# SMOOTH LINEARIZATION NEAR A FIXED POINT

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| 8 | Hans Weinberger | Optimal Numerical Approximation of a Linear Operator |
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| 10 | D. Fennell Evans<br>D.J. Mitchell<br>Suragit Mukherjee<br>Barry Ninham | Surfactant Diffusion; New Results and Interpretations |
| 11 | Leif Arkeryd | A Remark about the Final Aperiodic Regime for Maps on the Interval |
| 12 | Luis Magalhaes | Manifolds of Global Solutions of Functional Differential Equations |
| 13 | Kenneth Meyer | Tori in Resonance |
| 14 | C. Eugene Wayne | Surface Models with Nonlocal Potentials: Upper Bounds |
| 15 | K.A. Pericak-Spector | On Stability and Uniqueness of Fluid Flows through a Rigid Porous Medium |

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## Table of Contents

Abstract. In this paper we extend a theorem of Sternberg and Bileckii. We study a vector field, or a diffeomorphism, in the vicinity of a hyperbolic fixed point. We assume that the eigenvalues of the linear part A (at the fixed point) satisfy $Q^{th}$ order algebraic inequalities, where $Q > 2$, then there is $C^K$-linearization in the vicinity of the fixed point, where $K$ is the Q-smoothness of $A$. We give an explicit and simple algorithm for computing $K$. An illustrative example from celestial mechanics is included which shows that our main theorem is an improvement over previously known theories. We also show that if the vector field or diffeomorphism depends smoothly on a parameter, then the linearizing conjugation depends smoothly on the parameter.

## I. Introduction

The theory of linearization in dynamical systems represents one of the fundamental problems that arises in the study of ordinary differential equations. In this theory one begins with a nonlinear vector field defined near a fixed point, and one seeks sufficient conditions for the existence of a smooth curvilinear coordinate system with the property that the vector field is linear when written in terms of the new coordinate system. Given such a linearization theory, a natural question then is to determine the smoothness of the new curvilinear coordinate system. Also, if the new coordinate system is lacking in smoothness, one may try to determine the obstacles to smooth linearization. In this paper we will study these problems. We will show that satisfactory answers can be given, and that these answers are significantly better than previously known results.

In order to be more precise, let $X$ be a finite dimensional Banach space with norm $|\cdot|$. Let $U$ be an open set in $X$ and let $M \geq 1$. Consider two $C^M$-vector fields $f$ and $g$ on $U$ with a common fixed point $x_0 = 0$, i.e., $f(0) = g(0) = 0$. Let $N$ be a positive integer with $N \leq M$. We shall say that $f$ and $g$ are $C^N$-<u>conjugate near</u> $x = 0$ if there are neighborhoods $V_1$, $V_2$ with $0 \in V_i \subseteq U$ $(i = 1,2)$ and a $C^N$-diffeomorphism $H: V_1 \to V_2$ satisfying the following two properties: (i) $H(0) = 0$. (ii) Whenever $x(t)$ is a solution of $x' = f(x)$ with $x(t) \in V_1$ for $t$ in some interval $I$, then $y(t) = H(x(t))$ is a solution of $y' = g(y)$ for $t \in I$. Similarly, whenever $y(t)$ is a solution of $y' = g(y)$ with $y(t) \in V_2$ for $t \in I$, then $x(t) = H^{-1}(y(t))$ is a solution of $x' = f(x)$ for $t \in I$.

The mapping $y = H(x)$ above is referred to as a $C^N$-<u>conjugation</u> between $x' = f(x)$ and $y' = g(y)$. This definition extends to topological conjugacies (i.e., $N = 0$) by simply requiring $H$ to be a homeomorphism that satisfies (i) and (ii). For $N \geq 1$, statement (ii) above can be replaced by

$$g(H(x)) = DH(x)f(x) , \qquad x \in V_1$$

$$f(H^{-1}(y)) = DH^{-1}(y)g(y) , \qquad y \in V_2$$

where $DH$ and $DH^{-1}$ denote the respective Jacobian matrices.

In this paper we will study the question of a $C^N$-conjugation between

$$x' = Ax + F(x) = f(x) \tag{1.1}$$

and the associated linear equation

$$y' = Ay \tag{1.2}$$

where $A = Df(0)$ and $D^P F(0) = 0$ for $P = 0,1$. If Eq. (1.1) and Eq. (1.2) are $C^N$-conjugate near $x = 0$, then we shall say that $f$ <u>admits a $C^N$-linearization</u>.

Let $A$ be an $(n \times n)$ matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$ repeated with multiplicities and let $\sum(A) = \{\lambda_1,\ldots,\lambda_n\}$ denote the spectrum of $A$. Let $m = (m_1,\ldots,m_n)$ be a vector with nonnegative integer entries $m_1,\ldots,m_n$, and define $\gamma(\lambda,m)$ by

$$\gamma(\lambda,m) = \lambda - (m_1\lambda_1 + \ldots + m_n\lambda_n) ,$$

where $\lambda$ is a complex number. Let $|m| = m_1 + \ldots + m_n$.

We shall say that $A$ is <u>hyperbolic</u> if $\mathrm{Re}\lambda \neq 0$ for all $\lambda \in \sum(A)$. $A$ is said to be <u>stable</u> if $\mathrm{Re}\lambda < 0$ for all $\lambda \in \sum(A)$. $A$ is said to satisfy the <u>Sternberg condition of order</u> $N$, $N \geq 2$, if $\gamma(\lambda,m) \neq 0$ for all $\lambda \in \sum(A)$ and all $m$ satisfying $2 \leq |m| \leq N$. We shall say that $A$ satisfies the <u>strong</u> Sternberg condition of order $N$, if $A$ satisfies the Sternberg condition of order $N$ and

$$\mathrm{Re}\ \gamma(\lambda,m) \neq 0 \tag{1.3}$$

for all $\lambda \in \sum(A)$ and all $m$ with $|m| = N$. It is easy to see that if $A$ satisfies the strong Sternberg condition of order $N \geq 2$, then $A$ is hyperbolic.

Let $A$ be hyperbolic and let $\Sigma^+(A)$ or $\Sigma^-(A)$ denote, respectively, those eigenvalues $\lambda \in \Sigma(A)$ with $\mathrm{Re}\ \lambda > 0$ or $\mathrm{Re}\ \lambda < 0$. We shall say that $A$ is <u>strictly hyperbolic</u> if $A$ is hyperbolic and both $\Sigma^+(A)$ and $\Sigma^-(A)$ are nonempty. If $A$ is hyperbolic and $\Sigma^i(A) \neq \phi$, we define the <u>spectral spread</u> $\rho^i$ by

$$\rho^i = \frac{\max\{\ |\mathrm{Re}\ \lambda| : \lambda \in \Sigma^i(A)\ \}}{\min\{\ |\mathrm{Re}\ \lambda| : \lambda \in \Sigma^i(A)\ \}}$$

where $i = +$ or $-$.

Let $Q$ be a postive integer and let $A$ be hyperbolic. We define the <u>Q-smoothness</u> of $A$ to be the largest integer $K \geq 0$ such that:

(1) $Q - K\rho^- \geq 0$, if $\Sigma^+(A) = \phi$.

(2) $Q - K\rho^+ \geq 0$, if $\Sigma^-(A) = \phi$.

(3) There exist positive integers $M,N$ with $Q = M + N$, $M - K\rho^+ \geq 0$, $N - K\rho^- \geq 0$, when $A$ is strictly hyperbolic.

Since the spectral spreads are $> 1$, we see that the Q-smoothness of $A$ satisfies $K < \min(M,N)$ when $A$ is strictly hyperbolic.

The object of this paper is to prove the following two theorems concerning $C^K$-linearizations:

*Theorem 1. Let* $Q > 2$ *be an integer, and assume that* $F$ *is of class* $C^{3Q}$ *on* $U \subset X$ *with* $0 \in U$ *where* $D^P F(0) = 0$ *for* $P = 0,1$. *Let* $A$ *be strictly hyperbolic, and consider one of the following two assumptions:*

(A) *Assume that* $A$ *satisfies the strong Sternberg condition of order* $Q$.

(B) *Assume that* $D^P F(0) = 0$ *for* $0 < P < Q - 1$ *and that*

$$\mathrm{Re}\ \gamma(\lambda,m) \neq 0$$

*for all* $\lambda \in \sum(A)$ *and all* $m$ *with* $|m| = Q$.

*Under either assumption* (A) *or* (B), Eq. (1.1) *admits a* $C^K$*-linearization, where* $K$ *is the* Q-*smoothness of* $A$.

If $A$ is stable, then one can weaken the assumption on the smoothness of $F$. In particular we will prove the following result:

*Theorem 2. If* $A$ *is stable, then Theorem* 1 *remains valid when* $F$ *is of class* $C^{2Q}$.

*Remarks* 1. Sternberg (1957, 1958) studies the question of finding sufficient conditions that Eq. (1.1) admits a $C^s$-linearization. He showed that there is a function $V(s,\lambda_1,\ldots,\lambda_n) > 0$ with the property that if $A$ is hyperbolic and satisfies the Sternberg condition of order $N$ where $N > s + V$, then Eq (1.1) admits a $C^s$-linearization. While there are several alternate proofs of Sternberg's Theorem [cf. Chen (1963), Hartman (1964), Nelson (1969), Pugh (19**) and Takens (1971)], the implicit formulae of V are very complicated. See Hartman (1964, p. 257), for example. Our theorems assert that under the stronger assumption that (1.3) is valid, we can give sharper and similar estimates on the order of smoothness of the conjugation to the linear system. For the $C^0$-linearization theory see Grobman (1959, 1962), Hartman (1960ab, 1963, 1964) and Palmer (1980).

2. Our methods extend easily to the question of smooth linearization for diffeomorphisms in the vicinity of a fixed point. This extension, together with an associated application to the linearization of a vector field near a periodic orbit, is given in Section VII. Our theory for the hyperbolic case where $K = 1$ is similar to but not as strong as a theorem of Bileckii (1973, 1978). On the other hand for $K \geq 2$ we show that our results are stronger. (See Section IX. Also compare with Ise and Nagumo (1957).)

3. Because of Bileckii's Theorem for the $C^1$-linearization, we do not feel that our statement on the order of smoothness of the linearizing conjugation is the best possible estimate. However, for the strictly hyperbolic case, it is shown in Sell (1983a) that if A satisfies the strong Sternberg condition of order Q, where $Q = 2N$ or $2N + 1$, then one cannot expect the linearizing conjugation to be smoother than class $C^N$.

4. The assumptions about the smoothness of the vector field F in Theorems 1 and 2 have not been optimized. In the stable case, our methods suffice when F is of class $C^{Q + K}$, and in the strictly hyperbolic case one needs F to be of class $C^{Q + H + K}$, where $H = \max(M,N)$ and K is the Q-smoothness of A.

The assumption that $\gamma(\lambda,m) \neq 0$ for $2 \leq |m| \leq Q$ allows one to introduce a polynomial change of variables to eliminate the terms in Taylor series expansion of f with order between 2 and Q . (This implies that assumption (A) is stronger than assumption (B) in the above theorems.) As we shall see, the stronger assumption that $\text{Re}\,\gamma(\lambda,m) \neq 0$ for $|m| = Q$ allows us to eliminate the remainder term in the Taylor series expansion of $f$.

The argument which we present is based on the theory of nonlinear perturbations of linear equations with exponential dichotomies, cf. Coppel (1965). The change of variables we introduce gives rise to a related nonlinear differential equation on a different finite dimensional Banach space. The quantities $\gamma(\lambda,m)$, for $\lambda \in \sum(A)$ and $|m| = Q$, will arise as the eigenvalues of the associated linear equation, and Ineq. (1.3) will ensure that this linear equation has an exponential dichotomy.

In the next section we will present a heuristic proof of these theorems for the case $K = 1$. This will include a "derivation" of the associated nonlinear equation along with a brief review of the Taylor series expansion theory. The primary purpose of this section is to motivate the line of argumentation developed in Sections III and IV.

In Section V we will study the question of the smoothness of bounded solutions of small (nonlinear) perturbations of linear equations that have exponential dichotomies. The proof of Theorems 1 and 2 will be given in Section VI.

Our approach to the linearization problem is to show that the linearizing conjugation is a fixed point of a suitable integral operator. In this context we also show that the linearizing conjugtion varies smoothly in terms of a parameter when the coefficients of the vector field are smooth functions of the parameter. See Section VIII.

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II. Heurisitc Proof. Taylor Series Expansions.

In this section we shall present a heuristic proof of our theorems for the case $K = 1$. Let us begin with a brief review of Taylor's Theorem, cf. Dieudonne (1960) and Abraham and Robbin (1967).

Let $F:V_1 \to V_2$ be a smooth function where $V_1$ and $V_2$ are open sets in $X$. For $a \in V_1$ let $DF(a)$ denote the linear part of $F$ at $x = a$, i.e., $DF(a)$ is a linear map of $X$ into itself. For integers $P = 2,3,\ldots$ let $D^P F(a)$ denote the higher derivatives of $F$ at $x = a$. Thus $D^P F(a) \in L_P^s$, the space of multi-linear (P-linear) symmetric mappings of $X \times \ldots \times X$ (P-copies) into $X$, for $P = 2,3,\ldots$ .

The Taylor series expansion of $F$ near $a \in V_1$ is given by

$$F(x + a) = F_0(a) + F_1(a)x + F_2(a)\langle x\rangle^2 + F_3(a)\langle x\rangle^3 + \ldots \quad .$$

One then has $F_0(a) = F(a)$, $F_1(a) = DF(a)$ and $F_k(a) = (k!)^{-1}D^k F(a) \in L_k^s$ for $k = 2,3,\ldots$ . Also $\langle x\rangle^k$ denotes $x\ldots x$ (k-copies), and

$$F_2(a)\langle x\rangle^2 = F_2(a)\ \langle x,x\rangle$$

represents the value of $F_2(a)$ at the point $(x,x) \in X \times X$, with $F_k(a)\langle x\rangle^k$ defined similarly. It is convenient to define $L_0^s = X$ and $L_1^s = L(X,X)$, where the latter denotes the space of linear mappings of $X$ into $X$.

