ACCURATE COMPUTATIONS ON INERTIAL MANIFOLDS

M.S. JOLLY, R. ROSA, AND R. TEMAM

ABSTRACT. An algorithm for the computation of inertial manifolds is implemented. The effects of certain algorithm parameters are illustrated on an ordinary differential equation where an exact manifold is known. The algorithm is then applied to the Kuramoto-Sivashinsky equation to recover the high wave number components of elements on the global attractor. New error estimates are derived to include the effect of truncating the partial differential equation, as in a Galerkin approximation. The algorithm is adapted to compute an inertial manifold with delay and its performance compared to a shooting algorithm.

1. INTRODUCTION

This paper demonstrates how to accurately compute inertial manifolds for evolutionary equations. These manifolds are finite-dimensional, exponentially attracting, positively invariant, and smooth. In terms of a Fourier series representation of the solution to a partial differential equation, essentially the manifold provides a relation for the modes with high wave numbers in terms of those with low wave numbers. The existence of such a manifold typically hinges on whether there is a large enough gap in the spectrum of the linear part of the equation compared to the variation of the nonlinear part. A weaker gap condition appears in most constructive proofs of local (un)stable and center manifolds, since for local manifolds one may take a small enough neighborhood to reduce the strength of the nonlinear term and thereby achieve the gap condition. The accurate computation of such objects however, even when the manifolds are two-dimensional, poses some special difficulties (see [29], [2], [37] and for related issues regarding invariant tori [16]).

We implement here an algorithm developed in [59] for the computation of inertial manifolds. It provides a sequence of approximate inertial manifolds (AIMs) which converges to the actual inertial manifold. This sequence is distinguished from all other sequences of AIMs, except that in [11] in that convergence may be achieved with the dimension of the manifold held fixed. The others typically need to increase the dimension, in order to decrease the error from the inertial manifold. The sequence implemented here is distinguished from the earlier sequence in [11] in that the computational complexity of the algorithm is independent of the dimension of the manifold. If for a particular low-mode vector, the corresponding high-mode vector is desired, then this algorithm can be applied, without computing the high-mode vectors corresponding to other low-mode vectors. Its primary purpose then, is not the construction of all, or even a major portion of such a manifold. Rather, it is best suited for computing solutions *on* such manifolds, particularly, *backward in time*. By restricting the flow to an inertial manifold, one is effectively selecting low dimensional stable submanifolds that are involved in global bifurcations. This algorithm may be applied under less stringent conditions to local (un)stable manifolds and adapted to compute center manifolds [40].

Date: January 27, 1999.

1991 *Mathematics Subject Classification.* Primary: 34G20; Secondary 34D45, 35Q99, 47H20, 58F39.

Key words and phrases. Computation of inertial manifolds, approximate inertial manifolds, Kuramoto-Sivashinsky equation.

This work was supported in part by National Science Foundation, Grants Number DMS-9706903 and DMS-9705229, CNPq, Brasília, Brazil, and FAPERJ, Rio de Janeiro, Brazil. Computations were carried out on facilities made possible by NSF Grant Number CDA-9601632. Some of the work was completed at the Inst. for Mathematics and its Applications, University of Minnesota. and the Center for Nonlinear Studies, Los Alamos National Laboratory.

The general framework for the method is that of an evolutionary equation of the form

$$(1.1) \quad \frac{du}{dt} + Au = f(u), \quad u \in E, \quad u(0) = u_0,$$

where A is a linear operator and E is a Banach space. It is easiest to describe the approach in the particular case where A has complete set of eigenvectors associated with a sequence of positive eigenvalues and E is a Hilbert space. An inertial manifold is often sought as the graph of a function $\Phi : PE \rightarrow QE$, where $P = P_n$ is the projector onto the span of the first n eigenfunctions of A and $Q = Q_n = I - P$. The restriction of the flow to an inertial manifold yields the inertial form

$$(1.2) \quad \frac{dp}{dt} + Ap = Pf(p + \Phi(p)), \quad p(0) = p_0 \in PE,$$

which is a finite set of ordinary differential equations (ODEs) with the same long-time dynamics as the original evolutionary equation which may very well be infinite-dimensional.

The existence of an inertial manifold is known for a number of physical systems including the Kuramoto-Sivashinsky equation, the Cahn-Hilliard equation, and certain reaction diffusion equations in several space dimensions just to name a few (see [7], [48], [64], and references therein). A great variety of numerical schemes based on approximate inertial manifolds have been developed, analyzed and tested in the literature (see [18], [62], [25] and references therein). One issue is how much, if any, such methods can save in computing time. This article is not primarily concerned with this question. Instead we wish to demonstrate that one can accurately compute an inertial manifold so that the dimension of phase space can be kept quite small. Such a reduction can help in the geometric understanding of certain global bifurcations.

We begin in the next section with an example which demonstrates the use of an invariant (inertial) manifold to compute stable submanifolds involved in global bifurcations. In Section 3, we provide the general assumptions for both the existence of an inertial manifold, as well as the valid application of the algorithm in the next section. The algorithm itself is recalled in Section 4, along with several rigorous estimates on its convergence from [59]. An application is made to an ODE test case in Section 5. First an analytic treatment is carried out on this ODE to determine various quantities in the error estimates. Computations are then presented, demonstrating the significance of certain algorithm parameters and terms in the error estimates. This section serves to validate the code in a case where the exact solution is known as well as to highlight numerical issues. We then apply the method to the Kuramoto-Sivashinsky equation (KSE) in section 6. The computations in this case are compared to those of another sequence of approximate inertial manifolds (AIMs). To carry out the computations in Section 6, we are forced to truncate to a finite number of high modes. We provide error estimates for this effect in Section 7. Finally in Section 8 we adapt the algorithm for inertial manifolds to one for inertial manifolds with delay (IMDs) [13], [58], compare its performance against a shooting algorithm, and derive a modified Adams-Bashforth method based on the IMD.

2. A MOTIVATING EXAMPLE

We use below an ODE in $\mathbb{R}^3$ to illustrate how an inertial manifold naturally selects the relevant stable submanifolds involved in global bifurcations. To start, consider the simple system

$$(2.1) \quad \begin{aligned} \frac{dx}{dt} &= x - x^3 \\ \frac{d\eta}{dt} &= \eta - \eta^3 \\ \frac{d\zeta}{dt} &= -c\zeta. \end{aligned}$$

It is evident that (2.1) is a gradient system with the nine steady states $\{-1, 0, 1\} \times \{-1, 0, 1\} \times \{0\}$, the points on the corners $\{-1, 1\} \times \{-1, 1\} \times \{0\}$ being stable, the origin having an unstable manifold

of index two, and the rest having an unstable manifold of index one. The plane $\zeta = 0$ is an inertial manifold, and the x and η axes contain connecting orbits from the origin to the saddles. After the change of variables $\tilde{y} = \eta + x^2$, $z = x^2 + \zeta$ we have a system

$$(2.2) \quad \begin{aligned} \frac{dx}{dt} &= x - x^3 \\ \frac{d\tilde{y}}{dt} &= \tilde{y} + x^2 - (\tilde{y} - x^2)^3 - 2x^4 \\ \frac{dz}{dt} &= -cz + 2x(x - x^3) + cx^2, \end{aligned}$$

whose flow is conjugate to that of (2.1), but with an inertial manifold given by $z = \Phi(x, \tilde{y}) = x^2$.

Finally, in order to provide some feedback from the “high” mode z to the “low” modes $\{x, y\}$, we consider a modified version of (2.2) in which we add the term $z - x^2$ in the second equation so as to retain the exponential attraction to the inertial manifold and leave the flow on the manifold unperturbed

$$(2.3) \quad \begin{aligned} \frac{dx}{dt} &= x - x^3 \\ \frac{dy}{dt} &= y + x^2 - (y - x^2)^3 - 2x^4 + (z - x^2) \\ \frac{dz}{dt} &= -cz + 2x(x - x^3) + cx^2. \end{aligned}$$

A portion of the stable manifold for one of the saddles together with nearly all of the unstable manifold of the origin are shown in Figure 1, for the case $c = 2$. The steady states are indicated by small spheres. The curve which is partially hidden by the stable manifold is the trajectory, backwards in time, with an initial condition that is a perturbation (of size 0.1) of the saddle along its slow stable eigenspace. This represents one of a continuum of slow manifolds. Exactly one out of that family comprises the connecting orbit from the origin to the saddle, indicated here by the intersection of the two surfaces. The projection of this orbit onto the (x, y) -plane also coincides with the one-dimensional stable manifold of the corresponding saddle for the inertial form. It is clear that in order to pick out the distinguished slow manifold that is the connecting orbit, one needs more information than the linearization at the saddle provides. One possible way to find the connecting orbit is to pose an appropriate boundary value problem (see [17], and references therein). In many dynamical systems a stable manifold and an unstable manifold collide as a parameter is varied, to form connecting orbit(s). By visualizing the manifolds not only when they intersect, but also when they do not, we can begin to understand the geometric mechanism behind this global bifurcation. It is in the latter case, when the stable and unstable manifolds do not intersect, that the construction of the entire stable manifold (or at least a patch of it) is crucial.

The matter is further complicated in the case of a PDE. After spatial discretization the dimension of the full stable manifold at the saddle would typically be greater than two, and in fact increase with enhanced spatial accuracy. This is a situation where restricting the system to an inertial manifold can be quite useful. Doing so automatically picks out the relevant stable submanifold, which has low dimension and hence low complexity, so that it can be computed in a straightforward fashion, regardless of whether it consists of one or more connecting orbits. Precisely this situation arises with the Kuramoto-Sivashinsky equation. The stability of connecting orbits for the KSE was demonstrated in [9] by means of a reduction to an approximate inertial form (given by Ψ_2 as defined in (4.8)). That work extends the parameter range for stability previously obtained in [1] by perturbative techniques. The stable and unstable manifolds which collide to form this connecting orbit were visualized in [50] and found to be intimately involved in the formation of a completely different set of Silnikov orbits. The computations in [50] were carried out for the same approximate inertial form as used in [9]. It will be demonstrated in Section 6 (see Figure 10) that the algorithm implemented here allows one to compute the inertial form for the KSE much more accurately than done in [9] and [50], and earlier in [38], [62].

It is with this motivation that we compute the stable submanifold for (2.3) by use of an accurate inertial form. In Figure 2 we plot the relative error in the (y, z) -coordinates from (x^2, x^2) , which are those on the exact submanifold containing the connecting orbit, for the solution to the inertial form computed backwards in time with the initial condition being the perturbation (of size 0.1) of the saddle in its one-dimensional stable eigenspace. The index j refers to the number of iterations in the algorithm to compute the inertial manifold given in (4.6), (4.7). Recall that a similar perturbation in the one-dimensional slow stable eigenspace for the full equation produced a trajectory in Figure 1 which quickly diverged from the connecting orbit. Note that the maximum of the error in Figure 2 occurs (for $j = 5, 6$) at the initial condition. Thus to improve the computation of the connecting orbit in this approach we would reduce the size of the perturbation from the saddle, before considering further iterations of the algorithm. On the other hand, reducing the perturbation does not alleviate the divergence for the approach taken to produce the trajectory shown in Figure 1. Thus by restricting the flow to an accurate approximation of the inertial manifold we are forcing the lone stable eigenspace to be the relevant one.

3. GENERAL ASSUMPTIONS

The following assumptions guarantee both the existence of an inertial manifold and the convergence of the algorithm used here. While the framework below is somewhat technical, we will need to refer to each element described below when we validate the rate of convergence in our implementation. The generality is motivated by the application to PDEs, as the assumptions simplify considerably in the case of an ODE. We demonstrate the framework for both situations with several examples.

A1: The nonlinear term f in (1.1) is globally Lipschitz continuous from a Banach space E into another Banach space F ,

$$(3.1) \quad |f(u) - f(v)|_F \leq M_1 |u - v|_E, \quad \forall u, v \in E.$$

with

$$E \subset F \subset \mathcal{E};$$

the injections being continuous, each space dense in the following one, and $\mathcal{E}$ is also a Banach space. It follows that

$$|f(u)|_F \leq M_0 + M_1 |u|_E, \quad \forall u \in E,$$

for some $M_0 \geq 0$. (Actually $M_0 = |f(0)|_F$ is optimal.)

A2: The linear operator $-A$ generates a strongly continuous semigroup $\{e^{-tA}\}_{t \geq 0}$ of bounded operators on $\mathcal{E}$ such that

$$e^{-tA}F \subset E, \quad \forall t > 0.$$

A3: There exist two sequences of numbers $\{\lambda_n\}_{n=n_0}^{n_1}$, $\{\Lambda_n\}_{n=n_0}^{n_1}$, for some $n_0 \in \mathbb{N}$, $n_1 \in \mathbb{N} \cup \infty$, with $0 < \lambda_n \leq \Lambda_n$ for $n_0 \leq n \leq n_1$, and a sequence $\{P_n\}_{n=n_0}^{n_1}$ of finite dimensional projectors, such that if $Q_n = I - P_n$, then the following exponential dichotomies hold:

: $P_n \mathcal{E}$ is invariant under e^{-tA} , for $t \geq 0$, and $\{e^{-tA}|_{P_n \mathcal{E}}\}_{t \geq 0}$ can be extended to a strongly continuous group $\{e^{-tA}P_n\}_{t \in \mathbb{R}}$ of bounded operators on $P_n \mathcal{E}$ with

$$(3.2) \quad \begin{aligned} \|e^{-tA}P_n\|_{\mathcal{L}(E)} &\leq K_1 e^{-\lambda_n t}, & \forall t \leq 0, \\ \|e^{-tA}P_n\|_{\mathcal{L}(F, E)} &\leq K_1 \lambda_n^\alpha e^{-\lambda_n t}, & \forall t \leq 0, \end{aligned}$$

: $Q_n \mathcal{E}$ is positively invariant under e^{-tA} , for $t \geq 0$, with

$$(3.3) \quad \begin{aligned} \|e^{-tA}Q_n\|_{\mathcal{L}(E)} &\leq K_2 e^{-\Lambda_n t}, & \forall t \geq 0, \\ \|e^{-tA}Q_n\|_{\mathcal{L}(F, E)} &\leq K_2 (t^{-\alpha} + \Lambda_n^\alpha) e^{-\Lambda_n t}, & \forall t > 0, \end{aligned}$$

where $K_1, K_2 \geq 1$, and $0 \leq \alpha < 1$. We will at times drop the subscripts on the projectors P and Q for simplicity.

