A singular minimizer of a smooth strongly convex functional in three dimensions

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1 Introduction

We consider variational integrals of the form,

$$I(u) = \int_\Omega f(Du(x))dx, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set, $u\colon \Omega \to \mathbf{R}^m$ is a mapping belonging to $W^{1,2}(\Omega)$, $Du(x)$ denotes the gradient matrix of u at $x \in \Omega$, and f is a smooth strongly convex function with uniformly bounded second derivatives defined on the set $M^{m\times n}$ of all real $m \times n$ matrices. We recall that f is said to be strongly convex if there exists a constant $\nu > 0$, such that for all $\xi \in M^{m\times n}, X \in M^{m\times n}$, the inequality $f_{p^i_\alpha p^j_\beta}(X)\xi^i_\alpha \xi^j_\beta \geq \nu|\xi|^2$ holds. Here and in what follows we will be using Einstein's summation convention.

We shall consider the regularity of minimizers of I in $W^{1,2}(\Omega)$. Here by a minimizer we mean a function $u \in W^{1,2}(\Omega)$ such that for any smooth function $\phi\colon \Omega \to \mathbf{R}^m$ compactly supported in Ω the inequality $I(u+\phi) \geq I(u)$ holds. When f is strongly convex, it is not difficult to see that u is a minimizer of I if and only if u is a weak solution of the $Euler - Lagrange$ equation of I, i.e. u satisfies (in the sense of distributions)

$$\partial_\alpha f_{p^i_\alpha}(Du(x)) = 0, \qquad i = 1, \cdots m. \tag{1.2}$$

A classical result of C.B. Morrey ([Mo]) says that when $n = 2, m \geq 1$, every minimizer of $I(u)$ is regular. This is also the case when $n \geq 2, m = 1$ by celebrated results of De Giorgi([De1]) and Nash ([Na]). The methods used in the proof of De Giorgi and Nash can not be extended to the case $m \geq 2$ as shown by a counterexample of De Giorgi ([De2]). The first example of a nonsmooth minimizer for a smooth strongly convex functional of the type (1.1) was constructed by Nečas in high dimensions (see [Ne]). He considered the function $u\colon \mathbf{R}^n \to \mathbf{R}^{n^2}$ defined by

$$u_{ij} = \frac{x_i x_j}{|x|}, \tag{1.3}$$

and for large n constructed a strongly convex function f on $M^{n\times n^2}$ for which u is a minimizer of the corresponding functional I. Later Nečas, Hao and Leonardi ([HLN]) were able to modify this construction and make it work for $n \geq 5$. They used u given by

$$u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}. \tag{1.4}$$

Important counterexamples to regularity of solutions of elliptic systems which are not of the form (1.2) can be found in [GM] and [NJS]. For a comprehensive treatment of regularity questions we refer the reader to [Gi]. Interesting sufficient conditions for regularity are given in [Ko].

The purpose of this paper is to give a counterexample to regularity of weak solutions of (1.2) in the case $n = 3, m = 5$. We use exactly the same u defined by (1.4) and construct a smooth strongly convex function f such that u is a minimizer of I. The main idea of our construction is the following. Let $K = \{\nabla u(x), x \in \Omega\}$ be the set of gradients of u. We find a null Lagrangian L (see Definition 2.1 below) such that

$$\nabla L(X) = \nabla f(X), \qquad \forall X \in K \tag{1.5}$$

for a smooth strongly convex function f. Then u will satisfy the *Euler − Lagrange* equation of I automatically. To find the null Lagrangian we use the symmetries of the function u. We will see below that there is, up to a multiplicative factor, a unique quadratic null Lagrangian on $M^{5\times 3}$ which is invariant under the symmetries of the function u. It turns out that this null Lagrangian satisfies a necessary and sufficient condition for the existence of a strongly convex f satisfying (1.5).

2 Preliminaries

First we introduce some basic facts about null Lagrangians.

Definition 2.1 *(see [Ba1]) $L\colon M^{m\times n} \to \mathbf{R}$ is a null Lagrangian if for each smooth $u\colon \mathbf{R}^n \to \mathbf{R}^m$,*

$$\operatorname{div} \nabla L(\nabla u(x)) = 0. \tag{2.1}$$

We recall the following classical theorem about null Lagrangians (see [Da] or [BCO]).

Proposition 1 *Let $L\colon M^{m\times n} \to \mathbf{R}$, the following conditions are equivalent:*
i) L is a null Lagrangian.
ii) L is a linear combination of subdeterminants.
iii)L is rank-one affine, i.e. $t \to L(A + tB)$ is affine for each $A \in M^{m\times n}$ and each $B \in M^{m\times n}$ with rank $B = 1$.