Recall that a function $F:V_1 \to V_2$ is of class $C^N$ on $V_1$, where $V_1$ is an open set in $X$, if and only if for every $a \in V_1$ and $x$ sufficiently small one has

$$F(x + a) = F_0 + F_1 x + \dots + F_N\langle x\rangle^N + F_R \tag{2.1}$$

where $F_k \in L_k^s$ , $0 < k < N$ , is independent of $x$, continuous in a, and $F_R = F_R(x,a)$ satisfies

$$\lim_{x \to 0} |x|^{-N} F_R = 0 \tag{2.2}$$

cf. Glaeser (1958) and Abraham and Robbin (1967). Furthermore, if $F \in C^{N+1}$ , then the remainder term $F_R$ in Eq. (2.1) can be written in the form $F_R(x,a) = F_R^o\langle x\rangle^{N+1}$ where $F_R^o \in L^s{}_{N+1}$ is given by

$$F_R^o = \frac{1}{N!} \int_0^1 (1 - t)^N D^{N+1} F(tx + a)dt \ , \tag{2.3}$$

cf. Dieudonne (1969).

Let us now turn to the linearization problem for $K = 1$ . As a first step, consider a change of variables of the form

$$x = u + S\langle u\rangle^2 = u + S\langle u,u\rangle \tag{2.4}$$

where $u \in X$ and $S \in L_2^s$ . Next we will replace $u$ with $u = e^{tA}u_0$ (a solution of Eq. (1.2)) and $S$ by a time-varying function $S(t)$ so that the resulting function $x = x(t)$ given by Eq. (2.4) is a solution of Eq. (1.1). This implies, as we now show, that $S = S(t)$ must satisfy a related differential equation on $L_2^s$ . By differentiating Eq. (2.4) with respect to $t$ we get

$$\begin{aligned} x' = u' + \frac{d}{dt}(S\langle u\rangle^2) &= Ax + F(x) \\ &= Au + AS\langle u\rangle^2 + F(u + S\langle u\rangle^2) \end{aligned}$$

or equivalently

$$\frac{d}{dt}(S\langle u\rangle^2) = AS\langle u\rangle^2 + F(u + S\langle u\rangle^2) \; . \tag{2.5}$$

The left side of Eq. (2.5) is

$$\frac{d}{dt}(S\langle u\rangle^2) = \frac{d}{dt}(S\langle u,u\rangle) = S'\langle u,u\rangle + S\langle u',u\rangle + S\langle u,u'\rangle$$

$$= S'\langle u\rangle^2 + \{S,A\}\langle u\rangle^2 \; ,$$

where $\{S,A\} \varepsilon L_2^S$ is defined by

$$\{S,A\}\langle u,v\rangle = S\langle Au,v\rangle + S\langle u,Av\rangle \; .$$

Assume, for the moment, that there is a continuous function of $(u,S)$, $G(u,S) \varepsilon L_2^S$, with the property that

$$G(u,S)\langle u\rangle^2 = F(u + S\langle u\rangle^2) \; . \tag{2.6}$$

From Eq. (2.5) we then get the following differential equation on $L_2^S$,

$$S' = LS + G(u,S) \tag{2.7}$$

where $LS = AS - \{S,A\}$ .

Conversely, given any solution $S(t)$ of Eq. (2.7), one can easily verify that the function

$$x(t) = e^{At}u_0 + S(t)\langle e^{At}u_0, e^{At}u_0\rangle$$

is a solution of Eq. (1.1).

The important feature underlying this approach is the observation that the spectrum of the linear operator $L$ in Eq. (2.7) is

$$\textstyle\sum(L) = \{\gamma(\lambda,m) : \lambda \; \varepsilon \; \sum(A) \;\; \text{and} \;\; |m| = 2\} \; .$$

(See Lemma 12 below.) We see then that if $A$ satisfies the strong Sternberg condition of order 2, then the linear equation $S' = LS$ admits an exponential dichotomy. Next assume that the function $G(u,S)$ is bounded and Lipschitz-continuous in $S$ with a sufficiently small Lipschitz constant. Then there is a unique bounded continuous function $S(u)$ with the property that for every $u_0 \in X$ the function $S(e^{At}u_0)$ is a solution of Eq. (2.7), see Section V. Consequently the change of variables

$$x = H(u) = u + S(u)\langle u,u\rangle \tag{2.8}$$

is a $C^0$-conjugation between Eq. (1.1) and Eq. (1.2) near $u = 0$. Of course, we want to construct a $C^1$-conjugation. However because of the quadratic terms in (2.8), $H(u)$ can be smooth even when the continuous function $S(u)$ is nondifferentiable at $u = 0$.

Our proofs of Theorems 1 and 2 build on this general approach. In the strictly hyperbolic case the actual change of variables we use (even for $K = 1$) will be a variation on the theme described above. The modification is needed in order to show that the final change of variables $x = H(u)$ is smooth in a full neighborhood of $u = 0$. The key to implementing this approach consists of two lemmas which are developed in Section IV.

It may be helpful to expand a bit on the notation used for operators $S \in L_Q^S$, where $Q \geq 2$. As noted above, $S$ is a Q-linear symmetric mapping of $X \times \dots \times X$ (Q-copies) into $X$. We shall let

$$S\langle x^1,\dots,x^Q\rangle \tag{2.9}$$

denote the value of $S$ at the Q-tuple $\{x^1,\dots,x^Q\}$, where $x^i \in X$,

$1 \leq i \leq Q$ . If $x^i = x$ for $1 \leq i \leq Q$ , then we shall abbreviate (2.9) and write

$$S\langle x\rangle^Q = S\langle x,\dots,x\rangle \, . \tag{2.10}$$

Likewise if $x^i = u$ for $1 \leq i \leq Q - j$ and $x^i = v$ for $Q - j + 1 \leq i \leq Q$ then we shall write (2.10) as

$$S\langle u\rangle^{Q-j}\langle v\rangle^j = S\langle u,\dots,u,v,\dots,v\rangle \tag{2.11}$$

where $(Q - j)$ copies of u and j copies of v appear on the right side of (2.11). The expression $S\langle u\rangle^a \langle v\rangle^b \langle h\rangle^c$ for integers $a,b,c \geq 0$ is defined similarly.

Now assume that $X = U + V$ , where U and V are disjoint linear subspaces of X , and let $\{e_1,\dots,e_n\}$ be a basis for X with the property that $\{e_1,\dots,e_a\}$ is a basis for U and $\{e_{a+1},\dots,e_n\}$ is a basis for V . Any vector $x \in X$ can be written uniquely as $x = \sum_{i=1}^n x_i e_i$ , and we can identify x with the n-tuple of scalars $(x_1,\dots,x_n)$ .

Let $\alpha = (\alpha_1,\dots,\alpha_n)$ be an n-tuple of nonnegative integers and define $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$ . If $x = \sum_{i=1}^n x_i e_i$ we define the product $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ , and adopt the convention that $0^0 = 1$ . One then has

$$S\langle x\rangle^Q = \sum_{|\alpha|=Q} d(Q,\alpha)\, x^\alpha\, S_\alpha \tag{2.12}$$

where $d(Q,\alpha) = Q!/\alpha!$ is the multinomial coefficient,

$$S_\alpha = Se^\alpha = S\langle e_1,\dots,e_1, \dots ,e_n,\dots,e_n\rangle \, ,$$

and $e^\alpha$ denotes $\alpha_1$ copies of $e_1$, $\alpha_2$ copies of $e_2$ , ... ,$\alpha_n$ copies of $e_n$ occuring in the bracket $\langle\dots\rangle$ .

A similar decomposition is possible for (2.11) when $u \in U$ and $v \in V$. First we note that $u^\alpha = 0$ whenever there is an $i > a + 1$ with $\alpha_i > 1$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple define

$$\alpha(U) = (\alpha_1, \ldots, \alpha_a, 0, \ldots, 0)$$
$$\alpha(V) = (0, \ldots, 0, \alpha_{a+1}, \ldots, \alpha_n) .$$

Thus $\alpha = \alpha(U) + \alpha(V)$. Equation (2.11) then yields

$$S\langle u\rangle^{Q-j}\langle v\rangle^{j} = \sum d(Q - j, \alpha(U))d(j, \alpha(V))u^{\alpha(U)}v^{\alpha(V)}S_\alpha \qquad (2.13)$$

where the summation in (2.13) is over all n-tuples $\alpha$ with $|\alpha(U)| = Q - j$ and $|\alpha(V)| = j$.

## III. Preparatory Analysis

The proof of Theorem 2 (where $A$ is stable) is a variation of the argument used for Theorem 1. Therefore we will concentrate at first on the case where the matrix $A$ in

$$x' = Ax + F(x) \quad , \quad x \in X \; , \tag{1.1}$$

is strictly hyperbolic. Let $U$ and $V$ denote, respectively, the unstable and stable subspaces for the linear equation

$$w' = Aw \; . \tag{1.2}$$

By making a preliminary linear change of variables, if necessary, we can assume that the complementary spaces $U$ and $V$ are orthogonal with respect to some preassigned inner product on $X$. Thus $X = U + V$ and if $w = u + v$, where $u \in V$ and $v \in V$, then one has $|w|^2 = |u|^2 + |v|^2$. Also if $w = u + v$, then there is a $\theta \in R$ with the property that $|u| = |\cos \theta| \; |w|$ and $|v| = |\sin \theta| \; |w|$.

It will be convenient to introduce the following notation for a continuous function $f: X \to Y$, where $X$ and $Y$ are two Banach spaces:

$$\operatorname{Supp} f = \text{closure } \{x \in X : f(x) \neq 0\}$$

$$\|f\|_\infty = \sup\{|f(x)| : \; x \in X\}$$

$$|f|_\infty = \sup\{|f(x)| : \; |x| \le 1/5\}$$

$$|f|_{r,\infty} = \sup\{|f(x)| : \; |x| \le r\} \; ,$$

where $r > 0$. The last two norms will also be used for continuous functions defined in a suitable neighborhood of the origin in $X$.

Next we turn our attention to the nonlinear term $F(x)$ in Eq. (1.1). We assume that $F$ is of class $C^M$ on $X$, where $M \geq 2$.
Define $B$ by

$$B = \max\{|F|_\infty, |DF|_\infty, \dots, |D^M F|_\infty\} . \tag{3.1}$$

We now make two alterations on Eq. (1.1). The first is a rescaling of the variables, where x is replaced by $\varepsilon x$ and $\varepsilon$ is a positive real parameter. The second alteration is a truncation of $F$ so that

$$\operatorname{Supp} F \subseteq \{x \in X: \ |x| < r\} , \tag{3.2}$$

where $r$ will be chosen later and will satisfy $r \leq 1/5$. Since we will want to have control over the choice of the parameters $\varepsilon$ and $r$, it will be important to keep track of how these alterations affect the nonlinear function $F$.

For $0 < \varepsilon \leq 1$ we define

$$F^0(x,\varepsilon) = \varepsilon^{-1} F(\varepsilon x) .$$

Then we replace the nonlinear term $F(x)$ in Eq. (1.1) with $F^0(x,\varepsilon)$ to get

$$x' = Ax + F^0(x,\varepsilon) . \tag{1.$\varepsilon$}$$

For $0 < \varepsilon \leq 1$, Eq. (1.1) and Eq. (1.$\varepsilon$) are equivalent since one can change from (1.1) to (1.$\varepsilon$) by replacing $x$ with $\varepsilon x$.

By the chain rule one has

$$D^k F^0(x,\varepsilon) = \varepsilon^{k-1} D^k F(\varepsilon x) ,$$

where $D^k$ denotes the k-th derivative with respect to $x$. We can now prove the following result:

<u>Lemma 1. Let</u> $F$ <u>be of class</u> $C^M$ <u>where</u> $M \geq 4$ .

(A) <u>Then for</u> $0 < \varepsilon \leq 1$ <u>and</u> $0 \leq K \leq M$ <u>one has</u>

$$|D^K F^0|_\infty \leq \varepsilon^{K-1}|D^K F|_\infty \leq \varepsilon^{K-1} B \quad ,$$

<u>where</u> $B$ <u>is given by</u> (3.1).

(B) <u>Furthermore if</u> $F$ <u>satisfies</u> $D^P F(0) = 0$ <u>for</u> $0 \leq P \leq Q$ , <u>where</u> $2 \leq Q \leq M-1$ , <u>then one has</u>

$$|D^K F^0|_{r,\infty} \leq \varepsilon^Q \, B \, r^{Q+1-K} \tag{3.3}$$

<u>for</u> $0 \leq K \leq Q + 1$ , $0 < \varepsilon \leq 1$ <u>and</u> $r \leq 1/5$ .

<u>Proof</u>: Part (A) follows from the chain rule formula cited above. In order to prove (3.3) we claim that for $0 \leq K \leq Q + 1$ and $|x| \leq 1/5$ one has

$$|D^K F(x)| \leq \frac{1}{(Q+1-K)!} \, |D^{Q+1}F|_\infty |x|^{Q+1-K} \leq B|x|^{Q+1-K} \; . \tag{3.4}$$

Inequality (3.4) is established by induction and by noting that for $0 \leq K \leq Q$ one has

$$D^K F(x) = \int_0^1 \frac{d}{dt} \; D^K F(tx)dt = \int_0^1 D^{K+1}F(tx)x \; dt$$

since $D^K F(0) = 0$ . As a result of (3.4), one has

$$|D^K F(\varepsilon x)| \leq \varepsilon^{Q+1-K} \, B|x|^{Q+1-K} \; ,$$

for $|x| \leq 1/5$ . Since $r \leq 1/5$ , Ineq. (3.3) follows from Part (A). Q.E.D.

