The boundary value problems for
non-linear elliptic equations
and the maximum principle
for Euler-Lagrange equations

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In the present paper we investigate the boundary value problems for non-linear elliptic equations connected with infinite convex hypersurfaces. The asymptotic cone of such hypersurfaces is very important in our considerations. We study the problem of existence of solutions for Monge-Ampere equations on the entire space with prescribed asymptotic cone and the maximum principle and estimates for Euler-Lagrange equations. These problems have natural mutual connections.

§ 1. Infinite convex hypersurfaces and their asymptotic cones.

1.1 The main definitions

Let E^{n+1} be a $(n+1)$ - dimensional Euclidean space with Cartesian coordinates $x_1, x_2, \dots, x_n, z \equiv x_{n+1}$ and let E^n be the hyperplane $z = 0$ in E^{n+1} .

The graphs S_u of convex functions $z = u(x), x \in R^n$ are called complete infinite convex hypersurfaces in E^{n+1} . Thus from this definition it follows that we consider only convex hypersurfaces in E^{n+1} projected one-to-one on the entire hyperplane E^n .

Let H_u be the convex body defined by equation

$$z \geq u(x) \quad (1.1)$$

for all $x \in E^n$, where $u(x)$ is a convex function. Clearly

$$\partial H_u = S_u. \quad (1.2)$$

Let A be any point of H_u . The union of all closed rays starting from A and belonging to H_u is called the asymptotic cone of H_u at the point A . Let $\phi_\lambda : E^{n+1} \rightarrow E^{n+1}$ be the homothety of E^{n+1} with the center A and coefficient $\lambda > 0$. Then

$$K_u(A) = \lim_{\lambda \rightarrow 0} \phi_\lambda (H_u). \quad (1.3)$$

For every two points $A, A' \in H_u$ the convex cones $K_u(A)$ and $K_u(A')$ are congruent and can coincide by parallel translation in the direction of z -axis. The well known proof is based on the formula (1.3). Therefore we will denote the asymptotic cone $K_u(A)$ by K_u , if we do not need to use the special choice of the point A .

The asymptotic cone K_u is called non-degenerate if ∂K_u is projected one-to-one on the entire hyperplane E^n . For non-degenerate asymptotic cones it is convenient to replace them by their boundary. In this case the notation K_u will be used for the boundary of a solid asymptotic cone and we will say that

K_u is the asymptotic cone of the function $u(x)$.

1.2. Normal mapping and R-curvature of convex functions.

The normal mapping and R-curvature are the natural generalizations of the tangential mapping and the integral Gaussian curvature for arbitrary continuous convex functions (hyper surfaces). These concepts were introduced and studied by the author in [1] (more detailed exposition see in [2], [3]).

The normal mapping of every convex function $u(x)$ is constructed by supporting hyperplanes of its graph S_u .

Let P^n be a n -dimensional Euclidean space with Cartesian coordinates $p_1, p_2, \dots, p_n$ and let $p = (p_1, p_2, \dots, p_n)$ be a point of P^n . If

$$\alpha : z = p_1^0 x_1 + p_2^0 x_2 + \dots + p_n^0 x_n \quad (1.4)$$

is a supporting hyperplane of S_u , then the point

$$x(\alpha) = (p_1^0, p_2^0, \dots, p_n^0) \quad (1.5)$$

is called the normal image of the hyperplane α . For any set $e \in E^n$ the normal image $x_u(e)$ is defined as

$$x_u(e) = \bigcup_{\alpha} x(\alpha) \quad (1.6)$$

where α is any supporting hyperplane of S_u , whose intersection with S_u has points with projections in the set e .

We shall use the following well known properties of the normal mapping.

Property 1.

$$x_u(E^n) = x_{K_u}(E^n) \quad (1.7)$$

for every convex function $u(x)$ defined on the entire hyperplane E^n .

Property 2.

For every Borel subset e of E^n the set $x_u(e)$ is Lebesgue measurable.

Property 3.

Let $R(p) > 0$ be locally summable in E^n , then

$$\omega(R, u, e) = \int_{x_u(e)} R(p) \, dp \quad (1.8)$$

is a completely additive non-negative set function on the ring of Borel subsets of E^n .

If $u(x) \in C^2(E^n)$, then

$$\omega(R, u, e) = \int_e R(Du) \det(u_{ij}) \, dx$$

The set function $\omega(R, u, e)$ is called the R-curvature of a convex function $u(x)$.

Thus the R-curvature is the generalization of the Monge-Ampere operator

$R(Du) \det(u_{ij})$ for the set of all convex functions.

If

$$R(p) \equiv 1,$$

then

$$\omega(1, u, e) = \text{mes } x_u(e), \quad (1.9)$$

where the right part of (1.9) is the n -dimensional Lebesgue measure in the space p^n .

If $R(p) = (1 + |p|^2)^{-\frac{n+1}{2}}$, then $\omega(R, u, e)$ is the area (Lebesgue measure) of the image of e concerning the Gaussian mapping of S_u in the n -dimensional unit sphere S^n .

1.3 Remarks.

In this Subsection we consider a few properties of the normal mapping of convex functions.

Property 4. Let $u(x)$ be any convex function, then the function

$$v(x) = u(x) + \sum_{i=1}^n a_i x_i + b ,$$

where $a_1, a_2, \dots, a_n, b$ are constants, is convex and the set $x_v(e)$ can be obtained from $x_u(e)$ by the parallel translation of p_n on the vector $a = (a_1, a_2, \dots, a_n)$.

Property 5. the normal image of any convex cone K is a closed convex set in P^n whose dimension can take values $0, 1, 2, \dots, n$.

Property 6. If K is a non-degenerate convex cone, then $x_k(E^n)$ is a bounded closed n -dimensional convex domain with interior points.

The following Remarks follow from Properties 4,5,6.

It is sufficient to consider only convex functions $u : E^n \rightarrow R$ whose normal images contain the origin of P^n . If the asymptotic cone of such function is non-degenerate then we can additionally assume that the origin θ' of P^n is the interior point of $x_{K_u}(E^n)$. If we also assume that the vertex of this non-degenerate convex cone K_u is at the origin θ of E^n , then the equation of K_u is as follows

$$z = k(x) ,$$

where $K(x)$ is non-negative convex function in E^n and $k(x) = 0$ only at the point θ .

Since any point of E^n can be taken as origin, then it is sufficient to consider asymptotic non-degenerate convex cones whose vertices are projected in

the point $\theta \in E^n$. The equations of such cones are as follows

$$z = k(x) + b,$$

whose b is any constant

§2. The second boundary value problem for elliptic solutions of Monge-Ampere equations.

2.1 The statement of the second boundary value problem.

We consider elliptic solutions $u(x)$ of the Monge-Ampere equation

$$\det (u_{ij}) = f(x, u, Du) \quad (2.1)$$

in the entire space E^n . We assume that $f(x, u, p)$ is continuous in $E^n \times R \times P^n$. If $u(x)$ is a C^2 -elliptic solution of equation (2.1), then $u(x)$ is necessarily either a convex or a concave function. It is sufficient to consider only convex solutions of equation (2.1). We should impose the assumption of positiveness of $f(x, u, p)$ in $E^n \times R \times P^n$ if we investigate convex solutions of equation (2.1).

The convex function $u(x)$ is called a generalized solution of equation (2.1) if $u(x)$ satisfies (2.1) almost everywhere and the set function $\omega(1, u, e)$ is absolutely continuous. Note that

$$\omega(1, u, e) = \text{mes } x_u(e)$$

for any Borel subset e of E^n .

The statement of the second boundary value problem is as follows. Let K be a non-degenerate convex cone. Find wide sufficient conditions of existence of at least one generalized solution of equation (2.1) for which K is

the asymptotic cone.

This problem was formulated by Alexandrov and Pogorelov (Conference of Global Differential Geometry, Leningrad, 1958) and was solved by Bakelman in 1984 (see [4]). Up to 1984 only two special cases of equation (2.1) were investigated. The first one was studied by Alexandrov [5] in 1942 and it is related to existence and uniqueness of a general complete infinite hypersurface with prescribed area of Gaussian mapping. The second Theorem was established by Bakelman [7] in 1956 and is related to generalized solutions of the Monge-Ampere equation

$$\det (u_{ij}) = \frac{g(x)}{R(Du)} \quad (2.2)$$

(the particular case $R(Du) = [1+(Du)^2]^{\frac{n+1}{2}}$ corresponds to Alexandrov's Theorem.)

The necessary and sufficient condition of solvability of the second boundary value problem for equation (2.2) is as follows

$$\int_{E^n} g(x) dx = \int_{\chi_k(K^n)} R(p) dp \quad (2.3)$$

where $g(x) > 0$, $g(x) \in L(E^n)$, $R(p) > 0$, $R(p) \in L_{loc}(P^n)$ and $\chi_k(P^n)$ is the normal image of prescribed non-degenerate convex asymptotic cone K . Note that the desired solution is defined to within an additive constant.