A4: Equation (1.1) has a continuous semiflow $\{S(t)\}_{t \geq 0}$ in E , given by $S(t)u_0 = u(t)$, where $u = u(t)$ is the mild solution of (1.1) defined through the variation of constants formula:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(u(s))ds, \quad \forall t \geq 0.$$

A5: There exists $K_3 \geq 0$ independent of n such that

$$\|AP_n\|_{\mathcal{L}(E)} \leq K_3\lambda_n.$$

A6: For simplicity we assume that A is invertible.

A7: The *spectral gap condition* (SGC)

$$(3.4) \quad \Lambda_n - \lambda_n > 3M_1K_1K_2[\lambda_n^\alpha + (1 + \gamma_\alpha)\Lambda_n^\alpha], \quad \text{holds for some } n \in \mathbb{N},$$

where

$$\gamma_\alpha = \begin{cases} \int_0^{+\infty} e^{-r}r^{-\alpha}dr, & \text{if } 0 < \alpha < 1, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

In many PDE applications the spectrum of A consists of positive semi-simple eigenvalues

$$(3.5) \quad \mu_1 \leq \mu_2 \leq \dots, \quad \text{with } \mu_n \rightarrow +\infty,$$

associated with eigenvectors $w_1, w_2, \dots$, so that we may set $\lambda_n = \mu_n$, $\Lambda_n = \mu_{n+1}$, and

$$P_n = \text{proj. onto } \text{span}\{w_1, w_2, \dots, w_n\}.$$

In most applications the phase space can be taken to be a Hilbert space. For some parabolic equations, however, depending on the relationship between the dimension of the space, the order of the linear term and the strength of the nonlinear term, the appropriate phase space is merely a Banach space (see [47], for instance). Strictly speaking, one need only have the exponential dichotomy in **A3** hold for the particular choice of n satisfying the SGC in **A7**, though typically one has many choices for the splitting in **A3**, and the SGC holds only for *some* large n . In some of the cases we will consider here, one or more of the first eigenvalues are negative, so we simply neglect those and start with Λ_{n_0} as the first positive eigenvalue and choose λ_{n_0} to be some positive number smaller than Λ_{n_0} . If the eigenvalues were semisimple but complex, one would take λ_n and Λ_n to be the real part of the eigenvalues. The assumption that the projector P_n is finite-dimensional excludes the cases in which A has a continuous spectrum, but that is only because we want a finite dimensional inertial manifold in the end. The whole construction would go through if we allow P_n to be infinite-dimensional, and hence A to have bands with continuous spectrum, and we would end up with an infinite-dimensional ‘inertial’ manifold.

In most applications the global Lipschitz condition in (3.1) does not hold for the evolutionary equation in its original form. This can be corrected by annihilating the nonlinearity outside a ball in phase space. To do so, and be assured that none of the long-time dynamics are altered, we require in this situation that the evolutionary system be dissipative. This means there is an absorbing ball $B_\rho \subset E$, i.e., there exists $\rho > 0$ such that

$$|S(t)u_0|_E \leq \rho, \quad \forall t \geq t_0(|u_0|_E) \geq 0.$$

The prepared evolutionary equation is to have the nonlinearity (and hence the dynamics) unchanged inside the ball, and has the nonlinearity set to zero outside an even larger ball. We give a specific preparation procedure in Section 4.

We briefly recall below the construction of the *exact* inertial manifold as done in [60]. The discretization of this approach is what leads to the algorithm tested here. There are a number of earlier alternative approaches to prove the existence of inertial manifolds which are mostly based on either the Lyapunov-Perron method (c.f. [22]) or the graph transform method of Hadamard (c.f. [7]), and which construct the *entire* manifold at once, as the graph of a function $\Phi : P_n E \rightarrow Q_n E$. This is in contrast to the approach here, a variation of the Lyapunov-Perron method in which for a given $p_0 \in P_n E$, the corresponding image $\Phi(p_0)$ is found in terms of a single trajectory on the manifold. Thus it is a one-dimensional object

(the trajectory) which is discretized, rather than an n -dimensional object (the manifold). Earlier use of this approach can be found in [31], [32], [5], and [52].

The construction is described in two steps. First, we obtain an invariant manifold. Then the exponential attraction property is established, making the manifold in fact an inertial manifold. When (3.4) holds, we can find σ such that

$$(3.6) \quad \lambda_n + 2M_1K_1\lambda_n^\alpha < \sigma < \Lambda_n - 2M_1K_2(1 + \gamma_\alpha)\Lambda_n^\alpha.$$

For σ in this range, a single trajectory on an invariant manifold can be found as the fixed point $\varphi = \varphi(p)$ of a mapping $\mathcal{T}(\cdot, p)$ given by

$$(3.7) \quad \begin{aligned} \mathcal{T}(\varphi, p)(t) = & e^{-tA}p - \int_t^0 e^{-(t-s)A}Pf(\varphi(s))ds \\ & + \int_{-\infty}^t e^{-(t-s)A}Qf(\varphi(s))ds, \quad \forall t \leq 0, \quad \forall p \in PE, \end{aligned}$$

in the Banach space

$$\mathcal{F}_\sigma = \{\varphi \in C((-\infty, 0], E) ; \|\varphi\|_\sigma = \sup_{t \leq 0} (e^{\sigma t} |\varphi(t)|_E) < +\infty\}.$$

The entire manifold $\mathcal{M}$ is the collection of such trajectories given by $\mathcal{M} = \text{graph}\Phi$, where $\Phi : PE \rightarrow QE$ is defined by the fixed point φ of (3.7),

$$\Phi(p) = Q\varphi(p)(0), \quad \forall p \in PE.$$

The mapping $\mathcal{T}$ is a strict contraction in $\mathcal{F}_\sigma$, uniformly in PE , with

$$(3.8) \quad \|\mathcal{T}(\varphi_1, p) - \mathcal{T}(\varphi_2, p)\|_\sigma \leq \kappa_{n,\sigma} \|\varphi_1 - \varphi_2\|_\sigma, \quad \forall \varphi_1, \varphi_2 \in \mathcal{F}_\sigma, \quad \forall p \in PE,$$

where

$$(3.9) \quad \kappa_{n,\sigma} \equiv \frac{M_1K_1\lambda_n^\alpha}{\sigma - \lambda_n} + \frac{M_1K_2(1 + \gamma_\alpha)\Lambda_n^\alpha}{\Lambda_n - \sigma} < 1.$$

In order to show that the invariant manifold constructed as above is in fact an inertial manifold, all that remains is to establish the exponential attraction property. In fact the SGC in **A7** assures a stronger property sometimes referred to as *asymptotic completeness*, or *exponential tracking* [23]. This means that corresponding to any initial condition $u_0 \in E$ there exists a particular solution *on* the manifold, to which the trajectory through u_0 is attracted at an exponential rate. The so-called Relevance Theorem asserts the same property for the center manifold of a fixed point provided the fixed point is stable [3].

The proof of the asymptotic completeness in [60] requires that σ be taken in a narrower range,

$$(3.10) \quad \lambda_n + 3M_1K_1K_2\lambda_n^\alpha < \sigma < \Lambda_n - 3M_1K_1K_2(1 + \gamma_\alpha)\Lambda_n^\alpha.$$

In this case one has in fact that $\kappa_{n,\sigma} < 2/3$ so that the rate of contraction is guaranteed to be faster. Thus if we replace the gap condition in **A7** with the weaker condition

$$(3.11) \quad \Lambda_n - \lambda_n > 2M_1K_1\lambda_n^\alpha + 2M_1K_2(1 + \gamma_\alpha)\Lambda_n^\alpha,$$

we can satisfy (3.6), but not necessarily (3.10) and consequently have a invariant manifold $\mathcal{M} = \text{graph}\Phi$, which is not necessarily an inertial manifold. If (3.4) holds, then $\mathcal{M} = \text{graph}\Phi$ is indeed an inertial manifold.

We should mention that the same map in (3.7) is used in [52] but in a narrower framework where the contraction is obtained in a different space, leading to a spectral gap condition that is in fact weaker than (3.4) and which gives both the invariance and exponential attraction. When applicable, the approach in [52] can provide a manifold of smaller dimension than that provided by **A1-7**. Our error analysis for the discretization of the map in (3.7) will, however, use the assumptions in **A1-7**.

A related object of study is the global attractor, $\mathcal{A}$. Its existence is guaranteed provided the system is dissipative and the semiflow $S(t)$ is compact [30], [64]. In this case it can be defined by

$$\mathcal{A} = \bigcap_{t \geq 0} S(t)B_\rho.$$

This is the largest bounded invariant set for the system, and contains all steady states, periodic solutions, invariant tori, as well as the unstable manifolds associated with those objects. The global attractor must be contained in any inertial manifold, and thus provides a lower bound on the minimal dimension for all inertial manifolds. Unlike inertial manifolds, the global attractor can be quite complicated geometrically, and attract solutions at an algebraic rate.

4. THE ALGORITHM

To discretize the mapping $\mathcal{T}$, the function space $\mathcal{F}_\sigma$ is replaced by

$$\tilde{\mathcal{F}} = \{\psi : (-\infty, 0] \rightarrow E; \psi \text{ is piecewise constant with a finite number of discontinuities}\}.$$

Given a time step $\tau > 0$, number of steps $N \in \mathbb{N}$, and base point $p_0 \in PE$, we define $\mathcal{T}_\tau^N : \tilde{\mathcal{F}} \times PE \rightarrow \tilde{\mathcal{F}}$ by

$$(4.1) \quad \begin{aligned} \mathcal{T}_\tau^N(\psi, p_0)(t) = & e^{k\tau A} P p_0 - \int_{-k\tau}^0 e^{-(k\tau-s)A} P f(\psi(s)) ds \\ & + \int_{-\infty}^{-k\tau} e^{-(k\tau-s)A} Q f(\psi(s)) ds, \quad k = 0, 1, \dots, N \end{aligned}$$

for $t \in (-(k+1)\tau, -k\tau]$, if $k < N$, or $t \in (-\infty, -N\tau]$, if $k = N$.

Note that $\mathcal{T}_\tau^N$ acts as a sort of “projection” of $\mathcal{T}$ on $\tilde{\mathcal{F}}$ in that

$$\mathcal{T}_\tau^N(\psi, p_0)(-k\tau) = \mathcal{T}(\psi, p_0)(-k\tau).$$

We now consider the particular sequence of approximate inertial manifolds in [59] by assigning as the initial guess

$$(4.2) \quad \varphi^0(p_0)(t) = p_0, \quad \forall t \leq 0 \quad \forall p_0 \in PE$$

and choosing two sequences $\{\tau_j\}_{j \in \mathbb{N}}, \tau_j > 0$, and $\{N_j\}_{j \in \mathbb{N}}, N_j \in \mathbb{N}, N_j \geq 0$. Proceeding by Picard iteration we have

$$(4.3) \quad \varphi^j(p_0) = \mathcal{T}_{\tau_j}^{N_j}(\varphi^{j-1}(p_0), p_0), \quad \forall p_0 \in PE,$$

for $j = 1, 2, \dots$. Observe that τ_0 and N_0 are not relevant since $\varphi^0(p_0)$ is constant. The convergent sequence of approximate inertial manifolds is then given by

$$\mathcal{M}_j = \mathcal{M}_{j,n} = \text{graph } \Phi_j = \{p + \Phi_j(p); p \in P_n E\},$$

where

$$\Phi_j(p_0) = Q_n \varphi^j(p_0)(0), \quad \forall p_0 \in P_n E.$$

Again since the approximating trajectories φ^j are in $\tilde{\mathcal{F}}$, the integrals involved in the iteration process (4.3) can be explicitly calculated. For simplicity we denote

$$\varphi_k^j(p_0) = \varphi^j(p_0)(-k\tau_j), \quad k = 0, 1, \dots, N_j,$$

for $p_0 \in PE$ and $j = 0, 1, \dots$, so that $\Phi_j = Q\varphi_0^j$. For the case where $k\tau_j < N_{j-1}\tau_{j-1}$, we have

$$\begin{aligned}
(4.4) \quad \varphi_k^j(p_0) &= e^{k\tau_j A} P p_0 - \int_{-k\tau_j}^{-\ell_k \tau_{j-1}} e^{-(k\tau_j - s)A} P f(\varphi_{\ell_k}^{j-1}(p_0)) \, ds \\
&\quad - \sum_{\ell=0}^{\ell_k-1} \int_{-(\ell+1)\tau_{j-1}}^{-\ell\tau_{j-1}} e^{-(k\tau_j - s)A} P f(\varphi_{\ell}^{j-1}(p_0)) \, ds \\
&\quad + \int_{-\infty}^{-N_{j-1}\tau_{j-1}} e^{-(k\tau_j - s)A} Q f(\varphi_{N_{j-1}}^{j-1}(p_0)) \, ds \\
&\quad + \sum_{\ell=\ell_k+1}^{N_{j-1}-1} \int_{-(\ell+1)\tau_{j-1}}^{-\ell\tau_{j-1}} e^{-(k\tau_j - s)A} Q f(\varphi_{\ell}^{j-1}(p_0)) \, ds \\
&\quad + \int_{-(\ell_k+1)\tau_{j-1}}^{-k\tau_j} e^{-(k\tau_j - s)A} Q f(\varphi_{\ell_k}^{j-1}(p_0)) \, ds,
\end{aligned}$$

where ℓ_k is a nonnegative integer defined by

$$(4.5) \quad \begin{cases} -(\ell_k + 1)\tau_{j-1} < -k\tau_j \leq -\ell_k \tau_{j-1}, & \text{if } -(N_{j-1} + 1)\tau_{j-1} < -k\tau_j, \\ \ell_k = N_{j-1}, & \text{otherwise.} \end{cases}$$

The alternative case, where $k\tau_j \geq N_{j-1}\tau_{j-1}$ is actually simpler, as φ^{j-1} is constant over $(-\infty, k\tau_j]$. By a straightforward calculation of the integrals we have