In order to restrict the Supp F so that (3.2) is satisfied we will replace $F^0$ by

$$F_r(x,\varepsilon) = \alpha(|x|)F^0(x,\varepsilon) \tag{3.5}$$

where $\alpha$ is the scalar-valued function that is defined by Eq. (3.6) below and which satisfies

$$\text{Supp } \alpha \subseteq \{x \in X: \ |x| < r\} .$$

Define $p:R \to R$ by

$$p(t) = \begin{cases} 1 & , \ t \leq 0 \\ 1 - c \int_0^t s^M(s-1)^M ds & , \ 0 < t < 1 \\ 0 & , \ t \geq 1 \end{cases}$$

where $c^{-1} = \int_0^1 s^M(s-1)^M ds$ . Next define $\alpha : X \to R$ by

$$\alpha(|x|) = \alpha_r(|x|) = p(2r^{-1}|x| - 1) , \tag{3.6}$$

for $|x| \geq 0$ . Then one has $\alpha(|x|) = 1$ for $|x| \leq r/2$ and $\alpha(|x|) = 0$ for $|x| \geq r$ . Consequently the differential equation

$$x' = Ax + F_r(x,\varepsilon) \tag{1.$\varepsilon$,r}$$

agrees with (1.$\varepsilon$) for $|x| \leq r/2$ and $F_r$ satisfies (3.2). Furthermore if F is of class $C^M$ , then $F_r$ is of class $C^M$ in the x-variable. Also for $0 \leq P \leq M$ one has

$$D^P F_r(x,\varepsilon) = \sum_{k=0}^{P} \binom{P}{k} D^{P-k}\alpha \ D^k F^0(x,\varepsilon) , \tag{3.7}$$

Dieudonne (1969). In addition for $0 \leq k \leq M$ one has

$$\|D^k \alpha\|_\infty \leq 2^k r^{-k} \|D^k p\|_\infty . \tag{3.8}$$

Let us summarize the above in the following lemma.

Lemma 2. *Let* $F_r$ *be given by* Eq. (3.5) *where* $F$ *is of class* $C^M$, $M \geq 4$, *and* $D^P F(0) = 0$ *for* $0 \leq P \leq Q$ *with* $2 \leq Q \leq M-1$. *Then for* $0 \leq K \leq M$ *and* $r \leq 1/5$ *one has*

$$|D^K F_r|_{r,\infty} \leq 2\, B\, 3^K\, \rho\, \varepsilon^Q\, r^{-K+Q+1} ,$$

*where* $B$ *is given by* (3.1), $\rho = \max\{\|D^P p\|_\infty : \ 0 \leq P \leq M\}$ *and* $p$ *is the mapping defined before* Eq. (3.6).

Proof. The proof of this lemma is based on Eq. (3.7) and (3.8) as well as Lemma 1. As a consequence of these we have the following for $r \leq 1/5$ and $0 \leq K \leq Q + 1$:

$$
\begin{aligned}
|D^K F_r|_{r,\infty} &\leq \textstyle\sum_{k=0}^{K} \binom{K}{k} \|D^{K-k}\alpha\|_\infty\, |D^k F^0|_{r,\infty} \\
&\leq \textstyle\sum_{k=0}^{K} \binom{K}{k}\, 2^{K-k}\, r^{-K+k}\, \rho\, \varepsilon^Q\, B\, r^{Q+1-k} \\
&= B\, 3^K \rho\, \varepsilon^Q\, r^{-K+Q+1}
\end{aligned}
$$

since $\sum_{k=0}^{K} \binom{K}{k} 2^{K-k} = 3^K$ by the Binomial Theorem. For $Q + 2 \leq K \leq M$ we break the summation in the first line above into two sums and apply Lemma 1 with (3.8) to obtain

$$|D^K F_r|_{r,\infty} \leq \sum_{k=0}^{Q+1} \binom{K}{k} 2^{K-k} r^{-K+k} \rho \varepsilon^Q B r^{Q+1-k}$$

$$+ \sum_{k=Q+2}^{K} \binom{K}{k} 2^{K-k} r^{-K+k} \rho \varepsilon^{k-1} B$$

$$\leq B 3^K \rho \varepsilon^Q r^{-K+Q+1} + B 3^K \rho r^{-K+Q+2} \varepsilon^{Q+1}$$

$$\leq 2 B 3^K \rho \varepsilon^Q r^{-K+Q+1} .$$

since $Q \geq 2$, $0 < \varepsilon \leq 1$ and $r \leq 1$. Q.E.D.

The $C^K$-conjugation we will search for will have the form

$$x = g(w,S) = u + v + \sum_{j=0}^{Q} S_j \langle u \rangle^{Q-j} \langle v \rangle^j \tag{3.9}$$

where $u \in U$, $v \in V$, $S_0,\ldots,S_Q \in L_Q^s$. We will write $w = u + v$ and $S = (S_0,\ldots,S_Q)$. The tensors $S_0,\ldots,S_Q$ will depend on $w \in X$, but they will be chosen so that $S \in \sum^{Q+1}$ where

$$\textstyle\sum^{Q+1} = \{S = (S_0,\ldots,S_Q): \ |S_j| \leq 1 \ \text{ for } \ 0 \leq j \leq Q\} . \tag{3.10}$$

What we need to do here is to derive some elementary properties about the change of variables (3.9) in the vicinity of the origin $x = w = 0$.

First let $r$ satisfy

$$0 < r < \min(1/Q,\ 1/5) , \tag{3.11}$$

where $Q \geq 2$. Then observe that one has

$$(Q + 1)\, 2^Q r^Q < r/2 .$$

Consequently if $r$ satisfies (3.11) and $9/32 \leq a \leq 1/2$ then one has

$$(Q + 1)(1 + a)^Q r^Q < (Q + 1)(3/2)^Q \, r^Q < (Q + 1)(9/16) \, 2^Q \, r^Q$$
$$< (9/32)r < ar \tag{3.12}$$

The next statement is easily verified.

Lemma 3. Let $x$ satisfy Eq. (3.9) and let $w = u + v$. If $S \in \sum^{Q+1}$, and $|w| < r/2$, then $|x| < r$.

Also if $x$ is given by Eq. (3.9) and $w = u + v$, where $|u| = |\cos\theta||w|$, and $|v| = |\sin\theta||w|$, then one has

$$|x| > |u + v| - \sum_{j=0}^{Q} |S_j \langle u\rangle^{Q-j} \langle v\rangle^j|$$

$$> |w| - \sum_{j=0}^{Q} |S_j||w|^Q$$

The next lemma then follows from the last inequality and (3.12).

Lemma 4. Let $x$ satisfy Eq. (3.9) and let $w = u + v$. Let $r$ satisfy (3.11). If $S \in \sum^{Q+1}$ and $|w| = (1 + a)r$, where $9/32 < a < 1/2$, then $|x| > r$.

Next we will need some estimates on the size of the derivatives of the function $q$ which is defined by (3.9).

Lemma 5. Let $q$ be given by (3.9) where $S \in \sum^{Q+1}$. Assume that $r$ satisfies (3.11). Then the following are valid for $|w| < br$, where $b = 41/32$ :

(A) $|D_W^1 g|_{r,\infty} \leq a(Q)$

(B) $|D_W^K g|_{r,\infty} \leq b(Q,K)\, r^{Q-K}$ , $0 \leq K \leq Q$ ,

(C) $|D_W^K g|_{r,\infty} = 0$ , $Q + 1 \leq K$

(D) $|D_W^K D_S^1 g|_{r,\infty} \leq c(Q,K)\, r^{Q-K}$ , $0 \leq K$

(E) $D_W^K D_S^L g = 0$ , $2 \leq L$ and $0 \leq K$ .

*The terms* $a(Q)$, $b(Q,K)$ *and* $c(Q,K)$ *are constants which depend only on* $Q$ *and* $K$ *and not on* $r$ *for* $r \leq 1/5$.

*Proof*. We will use Eq. (2.13). Since $S \in \sum^{Q+1}$ one has $|S_j e^\alpha| \leq 1$ for $0 \leq j \leq Q$ and all n-tuples $\alpha$ . Also one has

$$|u^{\alpha(U)}\, v^{\alpha(V)}| \leq |w|^{|\alpha|} = |w|^Q$$

for every n-tuple $\alpha$ with $|\alpha| = Q$ . By using the facts that $r \leq 1/5$ , $|u| \leq |w|$ and $|v| \leq |w|$ , part (A) is easily checked. For part (B) we differentiate (2.13). Since one has

$$|D_W^K u^{\alpha(U)}\, v^{\alpha(V)}| \leq (\alpha!/K!)\, |w|^{Q-K} \qquad (3.13)$$

for $0 \leq K \leq Q$ and every n-tuple $\alpha$ with $|\alpha| = Q$, parts (B) and (C) are immediate. For parts (D) and (E) we first differentiate $S_j \langle u\rangle^{Q-j} \langle v\rangle^j$ with respect to $S$ . Again from (2.13) one has

$$D_S^1\, S_j\, \langle u\rangle^{Q-j}\, \langle v\rangle^j = \sum e\, u^{\alpha(U)}\, v^{\alpha(V)} \qquad (3.14)$$

where the summation in (3.14) is over all n-tuples $\alpha$ with $|\alpha(U)| = Q - j$ and $|\alpha(V)| = j$ , and the coefficients $e$ in (3.14) depend on $j$ and $\alpha$ and not on $r$ or $S$ . We then apply $D_w^k$ to (3.14) and use (3.13) to get Part (D). Finally Part (E) is a consequence of the fact that $g(w,S)$ is linear in $S$ . Q.E.D.

## IV. Factorization Lemmas.

In this section we will present two factorization lemmas which form the basis of our approach to the linearization problem. However before doing this we need to derive a formula for the partial derivatives of the composition of two smooth functions. Such formulae can be found, with varying degrees of explicitness, in Dieudonne (1974), Abraham and Marsden (1978) and especially Fraenkel (1978).

Let $E = H \circ g$ denote the composition of two smooth functions where $g = g(x_1, x_2)$ depends on two variables $x_1 \in X_1$, $x_2 \in X_2$ and $X_1$ and $X_2$ denote two finite dimensional Banach spaces. Let $D_1^K D_2^L E$ denote the Kth derivative with respect to $x_1$ and the Lth derivative with respect to $x_2$. Then $D_1^K D_2^L E$ is an element of $L_{K,L}^S(X_1, X_2; Y)$, the space of symmetric $(K,L)$-linear mapping of $X_1 \times \ldots \times X_1$ (K-copies), $X_2 \times \ldots \times X_2$ (L-copies) into $Y$ where $Y$ is a finite dimensional Banach space that contains the range of $H$.

*Lemma 6. Assume that* $H$ *and* $g$ *are of class* $C^M$. *Then the composition* $E = H \circ g$ *is of class* $C^M$ *and for* $1 \leq K + L \leq M$ *one has*

$$D_1^K D_2^L E = \sum A(K,L,k,\alpha) D^k H \ \Pi \ (D_1^p D_2^q g)^{\alpha(p,q)} \tag{4.1}$$

*where the following hold:*

(A) *The product* $\Pi$ *is taken over integers* $p$ *and* $q$ *satisfying* $0 \leq p \leq K$ *and* $0 \leq q \leq L$.

(B) $\alpha = \alpha(p,q)$ *is a matrix with nonnegative integer entries.*

(C) *The coefficients* $A(K,L,k,\alpha)$ *depend on* $K,L,k$ *and* $\alpha$ *and not on the functions* $H$ *or* $g$. *The largest of these coefficients and the number of nonzero coefficients are completely determined by* $K$ *and* $L$.

(D) $1 \leq k \leq K + L$ . (4.2)

(E) $\sum_{p,q} \alpha(p,q) = k$ (4.3)

(F) $\sum_{p,q} p\alpha(p,q) = K$ (4.4)

(G) $\sum_{p,q} q\alpha(p,q) = L$ . (4.5)

(H) The summation $\sum$ in (4.1) is taken over all $k$ and $\alpha$ subject to the conditions given above.

Proof. The precise value of the coefficient $A(K,L,k,\alpha)$ is not essential for our analysis, but it can be derived by using Fraenkel (1978). We will skip these details.

One proves the lemma by induction on $K + L$ . The validity of the lemma for $K + L = 1$ is elementary. For the induction step we assume that (4.1) is valid as stated and we then will verify it for $D_1^{K+1}D_2^LE$ and $D_1^KD_2^{L+1}E$ . Our argument will apply equally to both derivatives, therefore we will verify (4.1) for $D_1^{K+1}D_2^LE$ only. Let

$$T = (D^kH) \prod (D_1^p D_2^q g)^{\alpha(p,q)}$$

denote a typical term in (4.1), and extend $\alpha$ to satisfy $\alpha(K+1,q) = 0$ for $0 \leq q \leq L$. By Leibniz's formula, Dieudonne (1969) or Abraham and Marsden (1978), the derivative $D_1^1T$ will involve a sum of various terms. One such term is

$$(D^{k+1}H)(D_1^1D_2^0g) \prod (D_1^pD_2^qg)^{\alpha(p,q)} = (D^{k+1}H)(D_1^1D_2^0g)^{\alpha(1,0)+1} \hat{\prod} (D_1^pD_2^qg)^{\alpha(p,q)}$$

where $\hat{\Pi}$ denotes the product over $(p,q) \neq (1,0)$ and $0 \leq p \leq K+1$ , $0 \leq q \leq L$ . The new $\alpha$-matrix, which we denote by $\alpha^\circ$ , is then formed by replacing $\alpha(1,0)$ by $\alpha(1,0) + 1$ and leaving the other entries unchanged. It is easily seen that the above term then satisfies conditions (4.2-5) with $\alpha^\circ$ replacing $\alpha$ , $(k + 1)$ replacing $k$ and $(K + 1)$ replacing $K$ .

The other terms in $D_1^1T$ have the form

$$\alpha(p,q)(D^kH)(D_1^pD_2^qg)^{\alpha(p,q)-1}(D_1^{p+1}D_2^q\, g)^{\alpha(p+1,q)+1} \tilde{\Pi}\ (D_1^mD_2^\ell g)^{\alpha(m,\ell)}$$

where $\alpha(p,q) \geq 1$ and $\tilde{\Pi}$ denotes a product over $(m,\ell) \neq (p,q)$ or $(p + 1,q)$ with $0 \leq m \leq K + 1$, $0 \leq \ell \leq L$. The new $\alpha$-matrix, which we denote by $\alpha^\circ$ , is then given by

$$\alpha^\circ(p,q) = \alpha(p,q) - 1,$$
$$\alpha^\circ(p + 1,q) = \alpha(p + 1,q) + 1$$
$$\alpha^\circ(m,\ell) = \alpha(m,\ell) \quad , \qquad (m,\ell) \neq (p,q),(p + 1,q) .$$

Once again it is easily seen that the above terms satisfy conditions (4.3-5) with $\alpha^\circ$ replacing $\alpha$ , and $(K + 1)$ replacing $K$ . Q.E.D.

The first Factorization Lemma, which we derive next, will be used to prove Theorem 2.