The solution of the second boundary value problem for the general Monge-Ampere equation

$$\det (u_{ij}) = f(x, u, Du)$$

is essentially more difficult, because the simple necessary and sufficient condition (2.3) of the solvability of the same problem for equation (2.2) must be

replaced by the complicated implicit necessary condition

$$\text{mes } \chi_k(p^n) = \int_{E^n} f(x, u(x), Du(x)) dx . \quad (2.4)$$

Moreover, the application of fixed points Theorems contains difficulties because E^n is a non-compact set. The proof of Bakelman's recent existence Theorem for equation (20.1), presented in 2.3, is based on the construction of a new Monge-Ampere equation in some special introduced Banach space. This construction is significantly based on the properties of the asymptotic behavior of the function $f(x, u, p)$ and its first derivatives by $|x| \rightarrow +\infty$ and $|u| \rightarrow +\infty$.

2.2 The Second boundary problem for Monge-Ampere equations.

$$R(Du) \det(u_{ij}) = \phi(x).$$

Let $y(x) > 0$ be a summable function in E^n and let $R(p) > 0$ be a locally summable function in p^n .

Theorem 1. Let K be a non-degenerate convex cone in $E^{n+1} = E^n \times R$ and let

$$z = k(x), \quad x \in R^n$$

be the equation of K . Let

$$\int_{E^n} y(x) dx = \int_{\chi_k(E^n)} R(p) dp . \quad (2.5)$$

Then the second boundary value problem for the Monge-Ampere equation

$$\det(u_{ij}) = \frac{y(x)}{R(Du)} \quad (2.6)$$

has a generalized convex solution $u(x)$ and this solution is unique to within an additive constant.

As we noted in Subsection 2.1 this Theorem is proved by author in [7].

The proof of the statement of uniqueness to within an additive constant is based on the same ideas and facts as the proof of the uniqueness of convex generalized solutions of the Dirichlet problem for equation (2.6) (see [8], Chapter IV, §23).

The proof of the assertion of existence of generalized solutions made in Theorem 1 can be proved by the approximation of convex polyhedra. First we replace equation (2.6) by the equation

$$\omega(R, u, e) = \mu(e) \quad (2.7)$$

which can now be considered for all convex functions defined in E^n . In (2.7) $\mu(e)$ is any non-negative, completely additive set function subordinated to the condition

$$\mu(E^n) = \int_{x_k(E^n)} R(p) \, dp \quad (2.8)$$

If we consider the polyhedron case, then set functions $\mu(e)$ and prescribed convex cone K must satisfy the following necessary Assumptions.

Assumption A. The convex cone K is a non-degenerate convex polyhedral angle. We can assume in addition that the origin θ' of p^n lies inside $x_k(p^n)$ and therefore the equation of K has the form

$$z = k(x) + b \quad (2.9)$$

where $k(x)$ is a convex continuous piecewise linear homogeneous (of degree 1) function in E^n , satisfying the conditions $k(\theta) = 0$; $k(x) > 0$ if $x \neq \theta$, and b is any constant

Assumption B. The set function $\mu(e)$ satisfies the following condition: there

exists a finite system of points $A_1, A_2, \dots, A_m$ and a finite system of non-negative numbers $\mu_1, \mu_2, \dots, \mu_m$ such that

$$\mu(A_i) = \mu_i \quad (2.10)$$

and

$$\mu(e) = \sum_{A_i \in e} \mu(A_i) \quad (2.11)$$

for every subset e of E^n .

Theorem 1 can be reduced to the following Theorem for the case of convex polyhedra.

Theorem 2. If the polyhedral angle K satisfies Assumption A and the set function $\mu(e)$ satisfies Assumption B and if

$$\mu(E^n) = \int_{E^n} R(p) dp, \quad (2.12)$$

then there exist convex polyhedra

$$S_u : z = u(x), \quad x \in E^n$$

such that the vertices of S_u are projected only in the points $A_1, A_2, \dots, A_m$; K is the asymptotic cone of S_u and

$$\omega(R, u, A_i) = \mu_i, \quad i = 1, 2, \dots, m. \quad (2.13)$$

Moreover if $u_0(x)$ is a convex function defining one of such polyhedra, then all others can be obtained by the formula

$$u(x) = u_0(x) + C, \quad (2.14)$$

where C is an arbitrary constant.

Remarks. 1) Theorem 1 can be obtained by approximation of convex polyhedra which are constructed with help of Theorem 2. This is the traditional way now. In the theory of convex bodies and hypersurfaces the approximation method by polyhedra was developed by Minkowski, Alexandrov, Pogorelov and others (see [9], [6], [10], [11], [2], [8],). Therefore we note only that the conditions of Theorem 1 provide to obtain the sequence of convex polyhedra

$$S_{u_j} : z = u_j(x), x \in E^n$$

such that: a) $u_j(0) = 0$, $j = 1, 2, \dots, m$; b) all functions $u_j(x)$ satisfy the Lipschitz condition of degree 1 with the common constant; c) $\omega(R, u_j, e)$ converge weakly to the function $\mu(e)$ and

$$\lim_{j \rightarrow \infty} \omega(R, u_j, E^n) = \int_{X_k(E^n)} R(p) dp.$$

From convex polyhedra $u_j(x)$ it is possible to extract a convergent subsequence whose limit is a solution of the second boundary value problem for equation (2.7) (or equation (2.6)) with prescribed non-degenerate convex asymptotic cone. This completes the proof of Theorem 1.

Finally in this Subsection we present a new variant of the proof of Theorem 2 based on a variational method which up to now was not used in non-compact problems related to infinite convex hypersurfaces.

Proof of Theorem 2. We denote by $W^+(A_1, A_2, \dots, A_m, K)$ the set of all convex polyhedra, whose vertices are projected only in the points $A_1, A_2, \dots, A_m$ and whose asymptotic cones coincide with prescribed non-degenerate convex polyhedral angle K . Let T be the set of convex polyhedra $P: z = u(x)$ such that

- a) T is a subset of $W^+(A_1, A_2, \dots, A_m, K)$;
 b) If $u(x) \in T$, then the inequalities

$$0 \leq \omega(R, u, A_i) \leq \mu_i \quad (2.15)$$

hold for all $i = 2, 3, \dots, m$ and

$$\omega(R, u, A_1) = \int_{X_k(E^n)} R(p) dp - \sum_{i=2}^m \omega(R, u, A_i) \quad (2.16)$$

$$c) \quad u(A_1) = a_1, \quad (2.17)$$

where a_1 is an arbitrary fixed constant.

The set T is not empty, because the convex cone K with the vertex (A_1, a_1) belongs to T .

The system of real numbers

$$\xi_1 = u(A_1), \quad \xi_2 = u(A_2), \dots, \quad \xi_m = u(A_m), \quad (2.18)$$

taken from every convex polyhedron $u(x) \in W^+(A_1, A_2, \dots, A_m, K)$, defines this polyhedron one-to-one. The metric

$$d(u(x), v(x)) = \left\{ \sum_{i=1}^m [u(A_i) - v(A_i)]^2 \right\}^{1/2} \quad (2.19)$$

introduced in $W^+(A_1, A_2, \dots, A_m, K)$ shows that the mapping (2.17) isometrically maps $W^+(A_1, A_2, \dots, A_m, K)$ on some closed convex subset of the Euclidean space $R^m = \{\xi = (\xi_1, \xi_2, \dots, \xi_m)\}$. Since $W^+(A_1, A_2, \dots, A_m, K)$ can be identified with a corresponding closed subset of R^m , then T is also a closed subset of R^m . From (2.15-17) it follows that T is a bounded subset of R^m . Thus T is a compact set in R^m .

The function $f : T \rightarrow R$ acting by the formula

$$f(u) = \sum_{i=1}^m u(A_i) \quad (2.20)$$

is continuous in T . Hence

$$\inf_T f(u) = f_0 > -\infty$$

and there exists the polyhedron $u_0(x) \in T$ such that

$$f(u_0(x)) = f_0. \quad (2.21)$$

Now we prove that $u_0(x)$ is the desired solution of the second boundary value problem, satisfying the condition (2.17).