Case 1. $k\tau_j < N_{j-1}\tau_{j-1}$:

$$\begin{aligned}
(4.6) \quad \varphi_k^j(p_0) &= e^{k\tau_j A} P_n p_0 - A^{-1}(e^{(k\tau_j - \ell_k \tau_{j-1})A} P_n - P_n) f(\varphi_{\ell_k}^{j-1}(p_0)) \\
&\quad - \sum_{\ell=0}^{\ell_k-1} A^{-1}(e^{(k\tau_j - \ell \tau_{j-1})A} P_n - e^{(k\tau_j - (\ell+1)\tau_{j-1})A} P_n) f(\varphi_{\ell}^{j-1}(p_0)) \\
&\quad + A^{-1} e^{-(N_{j-1}\tau_{j-1} - k\tau_j)A} Q_n f(\varphi_{N_{j-1}}^{j-1}(p_0)) \\
&\quad + \sum_{\ell=\ell_k+1}^{N_{j-1}-1} A^{-1}(e^{-(\ell \tau_{j-1} - k\tau_j)A} - e^{-((\ell+1)\tau_{j-1} - k\tau_j)A}) Q_n f(\varphi_{\ell}^{j-1}(p_0)) \\
&\quad + A^{-1}(I - e^{-((\ell_k+1)\tau_{j-1} - k\tau_j)A}) Q_n f(\varphi_{\ell_k}^{j-1}(p_0)).
\end{aligned}$$

Case 2. $k\tau_j \geq N_{j-1}\tau_{j-1}$:

$$\begin{aligned}
(4.7) \quad \varphi_k^j(p_0) &= e^{k\tau_j A} P_n p_0 - A^{-1}(e^{(k\tau_j - N_{j-1}\tau_{j-1})A} P_n - P_n) f(\varphi_{N_{j-1}}^{j-1}(p_0)) \\
&\quad - \sum_{\ell=0}^{N_{j-1}-1} A^{-1}(e^{(k\tau_j - \ell \tau_{j-1})A} P_n - e^{(k\tau_j - (\ell+1)\tau_{j-1})A} P_n) f(\varphi_{\ell}^{j-1}(p_0)) \\
&\quad + A^{-1} Q_n f(\varphi_{N_{j-1}}^{j-1}(p_0)).
\end{aligned}$$

4.1. Comparison with other AIMs. Regardless of the choice of the sequences $\{\tau_j\}_{j \in \mathbb{N}}$, and $\{N_j\}_{j \in \mathbb{N}}$, the zero-th approximate inertial manifold is given by $\Phi_0 \equiv 0$, since $Q_n \varphi^0 \equiv 0$ for φ^0 given in (4.2). The first nontrivial approximation is given by $\Phi_1(p) = A^{-1} Q f(p)$. An alternative sequence of AIMs is defined implicitly (for large enough n , see [24],[21]) by the Q -component of the steady state equation

$$A p = Q f(p + q).$$

This alternative sequence is found recursively by

$$(4.8) \quad \Psi_{j,n}(p) = \Psi_j(p) = A^{-1} Q_n f(p + \Psi_{j-1}(p)), \quad \Psi_0(p) \equiv 0, \quad p \in P_n \mathbb{H}$$

Note that $\Phi_1 = \Psi_1$, which happens also to coincide with a manifold first introduced in [19] with a completely different derivation. For a number of PDE cases, in particular for the KSE and 2-D Navier-Stokes equations, it has been shown that the $\mathcal{N}_{j,n} = \text{graph } \Psi_{j,n}$ converge to the global attractor $\mathcal{A}$ in the sense that

$$(4.9) \quad \text{dist}_E(\mathcal{A}, \mathcal{N}_{j,n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

for *any* fixed j , even for $j = 0$ (see [39], [66]). The convergence in (4.9) is valid even in the absence of the SGC, i.e. it is independent of the existence of an actual inertial manifold. Much of the motivation behind nonlinear Galerkin methods based on such AIMs comes from an algebraic improvement in the estimates for the rate of convergence to the attractor for $j > 0$ compared to that for $j = 0$. In some cases, however, solutions to a PDE can be shown to be in the Gevrey class with respect to the spatial variable so that this improvement is on top of a decay rate in the wave number that is already exponential [28],[45]. A number of other AIMs have been introduced in an attempt to improve over the approximation provided by $\mathcal{N}_{j,n}$ (see for instance [63], [15], [10]), but in each case one must increase the dimension of the manifold in order to converge to the attractor.

The distinction we wish to emphasize between the sequences $\{\mathcal{M}_{j,n}\}$ and $\{\mathcal{N}_{j,n}\}$ (and many like the latter) is that in the case of $\{\mathcal{M}_{j,n}\}$ we may instead fix n , hence the dimension of the manifolds, and achieve convergence as $j \rightarrow \infty$. We restate below, in slightly more generality, the main convergence result proved in [59].

Theorem 4.1. *Assume that A1 through A6 hold and that the sequences $\{\tau_j\}_j$ and $\{N_j\}_j$ are chosen so that*

$$0 < \tau_j \leq c_1 \gamma^j \kappa_{n,\sigma}^j, \quad \forall j \in \mathbb{N},$$

with $\kappa_{n,\sigma}$ as defined in (3.9) for n satisfying (3.4) (resp. (3.11)), and

$$N_j \tau_j \geq c_2 j, \quad \forall j \in \mathbb{N},$$

for some constants $c_1, c_2 > 0$, some $0 < \gamma < 1$, and some σ satisfying (3.10) (resp. (3.6)). Then

$$d(\Phi, \Phi_j) \leq c_3 e^{-c_4 j}, \quad \forall j \in \mathbb{N},$$

for suitable constants $c_3, c_4 > 0$, and $\mathcal{M} = \text{graph } \Phi$ is an inertial (resp. invariant) manifold.

Thus if the SGC holds, then the convergence is to an inertial manifold. If one replaces the gap condition in A7 with the weaker condition (3.11) then the proof as in [59] for the convergence of the sequence $\{\mathcal{M}_{j,n}\}$ as $j \rightarrow \infty$ still goes through, but not the asymptotic completeness of, or even the exponential attraction to the limiting manifold. The limiting manifold in that case is merely guaranteed to be invariant. Finally, in the absence of (3.11), the convergence to the global attractor is still guaranteed, provided $n \rightarrow \infty$ as well [59].

In the original method of proof for the existence of inertial manifolds [22], the manifold is found as the fixed point of the mapping described by

$$(4.10) \quad \Theta_{j+1}(p_0) = \int_{-\infty}^0 e^{sA} Q f(\tilde{p} + \Theta_j(\tilde{p})) ds$$

where $\tilde{p} = \tilde{p}(s; p_0, \Theta_j)$ solves

$$(4.11) \quad \frac{d\tilde{p}}{dt} + A\tilde{p} = P f(\tilde{p} + \Theta_j(\tilde{p})), \quad \tilde{p}(0) = p_0.$$

In [11], a discretization of this mapping was presented, and shown to provide a convergent algorithm for computing inertial manifolds. Such a discretization was actually implemented in [57] to compute the manifold for a reaction diffusion equation over a portion of $P_n \mathcal{E}$ in a case where $n = 2$. Note that in order to obtain Θ_{j+1} requires global knowledge of Θ_j as one cannot predict where the trajectory in $P\mathcal{E}$ of the solution to (4.11) will go, i.e. where one will need to evaluate Θ_j . Practically speaking, one must interpolate the discrete representation of Θ_j , to solve (4.11) backward in time. Thus the complexity

of the computation grows dramatically with the dimension of the manifold n . Yet, if one is interested in computing the complete manifold (or at least a patch of it), the approach in [11] is probably more appropriate than the one used here.

If however, one is primarily interested in computing particular solutions on the manifold, there is no need to compute the manifold in its entirety. In this case the objective is to solve the initial value problem for the inertial form (1.2) so that at each time step, one needs only to compute the functional relation $q = \Phi(p)$ at a particular $p \in P\mathcal{E}$ (or several such p locations, in a multi-step scheme). To emphasize the difference with the approach in (4.10), note that to obtain φ_{j+1} via (3.7), one need only know φ_j , a one-dimensional object. Thus the complexity is essentially independent of the dimension of the manifold. Another application where one would prefer to apply (4.6), (4.7) is in post-processing via an AIM [26]. Roughly speaking, in that approach one evolves the solution via an adequately accurate Galerkin (or nonlinear Galerkin) approximation to obtain final value $p(t_1)$, and then recovers values for the high modes *only at the final time* from $q(t_1) = \Phi_a(p(t_1))$, where Φ_a is some appropriately chosen AIM. In this instance the enslavement relation is needed only at a single base point.

While Theorem 4.1 gives the main motivation for implementing this algorithm, the validation of a code for this purpose requires more details from the analysis. Specifically we need information about the convergence in $\tilde{\mathcal{F}}$. This is given in two lemmas below, which are proved in [59]. For both lemmas it is again assumed that **A1** through **A7** hold. The first ingredient is an a priori estimate on the variation in time of the exact solution on the manifold.

Lemma 4.2. *For $t < 0$ and τ such that $0 \leq \tau \leq -t$, we have that*

$$|\varphi(p_0)(t + \tau) - \varphi(p_0)(t)|_E \leq \tau \beta_{n,\sigma} e^{-\sigma t} (1 + |p_0|_E),$$

for all $p_0 \in P_n E$, where

$$(4.12) \quad \beta_{n,\sigma} = 2M_0 K_1 \lambda_n^\alpha + \frac{2K_1(K_3 \lambda_n + M_1 \lambda_n^\alpha)}{1 - \kappa_{n,\sigma}} \left\{ \frac{M_0 K_1}{\lambda_n^{1-\alpha}} + \frac{(1 + \gamma_\alpha) M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right\}.$$

It is not so much that we need to invoke the result of Lemma 4.2 than we need the quantity $\beta_{n,\sigma}$ which plays a central role in the error estimate for the algorithm in (4.6, 4.7). The reason we restate Lemma 4.2 here is to recall the purpose of $\beta_{n,\sigma}$. While the expression in (4.12) is adequate under the general assumptions **A1**–**A7**, it is conceivable that for a particular PDE, one might know more about the variation of solutions, in which case a smaller expression for $\beta_{n,\sigma}$ might be found. The next lemma measures the actual error after applying the mapping $\tilde{\mathcal{T}}$. It states precisely how $\beta_{n,\sigma}$ enters into the estimate. We do not have a contraction in the strict sense. Rather we have a contraction, plus a residual which decreases provided the time step τ decreases and the product $N\tau$ increases.

Lemma 4.3. *Let N be a nonnegative integer and $\tau > 0$, and assume σ and σ_0 satisfy (3.10) for a fixed $n \in \mathbb{N}$ with $\sigma_0 < \sigma$. Then for all $p_0 \in P_n E$ and all $\psi \in \tilde{\mathcal{F}}$, we have*

$$(4.13) \quad \begin{aligned} \|\varphi(p_0) - \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma &\leq \kappa_{n,\sigma} \|\varphi(p_0) - \psi\|_\sigma \\ &+ \left(\tau \beta_{n,\sigma} + \frac{\beta_{n,\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma - \sigma_0)N\tau} \right) (1 + |p_0|_E), \end{aligned}$$

where β is as in Lemma 4.2.

There are a number of numerical parameters that arise when applying the algorithm. For one of the computational examples to follow, we will examine how some of these parameters affect the error estimate through $\beta_{n,\sigma}$. Thus $\beta_{n,\sigma}$ serves as a guide to optimal parameter settings.

5. AN ODE TEST CASE

We consider another system of three ODEs, which in cylindrical coordinates is written as

$$(5.1) \quad \begin{aligned} \frac{dr}{dt} &= ar - ar^2 \\ \frac{d\theta}{dt} &= b \\ \frac{dz}{dt} &= -cz + r^2 \left[\frac{2a^2}{c}(1-r) + a \right], \end{aligned}$$

where $a > 0$, b and c are parameters to be chosen below. It is easy to verify that $\frac{d}{dt}(z - ar^2/c) = -c(z - ar^2/c)$ so that for $c > 0$ the paraboloid $z = ar^2/c$ is a two-dimensional inertial manifold.

5.1. Analytic treatment. Writing (5.1) in Cartesian coordinates and in the form of (1.1), we set $u = (x, y, z)^{\text{tr}}$

$$(5.2) \quad A = \begin{bmatrix} -a & b & 0 \\ -b & -a & 0 \\ 0 & 0 & c \end{bmatrix} \quad \text{and} \quad f(u) = \begin{pmatrix} -arx \\ -ary \\ \left(\frac{2a^2}{c} + a\right)r^2 - \frac{2a^2}{c}r^3 \end{pmatrix},$$

where $r = \sqrt{x^2 + y^2}$. In this ODE case we take $E = F = \mathcal{E} = \mathbb{R}^3$, and $\alpha = 0$ so that $\gamma_\alpha = 0$. We also take $n = 2$ with $\Lambda_n = c$, and $\lambda_n = 1$. We will determine c , so that the gap condition (3.4) holds for a particular choice of a and b .

Let

$$(5.3) \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For simplicity we will use the maximum norm, $\|u\| = \max\{|x|, |y|, |z|\}$. Then $\|P\| = \|Q\| = 1$ and the exponential dichotomies (3.2) and (3.3) are realized with

$$(5.4) \quad \begin{aligned} \|e^{-tA}P\| &\leq \sqrt{2}e^{at}, & \forall t \leq 0 \text{ (max. achieved at } t = \pi/4 + k\pi, k \in \mathbb{Z}) \\ \|e^{-tA}Q\| &= e^{-ct}, & \forall t > 0 \text{ (in fact } \forall t \in \mathbb{R}). \end{aligned}$$

Thus we may take $K_1 = \sqrt{2}$ and $K_2 = 1$.