<u>Lemma 7</u>. (First Factorization Lemma) <u>Let</u> $Q \geq 2$ <u>be an integer, and let</u> $M$ <u>and</u> $N$ <u>be positive integers with</u> $Q = M + N$. <u>Let</u> $H = \max(M,N)$, <u>and let</u> $F$ <u>be of class</u> $C^{3Q}$. <u>Define</u> $F_r(x,\varepsilon)$ <u>as in Section</u> III. <u>Assume that</u> $D^PF(0) = 0$ <u>for</u> $0 \leq P \leq Q$ . <u>Let</u> $U$ <u>and</u> $V$ <u>be complementary orthogonal subspaces in</u> $X$ <u>and let</u>

$$x = g(w,S) = u + v + \textstyle\sum_{j=0}^{Q} S_j\langle u\rangle^{Q-j}\langle v\rangle^j \qquad (4.6)$$

where $u \varepsilon U$, $v \varepsilon V$ and $S_0,\dots,S_Q \varepsilon L_Q^s$. Assume that $r$ satisfies

$$0 < r < \min(1/Q,1/5) . \tag{4.7}$$

Then there exist functions

$$G_j : X \times \Sigma^{Q+1} \to L_Q^s , \qquad 0 \le j \le Q ,$$

with $G_j = G_j(w;S) = G_j(u,v;S_0,\dots,S_Q)$, $w = u + v$ with $u \varepsilon U$, $v \varepsilon V$, and $S = (S_0,\dots,S_Q) \varepsilon \Sigma^{Q+1}$, satisfying the following properties:

(A) Each $G_j$ is of class $C^Q$, $0 \le j \le Q$.

(B) For $|x| \le r$ and $S \varepsilon \Sigma^{Q+1}$, one has

$$F_r(x,\varepsilon) = \textstyle\sum_{j=0}^{Q} G_j \langle u\rangle^{Q-j} \langle v\rangle^j \tag{4.8}$$

where $x$ is given by (4.6).

(C) For $0 \le j \le N - 1$ the functions $G_j$ have the form

$$G_j = G_j(u;S_0,S_1,\dots,S_j) ,$$

i.e. $G_j$ is independent of $v$ and of $S_k$ for $k > j$.

(D) For $N + 1 \le j \le Q$ the functions $G_j$ have the form

$$G_j = G_j(v;S_j,S_{j+1},\dots,S_Q) ,$$

i.e., $G_j$ is independent of $u$ and of $S_k$ for $k < j$.

(E) $G_j(w,S) = 0$ for $|w| \ge 3r/2$, $0 \le j \le Q$.

(F) There is a constant $C_1$, depending only on $Q$, and not on $F$, $\varepsilon$ or $r$, such that

$$\|D_w^K D_s^L G_j\|_\infty \leq C_1 B \varepsilon^Q r^{1-Q-H} \leq C_1 B \varepsilon^Q r^{1-2Q} \tag{4.9}$$

*for* $1 \leq K + L \leq Q$ *and* $0 \leq j \leq Q$ .

(G) *There is a constant* $C_2$ , *depending only on* $Q$ *and not on* $F$ , $\varepsilon$ *or* $r$ , *such that*

$$\begin{aligned} \|G_j\|_\infty &\leq C_2 B \varepsilon^Q r^{1-H} \leq C_2 B \varepsilon^Q r^{1-Q} \\ \|D_s^L G_j\|_\infty &\leq C_2 B \varepsilon^Q r^{Q-H} \leq C_2 B \varepsilon^Q \end{aligned} \tag{4.10}$$

*for* $1 \leq L$ *and* $0 \leq j \leq Q$ .

(H) *If the function* $F$ *is a* $C^R$*-function of a parameter* $\theta \in \Theta$ , *then the functions* $G_j$ *are also* $C^R$*-functions of* $\theta$ , $0 \leq R \leq \infty$ . *Furthermore if* *is compact, then the estimates* (4.9-10) *are uniform for* $\theta \in \Theta$ .

(The norms $\|.\|$ in (F-G) refer to the supremum for $w \in X$ and $S \in \sum^{Q+1}$ .)

*Proof.* Let $r$ satisfy (4.7). We begin by defining the set $\Omega$ , which will contain the supports of the functions $G_0,\ldots,G_Q$ . Specifically let $\Omega$ denote the set $(w,S)$ in $X \times \sum^{Q+1}$ , where $w = u + v$ with $u \in U$ , $v \in V$ , $S = (S_0,\ldots,S_Q)$ , $|w| \leq (3/2)r$ and

$$|x| = |u + v + \textstyle\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j} \langle v\rangle^j| \leq r .$$

Let $\Omega(S) = \{w \in X : (w,S) \in \Omega\}$ denote the fibres of $\Omega$ . Next define

$$E = E(w;S) = E(u,v;S_0,\ldots,S_Q) = F_r(u + v + \textstyle\sum_{j=0}^{Q} S_j\langle u\rangle^{Q-j} \langle v\rangle^j , \varepsilon)$$

for $(w,S) \in \Omega$ . Since $D^P F(0) = 0$ for $0 \leq P \leq Q$ it follows from (2.3) that

$$E(w;S) = \frac{1}{(Q-1)!} \left(\int_0^1 (1-t)^{Q-1} D_w^Q E(tw;S)dt\right)\langle w\rangle^Q$$

Furthermore one has

$$(D_w^Q\ E(tw;S))\langle w\rangle^Q = \sum_{j=0}^{Q} \binom{Q}{j} D_u^{Q-j}\ D_v^j\ E(tu,tv;S)\ \langle u\rangle^{Q-j}\langle v\rangle^j\ .$$

For $0 \leq j \leq Q$ define

$$H_j = H_j(u,v;S) = \frac{1}{(Q-1)!} \int_0^1 \binom{Q}{j}(1-t)^{Q-1}D_u^{Q-j}D_v^j\ E(tu,tv;S)dt\ . \tag{4.11}$$

One then has

$$F_r(u + v + \sum_{j=0}^{Q} S_j\langle u\rangle^{Q-j}\langle v\rangle^j,\varepsilon) = \sum_{j=0}^{Q} H_j\langle u\rangle^{Q-j}\ \langle v\rangle^j\ . \tag{4.12}$$

for $(w,S) \in \Omega$ . Since $F_r$ and $E$ are of class $C^{3Q}$, it follows from (4.11) that each $H_j$ is of class $C^{2Q}$ .

For $0 \leq j \leq N - 1$ and $0 \leq k + j \leq N - 1$ we define $D_v^k\ H_j^0$ by

$$D_v^k\ H_j^0 = D_v^k\ H_j\quad (\text{at}\quad v = 0\ )\ . \tag{4.13}$$

Similarly for $N + 1 \leq j \leq Q$ and $0 \leq k \leq j - N - 1$ we define $D_u^k H_j^0$ by

$$D_u^k\ H_j^0 = D_u^k\ H_j\quad (\text{at}\quad u = 0\ )\ . \tag{4.14}$$

For $j$ in the range $0 \leq j \leq N - 1$ , we use the Taylor series expansion of $H_j$ about $v = 0$ to get

$$H_j\langle u\rangle^{Q-j}\langle v\rangle^j = \sum_{k=0}^{N-1-j} \frac{1}{k!}\ D_v^k H_j^0\langle u\rangle^{Q-j}\langle v\rangle^{j+k} + H_{jR}\langle u\rangle^{Q-j}\langle v\rangle^N \tag{4.15}$$

where $H_{jR}\langle u\rangle^{Q-j}\langle v\rangle^N$ is the remainder term. By Eq. (2.3) one has

$$H_{jR} = \frac{1}{(N-1-j)!} \int_0^1 (1-s)^{N-1-j}\ D_v^{N-j}\ H_j(u,sv;S)\ ds. \tag{4.16}$$

By summing (4.15) for $0 \leq j \leq N - 1$ and by interchanging the order of summation one gets

$$\sum_{j=0}^{N-1} H_j\langle u\rangle^{Q-j}\langle v\rangle^j = \sum_{p=0}^{N-1} \left(\sum_{k=0}^{p} \frac{1}{k!} D_v^k H_{p-k}^0\langle u\rangle^k\right)\langle u\rangle^{Q-p}\langle v\rangle^p + \sum_{j=0}^{N-1} H_{jR}\langle u\rangle^{Q-j}\langle v\rangle^N . \tag{4.17}$$

By doing a similar analysis for $j$ in the range $N + 1 \leq j \leq Q$ one obtains

$$\sum_{j=N+1}^{Q} H_j\langle u\rangle^{Q-j}\langle v\rangle^j = \sum_{p=N+1}^{Q}\left(\sum_{k=0}^{Q-p} \frac{1}{k!} D_u^k H_{k+p}^0\langle v\rangle^k\right)\langle u\rangle^{Q-p}\langle v\rangle^p + \sum_{j=N+1}^{Q} H_{jR}\langle u\rangle^M\langle v\rangle^j , \tag{4.18}$$

where the term $H_{jR}$ in the last equation satisfy

$$H_{jR} = \frac{1}{(j - N - 1)!} \int_0^1 (1 - s)^{j-N-1} D_u^{j-N} H_j(su,v;S)ds . \tag{4.19}$$

By incorporating (4.17) and (4.18) into (4.12) we get

$$F_r(u + v + \sum_{j=0}^{Q} S_j\langle u\rangle^{Q-j}\langle v\rangle^j,\varepsilon) = \sum_{p=0}^{Q} G_p\langle u\rangle^{Q-p}\langle v\rangle^p , \tag{4.20}$$

where the terms $G_p$ in Eq. (4.20) are given by:

$$G_p = \sum_{k=0}^{p} \frac{1}{k!} D_v^k H_{p-k}^0\langle u\rangle^k , \quad 0 \leq p \leq N - 1 , \tag{4.21}$$

$$G_p = \sum_{k=0}^{Q-p} \frac{1}{k!} D_u^k H_{k+p}^0\langle v\rangle^k , \quad N + 1 \leq p \leq Q , \tag{4.22}$$

$$G_N = H_N + \sum_{j=0}^{N-1} H_{jR} \langle u\rangle^{N-j} + \sum_{j=N+1}^{Q} H_{jR}\langle v\rangle^{j-N} . \tag{4.23}$$

Since the functions $H_j$ are of class $C^{2Q}$, it follows from the above formulae that the functions

a) $G_p$ are of class $C^{2Q-p}$ for $0 \leq p \leq N-1$, and

b) $G_p$ are of class $C^{Q+p}$ for $N+1 \leq p \leq Q$.

c) $G_N$ is of class $C^{2Q-H}$ where $H = \max(M,N) \leq Q$.

Lemma 4 implies that the $H_j$'s and the $G_p$'s vanish for $41/32\ r \leq |w| \leq 3/2\ r$, so we define $G_p(w,S) = 0$ for $|w| > 3r/2$. (The value of the $G_p$'s when $S$ fails to satisfy $S \in \sum^{Q+1}$ is immaterial for our argument. One could extend the definition of the $G_p$'s to all of $X \times (L_0^s)^{Q+1}$ in any way so that $G_p$'s are of class $C^Q$ and have compact support.) This completes the proof of statements (A), (B) and (E) in the lemma.

In order to prove statements (C) and (D), we first observe that for $0 \leq j \leq N-1$ and $0 \leq k+j \leq N-1$, the function $D_v^k H_j^0$ depends only on $u$ and on $S_0, S_1, \ldots, S_k$. Consequently from Eq. (4.21) we see that for $0 \leq p \leq N-1$, the function $G_p$ depends only on $u$ and $S_0, \ldots, S_p$. Similarly for $N+1 \leq j \leq Q$ and $0 \leq k \leq j-N-1$, the functions $D_u^k H_j^0$ depend only on $v$ and on $S_{Q-k}, S_{Q-k+1}, \ldots, S_Q$. Consequently from Eq. (4.22) we see that for $N+1 \leq p \leq Q$, the function $G_p$ depends only on $v$ and $S_p, S_{p+1}, \ldots, S_Q$.

In order to estimate the derivatives of the $G_p$'s we adopt the following

<u>CONVENTION</u>: <u>We will use the terms</u> $C_3, C_4, C_5, \ldots$ <u>to refer to constants which depend only on</u> $Q$ <u>and not on</u> $F$, $\varepsilon$ <u>or</u> $r$.

The definition of the $G_p$'s is based on the higher order derivatives of the function $E$, which in turn is the composition of $F_r$ and $q$.

In order to estimate the derivatives of the $G_p$'s we shall use Lemma 6 and the formulae (4.11-23). For example, (4.11) yields

$$\|H_j\|_\infty \le C_3 \|D_w^Q E\|_\infty , \qquad 0 \le j \le Q .$$

Let us now apply Lemma 6 where $H = F_r$ and $g$ is given by (4.6). We claim that for $0 \le K + L \le 3Q$ one has

$$\|D_w^K D_s^L E\|_\infty \le C_4 B \varepsilon^Q r^{2Q-K} , \qquad 0 \le K , 1 \le L \tag{4.24}$$

$$\|D_w^K E\|_\infty \le C_5 B \varepsilon^Q r^{Q+1-K} , \qquad 0 \le K \tag{4.25}$$

Indeed by (4.1) we let

$$T = (D^k F_r) \Pi (D_w^p D_s^q g)^{\alpha(p,q)} \tag{4.26}$$

denote a typical term in the expansion for $D_w^K D_s^L E$. By Lemma 5(E) we see that we can restrict our attention to those terms $T$ for which $\alpha(p,q) = 0$ for $q \ge 2$. Now Lemma 5(B,D) implies that each of the factors $(D_w^p D_s^q g)^{\alpha(p,q)}$, with the exception of $(p,q) = (1,0)$, is bounded by $C_6 r^{(Q-p)\alpha(p,q)}$. By combining this with Lemma 2 and Lemma 5(A) we get

$$\|T\|_\infty \le C_7 B \varepsilon^Q r^{-k+Q+1} \hat{\Pi} r^{(Q-p)\alpha} \tag{4.27}$$

where $\hat{\Pi}$ denotes the product over all $(p,q) \ne (1,0)$ with $0 \le p \le K$, and $0 \le q \le L$. The exponent $e$ of $r$ in (4.27) is

$$e = -k + Q + 1 + \hat{\sum} (Q - p)\alpha(p,q)$$

where $\hat{\sum}$ is the summation over all $(p,q) \neq (1,0)$ with $0 \leq p \leq K$ and $0 \leq q \leq L$. By conditions (4.3-4) we get

$$e = -k + Q + 1 + Qk - Q\alpha(1,0) - K + \alpha(1,0)$$
$$= (Q - 1)(k - \alpha(1,0)) + Q + 1 - K \ .$$

If $k = \alpha(1,0)$, then by (4.3) all the other $\alpha$'s vanish. Consequently one has $K = k$ and $L = 0$ by (4.4-5). Also $e = Q + 1 - K$.