If our assertion is incorrect, then

$$\omega(R, u_0, A_s) < \mu_s \quad (2.22)$$

at least at one point A_s ($s = 2, 3, \dots, m$), because

$$\omega(R, u_0, A_s) \leq \mu_s$$

from the definition of the set T . Now consider the convex polyhedron

$$\tilde{P} : z = \tilde{u}(x) \in W^+(A_1, A_2, \dots, A_m, K)$$

such that

$$\tilde{u}(A_1) = a_1, \quad \tilde{u}(A_2) = u_0(A_2), \dots$$

$$\tilde{u}(A_{s-1}) = u_0(A_{s-1}), \quad \tilde{u}(A_s) = u_0(A_s) - \epsilon,$$

$$\tilde{u}(A_{s+1}) = u_0(A_{s+1}), \dots, \tilde{u}(A_m) = u_0(A_m),$$

where $\epsilon > 0$ is a sufficiently small number for which the inequality

$$\omega(R, \tilde{u}, A_s) < \mu_s \quad (2.23)$$

holds. Since

$$\sum_{i=1}^m \omega(R, \tilde{u}, a_i) = \int_{x_k(E^n)} R(p) dp \quad (2.24)$$

and

$$\omega(R, u, A_\ell) < \omega(R, u_0, A_\ell)$$

for $\ell = 1, 2, \dots, s-1, s+1, \dots, m$, then from (2.23-24) it follows that $\tilde{u}(x) \in T$.

But

$$\begin{aligned} f(u(x)) &= \sum_{i=1}^m \tilde{u}(A_i) = \sum_{i=1}^m u_0(A_i) - \epsilon = \\ &= f_0 - \epsilon < f_0. \end{aligned}$$

The last inequality is incompatible with equality (2.21). Hence $u_0(x)$ is the desired solution of the second boundary value problem for the equation (2.13).

Now let convex polyhedra P_1 and P_2 be the graphs of convex solutions $u_1(x)$, $u_2(x) \in W^+(A_1, A_2, \dots, A_m, K)$ of the equation (2.13), satisfying the additional condition

$$u_1(A_1) = u_2(A_1) = a_1.$$

If $u_1(x)$ and $u_2(x)$ are different functions, then according to Lemma proved in [6], Chapter IX, there exists at least one point A_j , where j is one of integers $1, 2, \dots, m$, such that the polyhedral angles $V_1 \subset P_1$ and $V_2 \subset P_2$ are projected in the point A_j and

$$V_1 \supset V_2 \quad \text{and} \quad V_1 \setminus V_2 \neq \emptyset.$$

Hence

$$\text{mes } x_{u_1}(A_j) > \text{mes } x_{u_2}(A_j).$$

Thus

$$\mu_j = \int_{x_{u_1}(A_j)} R(p) dp > \int_{x_{u_2}(A_j)} R(p) dp = \mu_j .$$

Therefore our assumption is incorrect. The proof of Theorem 2 is completed.

2.3. The second boundary value problem for general Monge-Ampere equations.

In this subsection we investigate the second boundary value problem for the Monge-Ampere equation

$$\det(u_{ij}) = f(x, u, Du) \quad (2.25)$$

in the class of convex generalized solutions. We divide the presentation into a few separate parts.

A) The main Assumptions.

A.1 Admissable convex cones. A nondegenerate convex cone in E^{n+1} is called admissable, if the equation of K is

$$z = k(x) \quad (2.26)$$

where $k(x)$ is a continuous, homogeneous of the order 1, convex function in E^n satisfying the conditions

$$a) \quad k(0) = 0 ,$$

$$b) \quad k(x) > 0 \text{ for all } x \neq 0 \text{ in } E^n .$$

It is sufficient to consider only admissable convex cones (see the final part of Section 1). We denote by K^* the normal image of a convex cone K , i.e.

$$K^* = x_k(E^n) . \quad (2.27)$$

For every admissible convex cone K the set K^* is closed, bounded and convex in the space p^n and the point θ' is an interior point of K^* , where θ' is the origin of p^n .

A.2 The properties of the function $f(x,u,p)$ and its derivatives.

The function $f(x,u,p)$ is continuous in $E^n \times R \times p^n$ together with its derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial p_i}$, ($i=1,2,\dots, n$) and

$$f(x,u,p) > 0, \quad (2.28)$$

$$\frac{\partial f(x,u,p)}{\partial u} > 0 \quad (2.29)$$

in $E^n \times R \times p^n$.

We also assume that the inequalities

$$\left| \frac{\partial f}{\partial u} \right| < \frac{C_0}{|x|^{n+2+\alpha}}, \quad (2.30)$$

$$\left| \frac{\partial f}{\partial p_i} \right| < \frac{C_1}{|x|^{n+\alpha}}, \quad (i=1,2,\dots,n) \quad (2.31)$$

hold for all $(x,u,p) \in E^n \times R \times K^*$ with $|x| \geq m_0$, where $\alpha = \text{const.} > 0$,

$C_0 = \text{const.} > 0$, $C_1 = \text{const.} > 0$, $m_0 = \text{const.} > 1$.

A.3 Estimators and their properties.

For every admissible convex cone K there exist two functions $\lambda_K(x,u)$ and $\phi_K(x,u)$, depending only on prescribed cone K , such that:

a) $\lambda_K(x,u)$ and $\phi_K(x,u)$ are positive and continuous in $E^n \times R$ and increase with respect to u for every fixed $x \in E^n$;

b) the inequalities

$$\lambda_K(x,u) < f(x,u,p) < \phi_K(x,u) \quad (2.32)$$

hold for all $x \in E^n$, $u \in R$, $p \in K^*$.

The functions $\lambda_K(x,u)$ and $\phi_K(x,u)$ are called estimators.

B) The statement of the main Theorem and the scheme of its proof.

Theorem 3. Let K be an admissible convex cone described in Assumption 1 and let $\lambda_K(x,u)$, $\phi_K(x,u)$ be estimators satisfying Assumption 2 (see part A of Subsection 2.3).

If there exist two numbers

$$-\infty < a_K < b_K < +\infty$$

such that

$$a) \int_{E^n} \phi_K(x, k(x) + q) dx < +\infty \quad (2.33)$$

for all $q \in [a_K, b_K]$;

$$b) \int_{E^n} \phi_K(x, k(x) + a_K) dx < \text{mes } K^*; \quad (2.34)$$

$$c) \inf_{\gamma \in K^*} \int_{E^n} \lambda_K(x, (x, \gamma) + b_K) dx > \text{mes } K^* \quad (2.35)$$

where $z = k(x)$ is the equation of the cone K and $(x, \gamma) = \sum_{i=1}^n \gamma_i x_i$,
 $x = (x_1, \dots, x_n) \in E^n$, $\gamma = (\gamma_1, \dots, \gamma_n) \in K^*$, then equation (2.25) has at least one generalized solution $u(x)$ with asymptotic cone K and

$$a_K < u(0) < b_K \quad (2.36)$$

The scheme of proof.

Since we consider unbounded convex functions on the entire space E^n , then the application of fixed point principles requires to construct the special functional space, where the second boundary value problem can be investigated. In this special functional space we study some modification of the Monge-Ampere operator $\det(u_{ij})$ and its inverse, which is induced by the original equation (2.25) and by prescribed admissible convex cone K . The final part of the proof for Theorem 3 is based on the application of Schauder principle to the inverse of the modified Monge-Ampere equation.

C) The functional space of the second boundary value problem. We denote by $C^0(E^n)$ the set of all continuous functions $u : E^n \rightarrow R$ and by $C^{0,1}(E^n)$ the subset of $C^0(E^n)$ consisting of all Lipschitz functions $u : E^n \rightarrow R$, i.e. $u(x) \in C^{0,1}(E^n)$ if and only if

$$L(u) = \sup_{x,y \in E^n} \frac{|u(y) - u(x)|}{|y - x|} < +\infty, \quad (2.37)$$

where $|y-x| = \text{dist}(x,y)$ in E^n .

Let $U_1 \subset U_2 \subset \dots \subset U_m \dots$ be the sequence of n -balls:

$$U_m : |x| \leq m, \quad (2.38)$$

$m = 1, 2, 3, \dots$. We denote by $\|u\|_m$ the number $\sup_{U_m} |u(x)|$ and by $\|u\|_A$ the number

$$\|u\|_A = |u(\theta)| + \sum_{m=1}^{\infty} \frac{\|u\|_m}{m^{2+\alpha}}, \quad (2.39)$$

where $\alpha = \text{const.} > 0$ (see Assumption 2).

Let A be the subset of $C^0(E^n)$ consisting of all functions $u(x)$ such that

$$\|u\|_A < +\infty.$$

Clearly

$$\|u\|_A < |u(0)| + L(u) \sum_{m=1}^{\infty} \frac{1}{m^{2+\alpha}} < +\infty$$

for every $u(x) \in C^{0,1}(E^n)$. On the other hand $\|e^{x_1 + \dots + x_n}\|_A = +\infty$. Thus A is a proper non-empty subset of $C^0(E^n)$.

The functional $\|u\|_A$ is a norm of the set A . Clearly A is a Banach space with respect to the norm. We denote by A_1 the subspace of A consisting of all functions $u(x) \in C^{0,1}(E^n)$.