From now on, let Ω be the unit ball in $\mathbf{R}^3$. Consider $u = (u_{ij}(x))$ given by

$$u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{|x|}{3}\delta_{ij}, \qquad i,j = 1,...,3.$$

Then for each $x \in \Omega, u(x) \in \{A \in M^{3\times 3}, A = A^t, trA = 0\} \cong \mathbf{R}^5$. For each $R \in SO(3)$ we have

$$u(Rx) = Ru(x)R^t = \rho_5(R)u(x),$$

where we denote by ρ_{2i+1} the unique irreducible representation of $SO(3)$ of dimension $2i+1$. This notation will be used throughout the paper. An easy calculation shows that

$$\nabla u(Rx) = \rho_5(R)\nabla u(x)R^t = \rho_5 \otimes \rho_3(R)\nabla u(x).$$

Lemma 2.1 *There exists a unique (up to multiplication by a scalar) quadratic invariant null Lagrangian L on $M^{5\times 3}$ which is invariant under the above action of $SO(3)$.*

Proof Consider the tensor space $T = \{a_{ijk} \in (\mathbf{R}^3)^{\otimes 3} | a_{ijk} = a_{jik}, a_{iik} = 0\}$. Clearly we have $T \cong \mathbf{R}^{15} \cong M^{5\times 3}$. By the Clebsch-Gordan formula (see [BD]), we know that

$$\rho_5 \otimes \rho_3 = \rho_7 \oplus \rho_5 \oplus \rho_3.$$

We now identify the quadratic null Lagrangians on $M^{5\times 3}$ with $\Lambda^2\mathbf{R}^3 \otimes \Lambda^2\mathbf{R}^5 \cong \mathrm{Hom}(\Lambda^2\mathbf{R}^3, \Lambda^2\mathbf{R}^5)$ and consider the representation σ of $SO(3)$ on $\mathrm{Hom}(\Lambda^2\mathbf{R}^3, \Lambda^2\mathbf{R}^5)$ induced by $\rho_3 \otimes \rho_5$. By classical group representation theory (see [BD]) we have

$$\sigma = \rho_9 \oplus \rho_7 \oplus \rho_5 \oplus \rho_5 \oplus \rho_3 \oplus \rho_1.$$

Therefore we see there is a unique one dimensional invariant subspace.

3 Constructions

3.1 Construction of L

Now we calculate explicitly the invariant quadratic null Lagrangian which will be denoted by L in what follows. (We slightly abuse the notation, since L is only determined up to a multiplicative factor.) Since we have $M^{5\times 3} = V_7 \oplus V_5 \oplus V_3$, where V_i is the $i-$dimensional irreducible invariant subspace.

We know from the classical invariant theory (see [We1]) that L must be of the following form:

$$L(A) = \alpha|X|^2 + \beta|Y|^2 + \gamma|Z|^2$$

where $A \in M^{5\times 3}$, $\quad A = X + Y + Z, \quad$ with $X \in V_7, Y \in V_5, Z \in V_3$.

We identify $M^{5\times 3}$ with $T = \{a_{ijk} \in (\mathbf{R}^3)^{\otimes 3} | a_{ijk} = a_{jik}, a_{iik} = 0\}$ in the obvious way. Now we use a classical procedure to decompose T into irreducible subspaces (see [We1]). We first decompose T into the trace-free part T' and its orthogonal supplement T_3, i.e. $T = T' \oplus T_3$. An easy calculation shows that the projection on T_3 is given by $a_{ijk} \to -\frac{1}{5}\delta_{ij}\eta_k + \frac{3}{10}\delta_{ki}\eta_j + \frac{3}{10}\delta_{jk}\eta_i$ with $\eta_k = a_{kii}, k = 1, 2, 3$. Then we decompose T' by using symmetrizations. We have $T' = T_1 \oplus T_2$, where the projection on T_1 is given by symmetrization, i.e. $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jki} + a_{kij})$; the projection on T_2 is given by $a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{jik} - a_{kji} - a_{kij})$, which corresponds to the following Young tableau:

1	2
3	

We remark that the antisymmetric part of any tensor in T is 0. We now identify T_1 with V_7, T_2 with V_5, T_3 with V_3.

We now use the condition that L has to vanish on rank-one matrices. These matrices correspond to the tensors in T which are of the form $a_{ijk} = c_{ij}\xi_k$, where $C = (c_{ij})$ is a trace-free symmetric matrix. A direct calculation of the norms of the projections a^l_{ijk} of a_{ijk} to T_l gives

$$a_{ijk} = a^1_{ijk} + a^2_{ijk} + a^3_{ijk},$$

with

$$|a^1_{ijk}|^2 = \frac{1}{3}|C|^2|\xi|^2 + \frac{2}{5}|C\xi|^2, \quad |a^2_{ijk}|^2 = \frac{2}{3}|C|^2|\xi|^2 - |C\xi|^2, \quad |a^3_{ijk}|^2 = \frac{3}{5}|C\xi|^2.$$

From this we see that, using the same notation as above, $L(A) = \alpha|X|^2 + \beta|Y|^2 + \gamma|Z|^2$ vanishes on rank one matrices if and only if

$$\alpha : \beta : \gamma = (-2) : 1 : 3.$$

For our purpose, we will take $\alpha = -2, \beta = 1, \gamma = 3$ in the following.