If $k \neq \alpha(1,0)$, then $k > \alpha(1,0)$ by (4.3). In this case one has

$$e \geq Q - 1 + Q + 1 - K = 2Q - K \geq Q + 1 - K$$

for $0 \leq L$ and

$$\|T\|_\infty \leq C_8 \, B \, \varepsilon^Q \, r^{2Q - K}$$

since $r \leq 1$. By incorporating this into (4.1) and summing one gets (4.24) and (4.25).

From Eq. (4.11) we see that only the derivative $D_w^Q E$ is used to determine the $H_j$'s. It follows then from (4.25) that $\|H_j\|_\infty \leq C_9 \, B \, \varepsilon^Q \, r$, $0 \leq j \leq Q$. Also from (4.11) one has

$$D_w^K \, D_s^L \, H_j = C_{10} \int_0^1 (1 - t)^{Q-1} \, D_w^K \, D_u^{Q-j} \, D_v^j \, D_s^L \, E \, dt \ .$$

Therefore it follows from (4.24 - 25) that

$$\|D_w^K \, D_s^L \, H_j\|_\infty \leq C_{11} \, B \, \varepsilon^Q \, r^{Q-K} \quad , \quad 0 \leq K \, , \, 1 \leq L \, , \tag{4.28}$$

$$\|D_w^K \, H_j\|_\infty \leq C_{11} \, B \, \varepsilon^Q \, r^{1-K} \quad , \quad 0 \leq K \, . \tag{4.29}$$

We see then that (4.28-29) gives bounds for the terms $D_v^k H_j^0$ and $D_u^K H_j^0$ appearing in (4.13-14). Similarly the remainder terms $H_{jR}$ appearing in (4.15) and (4.19) satisfy

$$\|H_{jR}\|_\infty < C_{12}\ B\ \varepsilon^Q\ r^{1+j-N}\ , \qquad 0 < j < N-1\ , \tag{4.30}$$

$$\|H_{jR}\|_\infty < C_{12}\ B\ \varepsilon^Q\ r^{1-j+N}\ , \qquad N+1 < j < Q\ . \tag{4.31}$$

By summing these various terms, we get

$$\|G_p\|_\infty < C_{13}\ B\ \varepsilon^Q\ r^{1-p} < C_{13}\ B\ \varepsilon^Q\ r^{1-N}\ , \qquad 0 < p < N-1,$$

$$\|G_p\|_\infty < C_{13}\ B\ \varepsilon^Q\ r^{1+p-Q} < C_{13}\ B\ \varepsilon^Q\ r^{1-M}\ , \qquad N+1 < p < Q\ .$$

$$\|G_N\|_\infty < C_{13}\ B\ \varepsilon^Q\ r^{1-H}\ .$$

This establishes the first inequality in (4.10).

In order to compute the derivatives of the $G_p$'s we use formulae (4.21-23). For example, for $0 < p < N-1$ and $0 < K < Q$ one has

$$D_w^K\ G_p = \textstyle\sum_{k=0}^{p}\ \frac{1}{k!}\ D_w^K\ D_v^k\ H_{p-k}^0 \langle u\rangle^k\ ,$$

with similar expressions valid for $N < p < Q$. By applying (4.29) and summing one obtains for $0 < K < Q$:

$$\|D_w^K\ G_p\|_\infty < C_{14}\ B\ \varepsilon^Q\ r^{1-K-p} < C_{14}\ B\ \varepsilon^Q\ r^{1-Q-N}\ , \qquad 0 < p < N-1. \tag{4.32}$$

$$\|D_w^K\ G_p\|_\infty < C_{14}\ B\ \varepsilon^Q\ r^{1+p-K-Q} < C_{14}\ B\ \varepsilon^Q\ r^{1-Q-M}\ , \quad N+1 < p < Q\ . \tag{4.33}$$

$$\|D_w^K G_N\|_\infty < C_{14}\ B\ \varepsilon^Q\ r^{1-Q-H}\ .$$

This establishes (4.9) for $L = 0$.

Estimating the derivatives $D_w^K\ D_s^L\ G_p$ for $L > 1$ is done similarly. For example for $0 \leqslant p \leqslant N - 1$ one has

$$D_w^K D_s^L G_p = \sum_{k=0}^{p} \frac{1}{k!} D_w^K D_s^L D_v^k H_{p-k}^0 \langle u\rangle^k . \qquad (4.34)$$

By applying (4.28) and summing we then obtain

$$\| D_w^K D_s^L G_p \|_\infty \leqslant C_{15} B \varepsilon^Q r^{Q-K-p} \leqslant C_{15} B \varepsilon^Q r^{1-N} \qquad (4.35)$$

for $0 \leqslant p \leqslant N - 1$, $0 \leqslant K + L \leqslant Q$ , $1 \leqslant L$ , and

$$\| D_w^K D_s^L G_p \|_\infty \leqslant C_{15} B \varepsilon^Q r^{p-K} \leqslant C_{15} B \varepsilon^Q r^{1-M} \qquad (4.36)$$

for $N + 1 \leqslant p \leqslant Q$ , $0 \leqslant K + L \leqslant Q$ and $1 \leqslant L$ . Also one has

$$\| D_w^K D_s^L G_N \|_\infty \leqslant C_{15} B \varepsilon^Q r^{Q-K-H} \leqslant C_{15} B \varepsilon^Q r^{1-H} \qquad (4.37)$$

for $0 \leqslant K + L \leqslant Q$ and $1 \leqslant L$. By combining (4.32-37) we obtain (4.9). As a consequence of (4.35-37) we also have

$$\| D_s^L G_p \|_\infty \leqslant C_{15} B \varepsilon^Q r^{Q - p} \leqslant C_{15} B \varepsilon^Q r^{Q - N} \qquad (4.38)$$

for $0 \leqslant p \leqslant N - 1$ , $1 \leqslant L$ ,

$$\| D_s^L G_p \|_\infty \leqslant C_{15} B \varepsilon^Q r^p \leqslant C_{15} B \varepsilon^Q r^{Q - M} \qquad (4.39)$$

for $N + 1 \leqslant p \leqslant Q$ and $1 \leqslant L$ , and from (4.37) one has

$$\| D_s^L G_N \|_\infty \leqslant C_{15} B \varepsilon^Q r^{Q-H}$$

for $1 \leqslant L$, which gives us the second inequality in (4.10).

The proof of statement (H) is straight-forward. The functions $H_j$ and $G_j$ only involve derivatives of $F = F(x,\theta)$ with respect to the x-variable. Therefore $G_j$ is as smooth as $F$ in the $\theta$-variable. The uniformity of the estimates (4.9-10) for $\theta$ in a compact set is achieved by redefining $B$ in (3.1) to be a sup for $|x| \leqslant 1/5$ and $\theta$ is the compact set. Q.E.D.

The next factorization lemma will be used to prove Theorem 2, where the matrix $A$ is assumed to be stable.

*Lemma 8.* (Second Factorization Lemma.) *Let* $Q \geq 2$ *be an integer, and let* $F$ *be of class* $C^{2Q}$. *Define* $F_r(x,\varepsilon)$ *as in Section III. Assume that* $D^P F(0) = 0$ *for* $0 \leq P \leq Q$ *and let*

$$x = g(v,S) = v + S\langle v\rangle^Q \tag{4.40}$$

*where* $v \in X$ *and* $S \in L_Q^s$. *Assume that* $r$ *satisfies*

$$0 < r < \min(1/Q, 1/5) . \tag{4.7}$$

*Then there is a function* $G = G(v,S)$

$$G : X \times \textstyle\sum^1 \to L_Q^s \quad ,$$

*with* $\sum^1 = \{S \in L_Q^s : \ |S| \leq 1\}$, *satisfying the following properties*:

(A) $G$ *is of class* $C^Q$.

(B) *For* $|x| \leq r$ *and* $S \in \sum^1$ *one has*

$$F_r(x,\varepsilon) = G(v,S)\langle v\rangle^Q \tag{4.41}$$

*where* $x$ *is given by* (4.41).

(C) $G(v,S) = 0$ *for* $v = 0$ *and for* $|v| \geq 3r/2$.

(D) *There is a constant* $C_1$, *depending only on* $Q$, *and not on* $F$, $\varepsilon$ *or* $r$, *such that*

$$\| D_v^K \, D_S^L \, G \|_\infty \leq C_1 \, B \, \varepsilon^Q \, r^{1-K} \tag{4.42}$$

*for* $1 \leq K + L \leq Q$.

(E) <u>There is a constant</u> $C_2$ , <u>depending only on</u> Q , <u>and not on</u> F , ε , <u>or</u> r , <u>such that</u>

$$\|G\|_\infty \leq C_2 \, B \, \varepsilon^Q \, r \qquad (4.43)$$

$$\|D_S^L \, G\|_\infty \leq C_2 \, B \, \varepsilon^Q \, r^Q \qquad (4.44)$$

(F) <u>If the function</u> F <u>is a</u> $C^R$<u>-function of a parameter</u> $\theta \in \Theta$ , <u>then the function</u> G <u>is also a</u> $C^R$<u>-function of</u> θ , $0 \leq R \leq \infty$ . <u>Furthermore, if is compact, then the estimates</u> (4.42-43) <u>are unfiform for</u> $\theta \in \Theta$ .

<u>Proof</u>. The argument here is a direct adaptation of the proof of Lemma 7. In the present case, one should view the space U as consisting of the zero vector only. Consequently $D_W^K = D_V^K$ . Only the term $H_Q$ remains, and this agrees with $G_Q$ , which we define to G . Since Q derivatives are lost in defining $H_Q$ , we see that when F is of class $C^{2Q}$ the function G of class $C^Q$ . Parts (A), (B) and (C) in this case now follow from the corresponding statements in Lemma 7.

By setting p = Q in (4.33) we get

$$\|D_V^K \, G \, \|_\infty \leq C_{14} \, B \, \varepsilon^Q \, r^{1-K}$$

for $0 \leq K \leq Q$ . Similarly (4.36) gives us

$$\|D_V^K \, D_S^L \, G\|_\infty \leq C_{15} \, B \, \varepsilon^Q \, r^{Q - K}$$

for $0 \leq K + L \leq Q$ and $1 \leq L$ . Putting these together we obtain (4.42 - 43). In order to get (4.44) we set p = Q in (4.39) to obtain

$$\|D_S^L \, G\| \leq C_{15} \, B \, \varepsilon^Q \, r^Q$$

for $1 \leq L$ . Finally it follows from (4.11) and the fact that $D^P F(0) = 0$ for $0 \leq P \leq Q$ , that one has $G(0,S) = 0$ . Q.E.D.

## V. Exponential Dichotomies and Smoothness.

As mentioned above, the proof of our main theorem will be based on the theory of smooth solutions of perturbations of linear differential equations with exponential dichotomies, a theory which we develop in this section. The basic problem is to study bounded solutions of the differential equation

$$S' = LS + R(u,t,S) \tag{5.1}$$

on a finite dimensional Banach space $X$, and the dependence of these solutions on the parameter $u$. We assume that $u \varepsilon U$, where $U$ is some parameter space, and $t \varepsilon R$. We will also assume that the linear operator $L$ is hyperbolic, which implies that there is a linear projection $P_0$ on $X$ and positive constants $k$ and $\nu$ so that

$$|W(t,s)| \leq k\, e^{-\nu|t - s|} \tag{5.2}$$

where

$$W(t,s) = \begin{cases} e^{Lt}\, P_0 e^{-Ls} , & s < t \\ e^{Lt}\, [I - P_0]e^{-Ls} , & t < s . \end{cases}$$

The solutions of Eq. (5.1) we seek will be fixed points $S = S(u,t)$, in the class $BC(U \times R,X)$ of bounded continuous mappings of $U \times R$ into $X$, of the operator

$$\mathfrak{J}\, S(u,t) = \int_{-\infty}^{\infty} W(t,s)R(u,s,S(u,s))ds. \tag{5.3}$$

We will assume that there are positive constants $A$, $B$ and $C$ such that

$$\begin{aligned} &|R(u,t,S)| \leq A + B|S| \\ &|R(u,t,S_1) - R(u,t,S_2)| \leq C|S_1 - S_2| \end{aligned} \tag{5.4}$$

for all $u,t,S,S_1$ and $S_2$. Furthermore we shall assume that $R$ is of class $C^Q$, where $Q$ will be specified below.

_Lemma 9. Let_ $R$ _be of class_ $C^Q$, $Q > 0$, _and assume that_ (5.2) _and_ (5.4) _are satisfied. If one has_

$$2kB < \nu , \qquad 2kC < \nu \tag{5.5}$$

_then there is a function_ $S \in BC(U \times R,X)$ _that is a fixed point of_ $\mathfrak{J}$. _Moreover_ $S$ _is uniquely determined by_ $|S|_\infty \leq (\nu - 2kB)^{-1}2kA$, _where_

$$|S|_\infty = \sup\{|S(u,t)|: u \in U , t \in R\} .$$

The basic idea is to show that $\mathfrak{J}$ maps the set

$$\{S \in BC(U \times R,X): \ |S|_\infty \leq (\nu - 2kB)^{-1}2kA\}$$

into itself and is a contraction on this set. Since this is a straight-forward argument we will omit the details. (See Coppel (1965).)

The next lemma addresses the question of the smoothness of the fixed point $S = S(u,t)$ of $\mathfrak{J}$. We formulate this as a local result which allows the derivatives $D_u^N S(u,t)$ to be singular at some points $u \in U$.