The Lipschitz constant is estimated in two stages. First we compute a local Lipschitz constant for f restricted to the cylinder of radius r . Treating f componentwise where $f(u) = (f_1(u), f_2(u), f_3(u))^{\text{tr}}$ and using the fact that

$$(5.5) \quad |r_1 - r_2| \leq \sqrt{2}\|Pu_1 - Pu_2\|$$

we find that

$$|f_1(u_1) - f_1(u_2)| = a|r_1x_1 - r_2x_2| \leq a|r_1||x_1 - x_2| + a|x_2||r_1 - r_2| \leq ar(1 + \sqrt{2})\|u_1 - u_2\|$$

with precisely the same final estimate for f_2 . Similarly for third component we have

$$\begin{aligned} |f_3(u_1) - f_3(u_2)| &\leq \left(\frac{2a^2}{c} + a\right)|r_1 + r_2||r_1 - r_2| + \frac{2a^2}{c}|r_1^2 + r_1r_2 + r_2^2||r_1 - r_2| \\ &= \left[\left(\frac{2a^2}{c} + a\right)|r_1 + r_2| + \frac{2a^2}{c}|r_1^2 + r_1r_2 + r_2^2|\right]|r_1 - r_2|. \end{aligned}$$

It follows that

$$(5.6) \quad \|f(u_1) - f(u_2)\| \leq \sqrt{2}d_1(r)\|u_1 - u_2\|, \quad \forall u_1, u_2 \text{ with } |r_1|, |r_2| \leq r,$$

where

$$(5.7) \quad d_1(r) = 2 \left(\frac{2a^2}{c} + a \right) r + 6 \frac{a^2}{c} r^2.$$

We also have

$$(5.8) \quad \|f(u)\| \leq d_0(r), \quad \forall u = (x, y, z)^{\text{tr}}, \text{ with } \sqrt{x^2 + y^2} \leq r$$

where

$$(5.9) \quad d_0(r) = \left(\frac{2a^2}{c} + a \right) r^2 + \frac{2a^2}{c} r^3.$$

The so-called prepared equation takes the form

$$(5.10) \quad \frac{du}{dt} + Au = f_\rho(u),$$

where

$$f_\rho(u) = \chi_\rho(r)f(u), \quad \forall u = (x, y, z)^{\text{tr}}$$

with

$$(5.11) \quad \chi_\rho(r) = \chi\left(\frac{r^2}{\rho^2}\right)$$

for some function $\chi \in C^1(\mathbb{R}^+)$ satisfying

$$(5.12) \quad \chi|_{[0,1]} = 1, \quad \chi|_{[2,\infty)} = 0, \quad 0 \leq \chi(s) \leq 1, \forall s \in [1, 2], \forall s \in \mathbb{R}^+.$$

Thus we will annihilate the nonlinear term outside a ball of radius $\sqrt{2}\rho$. In practice we take

$$(5.13) \quad \chi(s) = 2(s-1)^3 - 3(s-1)^2 + 1, \quad \text{for } s \in [1, 2]$$

which is easily seen to satisfy $|\chi'(s)| \leq 3/2$. Thus the flow within the ball of radius ρ for (5.10) coincides with that of (1.1). The preparation step is needed to meet the global Lipschitz condition in **A1**.

For the prepared nonlinearity, we have for some ξ between r_1^2/ρ^2 and r_2^2/ρ^2

$$\begin{aligned} \|f_\rho(u_1) - f_\rho(u_2)\| &= \|\chi_\rho(r_1)f(u_1) - \chi_\rho(r_2)f(u_2)\| \\ &= \|\left[\chi(r_1^2/\rho^2) - \chi(r_2^2/\rho^2)\right]f(u_1) + \chi_\rho(r_2)(f(u_1) - f(u_2))\| \\ &\leq |\chi'(\xi)| \left| \frac{r_1^2 - r_2^2}{\rho^2} \right| \|f(u_1)\| + \|f(u_1) - f(u_2)\| \\ &\leq \frac{3}{2\rho^2} |r_1 + r_2| |r_1 - r_2| d_0(\sqrt{2}\rho) + \sqrt{2}d_1(\sqrt{2}\rho) \|u_1 - u_2\| \\ &\leq \frac{3\sqrt{2}}{\rho} |r_1 - r_2| d_0(\sqrt{2}\rho) + \sqrt{2}d_1(\sqrt{2}\rho) \|u_1 - u_2\| \end{aligned}$$

where we assumed $r_i^2 \leq 2\rho^2$, $i = 1, 2$. If $r_i^2 \geq 2\rho^2$, $i = 1, 2$, then the left hand side is zero. If, say $r_1^2 \leq 2\rho_2^2 \leq r_2^2$, the argument above is valid with some ξ instead between r_1^2/ρ^2 and 2. The Lipschitz condition then reads as

$$\|f_\rho(u_1) - f_\rho(u_2)\| \leq M_\rho \|u_1 - u_2\|$$

where

$$(5.14) \quad M_\rho = \frac{6}{\rho} d_0(\sqrt{2}\rho) + \sqrt{2}d_1(\sqrt{2}\rho),$$

corresponds to M_1 for the prepared equation.

We now determine c to satisfy the SGC, which now reduces to

$$c - 1 > 6\sqrt{2}M_\rho.$$

For the particular case $a = 1$, $b = 0.1$, and $\rho = \sqrt{2}$ we obtain

$$d_0(\sqrt{2}\rho)|_{\rho=\sqrt{2}} = \frac{24}{c} + 4, \quad d_1(\sqrt{2}\rho)|_{\rho=\sqrt{2}} = \frac{32}{c} + 4,$$

so that

$$(5.15) \quad M_\rho|_{\rho=\sqrt{2}} = \sqrt{2} \left(\frac{104}{c} + 16 \right).$$

The SGC now reads as

$$c - 1 > \frac{1248}{c} + 192.$$

Solving for $c > 0$ we see that we can enforce the SGC by requiring that

$$c \geq 200 > \frac{1}{2} \left(193 + \sqrt{193^2 + 4 \cdot 1248} \right) \approx 199.3.$$

The error estimates will be improved, however, if we take c a bit larger than the minimum specified by the gap condition. For $c = 220$, we have by (5.15) that $M_\rho|_{\rho=\sqrt{2}} < 24$ and consequently that (3.10) holds for σ satisfying

$$(5.16) \quad 1 + 3 \cdot 24\sqrt{2} \approx 102.8 < \sigma < 118.1 \approx 220 - 3 \cdot 24\sqrt{2}.$$

One finds by applying elementary calculus to (3.9) that the contraction rate κ is a decreasing function of σ over the interval in (5.16), so we take $\sigma = 118$. We then have

$$\kappa = 24 \left(\frac{\sqrt{2}}{118 - 1} + \frac{1}{220 - 118} \right) < .53,$$

which is close to the infimum at the right endpoint. Since for this example $M_0 = 0$, we then have

$$(5.17) \quad \beta_{\sigma=118} \leq \frac{4(1+24)}{1-.47} < 213.$$

Thus we have as an estimate for the first residual term

$$(5.18) \quad R_1(\tau) \equiv \beta_\sigma \tau (1 + |y_0|_E) \leq 213\tau (1 + |y_0|_E).$$

To make small the second residual term

$$(5.19) \quad R_2(\sigma_0, N\tau) \equiv \frac{\beta_{\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma - \sigma_0)N\tau} (1 + |y_0|_E),$$

we choose $\sigma_0 = 103$. Thus working at $c = 220$ allows for a greater difference $\sigma - \sigma_0$ which helps decrease R_2 . Recalculating κ , and then β for this choice of σ_0 yields the estimate

$$(5.20) \quad R_2 \leq 14.5e^{-15N\tau} (1 + |y_0|_E)$$

5.2. Computational results. We carried out computations for (5.1) using two choices for the base point p_0 . The first choice is for p_0 specified by $r_0 = 1.2$ and $\theta_0 = 0$, so that the exact solution starting on the inertial manifold above the point p_0 , tends to ∞ backwards in time. In Figure 3 we plot three measurements of the error against the number of iterations for several combinations of values for c_1 and c_2 in

$$(5.21) \quad \tau_j = c_1 2^{-j}, \quad \text{and} \quad N_j = \frac{c_2}{c_1} j 2^j.$$

The three measurements are the theoretical error bound in (4.13), together with the error in an approximation of the σ -norm, and the error for the manifold itself. The approximation of the σ -norm is done by truncating in time to define for all $\psi \in \tilde{\mathcal{F}}$

$$\|\psi\|_{T,\sigma} = \sup_{T \leq t \leq 0} e^{\sigma t} |\psi(t)|_E.$$

	c_1	c_2/c_1	T	NFE
run1	.01	.5	.045	6.7×10^6
run2	.01	1	.09	2.7×10^7
run3	.1	1	.9	2.7×10^7

TABLE 1. Truncation time T , and total number of function evaluations (NFE)

We stop after nine iterations in each case and as a consequence take $T = N_9\tau_9$, which works out to the values shown in Table 1. This approximation is used in computing the theoretical bound as well. To compute the supremum, the explicit expression for the exact solution is used, as well as sampling on a grid of $\Delta t = 10^{-7}$, to detect possible suprema interior to the subintervals $((k+1)\tau_j, k\tau_j]$ for $k = 1, 2, \dots, N_j - 1$ and the interval $(T, N_j\tau_j]$. The time t_{sup} at which each supremum is attained by this computation is plotted in Figure 4 for each of the three cases.

Our interpretation is that the best result in Figure 3 (that for run2) corresponds to the case where t_{sup} is held small. This is especially true when the error is measured in the σ -norm. Note that this brings t_{sup} close to the time at which we ultimately wish to use the approximate solution, namely $t = 0$. The lack of monotonicity in the σ -norm for run1 is consistent with the fact that there is a residual in the error. We see from the same plot for run2 that increasing N_j by a factor of two serves to remove this kink, as one might expect from the form of the residual error R_2 in (5.19). The fact that the estimate for R_2 dominates that for R_1 , coupled with the fact that increasing N_j gave an improvement that is not reflected in the value of the estimate for R_2 , points to this term as one which might benefit from further analysis.

The consistency of the three types of plots in Figure 3, validates our computer code written to implement (4.6),(4.7). Figure 6 shows the approximate solution at each iterate, along with the exact solution for the run2. The approximate solutions are extended as a constant function to $T = .09$ in this case. They are not labeled individually, since the correspondence with the iteration step is indeed as one would guess. The initial condition is such that the exact solution of the unprepared equation tends to ∞ as $t \rightarrow -\infty$. The effect of preparing the nonlinearity outside the ball of radius $r = 1.414 \approx \sqrt{2}$, is felt only in run3, and not in run1, and run2, which is consistent with the longer time period of approximation for run3. This explains the poorer performance of run3, over relatively few iterations, despite the better error estimate. The method of preparation used in 5.1 inflates the Lipschitz constant over that of f restricted to the ball of radius $r = 1.414$ (see (5.14)). One can expect that a posteriori error estimates taking into account the smaller Lipschitz constant for run1 and run2, would be more consistent with the computational results. We conclude that even though the actual values of the residual estimates as shown in Figure 5 are not sharp enough to explain the differences in the σ -norm plots, the form is suggestive of how to affect these changes.

6. A PDE TEST CASE

The Kuramoto-Sivashinsky equation (KSE) we consider is often written as

$$(6.1) \quad \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

subject to periodic boundary conditions $u(x, t) = u(x+L, t)$, $L > 0$. A considerable amount of theoretical and computational work on the KSE can be found in the literature (see [1], [4], [34], and [51] for a small sampling). The dissipativity for the KSE was first established only in the invariant subspace of odd functions in [53]. In fact it was the KSE with periodic *and odd* boundary conditions that served as one of the first applications of the existence theory of inertial manifolds [20]. Dissipativity for the general periodic case has been established only somewhat recently, independently by Collet et al. [6], and Goodman [27] (see also the earlier work of Il'yashenko [35] and some simplifications in Pinto [55]).

This paved the way for the existence of a global attractor and an inertial manifold in the general case. Nevertheless in the computations which follow at the end of this section, *we restrict to the odd subspace*, in part, to cut the dimension of the inertial manifold in half.

The estimate of the dimension of the inertial manifold for the KSE was first estimated in [20], and later improved by [65] using the estimate for the absorbing ball from [6]. These estimates however, are of the form $\dim \leq cL^b$, where most of the effort is devoted to making the exponent b as small, yet the value of the constant c is not readily available. There is an arithmetic error in [65] which leads to a false estimate of $\dim \leq cL^{1.64}(\ln L)^{0.2}$ for the inertial manifold in the space L^2 , when in fact that analysis should yield $\dim \leq cL^{2.7}(\ln L)^{0.2}$.

We wish to apply the algorithm at the moderate value of $L = L^* = 4\sqrt{2}\pi$ and thus must actually *evaluate* the dimension. This calculation has been carried out in [41] to arrive at rigorous values for the dimension for the KSE prepared at radii ranging from that of a ball which can be shown to be absorbing, down to a much smaller one which nevertheless seems to contain the global attractor. We sketch briefly here the main steps involved.

Following the analysis in [6], we calculate a rigorous radius of an absorbing ball in L^2 to be $\rho_0 = 1433$ at $L = L^*$ (the same radius is valid for the general periodic case, even though the dimension of the inertial manifold will be larger). We also rework the analysis in [65], to obtain an estimate for ρ_1 , the radius of an absorbing ball in H^1 , in terms of ρ_0 . The equation is prepared in a manner quite different from that used in (5.1). Let $|\cdot|$, $|\cdot|_s$ denote the L^2 and H^s norms respectively. We show in [41] that

$$\left| u \frac{\partial u}{\partial x} - v \frac{\partial v}{\partial x} \right|_{-1} \leq \sqrt{2} \rho_0^{1/2} \rho_1^{1/2} |u - v| \quad \text{provided } |u|, |v| \leq \rho_0, \text{ and } |u|_1, |v|_1 \leq \rho_1,$$

and thus the nonlinear term $f(u) = u\partial u/\partial x$ when restricted to the ball in H^1 , is Lipschitz from the L^2 -topology to the H^{-1} -topology. We then invoke the following theorem of Valentine (see [36]).

Theorem 6.1. *Let E, F be Hilbert spaces and S a subset of E . For any Lipschitz function $g : S \rightarrow F$ there exists a Lipschitz function $\tilde{g} : E \rightarrow F$ such that $\tilde{g}|_S = g$ and $\text{Lip}(\tilde{g}) = \text{Lip}(g)$.*

Thus we can extend the restriction of f to the ball in H^1 to a globally Lipschitz function from L^2 to H^{-1} , keeping the Lipschitz constant fixed at $M = \sqrt{2} \rho_0^{1/2} \rho_1^{1/2}$. Numerical evaluation of both sides of the sharp gap condition in [52] leads to a minimal dimension of 292 (for $L = L^*$, odd case). If the estimate for the absorbing ball can be improved, so would be the estimate for the dimension of the manifold. This effect is displayed in Figure 7. One can also very well consider a possibly overly-prepared equation, where the nonlinear term is modified outside a ball of *arbitrary* radius ρ_0 , regardless of the rigorous estimate for the absorbing ball, much like in the case of a local invariant manifold. Any dynamics *entirely* within the ball of preparation are still preserved. On the other hand, we can expect trajectories of the KSE which are even just in part outside the ball to be altered in an indeterminable way.