Now we introduce the equivalence relation Γ in A setting $u(x) \Gamma v(x)$ if and only if $u(x) - v(x) = \text{const}$ in E^n . Clearly the factor space $B = A/\Gamma$ is again a Banach space with respect to the induced norm

$$\|\xi\|_B = \|\tilde{u}\|_A$$

where ξ is any element of B generated by the class of functions $\{u(x) + q\}$, where $u(x) \in A$ and q is any real number. We use the notation $u(x) = u(x) - u(0)$. We call the function $u(x) \in A$ a basic representative of the element $\xi \in B$, if $u(0) = 0$. We shall use the notation ξ_u for elements of the space B generated by basic representatives $u(x) \in A$. Clearly $\tilde{u}(x)$ is a basic representative for every $u(x) \in A$.

Every convex function $u(x)$ defined in E^n , whose graph has an admissible asymptotic cone, is an element of A_1 . Therefore $u(x) \in A$. This statement follows directly from the compactness of $x_u(E^n)$ in the space P^n .

Let T_k be a subset of B consisting of all elements $\xi \in B$ such that $\xi = \xi_u$, where $u(x)$ is any convex function defined in E^n and satisfying two conditions:

a) $u(\theta) = 0$;

b) the asymptotic cone of $u(x)$ is a fixed admissible cone K , considered in the second boundary value problem for equation (2.25).

The set T_k is not empty, since $\xi_k \in T_k$, where $z = k(x)$ is the equation of the admissible convex cone K mentioned above. The following lemmas are correct for any admissible convex cones K .

Lemma 1. T_k is a convex subset of B .

Lemma 2. T_k is a closed subset of B .

Lemma 3. T_k is a compact subset of B .

Proof of Lemma 1.

Let λ and μ be any positive numbers such that $\lambda + \mu = 1$. If ξ_f and ξ_g are any elements of T_k generated by the basic representatives $f(x)$ and $g(x)$, then functions $f(x)$ and $g(x)$ are convex in E^n , have one and the same asymptotic cone K , which is admissible, and $f(\theta) = g(\theta) = 0$.

Clearly the element $\lambda \xi_f + \mu \xi_g$ is generated by the function $\lambda f(x) + \mu g(x)$. Since $\lambda f(x) + \mu g(x)$ is convex and

$$\lambda f(\theta) + \mu g(\theta) = 0 ,$$

then

$$\lambda \xi_f + \mu \xi_g = \xi_{\lambda f + \mu g} . \quad (2.40)$$

If we prove that K is the asymptotic cone of $\lambda f(x) + \mu g(x)$, then from this fact and from equality (2.40) it follows that T_k is a convex set in B . Clearly the statement concerning the asymptotic cone of $\lambda f(x) + \mu g(x)$ is suf-

ficient to be established only for C^2 convex functions $f(x)$ and $g(x)$, because the case of general convex functions can be obtained by the simple approximation of corresponding C^2 convex functions.

Thus our convex functions $f(x)$ and $g(x)$, introduced in the beginning of the proof of Lemma 1, are twice differentiable in E^n .

Clearly

$$\text{grad} (\lambda f + \mu g) = \lambda \text{grad} f + \mu \text{grad} g \quad (2.41)$$

at any point $x_0 \in E^n$. From (2.41) it follows that

$$x(\gamma) = \lambda x(\gamma_1) + \mu x(\gamma_2),$$

where $\gamma, \gamma_1, \gamma_2$ are correspondingly the normal images of tangent hyperplanes of the graphs of the functions $\lambda f(x) + \mu g(x)$, $f(x)$ and $g(x)$ at the points $(x, \lambda f(x) + \mu g(x))$, $(x, f(x))$, $(x, g(x))$.

Since $x(\gamma_1), x(\gamma_2)$ are points of the convex set $x_k(E^n)$, then

$$x(\gamma) \in x_k(E^n). \quad (2.42)$$

If K_1 is the asymptotic cone of the convex function $\lambda f(x) + \mu g(x)$, then from (2.42) it follows that

$$x_{k'}(E^n) \subset x_k(E^n), \quad (2.43)$$

where $z = k'(x)$ is the equation of the asymptotic cone K' , whose vertex is at the point $\theta \in E^n$. From (2.43) we obtain

$$0 \leq k'(x) \leq k(x) \quad (2.44)$$

for all $x \in E^n$.

Now we prove that

$$k'(x) = k(x)$$

for all $x \in E^n$. Let ℓ be any axis in E^n , passing through the point θ , and s be the Cartesian coordinate in ℓ such that

$$|s| = [x_1^2 + x_2^2 + \dots + x_n^2]^{1/2}$$

for any point $x = (x_1, x_2, \dots, x_n) \in \ell$.

Let $\tilde{k}'(s)$, $\tilde{k}(s)$, $\tilde{f}(s)$, $\tilde{g}(s)$ be functions, which are generated by $k'(x)$, $k(x)$, $f(x)$, $g(x)$ on the axis ℓ .

Then

$$k'(s) = \begin{cases} -k'_1 s & \text{if } s \leq 0; \\ k'_2 s & \text{if } s > 0 \end{cases}$$

and

$$k(s) = \begin{cases} -k_2 s & \text{if } s \leq 0; \\ k_2 s & \text{if } s > 0, \end{cases}$$

where $0 \leq k'_1 \leq k_1$ and $0 \leq k'_2 \leq k_2$.

Thus we obtain the following chain of equalities

$$k'_2 + k'_1 = \int_{-\infty}^{+\infty} \left(\lambda \frac{d^2 f}{ds^2} + \mu \frac{d^2 g}{ds^2} \right) ds =$$

$$\lambda \int_{-\infty}^{+\infty} \frac{d^2 f}{ds^2} ds + \mu \int_{-\infty}^{+\infty} \frac{d^2 g}{ds^2} ds =$$

$$= \lambda(k_2 + k_1) + \mu(k_2 + k_1) = k_2 + k_1, \text{ because } \lambda \geq 0; \mu \geq 0 \text{ and } \lambda + \mu = 1.$$

$$\text{Thus } k'_2 + k'_1 = k_2 + k_1.$$

Since $0 \leq k'_2 \leq k_2$ and $0 \leq k'_1 \leq k_1$, then

$$k'_2 = k_2 \text{ and } k'_1 = k_1. \quad (2.45)$$

Hence

$$k'(x) = k(x) \quad (2.46)$$

for all $x \in E^n$, because ℓ is an arbitrary axis in E^n passing through θ .

From (2.46) it follows that

$$K' = K$$

and the proof of Lemma 1 is completed.

Proof of Lemma 2.

Let T_k be the closure of T_k in the space B . If $\xi \in T_k$, then there exist the elements $\xi_1, \xi_2, \dots, \xi_q, \dots$ of the set T_k such that

$$\lim_{q \rightarrow \infty} \|\xi_q - \xi\|_B = 0.$$

The basic representatives $u_q(x)$ of ξ_q are convex functions on E^n with one and the same asymptotic cone K , which is admissible, and $u_q(\theta) = 0$, $q = 1, 2, 3, \dots$. Clearly

$$\lim_{q \rightarrow \infty} \|u_q(x) - u(x)\|_A = 0,$$

where $u(x)$ is the basic representative of $\xi \in T_k$. Hence $u_q(x)$ converges uniformly to $u(x)$ in every closed ball $U_m: |x| \leq M$.

Thus $u(x)$ is a convex function with asymptotic cone K and $\xi \in T_K$. The proof of Lemma 2 is completed.

Proof of Lemma 3.

Let ξ be any element of T_K and $u(x)$ be the basic representative of ξ . Then $u(\theta) = 0$ and convex function $u(x)$ satisfies the Lipschitz condition in E^n with the constant $d_0 = \text{diam } K^*$. Thus

$$\begin{aligned} \|\xi\|_B = \|u(x)\|_A &= \sum_{m=1}^{\infty} \frac{\|u(x)\|_m}{m^{2+\alpha}} < \\ &< d_0 \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} = \text{const.} < +\infty. \end{aligned}$$

Let $\{u_\gamma(x)\}$ be the collection of basic representatives of $\{u_\gamma(x)\}$. From the last estimate it follows that there exists a sequence of convex functions $u_{\gamma_q}(x)$ which is subsequence of $\{u_\gamma(x)\}$ and which is uniformly convergent to some convex function $u_0(x)$ in every ball $U_m : |x| \leq m$. Clearly $u_0(\theta) = 0$ and K is the asymptotic cone of $u_0(x)$. Hence

$$\xi_{u_0} \in T_K.$$

If we establish that

$$\lim_{q \rightarrow \infty} \|u_{\gamma_q}(x) - u_0(x)\|_A = 0,$$

then the proof of Lemma 3 will be completed.