In the following we will assume $R$ to be of class $C^Q$, where $Q \geq 1$, and that (5.4-5) are satisfied. For $N = 1,2,\ldots,Q$ we define the set $U(N)$ as the collection of all $u \in U$ with the property that there is an open neighborhood $O(u)$ and positive constants $A(N,u)$ and $B(N,u)$ such that

$$\sup|D_u^{N-p}\, D_S^p\, R(v,t,S)| \leq A(N,u) \quad , \quad 0 \leq p \leq N - 1 \tag{5.6}$$

$$\sup|D_S^N\, R(v,t,S)| \leq B(N,u) < \nu(2k)^{-1} , \tag{5.7}$$

where the sup is taken over all $v \in O(u)$ , $t \in R$ and $|S| < (\nu - 2kB)^{-1}2kA$ . Note that for each $N = 1,2,\ldots,Q$ the set $U(N)$ is an open (perhaps empty) subset of U. Let $V(N) = U(1) \cap \ldots \cap U(N)$.

Lemma 10. Let R be of class $C^Q$ , $Q > 1$ , and assume that (5.2) and (5.4-5) are satisfied. Let $S(u,t)$ be the fixed point of $\mathfrak{J}$ described in Lemma 9. Then for $N = 1,2,\ldots,Q$ , the function $S(u,t)$ is of class $C^N$ in the u-variable for $u \in V(N)$. Furthermore if $O(u)$ is chosen so that (5.6-7) hold, then the Nth order derivatives of S are bounded in $O(u)$ by

$$(\nu - 2kB(N,u))^{-1} \; 2kA(N,u). \tag{5.8}$$

Proof. The proof is by induction on N . Let us first look at the case $N = 1$ . Let $u \in U(1)$ and let the neighborhood $O(u)$ and the constants $A(1,u)$, $B(1,u)$ be determined by the above definition. First we note that $S(v,t)$ is Lipschitz continuous in v for $v \in O(u)$ . Indeed if $S_0 \in BC(u \times R,X)$ is chosen so that $|S_0|_\infty < (\nu - 2kB)^{-1}2kA$ and

$$|S_0(v + h,t) - S_0(v,t)| < (\nu - 2kB(1,u))^{-1} \; 2kA(1,u)|h| \tag{5.9}$$

for $v + h$ , $v \in O(u)$ , then it is easily seen that (5.9) is satisfied by $\mathfrak{J} S_0$ . Consequently the fixed point $S(v,t)$ of $\mathfrak{J}$ also satisfies (5.9).

Next consider the first variational equation of (5.3), which is

$$\mathfrak{J}^{(1)}S^{(1)}(v,t) = \int_{\infty}^{\infty} W(t,s) \; R^{(1)}(v,s,S^{o}(v,s),S^{(1)}(v,s))ds, \tag{5.10}$$

where

$$R^{(1)}(v,s,S^{(0)},S^{(1)}) = D_vR(v,s,S^{(0)}) + D_SR(v,s,S^{(0)})S^{(1)} ,$$

$S^{(0)}(v,s) = S(v,s)$ is the fixed point of $\mathfrak{J} = \mathfrak{J}^{(0)}$ and $S^{(1)}(v,t)$ is the formal derivative of $S^{(0)}$ with respect to $v$. Since $R^{(1)}$ satisfies

$$|R^{(1)}(v,s,S^{(0)},S^{(1)})| \leq A(1,u) + B(1,u)|S^{(1)}|$$

$$|R^{(1)}(v,s,S^{(0)},S_1^{(1)}) - R^{(1)}(v,s,S^{(0)},S_2^{(1)})| \leq B(1,u)|S_1^{(1)} - S_2^{(1)}|$$

for $v \in O(u)$, $s \in R$, and since $2kB(1,u) < \nu$, it follows that Lemma 9 is applicable to the system

$$\mathfrak{J}^{(0)}S^{(0)}(v,t) = \int_{\infty}^{\infty} W(t,s)\ R(v,s,S^{(0)}(v,s))ds$$

$$\mathfrak{J}^{(1)}S^{(1)}(v,t) = \int_{\infty}^{\infty} W(t,s)R^{(1)}(v,s,S^{(0)}(v,s),S^{(1)}(v,s))ds$$

for $v \in O(u)$. Consequently $\mathfrak{J}^{(1)}$ has a unique fixed point $S^{(1)}(v,t)$ that satisfies

$$|S^{(1)}(v,t)| \leq (\nu - 2kB(1,u))^{-1}2kA(1,u) \tag{5.11}$$

for $v \in O(u)$. Now (5.9) implies that the difference quotient

$$|h|^{-1}\ [S_0(v + h,t) - S_0(v,t)]$$

satisfies the same bound for $v$, $v + h \in O(u)$. The usual arguments based on continuous dependence on parameters apply now as $|h| \to 0$, and we see that $S(v,t)$ is of class $C^1$ in $v$ for $v \in O(u)$. Furthermore one has $D_v S(v,t) = S^{(1)}(v,t)$. (See Hale (1969) and Hartman (1964).) Also (5.11) assures us that $D_V S$ satisfies the bound (5.8) on $O(u)$ for $N = 1$.

By restricting now to $u \in U(1) \cap U(2)$, the above argument applies to $S^{(1)}(v,t)$ for $v$ close to $u$. More generally, the verification of the induction step is now a straight-forward adaptation of the above ideas. Q.E.D.

## VI. Proof of Theorems

In this section we shall prove Theorems 3 and 4, which are stated below and which imply Theorems 1 and 2.

Algebraic reduction: The first step in the proof is a standard algebraic reduction cf. Poincare (1928), Siegel (1952), Sternberg (1957,1958) and Hartman (1964). Since the proof of the following lemma can be found in these references, we will not present it here.

Lemma 11. *Consider the differential equation*

$$x' = Ax + F(x) , \tag{6.1}$$

*where* $F \in C^M$, $M \geq 2$ *and* $D^P F(0) = 0$ *for* $P = 0,1$. *Assume that* $A$ *satisfies the Sternberg condition of order* $N$ *where* $2 \leq N \leq M$. *Then there is a* $C^\infty$*-conjugation* $y = H(x)$ *between* (6.1) *and* $y' = Ay + G(y)$ *where* $D^P G(0) = 0$ *for* $0 \leq P \leq N$.

As a result of Lemma 11 we see that it suffices to prove only the Statements (B) in Theorems 1 and 2. We direct our attention first to Theorem 1 where $A$ is strictly hyperbolic and $F \in C^{3Q}$ and satisfies $D^P F(0) = 0$ for $0 \leq P \leq Q$.

The linear operator $L$: Let $A$ be a given linear operator on $X$. This induces a linear operator $L = L(A)$ on $L_Q^S$ by the formula

$$LS = AS - \{S,A\} \tag{6.2}$$

where $S \in L_Q^S$ and $\{S,A\}$ is defined by

$$\{S,A\}\langle x_1,x_2,\ldots,x_Q\rangle = S\langle Ax_1,x_2,\ldots,x_Q\rangle + S\langle x_1,Ax_2,\ldots,x_Q\rangle + \ldots + S\langle x_1,x_2,\ldots,Ax_Q\rangle.$$

The next lemma describes the spectrum of $L$ in terms of the numbers $\gamma(\lambda,m)$ .

Lemma 12. Let $A$ be a linear operator on $X$ and define $L$ by Eq. (6.2). Then one has

$$\textstyle\sum(L) = \{\gamma(\lambda,m) : \lambda \in \sum(A) \text{ and } |m| = Q\} .$$

Proof. While this result is well-known we include a brief proof. First assume that the eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ of $A$ are distinct and let $\{x_1,\ldots,x_n\}$ denote the respective eigenvectors. For $m = (m_1,\ldots,m_n)$ define $x^m$ as the "vector" product $x^m = x_1^{m_1} \ldots x_n^{m_n}$ , where $x_1^{m_1}$ denotes $m_1$-copies of $x_1$ , etc. Now fix $m$ with $|m| = Q$ and let $i$ satisfy $1 \leq i \leq n$ . Define $S_m^i \in L_Q^s$ by

$$S_m^i \, x^p = \delta_{p,m} \, x_i \; ,$$

where $p = (p_1,\ldots,p_n)$ satisfies $|p| = Q$ , $\delta_{p,m} = 0$ for $p \neq m$ and $\delta_{m,m} = 1$ , and extend $S_m^i$ to be Q-linear and symmetric. One then has

$$L \, S_m^i \, x^p = AS_m^i \, x^p - \{S_m^i \, ,A\} \, x^p$$

$$= \delta_{p,m} \, \gamma(\lambda_i,m)x_i = \gamma(\lambda_i,m) \, S_m^i \, x^p \; ,$$

i.e. $S_m^i$ is an eigenvector of L with eigenvalue $\gamma(\lambda_i,m)$ . Since the collection

$$\{S_m^i : \; 1 \leq i \leq n \, , \; |m| = Q\}$$

forms a basis for $L_Q^s$ we see that Lemma 12 is valid in this case.

If $A$ does not have distinct eigenvalues, then one constructs a sequence $A_\nu$ with $A_\nu \to A$ as $\nu \to \infty$ and such that $A_\nu$ has distinct eigenvalues $\{\lambda_1^\nu,\ldots,\lambda_n^\nu\}$ . It follows from Kato (1976, Sect. II.5) that $\sum(A_\nu) \to \sum(A)$ as $\nu \to \infty$ . Since $L(A)$ is continuous in $A$ and since $\gamma(\lambda,m)$ is continuous in the $\lambda$'s , we have $\sum(L(A_\nu)) \to \sum(L(A))$ . Q.E.D.

<u>The exponential dichotomies</u>. Since the eigenvalues of $A$ satisfy $\mathrm{Re}\,\gamma(\lambda,m) \neq 0$ for all $\lambda \in \sum(A)$ and all $|m| = Q$, where $Q \geq 2$, it follows that $A$ and $L$ are hyperbolic. Consequently the linear equations

$$w' = A\,w \tag{6.3}$$

$$S' = L\,S \tag{6.4}$$

admit exponential dichotomies on $X$ and $L^s_Q$, respectively. Let

$$\Gamma = \min\{\,|\mathrm{Re}\,\gamma(\lambda,m)| : \gamma(\lambda,m) \text{ is an eigenvalue of } L\,\}$$

$$\Lambda^+ = \min\{\,\mathrm{Re}\,\lambda : \mathrm{Re}\,\lambda > 0 \text{ and } \lambda \text{ is an eigenvalue of } A\,\}$$

$$\Lambda^- = \min\{-\mathrm{Re}\,\lambda : \mathrm{Re}\,\lambda < 0 \text{ and } \lambda \text{ is an eigenvalue of } A\,\}.$$

In the case of (6.4) this means that there is a projection $P_0$ on $L^s_Q$ and positive constants $k$ and $\nu$ such that

$$|W(t,s)| \leq k e^{-\nu|t-s|} \tag{6.5}$$

where $W(t,s)$, the Green's function for (6.4), is given by

$$W(t,s) = \begin{cases} e^{Lt} P_0 e^{-Ls}, & s < t, \\ e^{Lt}[I - P_0]\, e^{-Ls}, & t < s. \end{cases} \tag{6.6}$$

The number $\nu$ appearing in (6.5) can be chosen to be any number in the interval

$$0 < \nu < \Gamma. \tag{6.7}$$

For (6.3) we let $\rho^+$ and $\rho^-$ denote the spectral spreads of $A$, where $A$ is strictly hyperbolic. We now modify the wording used to define the $Q$-smoothness of $A$ and let $K \geq 0$ denote the largest integer for which there are postive integers $M$ and $N$ such that $Q = M + N$, $K \leq \min(M,N)$ and

$$\Lambda^+ (M - K\rho^+) > -\Gamma, \quad \Lambda^- (N - K\rho^-) > -\Gamma. \tag{6.8}$$

We will prove the following:

<u>Theorem</u> 3. <u>Under the assumptions of Theorem</u> 1, Eq. (1.1) <u>admits a</u> $C^K$-<u>linearization where</u> $K$ <u>is given by</u> (6.8).

If $K$ satisfies (6.8), then it is evident that $K \geq$ Q-smoothness of A and consequently Theorem 3 implies Theorem 1.

Let $K$ satisfy (6.8) and fix $\alpha, \beta, \nu > 0$, $a > \rho^+$, $b > \rho^-$ (See Fig. 1) so that (6.5) and (6.7) hold and

$$\Sigma^-(A) \subseteq (-b\beta, -\beta), \quad \Sigma^+(A) \subseteq (\alpha, a\alpha), \tag{6.9}$$

and $\alpha (M - K a) > -\nu$, $\beta (N - K b) > -\nu$. Now choose $\sigma$ so that $0 < \sigma < \nu$ and

$$\alpha (M - K a) > -\nu + \sigma, \quad \beta (N - K b) > -\nu + \sigma. \tag{6.10}$$

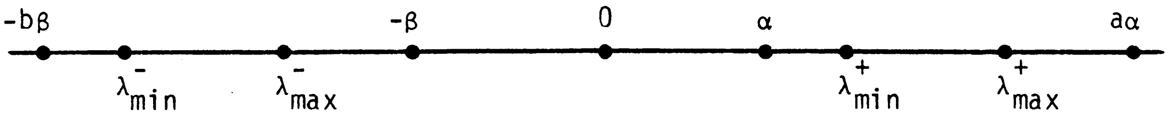


Figure 1. Spectrum of A.

$$\lambda^{\pm}_{min,max} = \min,\max \{ \operatorname{Re} \lambda : \lambda \in \Sigma^{\pm}(A) \}.$$

It follows from (6.9) that there is a projection $P$ on $X$ and a positive constant $k$ such that

$$|e^{At} P| \leq k\, e^{-\beta t} \qquad ,t \geq 0 \tag{6.11}$$

$$k^{-1} e^{-\beta t} \leq |e^{At} P| \leq k\, e^{-b\beta t} \qquad ,t \leq 0 \tag{6.12}$$

$$k^{-1} e^{\alpha t} \leq |e^{At} [I - P]| \leq k\, e^{a\alpha t} \qquad ,t \geq 0 \tag{6.13}$$

$$|e^{At} [I - P]| \leq k\, e^{\alpha t} \qquad ,t \leq 0 \tag{6.14}$$

where (for simplicity) we use the same $k$ in (6.5) and (6.11-14). Without any loss of generality we can assume that

$$U = \text{Range } (I - P) = \text{Unstable manifold}$$

$$V = \text{Range } (P) = \text{Stable manifold}$$

are orthogonal subspaces of $X$.

