^*}{x+a+\tau}. \tag{3.50}$$

By (3.43),

$$|\eta(x+a) - \eta(x)| \le \int_x^{x+a} |\eta'(\xi)| d\xi \le C^*(x+a)^\alpha a \le C^*(x+a)a^\alpha; \tag{3.51}$$

therefore (3.49)–(3.51) imply

$$|J_1| \le C^* a^\alpha. \tag{3.52}$$

Combining (3.46)–(3.52), we obtain

$$|\eta'(x+a) - \eta'(x)| = |\tilde{\lambda}'(x)| \le C^* a^\alpha. \tag{3.53}$$

This prove that if $\varepsilon \in (0, \varepsilon_K)$, then

$$\|\eta\|_{C^{1+\alpha}[0,\bar{x}_0+l]} \le C^{**}, \tag{3.54}$$

where the constant C^{**} is independent of K and τ.

We now choose $K = C^{**}$. Since the curve $\eta = T\gamma$ goes to $y = \varphi_0(x)$ uniformly as $\varepsilon \to 0$, it is easy to choose the remaining constants in the definition of X (as in [2]) so that T maps X into itself.

Obviously T is a continuous map from X to X (using $C^{1+\alpha}$ norm topology). For each $\tau > 0$, the image of T is contained in $C^{1+\beta}$ for any $\beta \in (\alpha, 1)$. Therefore T is compact. Hence T has a fixed point, by Schauder's fixed point theorem (see [4]). It is obvious that all the estimates of this section apply to the fixed point $\eta = T\eta$, where the estimates are independent of τ, therefore we can take a subsequence and pass the limit as $\varepsilon \to 0$ to obtain a solution (u, A) with one connected toner region. This proves Theorem 3.1.

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