The particular steps outlined above gave the best results of several explored in [41]. It is under the more general framework in **A1-7** however, that the error estimates in Section 4 were derived. To meet the latter criteria we express the KSE in the form of (1.1) by setting

$$(6.2) \quad Au = D^4u + D^2u + u, \quad f(u) = -uD u + u,$$

so that the eigenvalues of the linear part are

$$\mu_k = \left(\frac{2\pi}{L} k \right)^4 - \left(\frac{2\pi}{L} k \right)^2 + 1, \quad k = 1, 2, \dots$$

corresponding to a complete set of orthonormal eigenfunctions in L^2_{odd}

$$w_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L} kx\right).$$

Here we set $\lambda_n = \mu_n$ and $\Lambda_n = \mu_{n+1}$ for n to be determined by **A7**. (The treatment that produced Figure 7 used $Au = D^4u + D^2u$.)

We also select as spaces

$$(6.3) \quad E = H_{\text{odd}}^1, \quad F = \mathcal{E} = L_{\text{odd}}^2.$$

It is shown in [65] that the conditions **A1** through **A6** hold for the spaces $E, F, \mathcal{E}$ as in (6.3) if $K_1 = K_2 = (\frac{4}{3})^{1/4}$, and $\alpha = 1/4$. Furthermore it is shown in [41] that

$$|f(u) - f(v)| \leq d_1(r)|u - v|_1, \quad \text{provided } |u|_1, |v|_1 < r,$$

where

$$(6.4) \quad d_1(r) = \sqrt{2}\tilde{L}^{1/2}r + \tilde{L},$$

and $\tilde{L} = L/2\pi$.

We prepare the nonlinear term by applying Theorem 6.1 to obtain an extension of $f|_{B(0,r)}$ to a globally Lipschitz function $f_r : H_{\text{odd}}^1 \rightarrow L_{\text{odd}}^2$ such that $\text{Lip}(f_r) = d_1(r)$ and $f_r = f$ on $B(0, r)$, the ball of radius r in H_{odd}^1 . Thus we may take $M_0 = 0$ and $M_1 = d_1(r)$ in **A1**. In Figure 8 we plot the minimal dimension to satisfy the conditions **A1-7**, along with the data from Figure 7, for comparison. In the latter case, we plot against the value of ρ_1 which is computed in terms of ρ_0 , as in [65].

6.1. Computational results. Since we do not have an exact form for the inertial manifold we instead fix the length L and select several points on the global attractor, hence on any inertial manifold. We then try to reproduce the Q -component of each point using only the P -component. We consider first the case where u is the solution to the eight-dimensional Galerkin approximation, and proceed under the ansatz that there is a three-dimensional inertial manifold. Eight modes have been found to be the minimum needed for the Galerkin method to capture the correct dynamics; using more modes does not seem to change the qualitative features described below, nor introduce new elements to the global attractor [39], [42].

The computations are carried out on a renormalized version of the KSE

$$(6.5) \quad \frac{\partial v}{\partial \tau} + 4\frac{\partial^4 v}{\partial y^4} + \vartheta \left[\frac{\partial^2 v}{\partial y^2} + v \frac{\partial v}{\partial y} \right] = 0, \quad y \in [0, 2\pi],$$

where $\vartheta = L^2/\pi^2$ is taken to be the new parameter. The change of variables which relates (6.5) to (6.1) is given by

$$(6.6) \quad u(t, x) = lv\left(\frac{l^4}{4}t, lx\right) \quad \text{so that } |u|_1 = |u_x|_{L^2(0,L)} = l^{3/2}|v_y|_{L^2(0,2\pi)} = l^{3/2}|v|_1,$$

where $l = 2\pi/L$.

Several elements of the global attractor are shown in Figure 9 at $\vartheta = L^*/\pi^2 = 32$. Plotted here are most of the two-dimensional unstable manifold of the origin, along with the one-dimensional unstable manifold of the “mixed-mode” steady state labeled s_2 , and a stable limit cycle (in bold). The axes are $\sin(x)$, $\sin(2x)$, and $\sin(3x)$, with the two-dimensional unstable manifold of the origin being tangent to the first two axes. The spheres labeled s_1 and s_3 represent bimodal (as they are along the $\sin(2x)$ axis) node and saddle steady states respectively. There is a symmetric mixed-mode steady state which is hidden by the two-dimensional unstable manifold of the origin. We do not plot the one-dimensional unstable manifold of this second mixed-mode steady state, nor the two-dimensional unstable manifold of s_3 , to avoid complicating the picture. We can report, however, that the latter seems to have as its boundary s_3 and the limit cycle. Its shape is bowl-like, as indicated by the portion of the one-dimensional manifold of s_2 that approaches the limit cycle. Further extension of the two-dimensional unstable manifold of the origin suggests that it has as its boundary the unstable manifolds of mixed-mode steady states together with all of the steady states.

The convergence results for the sequences of AIMs given by Φ_j, Ψ_j , taken at the points p_1 (which is on the two-dimensional unstable manifold of the origin, not the one-dimensional manifold of s_2), p_2 (which is on the one-dimensional manifold of s_2), and p_3 (which is on the limit cycle) are shown in Figure 10.

Note that the plots for Ψ_j all level off starting at small values of j . Indeed, it has been rigorously shown that in some sense the error in Ψ_2 is comparable to that of the limiting function $\Psi_\infty \equiv \lim_{j \rightarrow \infty} \Psi_j$. On the other hand the error for the sequence Φ_j continues to improve through $j = 9$ (and should do so beyond, until round-off errors produce diminished returns). At a steady state, however, one expects the two sequences to perform comparably, since $\text{graph} \Psi_\infty$ contains all of the stationary solutions. This is confirmed by the results at s_2 shown in Figure 11.

To completely justify the computations we need to satisfy the gap condition, and thus consider a manifold of higher dimension. All of the elements of the global attractor described above lie within a ball given by $|v|_1 \leq 15$ in the phase space for (6.5), which by (6.6), corresponds to $|u|_1 \leq 15 \cdot 2^{3/2} \cdot 32^{-3/4} \approx 3.2$. We see from Figure 8 that for the KSE prepared at $\rho_1 = 3.2$, rigorous analysis provides an inertial manifold of dimension six, and the convergence estimates in Section 4 hold for a manifold of dimension 26.

With this in mind, we also consider a 64-mode Galerkin truncation of the KSE. We plot in Figure 12 the error from the “exact” Q_n -component at various values of n , for the analogue of point p_3 for this finer discretization. The two observations to make are that the errors decrease dramatically as n increases, but that saturation sets in quickly as j increases in the cases where $n = 30$ and $n = 40$. One explanation for this is that the point p_3 is not exactly on the limit cycle. It was computed so that the image of the Poincaré return map (to the plane defined by the first mode being zero) differed from p_3 in the 12th decimal place in the $|\cdot|_1$ norm. The Q_n -component itself decays rapidly in this norm, as expected from the Gevrey regularity of the solution and indicated in the enstrophy plot in Figure 13. For these reasons we also plot the $|\cdot|_1$ norm of the residual between successive iterations in Figure 14. The fact that the residual continues to decay through seven iterations suggests that the saturation in Figure 12 was indeed due to the inaccuracy of the point p_3 , rather than round-off error. The fact that the residual decays faster with increasing n is consistent with the convergence theory of the algorithm.

These computations have been carried out to compute invariant (perhaps inertial) manifolds for Galerkin approximations of the KSE. In the next section we analyze how these manifolds are connected to those for the PDE case.

7. EFFECT OF TRUNCATING THE HIGH MODES

Our aim in this section is to study the convergence to the exact inertial manifold of the family of AIMs obtained from that derived in [59] by a further truncation of the higher Q -modes. This modification is applied to the Kuramoto-Sivashinsky equation in Section 5 above. The implementation of the family of AIMs presented in [59] is fine for ODEs, as done in Section 4, but for PDEs the higher modes, the Q -modes, are infinite-dimensional, so that in general a further truncation is necessary. It seems reasonable to expect that the convergence to the exact inertial manifold still takes place provided the threshold for this truncation tends to infinity along with either the dimension, or the number of iterations. Our aim is not only to show that this is the case but also to quantify the error obtained with this further truncation even when the threshold for this truncation is kept bounded.

7.1. Error Estimates. In addition to the assumptions made in Section 2, for the analysis in this section we also require

A8 For $m > n$ we have $\Lambda_m > \Lambda_n$, $P_m P_n = P_n$ and, as a consequence, $Q_m Q_n = Q_m$.

The difference in the construction of the AIMs is that now we iterate the map $P_m \mathcal{T}_\tau^N$ instead of $\mathcal{T}_\tau^N$. The expression for the map $P_m \mathcal{T}_\tau^N$ is that for $\mathcal{T}_\tau^N$ in (4.1) with Q replaced by $P_m Q_n$, the other terms being unchanged. The same is true for the expressions (4.6) and (4.7) with respect to the “approximating trajectories” $\tilde{\varphi}_j$ obtained in this construction. More precisely, the AIMs are obtained in the following way:

We first choose sequences $\{\tau_j\}_{j \in \mathbb{N}}$, $\tau_j > 0$, $\{N_j\}_{j \in \mathbb{N}}$, $N_j \in \mathbb{N}$, $N_j \geq 0$, and also $\{m_j\}_{j \in \mathbb{N}}$, $m_j > n$. Then, we set

$$\tilde{\varphi}^0(p_0)(t) = p_0, \quad \forall t \leq 0, \forall p_0 \in P_n E,$$

and proceed by Picard iteration:

$$\tilde{\varphi}^j(p_0) = P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_0), p_0), \quad \forall p_0 \in P_n E.$$

Finally, we define the approximate inertial manifolds as the graphs

$$\tilde{\mathcal{M}}_j = \text{graph } \tilde{\Phi}_j = \{y + \tilde{\Phi}_j(y); y \in P_n E\},$$

of the maps $\tilde{\Phi}_j : P_n E \rightarrow P_{m_j} Q_n E$ defined by

$$\tilde{\Phi}_j(p_0) = P_{m_j} Q_n \tilde{\varphi}^j(p_0)(0), \quad \forall p_0 \in P_n E.$$

Since the spectral gap condition (3.4) is satisfied, an inertial manifold $\mathcal{M} = \text{graph } \Phi$ of a function $\Phi : P_n E \rightarrow Q_n E$ exists as described in Section 2, with $\Phi(p_0) = Q_n \varphi(p_0)$, where $\varphi(p_0)$ denotes the exact backward solution of (1.1) that passes through $p_0 + \Phi(p_0) \in \mathcal{M}$ at time $t = 0$, which is obtained as the fixed point of the map $\mathcal{T}(\cdot, p_0)$. We will need below the following estimate for $\varphi(p_0)$ which is proven in [59], eq. (2.14):

$$(7.1) \quad \|\varphi(y)\|_\sigma \leq \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 |p_0|_E \right].$$

The first result concerns the regularity of the AIMs:

Proposition 7.1. *For each $j \in \mathbb{N}$, the set $\tilde{\mathcal{M}}_j$ is a finite-dimensional topological submanifold of E .*

Proof: As for the map $\mathcal{T}$ (see (3.8)) it is easy to show that $\mathcal{T}_\tau^N$ is Lipschitz continuous with

$$\|\mathcal{T}_\tau^N(\psi_1, y) - \mathcal{T}_\tau^N(\psi_2, y)\|_\sigma \leq \kappa_{n,\sigma} \|\psi_1 - \psi_2\|_\sigma, \quad \forall \psi_1, \psi_2 \in \tilde{\mathcal{F}}, \quad \forall y \in P_n E,$$

where $\kappa_{n,\sigma}$ is given in (3.9), σ satisfies (3.10), $\tau > 0$, and N is any nonnegative integer.

From (3.2) we have that $\|P_n\|_{\mathcal{L}(E)} \leq K_1$. Let now $p_1, p_2 \in P_n E$ and $j \in \mathbb{N}$. Then,

$$\begin{aligned} \|\tilde{\varphi}^j(p_1) - \tilde{\varphi}^j(p_2)\|_\sigma &= \|P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_1) - P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_2), p_2)\|_\sigma \\ &\leq \|P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_1) - P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_2)\|_\sigma \\ &\quad + \|P_{m_j}\|_{\mathcal{L}(E)} \|\mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_2) - \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_2), p_2)\|_\sigma \\ &\leq \|P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_1) - P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_2)\|_\sigma \\ &\quad + K_1 \kappa_{n,\sigma} \|\tilde{\varphi}^{j-1}(p_1) - \tilde{\varphi}^{j-1}(p_2)\|_\sigma. \end{aligned}$$

It is easy to see that

$$\|P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_1) - P_{m_j} \mathcal{T}_{\tau_j}^{N_j}(\tilde{\varphi}^{j-1}(p_1), p_2)\|_\sigma \leq K_1 |p_1 - p_2|_E,$$

Thus,

$$\|\tilde{\varphi}^j(p_1) - \tilde{\varphi}^j(p_2)\|_\sigma \leq K_1 |p_1 - p_2|_E + K_1 \kappa_{n,\sigma} \|\tilde{\varphi}^{j-1}(p_1) - \tilde{\varphi}^{j-1}(p_2)\|_\sigma.$$

By iterating this last inequality, we obtain

$$\|\tilde{\varphi}^j(p_1) - \tilde{\varphi}^j(p_2)\|_\sigma \leq K_1 |p_1 - p_2|_E \sum_{\ell=0}^{j-1} K_1^\ell \kappa_{n,\sigma}^\ell + K_1^j \kappa_{n,\sigma}^j \|\tilde{\varphi}^0(p_1) - \tilde{\varphi}^0(p_2)\|_\sigma.$$