Let $\varepsilon > 0$ be any number. We fix a positive integer m_0 such that

$$\sum_{m=m_0}^{\infty} \frac{1}{m^{1+\alpha}} < \frac{\varepsilon}{4 \text{diam } K^*}$$

Then

$$\|u_{\gamma_q} - u_0\|_A = \sum_{m=1}^{m_0-1} \frac{\|u_{\gamma_q} - u_0\|_m}{m^{2+\alpha}} +$$

$$\begin{aligned}
 & + \sum_{m=m_0}^{\infty} \frac{\|u_{\gamma_q} - u_0\|_m}{m^{2+\alpha}} < \\
 & < (m_0 - 1) \|u_{\gamma_q} - u_0\|_m + 2 \text{diam } K * \sum_{m=m_0}^{\infty} \frac{1}{m^{1+\alpha}} < \\
 & < (m_0 - 1) \|u_{\gamma_q} - u_0\|_m + \frac{\varepsilon}{2}.
 \end{aligned}$$

Since $\lim_{q \rightarrow \infty} \|u_{\gamma_q} - u_0\|_m = 0$, then there exists a positive integer q_0 such that

$$(m_0 - 1) \|u_{\gamma_q} - u_0\|_m < \frac{\varepsilon}{2}$$

for $q > q_0$. Thus $\|u_{\gamma_q} - u_0\|_A < \varepsilon$ for $q > q_0$, where $\varepsilon > 0$ is any given number.

Lemma 3 is proved.

D) The proof of Theorem 3.

Now we return to the proof of Theorem 3, which is the main existence Theorem of §2.

Let ξ be any element of the set T_k and let $u(x)$ be its basic representative. We consider the collection of convex functions

$$u_a(x) = u(x) + a \quad (2.47)$$

where $a \in (-\infty, +\infty)$. The function

$$\begin{aligned}
 F_{u_a}(x) &= f(x, u_a(x), \nabla u_a(x)) = \\
 &= f(x, u(x) + a, \nabla u(x))
 \end{aligned}$$

is non-negative for all $x \in E^n$. Let $z = k(x)$ be the equation of the cone K .
Then

$$u_a(x) = u(x) + a \leq k(x) + a \quad (2.49)$$

for all $x \in E^n$. From Assumption A.3 it follows that

$$\begin{aligned} F_{u_a}(x) &\leq \phi_k(x, u(x) + a) \leq \\ &\leq \phi_k(x, k(x) + a) \end{aligned} \quad (2.50)$$

for all $x \in E^n$ and all real values of a .

From the conditions of Theorem 3 it follows that

$$\int_{E^n} F_{u_a}(x) dx \leq \int_{E^n} \phi_k(x, k(x) + a) dx < +\infty$$

for all $a \in [a_k, b_k]$.

From Assumption A.3 we obtain the inequality

$$F_{u_a}(x) \geq \inf_{\gamma \in K^*} \lambda_k(x, (x, \gamma) + a) \quad (2.51)$$

for all $x \in E^n$ and all $x \in R$.

Now we introduce the function

$$\psi(a) = \int_{E^n} F_{u_a}(x) dx \quad (2.52)$$

in $[a_k, b_k]$. Since

$$\psi(a) = \int_{E^n} f(x, u(x) + a, \nabla u(x)) dx \quad (2.53)$$

for $a \in [a_k, b_k]$, then from Assumptions A.2, A.3 and the conditions of Theorem 3 it follows that $\psi(a)$ is continuous, $\psi'(a)$ exists on $[a_k, b_k]$, and inequalities

$$\psi(a_k) \leq \int_{E^n} \phi_k(x, k(x) + a_k) dx < \text{mes } K^* \quad (2.54)$$

and

$$\psi(b_k) > \inf_{K^*} \int_{E^n} \lambda_k(x, (x, \gamma) + b_k) dx > \text{mes } K^*$$

hold. Since

$$\psi'(a) = \int_{E^n} \frac{\partial f(x, u(x) + a, \nabla u(x))}{\partial u} dx > 0,$$

then from (2.54) it follows that there exists only one number $a^* \in [a_k, b_k]$ such that

$$\psi(a^*) = \int_{E^n} F_{u_{a^*}}(x) dx = \text{mes } K^*. \quad (2.55)$$

Now we consider the second boundary value problem for equation

$$\det(z_{ij}) = F_{u_{a^*}}(x) \quad (2.56)$$

with prescribed asymptotic cone K .

Since all conditions of Theorem 1 are fulfilled, then this boundary value problem has only one convex generalized solution $z(x)$ satisfying the condition $z(0) = a^*$ and having K as the asymptotic cone of its graph.

Let $\tilde{u}(x) = u_{a^*}(x) - a^*$ and $\tilde{z}(x) = z(x) - z(0)$ be the basic representatives of the elements ξ_u and let η_z belong to the convex set T_K . Clearly the second boundary value problem for equation (2.56) and prescribed admissible convex cone K generate some operator

$$G : T_k \rightarrow T_k$$

such that $\eta_z = G(\xi_u)$.

The further considerations are devoted to the establishment of continuity and compactness of the operator G on the convex compact set T_k . They will permit to apply the Schauder fixed point Theorem to the operator $G: T_k \rightarrow T_k$. The existence Theorem 3 is the final result of these investigations.

Lemma 4. The functional $a^*: T_k \rightarrow \mathbb{R}$ is continuous.

Proof. Let ξ be any element of T_k and $u(x) \in A_1 \in A$ be a basic representative of ξ . Then $u(x)$ is a convex function, whose graph has K as its asymptotic cone, and $u(0) = 0$.

The real number $a^* = a^*(u)$ is the root of the equation

$$\int_{\mathbb{E}^n} f(x, u(x) + a, \nabla u(x)) dx = dx = \text{mes } K^* . \quad (2.57)$$

We proved above that this equation has only one root $a^* = a^*(u) \in [a_k, b_k]$.

Now it should be proved that $a^*(u_q)$ converges to $a^*(u)$ if

$$\lim_{q \rightarrow \infty} \|u_q - u\|_A = 0$$

and $\xi_{u_q}, \xi_u \in T_k$.

From (2.57) it follows that

$$\int_{\mathbb{E}^n} f(x, u(x) + a^*(u), \nabla u(x)) dx =$$

$$\int_{\mathbb{E}^n} f(x, u_q(x) + a^*(u_q), \nabla u_q(x)) dx = \text{mes } K^* .$$

Hence

$$\begin{aligned} & \int_{E^n} \{ [u(x) - u_q(x)] + [a^*(u) - a^*(u_q)] \} \left(\int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \right) dx = \\ & = - \sum_{i=1}^n \int_{E^n} \left[\frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right] \left(\int_0^1 \frac{\partial f}{\partial u_i} \Big|_{v_t} dt \right) dx, \end{aligned}$$

where $v_t(x) = (1-t)u_q(x) + tu(x) +$
 $+ (1-t)a^*(u_q) + ta^*(u)$

and

$$\frac{\partial v_t(x)}{\partial x_i} = (1-t) \frac{\partial u_q(x)}{\partial x_i} + t \frac{\partial u(x)}{\partial x_i},$$

$$0 < t < 1, i = 1, 2, \dots, n.$$

Since $a^*(u)$ and $a^*(u_q)$ do not depend on x , then from (2.58) we obtain

$$a^*(u) - a^*(u_q) =$$

$$\begin{aligned} & = - \frac{\int_{E^n} \{u(x) - u_q(x)\} \left(\int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \right) dx}{\int_{E^n} \left(\int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \right) dx} \\ & = - \frac{\int_{E^n} \sum_{i=1}^n \left(\frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right) \left[\int_0^1 \frac{\partial f}{\partial u_i} \Big|_{v_t} dt \right] dx}{\int_{E^n} \left(\int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \right) dx} \end{aligned}$$

Since $\frac{\partial f}{\partial u}$ is positive and continuous in $E^n \times R \times K^*$ (see Assumption A.2), then for any compact set Q in $E^n \times R \times K^*$ there exists a constant $h(Q) > 0$ such that

$$\frac{\partial f(x, u, p)}{\partial u} \geq h(Q) > 0.$$

For our purpose it is sufficient to consider the compact set

$$Q_0 = U_1 \times [\delta_1, \delta_2] \times K^*,$$

where U_1 is the unit ball $|x| \leq 1$ in E^n , and the numbers δ_1, δ_2 are determined by conditions

$$\delta_1 \leq v_t(x) \leq \delta_2.$$

The finite values of δ_1, δ_2 depend only on $\|u(x)\|_1$, numbers a_k, b_k and the integer $N_1 > 0$ such that

$$\|u_q(x) - u(x)\|_A < 1$$

if $q \geq N_1$. If $x \in U_1$, then the point

$$(v_t(x), \nabla v_t(x)) \in [\delta_1, \delta_2] \times K^*$$

for all $t \in [0, 1]$, because $\nabla v_t(x) = (1-t)\nabla u(x) + t\nabla u_q(x) \in K^*$ *)

Here we take into account that $\nabla u(x)$ and $\nabla u_q(x)$ are points of the convex set K^*). Thus we obtain the inequality

$$\int_{E^n} \left(\int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \right) dx \geq h(Q_0) \text{mes } Q_0 > 0 \quad (2.59)$$

*) Since $u(x)$, $u_q(x)$ and $v_t(x)$ are convex functions, the notation $\nabla u(x)$ etc. is used also for supporting hyperplanes if there does not exist a tangent hyperplane of the graph of $u(x)$ at the point $(x, u(x))$.

for all $q \geq N_1$.