3.2 The construction of f

We recall that $K = \{\nabla u(x), x \in \Omega\} = \{\nabla u(x), x \in S^2\} \subset M^{5\times 3}$, where u is defined by (1.4), and where we have identified the 3×3 trace-free symmetric matrices with $\mathbf{R}^5$. A necessary condition for the existence of a strongly convex function f satisfying (1.5) is that there exist $\delta_0 > 0$, such that

$$\nabla L(X)\cdot(Y-X) \le -\delta_0|Y-X|^2 \qquad \forall X, Y \in K. \tag{3.1}$$

We will see this condition is satisfied.

Lemma 3.1 *For any $X = \nabla u(x), Y = \nabla u(y) \in K$, where $x, y \in S^2$, we have*

$$L(\nabla u(x) - \nabla u(y)) \ge 8|x-y|^2.$$

Proof: First we note that we have the following decomposition for $\nabla u(x) \in K, x \in S^2$.

$$u_{ijk} = u^1_{ijk} + u^2_{ijk} + u^3_{ijk},$$

where

$$\begin{aligned} u^1_{ijk} &= -x_i x_j x_k + \frac{1}{5}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}), \\ u^2_{ijk} &= 0, \\ u^3_{ijk} &= \frac{4}{5}(x_i\delta_{jk} + x_j\delta_{ik} - \frac{2}{3}x_k\delta_{ij}). \end{aligned}$$

and

$$|u^1_{ijk}|^2 = \frac{2}{5}, \quad |u^3_{ijk}|^2 = \frac{64}{15}.$$

Hence $\qquad L(\nabla u(x)) \equiv 12 \qquad \forall x \in S^2.$

Since L is quadratic, we have

$$L(\nabla u(x) - \nabla u(y)) = 2L(\nabla u(x)) - 2L(\nabla u(x), \nabla u(y)),$$

where we slightly abuse the notation by using L also for the symmetric bilinear form corresponding to the quadratic form L.

$$\begin{aligned} L(\nabla u(x), \nabla u(y)) &= -2u^1_{ijk}(x)\cdot u^1_{ijk}(y) + 3u^3_{ijk}(x)\cdot u^3_{ijk}(y) \\ &= -2\left(-x_i x_j x_k + \frac{1}{5}(x_i\delta_{jk} + x_j\delta_{ki} + x_k\delta_{ij})\right)\cdot \\ &\quad \left(-y_i y_j y_k + \frac{1}{5}(y_i\delta_{jk} + y_j\delta_{ik} + y_k\delta_{ij})\right) \\ &\quad +3\left(\frac{4}{5}\right)^2 (x_i\delta_{jk} + x_j\delta_{ki} - \frac{2}{3}x_k\delta_{ij})\cdot(y_i\delta_{jk} + y_j\delta_{ik} - \frac{2}{3}y_k\delta_{ij}) \\ &= -2\langle x, y\rangle^3 + 14\langle x, y\rangle. \end{aligned}$$

Let $t = \langle x, y\rangle$. Then $-1 \le t \le 1$, and we have

$$\begin{aligned} L(\nabla u(x) - \nabla u(y)) &= 2L(\nabla u(x)) - 2L(\nabla u(x), \nabla u(y)) \\ &= 2(1-t)\left(-2(1+t+t^2)+14\right) \\ &\ge 16(1-t) \\ &= 8|x-y|^2. \end{aligned}$$

The proof of Lemma 3.1 is finished.

We have $L(X) = 12$ for all $X \in K$ and therefore Lemma 3.1 gives

$$\begin{aligned} \nabla L(\nabla u(x)) \cdot (\nabla u(y) - \nabla u(x)) &= -L(\nabla u(x) - \nabla u(y)) \\ &\quad + L(\nabla u(x)) + L(\nabla u(y)) - 2L(\nabla u(x)) \\ &= -L(\nabla u(x) - \nabla u(y)) \\ &\le -8|x-y|^2. \end{aligned}$$

Since we have

$$\frac{21}{4}|x-y|^2 \le |X-Y|^2 \le \frac{20}{3}|x-y|^2$$

for $X = \nabla u(x), Y = \nabla u(y)$, we see that the condition (3.1) is satisfied.