<u>The parameters</u> $\varepsilon$ <u>and</u> $r$ : Next we fix the parameters $\varepsilon$ and $r$. With $F$ given so that $D^P F(0) = 0$ for $0 \leq P \leq Q$, we define $F_r(x,\varepsilon)$ as in Section III. Then replace Eq. (6.1) by

$$x' = Ax + F_r(x,\varepsilon) . \qquad (6.\varepsilon,r)$$

Next we choose the functions

$$G_j: \ X \times \Sigma^{Q+1} \to L_Q^s , \qquad 0 \leq j \leq Q$$

by Lemma 7. Let $B$ be given by (3.1) and determine the constants $C_1$ and $C_2$ so that (4.9-10) hold. Next fix $\delta$ so that $0 < \delta < 1$ and let $\varepsilon = r$ be chosen so that

$$0 < r < \min(1/Q,1/5) \qquad (6.15)$$

$$C_2 B \ \varepsilon^Q \ r^{1-Q} = C_2 B \ \varepsilon^1 \leq \nu\delta/2k. \qquad (6.16)$$

<u>The nonlinear equations on</u> $L_Q^s$ : We now make a change of variables

$$x = g(w,s) = u + v + \textstyle\sum_{j=0}^{Q} S_j \ \langle u\rangle^{Q-j} \langle v\rangle^j \qquad (6.17)$$

where $u \in U$, $v \in V$ and $S = (S_0,\ldots,S_Q) \in \Sigma^{Q+1}$. Next we replace $u$ and $v$ by $e^{At}u_0$ and $e^{At}v_0$, where $u_0 \in U$ and $v_0 \in V$. Also we replace $S$ by a time-varying function $S(t)$ so that the resulting function $x = x(t)$ given by (6.17) is a solution of (6.$\varepsilon$,r). As we next show, the entries $S_0(t),\ldots,S_Q(t)$ must satisfy a related differential equation on $L_Q^s$ while $|x(t)| < r$. By differentiating (6.17) with respect to $t$ we get

$$x' = u' + v' + \frac{d}{dt}\left(\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j} \langle v\rangle^{j}\right) = Ax + F_r(x,\varepsilon) .$$

Since $u' = Au$ and $v' = Av$ this becomes

$$\frac{d}{dt}\left(\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j}\langle v\rangle^{j}\right) = A\left(\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j}\langle v\rangle^{j}\right) + F_r(x,\varepsilon) .$$

If $L$ is given by (6.2), then it follows from (4.8) that

$$\sum_{j=0}^{Q} S_j' \langle u\rangle^{Q-j}\langle v\rangle^{i} = L\left(\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j}\langle v\rangle^{j}\right) + F_r(x,\varepsilon)$$

$$= L\left(\sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j}\langle v\rangle^{j}\right) + \sum_{j=0}^{Q} G_j(w,S)\langle u\rangle^{Q-j}\langle v\rangle^{j} .$$

Therefore $S_j = S_j(t)$ is a solution of

$$S_j' = LS_j + G_j(w,S) , \qquad 0 < j < Q , \tag{6.18}$$

Conversely if $S_j = S_j(t)$ are solutions of (6.18) with $|S_j(t)| < 1$ for $t$ in some interval $I$ and if the corresponding function $x = x(t)$ given by (6.17) satisfies $|x(t)| < r$ for $t \varepsilon I$, then $x(t)$ is a solution of $(6.\varepsilon,r)$. (We will also use the notation $G_j(u,v,S)$ in place of $G_j(w,s)$ in (6.18), where $w = u + v$.)

Primary objective: Let us return to the change of variables given by (6.17). We will apply the theory of Section V to the system of equations (6.18) to find appropriate bounded continuous functions $S_j = S_j(w)$

$$S_j\colon\ X \to L_Q^s \quad , \qquad 0 \leq j \leq Q \, .$$

We will then replace $S_j$ in (6.18) by $S_j(w)$ to obtain

$$x = w + \Phi(w) = u + v + \textstyle\sum_{j=0}^{Q} S_j(w)\ \langle u\rangle^{Q-j}\ \langle v\rangle^{j} \, . \tag{6.19}$$

where $w = u + v$ . The next lemma will be needed to show that the change of variables in (6.19) is of class $C^K$ where $K$ satisfies (6.8).

For the purpose of proving the following lemma, we adopt a standard notation for partial derivatives. Let $h(w) = h(u,v)$ be a sufficiently smooth function defined on an open set in $X$. An n-tuple of nonnegative integers $\alpha = (\alpha_1,\ldots,\alpha_n)$ will be called an <u>index</u>. We define

$$D_w^{\alpha}\, h = \frac{\partial^{|\alpha|}}{\partial w_1^{\alpha_1} \ldots \partial w_n^{\alpha_n}}\, h \, .$$

For indicies $\alpha$ and $\beta$ of the form $\alpha = \alpha(U)$, $\beta = \beta(V)$ (See Section II) define

$$D_u^{\alpha} D_v^{\beta} h = \frac{\partial^{|\alpha|+|\beta|}}{\partial u_1^{\alpha_1} \ldots \partial u_a^{\alpha_a}\, \partial v_{a+1}^{\beta_{a+1}} \ldots \partial v_n^{\beta_n}}\, h \, . \tag{6.20}$$

The total partial derivative $D_u^p\, D_v^q\, h$, where $p$ and $q$ are nonnegative integers, is a tensor whose components are given by (6.20) where $|\alpha| = p$ and $|\beta| = q$.

For two indicies $\alpha$ and $\beta$ we will write $\alpha \leq \beta$ provided $0 \leq \alpha_i \leq \beta_i$ for all i. If $\alpha$, $\beta$, and $\gamma$ are indicies with $0 \leq \gamma$, $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$. Also if $0 \leq \beta - \alpha \leq \gamma$ then $|\gamma - \beta + \alpha| = |\gamma| - |\beta| + |\alpha|$.

Lemma 13. Let $x = w + \Phi(w)$ be given by (6.19) where $S_j\colon\ X \to L_Q^s$ denote bounded continuous functions for $0 \le j \le Q$. Assume that there is an integer $K \ge 0$ and $\mu > 0$, $\tau > 0$ such that $M - Ka + \mu > 0$, $N - Kb + \tau > 0$ and $K \le \min(M,N)$. This change of variables is of class $C^K$ when the following three conditions are satisfied:

(1) For $0 \le j \le N-1$, the function $S_j = S_j(u)$ depends only on $u \in U$ and is of class $C^K$ for $u \neq 0$, and for every index $p = (p_1,\dots,p_a,0,\dots,0)$ with $0 \le |p| \le K$ there is a constant $d_1$ such that

$$|u|^{|p|a-\mu}\,|D_u^p\, S_j(u)| \le d_1 \tag{6.21}$$

in a punctured neighborhood of $u = 0$.

(2) For $N+1 \le j \le Q$, the function $S_j = S_j(v)$ depends only on $v \in V$ and is of class $C^K$ for $v \neq 0$, and for every index $q = (0,\dots0,q_{a+1},\dots,q_n)$ with $0 \le |q| \le K$ there is a constant $d_2$ such that

$$|v|^{|q|b-\tau}\,|D_v^q\, S_j(v)| \le d_2 \tag{6.22}$$

in a punctured neighborhood of $v = 0$.

(3) The function $S_N(u,v)$ is of class $C^K$ in $u$ and $v$ when $u \neq 0$ and $v \neq 0$. Furthermore for any indices

$$p = (p_1,\dots,p_a,0,\dots,0) \quad , \quad q = (0,\dots,0,q_{a+1},\dots,q_n)$$

with $0 \le |p| + |q| \le K$ there is a constant $d_3$ such that

$$|u|^{|p|\ a-\mu}\ |v|^{|q|\ b-\tau}\ |D_u^p\ D_v^q\ S_N\ (u,v)| \leq d_3\ , \quad (u \neq 0\ ,\ v \neq 0)\ . \tag{6.23}$$

Proof: We will show that each of the functions $\Phi^j$ defined by $\Phi^j(u,v) = S_j(w)\ \langle u\rangle^{Q-j}\langle v\rangle^j$ , $0 \leq j \leq Q$ , is of class $C^K$ in a neighborhood of $u = v = 0$ . It follows from (2.13) that each $\Phi^j$ can be written in the form

$$\Phi^j = \sum s_j^{\alpha,\beta}\ u^\alpha\ v^\beta$$

where $\alpha = (\alpha_1,\ldots,\alpha_a,0,\ldots,0)$ , $\beta = (0,\ldots,0,\beta_{a+1},\ldots,\beta_n)$ , $|\alpha| = Q - j$ , $|\beta| = j$ , and $s_j^{\alpha,\beta}$ denote the components of $S_j$ . It will suffice to show that each of the functions

$$g = s_j^{\alpha,\beta}\ u^\alpha\ v^\beta = su^\alpha v^\beta\ ,$$

where $s = s_j^{\alpha,\beta}$ is a real-valued function satisfying either (1), (2) or (3), depending on the choice of $j$ , is of class $C^K$ in a neighborhood of $u = v = 0$ .

Let us first look at the case where $j = N$ . Then one has $|\alpha| = M$ , $|\beta| = N$ . Now consider indicies $p = (p_1,\ldots,p_a,0,\ldots,0)$ , $q = (0,\ldots,0,q_{a+1},\ldots,q_n)$ where $0 \leq |p| + |q| \leq K$ . By Laplace's Formula one has

$$\begin{aligned} D_u^pD_v^qg = D_u^pD_v^qsu^\alpha v^\beta &= \sum e\ (D_u^r\ D_v^t\ s)(D_u^{p-r}\ u^\alpha)(D_v^{q-t}\ v^\beta) \\ &= \sum \hat{e}\ (D_u^r\ D_v^t\ s)\ u^{\alpha-p+r}\ v^{\beta-q+t} \end{aligned} \tag{6.24}$$

where the coefficients $e$, $\hat{e}$ depend on $p$, $q$, $r$, $t$, $\alpha$ and $\beta$ (but not on $u$ and $v$) and the summation $\sum$ is over all indicies $r$ and $t$ with $0 \leq r \leq p$ , $0 \leq t \leq q$ and with $p - r \leq \alpha$ and $q - t \leq \beta$ .

We then know that

$$\begin{aligned} |u^{\alpha-p+r}| &\leq |u|^{|\alpha|-|p|+r} \leq |u|^{M-|p|a+|r|a} \\ &\leq |u|^{M-Ka+|r|a+\mu-\mu} \leq |u|^{M-Ka+\mu}\ |u|^{|r|a-\mu}\ . \end{aligned}$$

Similarly one has

$$|v^{\beta-q+t}| \leq |v|^{N-Kb+\tau} |v|^{|t|a-\tau}.$$

Therefore by part (3) of the lemma there is a constant $d_3$ such that

$$|(D_u^r D_v^t s)u^{\alpha-p+r} v^{\beta-q+t}| \leq d_3|u|^{M-Ka+\mu} |v|^{N-Kb+\tau}.$$

Since $M - Ka + \mu > 0$ and $N - Kb + \tau > 0$ it follows from (6.24) that $D_u^p D_v^q g$ has a unique continuous extension to the linear manifolds $u = 0$ and $v = 0$, viz. $D_u^p D_v^q g = 0$ when $u = 0$ or $v = 0$. Similarly for $0 \leq j \leq N - 1$ and $N + 1 \leq j \leq Q$ we conclude that $D_u^p D_v^q g = 0$ is the continuous extension of $D_u^p D_v^q g$ to the manifolds $u=0$ and $v=0$.

In order to complete the proof we need to show that the remainder term in the Taylor series expansion of $\Phi^j$ satisfies (2.2) with $N$ replaced by $K$. For this purpose it is convenient to divide the argument into 4 parts:

(i) $w = 0$

(ii) $w = u$, $u \neq 0$

(iii) $w = v$, $v \neq 0$

(iv) $w = u + v$, $u \neq 0$ and $v \neq 0$.

We will show that in each of the four cases above and for $0 \leq j \leq Q$, one has

$$\Phi^j(u + h, v + k) = \Phi^j_{p,q}\langle h\rangle^p\langle k\rangle^q + \Phi^j_R$$

the summation $\sum$ is over integers $0 \leq p + q \leq K$, the coefficient $\Phi^j_{p,q}$ is the

the total derivative $D_u^p D_v^q \Phi^j$ (which, we have seen, is continuous in w), and $\Phi_R^j$ satisfies

$$\lim \|(h,k)\|^{-K} \Phi_R^j = 0 \tag{6.25}$$

where $\|(h,k)\| = \max(|h|,|k|)$ and the limit in (6.25) is taken as $(h,k) \to (0,0)$. The main problem is to verify (6.25).

The argument for each of the following cases:

1) $w = u + v$ , $u \neq 0$ ; $\quad 0 \leq j \leq N - 1$

2) $w = u + v$ , $v \neq 0$ ; $\quad N + 1 \leq j \leq Q$

3) $w = u + v$ , $u \neq 0$ and $v \neq 0$ ; $\quad j = N$

is the same. By Eq. (2.13) we see that $\Phi^j$ is a sum of terms, and that each term is the product of $C^K$-functions near $w$ . Hence $\Phi^j$ is of class $C^K$ near $w$ .

For the case

4) $w = 0$ ; $0 \leq j \leq Q$

we write $\Phi^j = S_j(0) \langle h\rangle^{Q-j}\langle k\rangle^j + \Phi_R^j$ where $\Phi_R^j = [S_j(w) - S_j(0)]\langle h\rangle^{Q-j}\langle k\rangle^j$. Since $Q \geq K$ and $S_j(w)$ is continuous at $w = 0$ we see that $\Phi_R^j$ satisfies (6.25).