According to Assumption A.2 the inequalities

$$0 < \frac{\partial f}{\partial u} \leq \frac{C_0}{|x|^{n+2+\alpha}}$$

hold for all $x \in E^n$ with $|x| > m_0$, $u \in K$ and $p \in K^*$, where $\alpha = \text{const.} > 0$ and $m_0 = \text{const.} > 1$; without losing generality we can assume that m_0 is any positive integer more than $(2^{1/2+\alpha} - 1)^{-1}$. Let

$$I_1 = \left| \int_{E^n} \{ [u(x) - u_q(x)] \cdot \int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \} dx \right|.$$

Then

$$\begin{aligned} I_1 &\leq \int_{|x| \leq m_0} \{ |u(x) - u_q(x)| \int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \} dx + \\ &+ \int_{|x| > m_0} \{ |u(x) - u_q(x)| \cdot \int_0^1 \frac{\partial f}{\partial u} \Big|_{v_t} dt \} dx. \end{aligned}$$

Let

$$C_2 = \sup_{|x| \leq m_0} \frac{\partial f}{\partial u} \Big|_{v_t} < +\infty. \quad (2.61)$$

Clearly C_2 depends only on $\|u(x)\|_{m_0}$, K^* and numbers a_k, b_k . Since $u(x)$ and $u_q(x)$ are basic representatives of elements of the set T_k , then

$$u(\theta) = u_q(\theta) = 0 \quad (2.62)$$

and

$$|u(x)| \leq |x| \cdot \text{diam } K^*, \quad (2.63)$$

$$|u_q(x)| \leq |x| \cdot \text{diam } K^*.$$

From (2.60 - 2.63) it follows that

$$I_1 \leq C_2 \|u(x) - u_q(x)\|_{m_0} + \\ + \frac{C_0 \sigma_{n-1}}{2+\alpha} \sum_{m=m_0}^{\infty} \|u(x) - u_q(x)\|_{m+1} \left(\frac{1}{m^{2+\alpha}} - \frac{1}{(m+1)^{2+\alpha}} \right),$$

where σ_{n-1} is the area of the unit sphere S^{n-1} . Since

$$\|u(x) - u_q(x)\|_{m_0} \leq m_0^{2+\alpha} \|u(x) - u_q(x)\|_A$$

and

$$\frac{1}{(m+1)^{2+\alpha}} > \frac{1}{m^{2+\alpha}} - \frac{1}{(m+1)^{2+\alpha}}$$

for $m > m_0 > (2^{\frac{1}{2+\alpha}} - 1)^{-1}$, then

$$I_1 \leq [C_2 m_0^{2+\alpha} + \frac{C_0 \sigma_{n-1}}{2+\alpha}] \|u(x) - u_q(x)\|_A, \quad (2.64)$$

where the integer m_0 satisfies the inequality

$$m_0 > (2^{\frac{1}{2+\alpha}} - 1)^{-1}$$

and constants C_0 and C_2 are independent of q .

Let

$$I_2 = \left| \int_{E^n} \left\{ \sum_{i=1}^n \left[\frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right] \left(\int_0^1 \frac{\partial f}{\partial u_j} \bigg|_{v_t} dt \right) \right\} dx \right|$$

Clearly

$$I_2 \leq \sum_{i=1}^n \int_{R^n} \left\{ \left| \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right| \left(\int_0^1 \left| \frac{\partial f}{\partial u} \right|_{v_t} dt \right) \right\} dx.$$

According to Assumption A.2 the inequalities

$$\left| \frac{\partial f}{\partial u_i} \right| \leq \frac{C_1}{|x|^{n+\alpha}}, \quad i = 1, 2, \dots, n \quad (2.66)$$

hold for all $x \in E^n$ with $|x| > m_0$, $u \in R$, and $p \in K^*$. The functions $\frac{\partial u_q(x)}{\partial x_i}$ converge to $\frac{\partial u(x)}{\partial x_i}$ almost everywhere in E^n and

$$\nabla u_q(x) \in K^*, \quad \nabla u(x) \in K^* \quad (2.67)$$

for all $x \in K^*$.

From (2.65-67) it follows that

$$\begin{aligned} I_2 \leq \sum_{i=1}^n \left(\sup_{|x| \leq m} \int_0^1 \left| \frac{\partial f}{\partial u_i} \right|_{v_t} dt \right) \cdot \int_{|x| \leq m} \left| \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right| dx + \\ + 2C_1 n \text{diam } K^* \int_{|x| > m} \frac{dx}{|x|^{n+\alpha}}, \end{aligned} \quad (2.68)$$

where $m > m_0$ is an arbitrary positive integer. Since

$$\int_{|x| > m} \frac{dx}{|x|^{n+\alpha}} = \frac{\sigma_{n-1}}{\alpha m^\alpha},$$

then we can find $m^* > 0$ such that for every integer $m > \max \{m_0, m^*\}$ the inequality

$$2C_1 n \text{diam } K^* \frac{\sigma_{n-1}}{\alpha m^\alpha} < \frac{\varepsilon}{2} \quad (2.69)$$

holds, where $\varepsilon > 0$ is a given positive arbitrary number. We fix some integer $m > \max \{m_0, m^*\}$. Then

$$\sup_{|x| \leq m} \int_0^1 \left| \frac{\partial f}{\partial u_i} \right|_{v_t} dt \leq C < +\infty$$

($i=1, 2, \dots, n$), where the constant C_m depends only on m, a_k, b_k and $\text{diam } K^*$.

Really $\frac{\partial f}{\partial u_i}$ are continuous functions according to Assumption A_2 and we consider the supremum of $\left| \frac{\partial f}{\partial u_i} \right|$ in the compact set:

$$|x| \leq m, a_k - |x| \cdot \text{diam } K^* \leq u \leq b_k + |x| \text{diam } K^*,$$

Since $u(x)$ and $u_q(x)$ are convex functions, then

$$\lim_{q \rightarrow \infty} \int_{|x| \leq m} \left| \frac{\partial u(x)}{\partial x_j} - \frac{\partial u_q(x)}{\partial x_j} \right| dx = 0$$

Therefore we can find N_2 such that

$$\sum_{i=1}^n \left(\sup_{|x| \leq m} \int_0^1 \left| \frac{\partial f}{\partial u_j} \right|_{v_t} dt \right) \int_{|x| \leq m} \left| \frac{\partial u(x)}{\partial x_j} - \frac{\partial u_q(x)}{\partial x_j} \right| dx < \frac{\varepsilon}{2}.$$

if $q > N_2$.

Thus

$$I_2 < \varepsilon, \quad (2.70)$$

if $q > N_2$.

Now from (2.59), (2.64), (2.70) it follows that

$$\lim_{q \rightarrow \infty} a^*(u_q) = a^*(u)$$

if $\|u_q - u\|_A \rightarrow 0$. Lemma 4 is proved.

Lemma 5. The operator $G: T_k \rightarrow T_k$ is continuous.

Proof. Let the sequence $\xi_q \in T_k$ converge to the element $\xi_0 \in T_k$ in the space B . We should prove that

$$\lim_{q \rightarrow \infty} \| \eta_q - \eta_0 \|_B = 0, \quad (2.71)$$

where $\eta_q = G(\xi_q)$ and $\eta_0 = G(\xi_0)$.

Since T_k is compact subset of B (see Lemma 3), then there exists a subsequence η_{q_i} convergent to some element $\bar{\eta} \in B$, i.e.

$$\lim_{j \rightarrow \infty} \| \eta_{q_j} - \bar{\eta} \|_B = 0 .$$

Since T_k is closed in B , then $\eta \in T_k$. It is well known that the set functions $\omega(1, \bar{v}_{q_j}, e)$ converge weakly to the set function $\omega(1, v, e)^{*)}$ in E^n , where $v_{q_j}(x)$ and $v(x)$ are representatives of η_{q_j} and η satisfies the conditions

$$v_{q_j}(\theta) = a^*(u_{q_j}); \quad v(\theta) = a^*(u_0)$$

On the other hand all the functions

$$F_{u_{q_j}} + a^*(x) = f(x, u_{q_j} + a^*(u_{q_j}), \nabla u_{q_j})^{**}) \quad (2.72)$$

are non-negative and satisfy the inequality

$$F_{u_{q_j}} + a^*(x) \leq \phi_k(x, k(x) + b_k) \quad (2.73)$$

for all $x \in E^n$, where $z = k(x)$ is the equation of admissible convex cone K (see Assumption A.1), prescribed for all functions $u_{q_j}(x)$.

Note that the following facts are fulfilled:

$$a) \quad \lim_{j \rightarrow \infty} \| u_{q_j} - u_0 \|_A = 0 ;$$

*) $\omega(1, u, e)$ is the R-curvature of a convex function $u(x)$ with $R(p) \equiv 1$ for all $p \in p^n$.