Since also

$$\|\tilde{\varphi}^0(p_1) - \tilde{\varphi}^0(p_2)\|_\sigma = \|P_n(p_1 - p_2)\|_\sigma \leq K_1 |p_1 - p_2|_E,$$

we find that

$$\|\tilde{\varphi}^j(p_1) - \tilde{\varphi}^j(p_2)\|_\sigma \leq K_1 |p_1 - p_2|_E \sum_{\ell=0}^j K_1^\ell \kappa_{n,\sigma}^\ell.$$

Since $\tilde{\Phi}_j(y) = P_{m_j} Q_n \tilde{\varphi}^j(y)(0)$, for all $y \in P_n E$, we obtain

$$|\tilde{\Phi}_j(p_1) - \tilde{\Phi}_j(p_2)|_E \leq K_1 |p_1 - p_2|_E \sum_{\ell=0}^j K_1^\ell \kappa_{n,\sigma}^\ell.$$

Therefore, since P_n is a finite-dimensional projector, we deduce that $\tilde{\mathcal{M}}_j = \text{graph } \tilde{\Phi}_j$ is a finite-dimensional Lipschitz submanifold of E . $\square$

Remark 7.2. *It is possible to extend the above result to show that the whole map $P_m \mathcal{T}_\tau^N$ is Lipschitz continuous with a Lipschitz constant $\tilde{\kappa}_{n,\sigma}$ similar to $\kappa_{n,\sigma}$, and that under an extra condition similar to the spectral gap condition (3.4) this constant $\tilde{\kappa}_{n,\sigma}$ is also strictly less than 1 and the Lipschitz constants of the maps $\tilde{\Phi}_j$ are bounded uniformly by $K_1/(1 - \tilde{\kappa}_{n,\sigma})$.*

For the convergence of the AIMs, we proceed similarly to the proof in [59]. The first step concerns only the exact solution, hence, we borrow it unmodified from [59]; it is Lemma 4.2 appearing in Section 3 above. The second step is to estimate how the map $P_m \mathcal{T}_\tau^N$ brings elements in $\tilde{\mathcal{F}}$ closer to the exact orbits on the inertial manifold:

Lemma 7.3. *Let N be a nonnegative integer, $m \in \mathbb{N}$ with $m > n$, and $\tau > 0$, and assume σ and σ_0 satisfy (3.10) with $\sigma_0 < \sigma$. Then for all $p_0 \in P_n E$ and all $\psi \in \tilde{\mathcal{F}}$, we have*

$$(7.2) \quad \begin{aligned} \|\varphi(p_0) - P_m \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma &\leq \kappa_{n,\sigma} \|\varphi(p_0) - \psi\|_\sigma \\ &+ \left(\tau \beta_{n,\sigma} + \frac{\beta_{n,\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma - \sigma_0)N\tau} + \tilde{\beta}_{n,\sigma} \frac{\Lambda_m^\alpha}{\Lambda_m - \sigma} \right) (1 + |p_0|_E), \end{aligned}$$

where $\beta_{n,\sigma}$ and β_{n,σ_0} are as in Lemma 4.2, and

$$(7.3) \quad \tilde{\beta}_{n,\sigma} = K_2(1 + \gamma_\alpha) \left\{ M_0 + \frac{M_1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right] \right\}.$$

Proof: By triangulation,

$$(7.4) \quad \|\varphi(p_0) - P_m \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma \leq \|\varphi(p_0) - \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma + \|\mathcal{T}_\tau^N(\psi, p_0) - P_m \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma.$$

For the second term we note that

$$\mathcal{T}_\tau^N(\psi, p_0)(t) - P_m \mathcal{T}_\tau^N(\psi, p_0)(t) = Q_m \mathcal{T}_\tau^N(\psi, p_0)(t) = \int_{-\infty}^{-k\tau} e^{-(k\tau-s)A} Q_m f(\psi(s)) ds,$$

for $t \in (-(k+1)\tau, -k\tau]$, $k < N$, if $-N\tau < t$, and $t \in (-\infty, -N\tau]$, $k = N$, if $t \leq -N\tau$. Thus, using (3.3),

$$\begin{aligned} &e^{-\sigma t} \left| \int_{-\infty}^{-k\tau} e^{-(k\tau-s)A} Q_m f(\varphi(p_0)(s)) ds \right|_E \\ &\leq e^{-\sigma k\tau} K_2 \int_{-\infty}^{-k\tau} ((-k\tau - s)^{-\alpha} + \Lambda_m^\alpha) e^{-\Lambda_m(-k\tau-s)} (M_0 + M_1 |\varphi(p_0)(s)|_E) ds \\ &\leq (M_0 + M_1 \|\varphi(p_0)\|_\sigma) K_2 \int_{-\infty}^{-k\tau} ((-k\tau - s)^{-\alpha} + \Lambda_m^\alpha) e^{-(\Lambda_m - \sigma)(-k\tau-s)} ds. \end{aligned}$$

Since

$$\int_{-\infty}^{-k\tau} (-k\tau - s)^{-\alpha} e^{-(\Lambda_m - \sigma)(-k\tau - s)} ds = \frac{\gamma_\alpha}{(\Lambda_m - \sigma)^{1-\alpha}},$$

and

$$\int_{-\infty}^{-k\tau} e^{-(\Lambda_m - \sigma)(-k\tau - s)} ds = \frac{1}{\Lambda_m - \sigma},$$

we find that

$$\int_{-\infty}^{-k\tau} ((-k\tau - s)^{-\alpha} + \Lambda_m^\alpha) e^{-(\Lambda_m - \sigma)(-k\tau - s)} ds = \frac{\gamma_\alpha \Lambda_m^\alpha + (\Lambda_m - \sigma)^\alpha}{\Lambda_m - \sigma} \leq \frac{(1 + \gamma_\alpha) \Lambda_m^\alpha}{\Lambda_m - \sigma}.$$

Therefore,

$$e^{-\sigma t} \left| \int_{-\infty}^{-k\tau} e^{-(k\tau - s)A} Q_m f(\varphi(p_0)(s)) ds \right|_E \leq (M_0 + M_1 \|\varphi(p_0)\|_\sigma) \frac{K_2(1 + \gamma_\alpha) \Lambda_m^\alpha}{\Lambda_m - \sigma}.$$

Using now (7.1),

$$\begin{aligned} & e^{-\sigma t} \left| \int_{-\infty}^{-k\tau} e^{-(k\tau - s)A} Q_m f(\varphi(p_0)(s)) ds \right|_E \\ & \leq \left(M_0 + M_1 \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 |p_0|_E \right] \right) \frac{K_2(1 + \gamma_\alpha) \Lambda_m^\alpha}{\Lambda_m - \sigma} \\ & \leq \left(M_0 + \frac{M_1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right] \right) \frac{K_2(1 + \gamma_\alpha) \Lambda_m^\alpha}{\Lambda_m - \sigma} (1 + |p_0|_E) \\ & = \tilde{\beta}_{n,\sigma} \frac{\Lambda_m^\alpha}{\Lambda_m - \sigma} (1 + |p_0|_E), \end{aligned}$$

where $\tilde{\beta}_{n,\sigma}$ is given by (7.3). Hence,

$$\|\mathcal{T}_\tau^N(\psi, p_0) - P_m \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma \leq \tilde{\beta}_{n,\sigma} \frac{\Lambda_m^\alpha}{\Lambda_m - \sigma} (1 + |p_0|_E).$$

Insert this estimate into (7.4) to find that

$$\|\varphi(p_0) - P_m \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma \leq \|\varphi(p_0) - \mathcal{T}_\tau^N(\psi, p_0)\|_\sigma + \tilde{\beta}_{n,\sigma} \frac{\Lambda_m^\alpha}{\Lambda_m - \sigma} (1 + |p_0|_E).$$

Taking into account the estimate of Lemma 4.3 we finally obtain (7.2). $\square$

We may now estimate the distance from each “approximating trajectory” $\tilde{\varphi}^j(p_0)$ to the exact trajectory $\varphi(p_0)$ by iterating the estimate in Lemma 7.3.

Lemma 7.4. *Let σ and σ_0 satisfy (3.10) with $\sigma_0 < \sigma$. Then*

$$\begin{aligned} (7.5) \quad & \sup_{p_0 \in P_n E} \frac{\|\varphi(p_0) - \tilde{\varphi}^j(p_0)\|_\sigma}{1 + |p_0|_E} \leq \kappa_{n,\sigma}^j \left(K_1 + \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right] \right) \\ & + \sum_{\ell=0}^{j-1} \kappa_{n,\sigma}^\ell \left(\tau_{j-\ell} \beta_{n,\sigma} + \frac{\beta_{n,\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma - \sigma_0)N_{j-\ell}\tau_{j-\ell}} + \tilde{\beta}_{n,\sigma} \frac{\Lambda_{m_{j-\ell}}^\alpha}{\Lambda_{m_{j-\ell}} - \sigma} \right), \end{aligned}$$

where $\beta_{n,\sigma}$ and β_{n,σ_0} are as in Lemma 4.2, and $\tilde{\beta}_{n,\sigma}$ as in Lemma 7.3.

Proof. Just iterate estimate (7.2) and then use the fact that

$$\begin{aligned} \|\varphi(p_0) - \tilde{\varphi}^0(p_0)\|_\sigma &\leq \|\varphi(p_0)\|_\sigma + \|\tilde{\varphi}^0(p_0)\|_\sigma = \|\varphi(p_0)\|_\sigma + \|P_n p_0\|_\sigma \quad (\text{by (7.1)}) \\ &\leq \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 |p_0|_E \right] + K_1 |p_0|_E \\ &\leq \left(K_1 + \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right] \right) (1 + |p_0|_E). \end{aligned}$$

□

Thanks to Lemma 7.4, we can obtain the convergence of the manifolds $\tilde{\mathcal{M}}_j = \text{graph } \tilde{\Phi}_j$ to the inertial manifold $\mathcal{M} = \text{graph } \Phi$ in a suitable topology. The topology we consider is the one defined by the metric

$$d(\Phi, \tilde{\Phi}_j) = \sup_{p_0 \in P_n E} \frac{|\Phi(p_0) - \tilde{\Phi}_j(p_0)|_E}{1 + |p_0|_E}.$$

We then have

Theorem 7.5. *Assume A1 to A8 hold. Assume also that $\Lambda_m \rightarrow +\infty$ as $m \rightarrow +\infty$. Suppose the sequences $\{\tau_j\}_j$, $\{N_j\}_j$ and $\{m_j\}_j$ are chosen so that $\tau_j \rightarrow 0$, $N_j \tau_j \rightarrow +\infty$, and $m_j \rightarrow +\infty$ as $j \rightarrow +\infty$, with $m_j > n$. Then*

$$d(\Phi, \tilde{\Phi}_j) \rightarrow 0, \quad \text{as } j \rightarrow +\infty.$$

Proof. The result follows directly from Lemma 7.4. □

For the exponential convergence of the manifolds the sequences $\{\tau_j\}_j$, $\{N_j\}_j$ and $\{m_j\}_j$ have to be chosen appropriately:

Theorem 7.6. *Assume A1 to A8 hold. Assume also that it is possible to choose a sequence $\{m_j\}_j$ with $m_j > n$ and such that*

$$(7.6) \quad \Lambda_{m_j}^{1-\alpha} \geq \frac{c_0}{\delta^j}, \quad \forall j \in \mathbb{N},$$

for some $c_1 > 0$ and some $0 < \delta < 1$. Suppose that the sequences $\{\tau_j\}_j$ and $\{N_j\}_j$ are chosen so that

$$(7.7) \quad 0 < \tau_j \leq c_1 \gamma^j, \quad \forall j \in \mathbb{N},$$

and

$$(7.8) \quad N_j \tau_j \geq c_2 j, \quad \forall j \in \mathbb{N},$$

for some $c_1, c_2 > 0$ and some $0 < \gamma < 1$. Let now σ and σ_0 satisfy (3.10) with $\sigma < \sigma_0$. Let

$$\epsilon = \min\{\kappa_{n,\sigma}, \max\{\gamma, e^{-c_2(\sigma-\sigma_0)}, \delta\}\},$$

which is less than 1. Then for any η with $\epsilon < \eta < 1$, there exists a $c_\eta \geq 0$ such that

$$(7.9) \quad d(\Phi, \tilde{\Phi}_j) \leq c_\eta \eta^j, \quad \forall j \in \mathbb{N}.$$

Proof. From (7.5), we have

$$d(\Phi, \tilde{\Phi}_j) \leq c_3 \kappa_{n,\sigma}^j + \sum_{\ell=0}^{j-1} \kappa_{n,\sigma}^\ell \left(\tau_{j-\ell} \beta_{n,\sigma} + \frac{\beta_{n,\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma-\sigma_0)N_{j-\ell}\tau_{j-\ell}} + \tilde{\beta}_{n,\sigma} \frac{\Lambda_{m_{j-\ell}}^\alpha}{\Lambda_{m_{j-\ell}}^{1-\alpha} - \sigma} \right),$$

where we may take

$$(7.10) \quad c_3 = K_1 + \frac{1}{1 - \kappa_{n,\sigma}} \left[\frac{M_0 K_1}{\lambda_n^{1-\alpha}} + (1 + \gamma_\alpha) \frac{M_0 K_2}{\Lambda_n^{1-\alpha}} + K_1 \right].$$

From (7.6), (7.7), and (7.8), we can bound the terms inside the parentheses as follows:

$$\begin{aligned} \tau_{j-\ell} \beta_{n,\sigma} &\leq c_4 \gamma^{j-\ell}, \\ \frac{\beta_{n,\sigma_0}}{\sigma - \sigma_0} e^{-(\sigma - \sigma_0) N_{j-\ell} \tau_{j-\ell}} &\leq c_4 e^{-\zeta(\sigma - \sigma_0)(j-\ell)}, \text{ and} \\ \tilde{\beta}_{n,\sigma} \frac{\Lambda_{m_{j-\ell}}^\alpha}{\Lambda_{m_{j-\ell}} - \sigma} &\leq \tilde{\beta}_{n,\sigma} \frac{\Lambda_{m_{j-\ell}}^{\alpha-1}}{1 - \frac{\sigma}{\Lambda_{m_{j-\ell}}}} \leq \tilde{\beta}_{n,\sigma} \frac{\Lambda_{m_{j-\ell}}^{\alpha-1}}{1 - \frac{\sigma}{\Lambda_n}} \leq c_4 \delta^{j-\ell}. \end{aligned}$$

with

$$(7.11) \quad c_4 = \max\{\beta_{n,\sigma} c_1, \frac{\beta_{n,\sigma}}{\sigma - \sigma_0}, \frac{\tilde{\beta}_{n,\sigma}}{c_0(1 - \sigma/\Lambda_n)}\}.$$