It turns out that (3.1) together with the fact that L is constant on K is also sufficient for the existence of a strongly convex function satisfying (1.5). A natural attempt to make such an extension would be to take the convex hull of K and consider a modification of the corresponding Minkowski functional. However, since the convex hull of K may not be smooth at K, we need to slightly modify this construction.

We fix $\epsilon > 0$ (the exact value will be specified later) and for each $X \in K$, consider the 10 dimensional ball of radius $r_\epsilon = \epsilon|\nabla L(X)| = \epsilon\sqrt{160}$ passing through X centered at $X' = X - \nabla L(X)\epsilon$. We will denote the ball as B_{X',r_ϵ}.

Lemma 3.2 *When ϵ is sufficiently small we have*

$$\nabla L(X)(\tilde{Y} - X) \le -\frac{1}{2}|\tilde{Y} - X|^2, \tag{3.2}$$

for each $X \in K$ and each $\tilde{Y} \in B_{Y',r_\epsilon}$, where B_{Y',r_ϵ} is defined above, with Y being an arbitrary point of K.

Proof: The inequality

$$|\tilde{Y} - Y'|^2 \le \epsilon^2 |\nabla L(Y)|^2$$

gives

$$\nabla L(Y) \cdot (\tilde{Y} - Y) \le -\frac{1}{2\epsilon}|\tilde{Y} - Y|^2$$

Hence

$$\begin{aligned}\nabla L(X)\cdot(\tilde{Y}-X) &= (\nabla L(X)-\nabla L(Y))\cdot(\tilde{Y}-Y)+\nabla L(Y)\cdot(\tilde{Y}-Y)\\ &\quad +\nabla L(X)\cdot(Y-X)\\ &\le 10|Y-X||\tilde{Y}-Y|-\frac{1}{2\epsilon}|\tilde{Y}-Y|^2-\frac{6}{5}|Y-X|^2,\end{aligned}$$

and the statement follows easily.

Let $S=\cup_{X\in K}B_{X',r_\epsilon}$. When ϵ is small, the boundary of S is smooth by elementary results about tubular neighborhoods (see [Hi] or [We2]). Lemma 3.2 implies that (for sufficiently small ϵ) all the eigenvalues of the second fundamental form of ∂S are negative and bounded above uniformly on K by a negative constant γ (i.e the principle curvatures $k_i(X)\le\gamma<0,\forall i$ and $\forall X\in K$). Since ∂S is smooth, we conclude that ∂S is locally strongly convex at any point of $U\cap\partial S$, where U is a small neighborhood of K.

Now take G to be the convex hull of S in $V_7\oplus V_3$. Using Lemma 3.2 and the fact that ∂S is smooth and locally strongly convex in $U\cap\partial S$, we infer that $U\cap\partial G=U\cap\partial S$ when the neighborhood U of K is choosen to be sufficiently small. Let

$$F_1(X)=\min\{t\ge 0, X\in tG\},\quad F(X)=12F_1^2(X).$$

Then F is smooth and strongly convex in U (see [Ro]), and $\nabla L(X)=\nabla F(X)$ for each $X\in K$. Let ϕ be a smooth non-negative mollifier with support in B_1 and let

$$F_\delta=\phi_\delta * F,$$

where $\phi_\delta(x)=\delta^{-n}\phi(\frac{x}{\delta})$. Define

$$H_{\delta,\tau}(X)=F_\delta+\tau|X|^2.$$

Let $0\le\eta\le 1$ be a smooth cut-off function satisfying $\eta=1$ in U', and $\eta=0$ outside U, where U' is an open neighborhood of K satisfying $\bar{U}'\subset U$.

Now define

$$H=(1-\eta)H_{\delta,\tau}+\eta F.$$

A straightforward calculation shows that H is a strongly convex function on $V_7\oplus V_3$ when δ and τ is small enough. Now take $f(A)=H(X+Y)+|Z|^2$ to be our final function, where $A\in M^{5\times 3}$, $A=X+Y+Z$, with $X\in V_7, Y\in V_5, Z\in V_3$. We know that f coincide with $F(X+Y)+|Z|^2$ in the neighborhood of K, thus $\nabla L(X)=\nabla f(X)$ for all $X\in K$ holds and f is a smooth strongly convex function everywhere. This proves the following theorem:

Theorem 1 *Let* $\Omega = \{x \in \mathbf{R}^3, |x| < 1\}$ *and let* $u\colon \Omega \to \mathbf{R}^5$ *be defined by* $u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{3}\delta_{ij}$, $i, j = 1, \cdots 3$, *where we identify the* 3×3 *symmetric trace-free matrices with* $\mathbf{R}^5$. *Then* u *is a minimizer of* $I(u) = \int_\Omega f(Du(x))$, *where* f *is the smooth strongly convex function defined above.*

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