**) The functions $u_{q_j}(x)$ and $u_0(x)$ are basic representatives of the elements ξ_{q_j} and ξ_0 .

- b) $\nabla u_{q_j}(x) \in K^*$ for all $x \in E^n$ and all integers q_j ;
- c) $\frac{\partial u_{q_j}}{\partial x_i}$ converges to $\frac{\partial u_0}{\partial x_i}$ almost everywhere in E^n , ($i=1,2,\dots,n$);
- d) $\phi_M(x, k(x) + b_k)$ is a non-negative summable function in E^n ;
- e) Estimates (2.59) are correct for the functions $u_{q_j}(x)$ and $u_0(x)$.

Now we use the well known Lebesgue Theorem and obtain

$$\begin{aligned} \omega(1, v, e) &= \lim_{q_j \rightarrow \infty} \omega(1, u_{q_j}, e) = \\ &= \lim_{q_j \rightarrow \infty} \int_e F_{u_{q_j}} + a^*(u_{q_j})(x) dx = \int_e F_{u_0} + a^*(u_0)(x) dx, \end{aligned}$$

where e is any Borel subset of E^n . Note that we used continuity of the functional $a^*: T_K \rightarrow R$ in these equalities.

Thus $v(x)$ is a convex generalized solution of the equation

$$\det \left(\frac{\partial^2 \bar{v}}{\partial x_i \partial x_j} \right) = F_{u_0} + a^*(u_0)(x)$$

and the function $v_0(x)$ is also a convex generalized solution of the equation

$$\det \left(\frac{\partial^2 v_0}{\partial x_i \partial x_j} \right) = F_{u_0} + a^*(u_0)(x).$$

Since

$$v(\theta) = v_0(\theta) = a^*(u_0)$$

and the admissible convex cone K is the asymptotic cone for both functions $\bar{v}(x)$ and $v_0(x)$, then $\bar{v}(x) = v_0(x)$ for all $x \in E^n$.

Thus (20.71) is correct. Lemma 5 is proved.

Now we can finish the proof of Theorem 3. Since T_k is a compact set in B , Then $G(T_k)$ is also compact in B . Moreover $G(T_k) \subset T_k$. Hence the operator G has at least one fixed point $\xi \in T_k$. But the function $u(x) + a^*(\tilde{u}(x))$ is the representative for both elements ξ and $G(\xi)$. Therefore $u(x)$ is the desired solution of the second boundary value problems for equation

$$\det(u_{ij}) = f(x, u, Du).$$

Theorem 3 is proved.

§3. The maximum principle and estimates of solutions for elliptic Euler-Lagrange equations.

The concept and techniques of the global theory of convex hypersurfaces and functions can successfully be applied to the Dirichlet problem for elliptic Euler-Lagrange equations. In the present Section we establish the sharp interlocked necessary and sufficient conditions of two-sided estimates for solutions of such equations. These conditions are not subjected to a number of essential traditional limitations (see for example [12], [13], [14]). The interesting special cases of such Euler-Lagrange equations arise in elasticity-plasticity and global differential geometry problems (see [15], [16], [17]).

3.1 The main Assumptions. Let $F(x, u; p)$ be a C^2 -function in $\Omega \times \mathbb{R} \times \mathbb{P}^n$, where Ω is a bounded open domain in E^n , and $\partial\Omega$ is a closed C^2 -hypersurface. It is well known that any C^2 -function $u(x)$ minimizing (maximizing) the multiple integral

$$I(u) = \int_{\Omega} F(x, u, Du) dx \quad (3.1)$$

is necessarily a solution of the Euler-Lagrange equation

$$\sum_{i=1}^n \frac{d}{dx_i} \left(\frac{\partial F}{\partial p_i} \right) - \frac{\partial F}{\partial z} = 0. \quad (3.2)$$

This equation can also be written in the following form

$$\sum_{i,k=1}^n \frac{\partial^2 F}{\partial p_i \partial p_k} u_{ik} + \sum_{i=1}^n \frac{\partial^2 F}{\partial p_i \partial u} u_i + \quad (3.3)$$

$$+ \sum_{i=1}^n \frac{\partial^2 F}{\partial p_i \partial x_i} - \frac{\partial F}{\partial z} = 0 .$$

Equation (3.3) is elliptic if and only if the quadratic form

$$\sum_{i,k=1}^n \frac{\partial^2 F(x,u,p)}{\partial p_i \partial p_k} \xi_i \xi_k \quad (3.4)$$

is positive (negative) definite in $\bar{\Omega} \times R \times P^n$. The Dirichlet problem for wide classes of Euler-Lagrange equations is investigated by many authors (see [12], [18], [13], [16], [15], and other papers *).

Assumption A.4. Let $F(x,u,p)$ be a C^2 -function in $\bar{\Omega} \times R \times P^n$ and let the quadratic form (3.4) is positive definite in $\Omega \times R \times P^n$. Then there exist functions $\phi(p)$, $\phi(x,u)$, $R_1(p)$, $R_2(p)$, $y_1(x,u)$, $y_2(x,u)$ satisfying the following conditions:

1) $\phi(p)$ is a strictly convex C^2 -function in P^n *) and $\lambda(x,u)$ is positive and the inequality

$$\lambda(x,u) \det \left(\frac{\partial^2 \phi(p)}{\partial p_i \partial p_j} \right) < \det \left(\frac{\partial^2 F(x,u,p)}{\partial p_i \partial p_j} \right) \quad (3.5)$$

holds for every point $(x,u,p) \in \Omega \times R \times P^n$;

2) The functions $R_1(p)$, $R_2(p)$ are positive and locally summable with degree n in P^n , the functions $y_1(x,u)$, $y_2(x,u)$ are non-negative in $\Omega \times R$ and the inequalities

$$-\frac{g_1(x,u)}{R_1(p)} < D(x,u,p) < \frac{g_2(x,u)}{R_2(p)} \quad (3.6)$$

hold for every point $(x,u,p) \in \Omega \times R \times P^n$, where

$$D(x,u,p) = - \sum_{i=1}^n \frac{\partial^2 F(x,u,p)}{\partial p_i \partial u} p_i - \quad (3.7)$$

$$- \sum_{i=1}^n \frac{\partial^2 F(x,u,p)}{\partial p_i \partial x_i} + \frac{\partial F(x,u,p)}{\partial u}$$

3) The function

$$\psi_1(x,u) = \frac{g_1^n(x,u)}{n^n \lambda(x,u)} \quad (3.8)$$

and

$$\psi_2(x,u) = \frac{g_2^n(x,u)}{n^n \lambda(x,u)} \quad (3.9)$$

are non-decreasing with respect to u for every fixed $x \in \Omega$.

3.2. Estimates of solutions of the Dirichlet problems for the Euler-Lagrange equations.

We consider the Dirichlet problem

$$\sum_{i=1}^n \frac{d}{dx_i} \left(\frac{\partial F}{\partial p_i} \right) - \frac{\partial F}{\partial u} = 0 \quad (3.10)$$

$$u|_{\partial\Omega} = h(X) \in C(\partial\Omega) . \quad (3.11)$$

In Subsections 3.2 and 3.3 it is sufficient to consider only open bounded domains with continuous $\partial\Omega$.

*) A convex C^2 -function $\phi(p)$ is called strictly convex, if $\det \left(\frac{\partial^2 \phi(p)}{\partial p_i \partial p_j} \right) > 0$ in P^n .

Let $M = \sup_{\partial\Omega} h(x)$, $m = \inf_{\partial\Omega} h(x)$. Clearly $-\infty < m \leq M < +\infty$.

Theorem 4. Let $u(x) \in V_2^n(\Omega) \cap C(\bar{\Omega})^*$ be a solution of the Dirichlet problem (3.10-11) and let all conditions of Assumption 4 be fulfilled.

Then the inequalities

$$\int_{\Omega} \psi_1(x, m) dx < \int_{K^*} \tilde{R}_1^n(q) dq, \quad (3.12)$$

$$\int_{\Omega} \psi_2(x, M) dx < \int_{K^*} \tilde{R}_2^n(q) dq \quad (3.13)$$

provide the estimates

$$m - C(R_2, \omega_2) \text{diam } \Omega \leq u(x) \leq M + C(R_2, \omega_1) \text{diam } \Omega \quad (3.14)$$

for all $x \in \bar{\Omega}$, where

$$\omega_1 = \int_{\Omega} \psi_1(x, m) dx, \quad \omega_2 = \int_{\Omega} \psi_2(x, M) dx \quad (3.15)$$

and the values of the constants $C(R_i, \omega_i)$ ($i=1,2$) will be described in the end of the proof of Theorem 4.