Therefore,

$$d(\Phi, \tilde{\Phi}_j) \leq c_3 \kappa_{n,\sigma}^j + c_4 \sum_{\ell=0}^{j-1} \kappa_{n,\sigma}^\ell \left(\gamma^{j-\ell} + e^{-\zeta(\sigma - \sigma_0)(j-\ell)} + \delta^{j-\ell} \right).$$

Thus,

$$d(\Phi, \tilde{\Phi}_j) \leq c_3 \kappa_{n,\sigma}^j + c_5 \sum_{\ell=0}^{j-1} \kappa_{n,\sigma}^\ell \epsilon^{j-\ell},$$

for $c_5 = 3c_4$ and $0 < \tilde{\epsilon} < 1$ given by

$$\tilde{\epsilon} = \max\{\gamma, e^{-\zeta(\sigma - \sigma_0)}, \delta\}.$$

Let η be such that $\eta > \epsilon$. Then,

$$\sum_{\ell=0}^{j-1} \kappa_{n,\sigma}^\ell \tilde{\epsilon}^{j-\ell} = \sum_{\ell=0}^{j-1} \left(\frac{\kappa_{n,\sigma}}{\eta} \right)^\ell \eta^\ell \left(\frac{\tilde{\epsilon}}{\eta} \right)^{j-\ell} \eta^{j-\ell} = \eta^j \sum_{\ell=0}^{j-1} \left(\frac{\epsilon}{\eta} \right)^{j-\ell} \leq \eta^j \frac{\eta}{\eta - \epsilon}.$$

Hence,

$$d(\Phi, \tilde{\Phi}_j) \leq c_3 \kappa_{n,\sigma}^j + c_5 \frac{\eta}{\eta - \epsilon} \eta^j \leq c_\eta \eta^j,$$

for

$$(7.12) \quad c_\eta = c_3 + c_5 \frac{\eta}{\eta - \epsilon}.$$

□

Remark 7.7. Note from the proof of Theorem 7.6 that in fact

$$\sup_{p_0 \in P_n E} \frac{\|\varphi(p_0) - \tilde{\varphi}^j(p_0)\|_\sigma}{1 + |p_0|_E} \leq c_\eta \eta^j, \quad \forall j \in \mathbb{N}.$$

Remark 7.8. In case the nonlinear term f is C^1 , it can be further proved that the AIMs are of class C^1 and converge in the C^1 norm on bounded sets to the exact inertial manifold. In case the derivative of the nonlinear term f is Hölder continuous, the AIMs can be chosen so that the C^1 convergence is exponential.

7.2. Application to the KSE. We saw in the previous section that with $r = 3.2$, **A1-7** hold for $n \geq 26$. The condition **A8** also holds, trivially. Here we fix $n = 40$, and observe that κ is a decreasing function of σ over the interval in (3.10). Thus we select σ and σ_0 to be the right and left endpoints (minus .001 and plus .001, respectively), in the calculation of $\beta_{n,\sigma}$, β_{n,σ_0} from (4.12), and $\tilde{\beta}_{n,\sigma}$ from (7.3). We consider a fixed truncation $m_j = m = 64$, and plot in Figure 15 the total error as in estimated in (7.5) along with the individual terms denoted e1 through e4, consecutively from left to right (the last three are in the summation). The plots are for two algorithm settings, set1: $c_1 = .001$ $c_2/c_1 = 1$ (as in the pervious runs for the KSE) and set2: $c_1 = .0001$ $c_2/c_1 = 10$. For the first setting, the term e2 dominates by about a factor of ten suggesting that c_1 should be decreased by just this amount. By simultaneously increasing c_2/c_1 by the same factor in set2, we leave the terms e1, e3, and e4 unchanged and arrive at a total error which is fairly evenly balanced in all the terms. Note that e4 actually increases with the number of iterations since m_j is fixed in this case.

8. COMPUTING INERTIAL MANIFOLDS WITH DELAY

Related to the notion of an inertial manifold is that of an inertial manifold with delay (IMD) recently introduced in [12]. This manifold enslaves the high modes at the present time in terms of not only the low modes at the present, but also the high modes at a time in the recent past. It is sought as the graph of a function of the form

$$(8.1) \quad q(t) = \psi(p(t), q(t - T)),$$

and unlike an inertial manifold, is infinite dimensional. Here t is arbitrary, so by translation we may consider the manifold as given by the relation

$$(8.2) \quad q_0 = \psi(p_0, q_{-T}).$$

The conditions under which this result holds are significantly more relaxed than those for an inertial manifold, or for that matter the approximation of the global attractor by a sequence of finite dimensional manifolds. We recall how it is stated in [13]. To be consistent with [13] we assume in this section that the given phase space $\mathcal{E}$ for (1.1) is a Hilbert space, and that A is an unbounded self-adjoint, positive operator, with compact inverse and eigenvalues $\{\mu_k\}_{k=1}^\infty$ as in (3.5) which also satisfy

A9: μ_{k+1}/μ_k is bounded as $k \rightarrow \infty$.

The projectors P_n , Q_n are then defined in terms of the eigenspaces as in the KSE case. We will consider the domain of A^α , denoted $D(A^\alpha)$, with norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|$, where $\|\cdot\|$ is the norm of $\mathcal{E}$.

Theorem 8.1. *Suppose **A1**, **A2**, **A3** and **A9** hold with $\lambda_n = \mu_n$, $\Lambda_n = \mu_{n+1}$, $E = D(A^\alpha)$ and $F = \mathcal{E}$. Then there exist c_6 and c_7 depending only on A and f such that for any T satisfying*

$$(8.3) \quad M_1 \lambda_n^\alpha T \leq c_6, \quad M_1 T^{1-\alpha} \leq c_7,$$

and for any $p_0 \in P_n E$, $q_{-T} \in Q_n E$, there exists a unique solution of (1.1) defined on $[-T, +\infty)$ such that

$$P_n u(0) = p_0, \quad Q_n u(-T) = q_{-T}.$$

Moreover

$$(p_0, q_{-T}) \mapsto \psi(p_0, q_{-T}) = Q_n u_0,$$

defines a C^1 mapping from $P_n E \times Q_n E$ to $Q_n E$, and for $(p_0^i, q_{-T}^i) \in P_n E \times Q_n E$, $i = 1, 2$ we have

$$(8.4) \quad \|\psi(p_0^1, q_{-T}^1) - \psi(p_0^2, q_{-T}^2)\|_\alpha \leq \frac{1}{2} \|p_0^1 - p_0^2\|_\alpha + 2e^{-\nu T} \|q_{-T}^1 - q_{-T}^2\|_\alpha,$$

where $\nu = \min(\mu_{n+1}, \mu_n + 16M_1\mu_n^\alpha)$.

The existence proof for the IMD is similar to that for the inertial manifold except that the expression for the contraction mapping in (3.7) is modified to

$$(8.5) \quad \begin{aligned} \mathcal{T}_{-T}(\psi, p_0)(t) &= e^{-tAP} p_0 - \int_t^0 e^{-(t-s)A} P f(\psi(s)) ds \\ &+ e^{-(T+t)AQ} q_{-T} + \int_{-T}^t e^{-(t-s)A} Q f(\psi(s)) ds, \quad \text{for } -T \leq t \leq 0, \quad \forall y \in PE, \end{aligned}$$

for ψ in the Banach space $C([-T, 0], E)$.

Remark 8.2. *We differ slightly in our formulation of $\mathcal{T}_{-T}$ compared to that in [13], where the role of ψ is played by a function $\xi : [-T, 0] \rightarrow Q_n E$ so that the mapping is explicit only for the high modes component and $y(t)$ is implicitly updated as the solution of*

$$\frac{dy}{dt} + Ay = P_n(y + \xi(t)), \quad y(0) = p_0.$$

Mathematically the two are equivalent, but (8.5) is already set up for the approximation as in the case of (3.7).

A major source of computational complexity in implementing the algorithm given by (4.6), (4.7) is that one must discretize in time back to $-\infty$. In discretizing (8.5) we need only deal with time on the interval $[-T, 0]$, for what in fact are small delays T . In addition, the assumptions needed for the IMD are weaker than those for an inertial manifold. In particular, there is no gap condition to meet, which allows us to use a smaller value of n than for the inertial manifold, and means that a system like the 2D NSE has an IMD. One needs to take T small enough, but this actually helps in the convergence of the algorithm. In fact the smaller the T , the better the convergence, as we will demonstrate below. On the other hand, to employ the IMD in a numerical scheme to save computational effort, one cannot take T too small either. We will discuss this issue further in the computational subsection that follows.

One simplification is that an analogue of the second case in the algorithm for inertial manifolds (4.7) is not needed. The discretized version of (8.5) reads as,

$$(8.6) \quad \begin{aligned} \psi_k^j(p_0) &= e^{k\tau_j A} P_n p_0 - A^{-1}(e^{(k\tau_j - \ell_k \tau_{j-1})A} P_n - P_n) f(\psi_{\ell_k}^{j-1}(p_0)) \\ &- \sum_{\ell=0}^{\ell_k-1} A^{-1}(e^{(k\tau_j - \ell \tau_{j-1})A} P_n - e^{(k\tau_j - (\ell+1)\tau_{j-1})A} P_n) f(\psi_{\ell}^{j-1}(p_0)) \\ &+ e^{-(T+k\tau_j)A} Q_n q_{-T} \\ &+ \sum_{\ell=\ell_k+1}^{N_{j-1}-1} A^{-1}(e^{-(\ell \tau_{j-1} - k\tau_j)A} - e^{-((\ell+1)\tau_{j-1} - k\tau_j)A}) Q_n f(\psi_{\ell}^{j-1}(p_0)) \\ &+ A^{-1}(I - e^{-((\ell_k+1)\tau_{j-1} - k\tau_j)A}) Q_n f(\psi_{\ell_k}^{j-1}(p_0)). \end{aligned}$$

where ℓ_k is as in (4.5). Indeed, once a code for the algorithm in (4.6), (4.7) for computing the inertial manifold is developed, it is a simple matter to adapt it for (8.6).

Alternatively, one may consider a shooting algorithm for determining q_0 , given p_0 and q_{-T} . This amounts to finding p_{-T} such that

$$(8.7) \quad G(p_{-T}) \equiv PS(T)u_{-T} - p_0 = 0, \quad \text{where } u_{-T} = p_{-T} + q_{-T}.$$

A sketch in Figure 16 illustrates the shooting algorithm: a point in one affine space (horizontal dotted line), must be mapped under the semiflow to another affine space (vertical dotted line).

8.1. Computational results. We apply Newton's method to solve (8.7) for (5.1) (with $u_0 = (.5, .5, 1)$, and q_{-T} determined by the exact solution). The Jacobian matrix $\partial G / \partial p_{-T}$ is computed by solving (1.1) and the linearized version of (1.1) simultaneously using as initial conditions u_{-T} , and the identity, respectively. This is carried out using the variable-step ODE solver ODESSA [43], which employs variable-order backwards difference formulas, and is designed specifically for the computation of sensitivities (in this case, with respect to the initial condition). In ODESSA, one specifies local error tolerances, and then the time step is automatically adapted to try to meet that objective. The convergence for both the Newton algorithm and iterations of the mapping in (8.6) are compared in Figure 17. For the Newton case this is done for various tolerance settings in ODESSA. To be specific, with the relative local error tolerance `rtol` fixed to be 10^{-d} , we complete three Newton iterations in separate runs with $d = 6, 8$, and 10 . The error in solving the ODE is the most natural explanation for the fact that the Newton plots are not monotonic. This is remedied in a composite run, in which we set $d = 6$ on the first iteration, $d = 8$ on the second, and $d = 10$ on the third. The absolute local error tolerance in each instance is taken to be `atol` = `.001 · rtol`). The results for the same experiment, but with a smaller choice for T are plotted in Figure 18. We observe that the approach in (8.6) benefits more from the tacit improvement in the initial guess than does the Newton-shooting approach, and that for either choice of T , the latter takes more function evaluations to achieve the same error. Of course, one could compute the value of p_{-T} by other means, such as collocation, and perhaps find a reversal of the efficiency comparison. The point here is that the algorithm given by (8.6) is at least competitive with a straightforward shooting algorithm for the computation of the IMD.

A more fundamental question is: what would be the computational application of the IMD? One possibility is to develop alternative integration schemes in which the high modes are updated not by conventional means, but by using the low mode at the current time step, and the high mode at the previous step in the functional relationship (8.1). Certain well-known schemes are easily modified for this purpose. For instance the second-order Adams-Bashforth scheme applied to the system $du/dt = f(u) = -Au + F(u)$, using a time step h can be written as

$$(8.8) \quad \begin{aligned} u_0 &= u(0), \quad u_1 \approx u(h), \\ u_{i+1} &= u_i + \frac{h}{2}[3f_i - f_{i-1}], \quad i \geq 2, \end{aligned}$$

where $f_i = f(u_i)$, and the approximation u_1 is found by a second-order Runge-Kutta method. Alternatively we could approximate the solution to the reduced system

$$dp/dt = g(p(t), q(t - T)) = -Ap(t) + PF(p(t) + \psi(p(t), q(t - T)))$$

with

$$(8.9) \quad \begin{aligned} g_0 &= Pf(u(0)), \quad g_1 = Pf(u_1), \\ p_{i+1} &= p_i + \frac{T}{2}[3g_i - g_{i-1}], \quad i \geq 2, \end{aligned}$$

where $g_i = g(p_i, q_{i-1})$, and $q_i = \psi(p_i, q_{i-1})$. Starting with $i = 3$, we first compute $\psi(p_i, q_{i-1})$, (using q_{i-2} as an initial guess), substitute to obtain $g(p_i, q_{i-1})$, and thus p_{i+1} . The estimate on the variation in the IMD provided by (8.4) could be used to obtain error estimates.