Proof. First of all we explain the notations in the right parts of inequalities (3.12), (3.13). K^* is the normal image of the asymptotic cone K of the strictly convex C^2 -function $\phi(p)$; $x_\phi: p^n \rightarrow Q^n$ is the tangential mapping, generated by the convex function $\phi(p)$;

$$\tilde{R}_i(q) = R_i(x_\phi^{-1}(q)), \quad (i=1,2) \quad (3.16)$$

and dq is the element of the Lebesgue measure in the Euclidean space Q^n .

*) $V_2^n(\Omega)$ is the Sobolev space with functions having all second generalized derivatives summable with degree n in Ω .

Let $G = \text{Co}\Omega$ and $\overline{G}$ be the closure of G . Clearly $\overline{G} = \overline{\text{Co}\Omega}$, where $\overline{\text{Co}\Omega}$ is the closed convex hull of Ω .

Let $H_m = \overline{G} \times \{M\}$ and $H_m = \overline{G} \times \{m\}$. We construct the convex body Z which is the closed convex hull of the set $H_m \cup H_m \cup S_u$, where S_u is the graph of $u(x)$, $x \in \overline{\Omega}$.

Clearly the part of ∂Z , which are projected in the open convex domain G consists of the graphs S_{v_1} and S_{v_2} , where $v_1(x)$ is a concave function in G .

Since

$$m < u|_{\partial\Omega} < M \quad (3.17)$$

and

$$v_2(x) < u(x) < v_1(x) \quad (3.18)$$

for all $x \in G$, then it is sufficient only to establish the inequalities

$$v_2(x) > m - C(R_2, \omega_2) \text{diam } \Omega \quad (3.19)$$

and

$$v_1(x) < M + C(R_1, \omega_1) \text{diam } \Omega. \quad (3.20)$$

Moreover, it is sufficient to establish only (3.19), because (3.20) can be proved in the same way as (3.19). Our further considerations follow the proof of our analogous Theorems for solutions of the Dirichlet problem for elliptic quasilinear equations with the corresponding changes and developments. (see for example my papers [3], [15], [17]). Clearly the interesting case is only that, when there exist the points $x \in \Omega$ such that

$$u(x) < m \quad (3.21)$$

We denote by $Q_2 \subset \Omega$ the set, where

$$v_2(x) = u(x).$$

Clearly Q_2 is a Lebesgue measurable set and

$$\text{mes } \overline{Q}_2 = \text{mes } Q_2,$$

because $\overline{Q}_2 \subset \overline{\Omega}$, $\text{mes } \partial\Omega = 0$ and

$$Q_2 = \overline{Q}_2 \setminus \partial\Omega.$$

The inequality

$$\begin{aligned} & n[\det(u_{ij})]^{\frac{1}{n}} [\det(\frac{\partial^2 F}{\partial p_i \partial p_j})]^{\frac{1}{n}} < \\ & < \sum_{i=1}^n \frac{\partial^2 F}{\partial p_i \partial p_j} u_{ij} = D(x, u(x), \nabla u(x)) \end{aligned}$$

hold at any point $x \in Q_2$. Applying Assumption A.4 to inequality (3.22) we obtain

$$\begin{aligned} & R_2^n(\nabla u(x)) \det \frac{\partial^2 \phi(\nabla u(x))}{\partial p_i \partial p_j} \det(u_{ij}(x)) < \\ & < \frac{g_2(x, u)}{n^n \lambda(x, u)} = \psi_2(x, u(x)) < \psi_2(x, M) \end{aligned} \quad (3.23)$$

at every point $x \in Q_2$.

Since $u(x) \in V_2^n(\Omega) \subset C(\overline{\Omega})$, then the following formula

$$\begin{aligned} & \omega(R_2^n \det(\frac{\partial^2 \phi}{\partial p_i \partial p_j}), v_2, G) = \\ & = \int_{Q_2} R_2^n(\nabla u(x)) \det(\frac{\partial^2 \phi(\nabla u(x))}{\partial p_i \partial p_j}) \det(u_{ij}(x)) dx \end{aligned}$$

is established in [3] for the total $R_2^n \det(\frac{\partial^2 \phi}{\partial p_i \partial p_j})$ - curvature of the convex function $v_2(x)$. Thus from (3.24) and (3.23) it follows that

$$\omega(R_2^n \det(\frac{\partial^2 \phi}{\partial p_i \partial p_j}), v_2, u) \leq \int_{\Omega} \psi_2(x, M) dx = \omega_2 \quad (3.25)$$

Note that

$$\begin{aligned} \int_{p^n} R_2^n(p) \det(\frac{\partial^2 \phi}{\partial p_i \partial p_j}) dp &= \int_{x_\phi(p^n)} \tilde{R}_2^n(q) dq = \\ &= \int_{K^*} \tilde{R}_2^n(q) dq. \end{aligned} \quad (3.26)$$

where all notations were explained in the beginning of the proof of Theorem 4.

From Bakelman's Theorem (see [3], [15]), inequality (3.25) and the conditions of Theorem 4 (see inequality (3.13)) it follows the validity of the estimate

$$m - C(R_2, \omega_2) \text{diam } \Omega \leq v_2(x) \leq u(x) \quad (3.27)$$

for all $x \in \overline{\Omega}$.

Now we describe the way of computing of the constant $C(R_2, \omega_2)$ (see [3])

First we consider the function

$$\tau_{R_2}(p) = \int_{|p| \leq p} R_2^n(p) \det(\frac{\partial^2 \phi(p)}{\partial p_i \partial p_j}) dp \quad (3.28)$$

for all $p \in [0, +\infty)$. Since $\tau_{R_2}(p)$ is continuous and strictly increasing, then there exists its inverse $p = T_{R_2}(\tau)$ for all $\tau \in [0, A(R_2))$, where

$$A(R_2) = \int_{K^*} \tilde{R}_2^n(q) dq.$$

Now $C(R_2, \omega_2)$ is defined by the formula

$$C(R_2, \omega_2) = T_{R_2}(\omega_2),$$

$$\text{where } \omega_2 = \int_{\Omega} \psi_2(x, M) dx < A(R_2).$$

The estimate

$u(x) \leq v_1(x) \leq M + C(R_1, \omega_1) \text{ diam } \Omega$ can be obtained in the similar way.

The proof of Theorem 4 is completed.

3.3 The important special cases and examples.

A) The geometric maximum principle.

If the function F depends on x and u , then $\omega_1 = \omega_2 = 0$. Clearly $C(R_1, 0) = C(R_2, 0) = 0$. Then from (3.14) it follows that

$$m \leq u(x) \leq M$$

for all $x \in \overline{\Omega}$.

Therefore Theorem 3 is called the geometric maximum principle.

B) The Euler-Lagrange equations for the functionals

$$I(u) = \int_{\Omega} [F(\nabla u) + f(x, u)] dx.$$

The Euler-Lagrange equation for the functionals

$$I(u) = \int_{\Omega} [F(\nabla u) + f(x, u)] dx$$

has the following form

$$\sum_{i,k=1}^n \frac{\partial^2 F(\nabla u)}{\partial p_i \partial p_j} u_{ik} = \frac{\partial f(x, u)}{\partial u}. \quad (3.29)$$

According to Assumption A.4

$F(p) = \phi(p)$ is a strictly convex function in p^n ; $\phi(x, u) \equiv 1$ and $\frac{\partial^2 f(x, u)}{\partial u^2} > 0$ in $\bar{\Omega} \times \mathbb{R}$ (i.e. $f(x, u)$ is a convex C^2 -function of u for every fixed $x \in \bar{\Omega}$). Let

$$\frac{\partial f(x, u)}{\partial u} = f_u^+(x, u) - f_u^-(x, u)$$

where $f_u^+(x, u) > 0$ and $f_u^-(x, u) > 0$ are positive and negative parts of $\frac{\partial f(x, u)}{\partial u}$.

Thus inequalities (3.12) and (3.13) (see the statement of Theorem 3) can be written in the following form

$$\frac{1}{n^n} \int_{\Omega} [f_u^-(x, m)]^n dx < \text{mes } K^* \quad (3.12')$$

and

$$\frac{1}{n^n} \int_{\Omega} [f_u^+(x, m)]^n dx < \text{mes } K^* \quad (3.13')$$

Since

a finite positive number if the asymptotic cone of $F(p)$ is non-degenerate;

$\text{mes } K^* =$
 $+\infty$ if this cone degenerate,

then the inequalities (3.12') and (3.13') are non-trivial restrictions only for the case of a non-degenerate asymptotic cone of $F(p)$.

The special case

$$f(x, u) = f(x) \cdot u,$$

which realted to many applications (see [3], [14], [13], [15]), leads to the inequalities

$$\frac{1}{n^n} \int_{\Omega} f_{\pm}^n(x) dx < \text{mes } K^*.$$

The last inequalities provide estimates of solutions of the Dirichlet problem for elliptic equations

$$\sum_{i,j=1}^n \frac{\partial^2 F(\nabla u)}{\partial p_i \partial p_j} u_{ijk} = f(x).$$

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