From the point of view of dynamical systems, however, we do not see a particular advantage for using a modified scheme based on the IMD. The reason is that it does not seem at all suitable for integrating backwards in time, unlike the inertial manifold. The fact that the IMD can be used to compute *any* solution of (1.1) means that it inherits the general ill-posedness of the original PDE problem in negative time. It seems then that the only reason one might compute a solution in this way is to save in computational effort. That might be achieved by being able to take a larger time step T in (8.9) than the time step h in (8.8), but must be balanced against the extra effort to compute ψ . We will not

speculate here as to how such an effort might ultimately pay off. We simply conclude that if an inertial manifold with delay is desired, then (8.6) is a viable algorithm to compute it.

FIGURE 1. Phase portrait for (2.3).

FIGURE 2. Relative error in computing stable (sub)manifold using the computed inertial form for (2.3).

FIGURE 3. Error vs. # of iterations. The three uppermost plots are the theoretical estimates, the three lowest plots are for $\|\Phi_j(p_0) - \Phi(p_0)\|$, and the three in the middle are for $\|\varphi^j - \varphi\|_{T,\sigma}$

FIGURE 4. t_{sup} vs. # of iterations.

FIGURE 5. Residual terms vs. iteration.

 FIGURE 6. $Q\varphi_j$, $j = 1, 2, \dots, 9$, and $Q\varphi$ for (5.1).

FIGURE 7. Minimal dimension of an inertial manifold for prepared KSE.

FIGURE 8. Minimal dimension under two different sets of assumptions.

FIGURE 9. Phase portrait for the KSE at $L = L^*$.

FIGURE 10. Relative error for Φ_j , Ψ_j at the points p_1, p_2 , and p_3 . The three upper plots are for Ψ_j , and the three lower plots are for Φ_j .

FIGURE 11. Relative error for Φ_j , Ψ_j at the steady state s_2

FIGURE 12. Gradient error from “exact” high-mode component of the point $p3$. FIGURE 13. Enstrophy of the point $p3$.

FIGURE 14. Gradient norm of residue.

 FIGURE 15. Total error estimate and individual terms in (7.5) for set1: $c_1 = .001$, $c_2/c_1 = 1$ and set2: $c_1 = .0001$, $c_2/c_1 = 10$

FIGURE 16. Schematic illustration of shooting algorithm to compute an IMD.

FIGURE 17. Error vs. # function evaluations (of the “right hand side” of (1.1): $-Au + f(u)$), with $T = .001$.

FIGURE 18. Error vs. # function evaluations with $T = .0001$.

REFERENCES

- [1] D. Armbruster, J. Guckenheimer, and P. Holmes. Kuramoto-Sivashinsky dynamics on the center-unstable manifold. *SIAM J. Appl. Math.*, 49(3):676–691, 1989.
- [2] H. W. Broer, H. M. Osinga, and G. Vegter. Algorithms for computing normally hyperbolic invariant manifolds. *Z. Angew. Math. Phys.*, 48(3):480–524, 1997.
- [3] J. Carr. *Applications of centre manifold theory*, volume 35 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1981.
- [4] H.-C. Chang. Traveling waves on fluid interfaces: normal form analysis of the Kuramoto-Sivashinsky equation. *Phys. Fluids*, 29(10):3142–3147, 1986.
- [5] S.-N. Chow and K. Lu. Invariant manifolds for flows in banach spaces. *J. Differential Equations*, 74:285–317, 1988.
- [6] P. Collet, J.-P. Eckmann, H. Epstein, and J. Stubbe. A global attracting set for the Kuramoto-Sivashinsky equation. *Comm. Math. Phys.*, 152(1):203–214, 1993.
- [7] P. Constantin, C. Foias, B. Nicolaenko, and R. Temam. *Integral manifolds and inertial manifolds for dissipative partial differential equations*, volume 70 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.
- [8] P. Constantin, C. Foias, B. Nicolaenko, and R. Temam. Spectral barriers and inertial manifolds for dissipative partial differential equations. *J. Dynamics Differential Equations*, 1(1):45–73, 1989.
- [9] S. P. Dawson and A. M. Mancho. Collections of heteroclinic cycles in the Kuramoto-Sivashinsky equation. *Phys. D*, 100(3-4):231–256, 1997.
- [10] A. Debussche and M. Marion. On the construction of families of approximate inertial manifolds. *J. Differential Equations*, 100(1):173–201, 1992.
- [11] A. Debussche and R. Temam. Convergent families of approximate inertial manifolds. *J. Math. Pures Appl. (9)*, 73(5):489–522, 1994.
- [12] A. Debussche and R. Temam. Inertial manifolds with delay. *Appl. Math. Lett.*, 8(2):21–24, 1995.
- [13] A. Debussche and R. Temam. Some new generalizations of inertial manifolds. *Discrete Contin. Dynam. Systems*, 2(4):543–558, 1996.
- [14] A. Debussche and R. Temam. A convergent family of approximate inertial manifolds. *Appl. Math. Lett.*, 6(1):87–90, 1993.
- [15] F. Demengel and J.M. Ghidaglia. Inertial manifolds for partial differential evolution equations under time-discretization: existence, convergence, and applications. *J. Math. Anal. and Appl.*, 155:177–225, 1991.
- [16] L. Dieci and J. Lorenz. Computation of invariant tori by the method of characteristics. *SIAM J. Num. Anal.*, 32:1436–1474, 1995.
- [17] E. J. Doedel, M. J. Friedman, and B. I. Kunin. Successive continuation for locating connecting orbits. *Numer. Algorithms*, 14(1-3):103–124, 1997. Dynamical numerical analysis (Atlanta, GA, 1995).
- [18] T. Dubois, F. Jauberteau, and R. Temam. Solution of the incompressible Navier-Stokes equations by the nonlinear Galerkin method. *J. Sci. Comput.*, 8(2):167–194, 1993.
- [19] C. Foias, O. Manley, and R. Temam. Modelling of the interaction of small and large eddies in two-dimensional turbulent flows. *RAIRO Modél. Math. Anal. Numér.*, 22(1):93–118, 1988.
- [20] C. Foias, B. Nicolaenko, G. R. Sell, and R. Temam. Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension. *J. Math. Pures Appl. (9)*, 67(3):197–226, 1988.
- [21] C. Foias and J.-C. Saut. Remarques sur les équations de Navier-Stokes stationnaires. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10(1):169–177, 1983.
- [22] C. Foias, G. R. Sell, and R. Temam. Inertial manifolds for nonlinear evolutionary equations. *J. Differential Equations*, 73(2):309–353, 1988.
- [23] C. Foias, G. R. Sell, and E. S. Titi. Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations. *J. Dynamics Differential Equations*, 1(2):199–244, 1989.
- [24] C. Foias and R. Temam. Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(1):28–63, 1978.
- [25] B. García-Archilla and J. de Frutos. Time integration of the non-linear Galerkin method. *IMA J. Numer. Anal.*, 15(2):221–244, 1995.
- [26] B. García-Archilla, J. Novo, and E.S. Titi. Postprocessing the Galerkin method: a novel approach to approximate inertial manifolds. *SIAM J. Numer. Anal.*, 35(3):941–972 (electronic), 1998.
- [27] J. Goodman. Stability of Kuramoto-Sivashinsky and related systems. *Comm. Pure Appl. Math.*, 47:293–306, 1994.
- [28] M. Graham, P. Steen, and E.S. Titi. Computational efficiency and approximate inertial manifolds for a Bénard convection system. *J. Nonlin. Sci.*, 3:153–167, 1993.
- [29] J. Guckenheimer and P. Worfolk. Dynamical systems: some computational problems. In *Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992)*, volume 408 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–277. Kluwer Acad. Publ., Dordrecht, 1993.
- [30] J. K. Hale. *Asymptotic behavior of dissipative systems*, volume 25 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

- [31] J. K. Hale and C. Perelló. The neighborhood of a singular point of functional differential equations. *Contributions to Differential Equations*, 3:351–375, 1964.
- [32] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.
- [33] J. M. Hyman and B. Nicolaenko. The Kuramoto-Sivashinsky equation: a bridge between PDEs and dynamical systems. *Phys. D*, 18(1-3):113–126, 1986. Solitons and coherent structures (Santa Barbara, Calif., 1985).
- [34] J. M. Hyman, B. Nicolaenko, and S. Zaleski. Order and complexity in the Kuramoto-Sivashinsky model of weakly turbulent interfaces. *Phys. D*, 23(1-3):265–292, 1986. Spatio-temporal coherence and chaos in physical systems (Los Alamos, N.M., 1986).
- [35] Ju. S. Ilyashenko. Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation. *J. Dynamics Differential Equations*, 4:585–615, 1992.
- [36] V. I. Istratescu. *Fixed point theory*. D. Reidel Publishing Co., Dordrecht, 1981. An introduction, With a preface by Michiel Hazewinkel.
- [37] M. E. Johnson, M. S. Jolly, and I. G. Kevrekidis. Two-dimensional invariant manifolds and global bifurcations: some approximation and visualization studies. *Numer. Algorithms*, 14(1-3):125–140, 1997. Dynamical numerical analysis (Atlanta, GA, 1995).
- [38] M. S. Jolly, I. G. Kevrekidis, and E. S. Titi. Approximate inertial manifolds for the Kuramoto-Sivashinsky equation: analysis and computations. *Phys. D*, 44(1-2):38–60, 1990.
- [39] M. S. Jolly, I. G. Kevrekidis, and E. S. Titi. Preserving dissipation in approximate inertial forms for the Kuramoto-Sivashinsky equation. *J. Dynamics Differential Equations*, 3(2):179–197, 1991.
- [40] M. S. Jolly and R. Rosa. Computations on center manifolds (in preparation). 1998.
- [41] M. S. Jolly, R. Rosa, and R. Temam. Evaluating the dimension of an inertial manifold for the Kuramoto-Sivashinsky equation (in preparation). 1998.
- [42] I. G. Kevrekidis, B. Nicolaenko, and J. C. Scovel. Back in the saddle again: a computer assisted study of the Kuramoto-Sivashinsky equation. *SIAM J. Appl. Math.*, 50(3):760–790, 1990.
- [43] J. Leis and M. Kramer. The simultaneous solution and sensitivity analysis of systems described by ordinary differential equations. *ACM Trans. Math. Software*, 14(1):45–60, 1988.
- [44] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
- [45] X. Liu. Gevrey class regularity and approximate inertial manifolds for the Kuramoto-Sivashinsky equation. *Physica D*, 50:135–151, 1991.
- [46] F. Ma and T. Küpper. A numerical method to calculate center manifolds of ODE's. *Appl. Anal.*, 54(1-2):1–15, 1994.
- [47] R. Mañé. Reduction of semilinear parabolic equations to finite dimensional C^1 flows. pages 361–378. *Lecture Notes in Math.*, Vol. 597, 1977.
- [48] J. Mallet-Paret and G. R. Sell. Inertial manifolds for reaction diffusion equations in higher space dimensions. *J. Amer. Math. Soc.*, 1(4):805–866, 1988.
- [49] M. Marion. Approximate inertial manifolds for reaction diffusion equations in high space dimension. *J. Dynamics and Diff. Eqs.*, 1:245–267, 1989.
- [50] M. S. Jolly M. E. Johnson and I. G. Kevrekidis. In preparation. 1998.
- [51] D. Michelson. Stability of the Bunsen flame profiles in the Kuramoto-Sivashinsky equation. *SIAM J. Math. Anal.*, 27(3):765–781, 1996.
- [52] M. Miklavčič. A sharp condition for existence of an inertial manifold. *J. Dynamics Differential Equations*, 3(3):437–456, 1991.
- [53] B. Nicolaenko, B. Scheurer, and R. Temam. Some global dynamical properties of the Kuramoto-Sivashinsky equations: nonlinear stability and attractors. *Phys. D*, 16(2):155–183, 1985.
- [54] B. Nicolaenko, B. Scheurer, and R. Temam. Some global dynamical properties of a class of pattern formation equations. *Comm. Partial Differential Equations*, 14(2):245–297, 1989.
- [55] F. Pinto. Nonlinear stability and dynamical properties for a Kuramoto-Sivashinsky equation in space dimension two. *Discrete and Continuous Dynamical Systems*, 1998.
- [56] Y. Pomeau and P. Manneville. Stability and fluctuations of spatially periodic flow. *Physique Lett.*, 40:609–612, 1979.
- [57] J. Robinson. Computing inertial manifolds. *SIAM J. Num. Anal.*
- [58] J. Robinson. Inertial manifolds with and without delay. *Discrete Contin. Dynam. Systems*.
- [59] R. Rosa. Approximate inertial manifolds of exponential order. *Discrete and Continuous Dynamical Systems*, 1:421–448, 1995.
- [60] R. Rosa and R. Temam. Inertial manifolds and normal hyperbolicity. *ACTA Applicandae Mathematicae*, 45:1–50, 1996.
- [61] R. D. Russell, D. M. Sloan, and M. R. Trummer. On the structure of Jacobians for spectral methods for nonlinear partial differential equations. *SIAM J. Sci. Statist. Comput.*, 13(2):541–549, 1992.

- [62] R. D. Russell, D. M. Sloan, and M. R. Trummer. Some numerical aspects of computing inertial manifolds. *SIAM J. Sci. Comput.*, 14(1):19–43, 1993.
- [63] R. Temam. Induced trajectories and approximate inertial manifolds. *RAIRO Modél. Math. Anal. Numér.*, 23(3):541–561, 1989. Attractors, inertial manifolds and their approximation (Marseille-Luminy, 1987).
- [64] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, volume 68 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1997.
- [65] R. Temam and X. Wang. Estimates on the lowest dimension of inertial manifolds for the Kuramoto-Sivashinsky equation in the general case. *Differential Integral Equations*, 7(3-4):1095–1108, 1994.
- [66] E.S. Titi. On approximate inertial manifolds to the Navier-Stokes equations. *J. Math. Anal. Appl.*, 149(2):540–557, 1990.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405 USA
E-mail address: `msjolly@indiana.edu`

INSTITUTO DE MATEMATICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, RIO DE JANEIRO, RJ 21945-970 BRAZIL
E-mail address: `rrosa@labma.ufrj.br`

LABORATOIRE D'ANALYSE NUMÉRIQUE, UNIVERSITÉ PARIS SUD, ORSAY 91405 FRANCE, AND THE INSTITUTE FOR SCIENTIFIC COMPUTING AND APPLIED MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405 USA
E-mail address: `temam@indiana.edu`