For the four remaining cases

5) $w = u + v$ , $v \neq 0$ ; $\quad 0 \leq j \leq N - 1$

6) $w = u + v$, $u \neq 0$ ; $\quad N + 1 \leq j \leq Q$

7) $w = u + v$, $u = 0$ and $v \neq 0$ ; $\quad j = N$

8) $w = u + v$, $u \neq 0$ and $v = 0$ ; $\quad j = N$

the arguments are similar. Let us look for example at case 8) for $j = N$.

$$\Phi^j(u + h,k) = S_j(u + h,k)\langle u + h\rangle^M\langle k\rangle^N$$
$$= S_j(u,0)\langle u\rangle^M\langle k\rangle^N + \Phi_R^j \quad ,$$

where the last equation defines $\Phi_R^j$. Now one has

$$\Phi_R^j = S_j(u + h,k)\langle u + h\rangle^M\langle k\rangle^N - S_j(u,0)\langle u\rangle^M\langle k\rangle^N$$
$$= \sum_{p=1}^M \binom{M}{p} S_j(u,0)\langle h\rangle^p\langle u\rangle^{M-p}\langle k\rangle^N$$
$$+ [S_j(u + h,k) - S_j(u,0)]\langle u + h\rangle^M\langle k\rangle^N \ .$$

By using the continuity of $S_j$ and the fact that $K < \min(M,N)$, we see that $\Phi_R^j$ satisfies (6.25).

<u>Solutions of Equation</u> (6.18): We now assume that $\mathrm{Re}\ \gamma(\lambda,m) \neq 0$ for all $\lambda \in \sum(A)$ and all $|m| = Q$. The $(Q + 1)$-equations given by (6.18) can be written as a single system

$$S' = (\mathrm{diag}\ L)S + G(w,S) \tag{6.26}$$

where $S = (S_0,\ldots,S_Q)$, $G = (G_0,\ldots,G_Q)$ and $\mathrm{diag}\ L$ is a square matrix with $(Q + 1)$-blocks of $L$ on the main diagonal and 0's elsewhere. Let $\underline{W} = \mathrm{diag}\ W$ be the Green's function for the linearized equation

$$S' = (\mathrm{diag}\ L)S \tag{6.27}$$

where $W$ is given by (6.6).

We seek solutions of (6.26) in the form

$$S = (S_0,\ldots,S_Q) = S(w)$$

where $S$ is a fixed point of the operator

$$\mathfrak{J}\, S(w) = \int_{-\infty}^{\infty} \underline{W}(0,s)G(w \cdot s, S(w \cdot s))ds \tag{6.28}$$

in the class $BC(X,(L_Q^s)^{Q+1})$. In (6.28) the function $w \cdot t = e^{At}w$ is the solution of (6.3) passing through $w \in X$, and $w = u + v$ where $u \in U$ and $v \in V$.

We want to apply Lemma 9 to the operator $\mathfrak{J}$ given by (6.28). In order to do this we define

$$|S| = \max\{|S_j| : 0 \leq j \leq Q\}$$

$$|D_w^K \;\; D_s^L \, G| = \max\{|D_w^K \;\; D_s^L \, G_j| : 0 \leq j \leq Q\}$$

for $0 \leq K + L$. By (4.10), (6.5), (6.7) and (6.16) we see that if $S \in BC(X, \sum^{Q+1})$ then

$$\| \mathfrak{J}\, S(w) \|_\infty \leq \frac{2k}{\nu} \; C_2 \; B \; \varepsilon^Q \; r^{1-Q} \leq \delta < 1. \tag{6.29}$$

Also by (4.10) one has

$$\begin{aligned} |(G(w,S_1) - G(w,S_2)| &\leq \| D_s^1 \; G \|_\infty \; |S_1 - S_2| \\ &\leq C_2 \; B \; \varepsilon^Q \; |S_1 - S_2| \\ &\leq C_2 \; B \; \varepsilon \; |S_1 - S_2| \; . \end{aligned}$$

Since $2k \; C_2 \; B \; \varepsilon \leq \delta\nu < \nu$ by (6.16) we see that (5.5) is valid. Therefore by Lemma 9, Eq. (6.28) has a unique bounded continuous solution

$$S(w) = (S_0(w), S_1(w), \ldots, S_Q(w)) \; . \tag{6.30}$$

Also by (6.29) one has

$$|S(w)| \leq \delta < 1 \tag{6.31}$$

for all $w = u + v \varepsilon X$ .

Because of the "triangular" form of the function $G$ as described by Lemma 7(C,D), we see that the function $S_0(w)$ in (6.30) is the unique bounded continuous solution of

$$\mathcal{J}_0 S_0(u) = \int_{-\infty}^{\infty} W(0,s) G_0(u \cdot s, S_0(u \cdot s))ds . \tag{6.32}$$

Therefore $S_0(w)$ is independent of $v$ since $G_0$ is independent of $v$. Similarly $S_1(w)$ is independent of $v$. In fact for $0 \leq j \leq N - 1$, the functions $S_j(w)$ are independent of $v \varepsilon V$. We shall write them in the form $S_j = S_j(u)$. For the same reasons we see that for $N + 1 \leq j \leq Q$, the functions $S_j = S_j(v)$ are independent of $u \varepsilon U$.

Fixed Points of $\mathcal{J}$ as Solutions of (6.26): Let $S = S(w) = S(u,v)$ be the fixed point of $\mathcal{J}$ described in the last two paragraphs and consider $S(w \cdot t) = S(u \cdot t, v \cdot t)$ as a function of $t$. One then has

$$S(u \cdot t, v \cdot t) = \int_{-\infty}^{\infty} \underline{W}(0,s)\ G(u \cdot (t+s),\ v \cdot (t+s), S(u \cdot (t+s),\ v \cdot (t+s))$$

$$= \int_{-\infty}^{\infty} \underline{W}(t,\xi)\ G(u \cdot \xi, v \cdot \xi, S(u \cdot \xi, v \cdot \xi)$$

where the change of variables $\xi = t + s$ was used in the last equality. By differentiating the last expression with respect to $t$, we see that $S(w \cdot t)$ is a solution of (6.26) where $w$ is replaced by $w \cdot t$. By (6.31) one has $|S(w \cdot t)| \leq \delta < 1$ for all $w \varepsilon X$ and $t \varepsilon R$.

Conversely let $\phi(t)$ be any solution of (6.26) (where $w$ is replaced by $w \cdot t$) and assume that $\phi$ satisfies $|\phi(t)| < 1$ for all $t \in R$. We then claim that $\phi(t) = S(w \cdot t)$. It is easy to verify this by applying the uniqueness portion of the Contraction Mapping Theorem to the operator

$$F\,\phi(t) = \int_{-\infty}^{\infty} \underline{W}(t,s)\; G(u\cdot s, v\cdot s, \phi(s))ds$$

on the class of continuous mappings $\phi : R \to (L_Q^s)^{Q+1}$ that satisfy $|\phi(t)| < 1$ for all $t \in R$.

Now make the change of variables

$$x = w + \Phi(w) = u + v + \textstyle\sum_{j=0}^{Q} S_j(w)\; \langle u\rangle^{Q-j}\; \langle v\rangle^{j} \tag{6.33}$$

where $w = u + v$, $u \in U$, $v \in V$ and $S = (S_0, \ldots, S_Q)$ denotes the fixed point of $\mathfrak{J}$. If we replace $w$ by $w \cdot t = e^{At}\, w$, where $|w| < (3/2)r$, and define $x(t) = w \cdot t + \Phi(w \cdot t)$ then we have the following facts as a consequence of Lemmas 3,4 and Eq. (4.8):

(1) If $|x(t)| < r$ for $t$ in an interval $I$, then $x(t)$ is a solution of $(6.\varepsilon, r)$ for $t \in I$.

(2) If $|w \cdot t| < r/2$ for $t$ in an interval $I$, then $x(t)$ is a solution of $(6.\varepsilon, r)$ for $t \in I$.

Next we will show that the change of variables in (6.33) is of class $C^K$ for $w$ near $w = 0$, where $K$ is given by (6.8). Assuming this smoothness we next see that $D\Phi(0) = 0$. Consequently by the Inverse Function Theorem the change of variables (6.33) describes a $C^K$-conjugation between $(6.\varepsilon, r)$ and (6.3) near $x = w = 0$.

<u>Smoothness of Solutions</u>: The last step in the argument is to verify the smoothness of the solution

$$S = (S_0(u),\ldots,S_{N-1}(u),S_N(u,v),S_{N+1}(v),\ldots,S_Q(v))$$

of Eq. (6.26) and (6.28) as described in Lemma 13.

First let us consider $S_j = S_j(u)$ for $0 \leq j \leq N-1$. We will now show that $S_j(u)$ is of class $C^K$ for $u \neq 0$, with the aid of Lemma 10. If $u \in U$, the unstable manifold, and $u \neq 0$ then $|u \cdot t| \to +\infty$ as $t \to +\infty$. Consequently there is a $T = T(u) \geq 0$ such that $|u \cdot t| > 3r/2$ for $t > T$. It follows from (6.13) that for all $u \in U$, $u \neq 0$,

$$|e^{At}[I - P]u| = |u \cdot t| > \frac{3r}{2} \qquad , t > T$$

where $T$ is defined by

$$k^{-1}e^{\alpha T} = \frac{3r}{2} \cdot \frac{1}{|u|} . \tag{6.34}$$

Let $j$ be fixed, $0 \leq j \leq N - 1$, and define $R$ by

$$R(u,t,S) = G_j(e^{At}u,S) = G_j(e^{At}[I - P]u,S)$$

Recall that $G_j$ depends only on $u$ and $S_0,S_1,\ldots,S_j$. By Lemma 7(E) one has

$$D_u^p\, D_s^q\, G_j\, (e^{At}\, [I-P]\, u\, ,S) = 0 \qquad , \qquad t > T . \tag{6.35}$$

where $T$ is given by (6.34) and $0 \leq p + q \leq Q$. Since one has

$$|D_u^p\, D_s^q\, R\,| \leq |D_u^p\, D_s^q\, G_j|\; |e^{At}[I - P]|^p$$

it follows from (6.13-14), (6.34-35) and (4.9) that

$$\sup\, |D_u^p\, D_s^q\, R| \leq C_1\, B\, \varepsilon^Q\, r^{1-2Q}\, k^p\, e^{a\alpha Tp} \leq 2^{ap}C_1\, B\varepsilon^Q r^{ap+1-2Q}\, k^{p+ap}\, |u|^{-ap} \tag{6.36}$$

where the supremum is taken over $t \in R$ and $S \in \sum^{Q+1}$. Since (6.35) remains valid when $u$ is replaced by $(u + h) \varepsilon U$ where $h$ is small, it follows that (6.36) holds uniformly in a neighborhood of $u$ in $U$. It follows that (5.6) is valid in open neighborhood $O(u)$, for $1 \leq p + q \leq Q$.

Let us next consider $D_S^q R$ for $1 \leq q \leq Q$. Since one has $|D_S^L R| \leq |D_S^L G_j|$

it follows from (4.10), and (6.16) that

$$\sup |D_S^L R| \leq C_2 B \varepsilon^Q \leq \nu\delta/2k \tag{6.37}$$

where the supremum is taken over $t \varepsilon R$ and $S \varepsilon \sum^{Q+1}$. Furthermore (6.37) remains valid uniformly in a neighborhood of $u$ in $U$, since (6.35) remains valid when $u$ is replaced by $(u + h) \varepsilon U$, where $h$ is small. Therefore (5.7) is valid, and its follows from Lemma 10 that $S_j = S_j(u)$ is of class $C^Q$ when $u \neq 0$, for $0 \leq j \leq N - 1$.

The argument that $S_j = S_j(v)$ is of class $C^Q$ for $v \neq 0$, when $N + 1 \leq j \leq Q$, is similar. We omit these details.

Let us now consider $S_N = S_N(u,v)$ for $u \neq 0$ and $v \neq 0$. It follows from (6.12) and (6.13) that

$$|e^{At}w|^2 = |e^{At}[I-P]u|^2 + |e^{At}Pv|^2 > \left(\frac{3r}{2}\right)^2, \qquad t < T_1 \text{ and } T_2 < t$$

where $T_1$ and $T_2$ are defined by

$$k^{-1}e^{-\beta T_1} = \frac{3r}{2}\frac{1}{|v|}, \qquad k^{-1}e^{\alpha T_2} = \frac{3r}{2}\frac{1}{|u|} \tag{6.38}$$

Define

$$R(u,v,t,S) = G_N(e^{At}[I - P]u, e^{At} Pv,S) .$$

By Lemma 7(E) one has

$$D_u^p D_v^q D_s^r G_N(e^{At}[I - P]u, e^{At}Pv,S) = 0 \tag{6.39}$$

for $t < T_1$ and $T_2 < t$ provided $0 \leq p + q + r \leq Q$.

Since one has

$$|D_u^p\, D_v^q\, D_s^r\, R| \leq |D_u^p\, D_v^q\, D_s^r\, G_N|\ |e^{At}[I - P]|^p\ |e^{At}P|^q$$

it follows from (4.9), (6.11-14) and (6.38-39) that

$$\begin{aligned} \sup\ |D_u^p\, D_s^q\, D_s^r\, R| &\leq C_1\ B\ \varepsilon^Q\ r^{1-2Q}\ k^p e^{ap\alpha T_2}\ k^q e^{-bq\beta T_1} \\ &\leq 2^{ap+bq} C_1\ \varepsilon^Q\ r^{ap+bq+1-2Q}\ k^{p+q+ap+bq}\ |u|^{-ap}|v|^{-bq} \end{aligned} \tag{6.40}$$

where the supremum is taken over all $t \varepsilon R$ and $S \varepsilon \sum^{Q+1}$. Since (6.39) remain valid in a neighborhood of $w$, it follows that (6.40) holds uniformly in a neighborhood of $w$. This establishes (5.6). Since $|D_s^L\ R| \leq |D_s^L\ G_j|$ it follows (as above) that

$$\sup\ |D_S^L\ R| \leq C_2\ B\ \varepsilon^Q \leq \nu\delta/2k \tag{6.41}$$

uniformly in a neighborhood of $w$, which establishes (5.7). Hence $S_N$, is of class $C^Q$ in $w = u + v$ when $u \neq 0$ and $v \neq 0$.

In order to complete the proof of smoothness of the conjugacy we need to verify that the functions $S_j(u)$, $0 \leq j \leq N - 1$, $S_j(v)$, $N + 1 \leq j \leq Q$ and $S_N(u,v)$ satisfy (6.21-23) for a suitable choice of $\mu$ and $\tau$. Let $K$ be given by (6.8) and let (6.10) be valid. Now set $\mu = \alpha^{-1}(\nu - \sigma)$ and $\tau = \beta^{-1}(\nu - \sigma)$. Then one has $M - Ka + \mu > 0$, $N - Kb + \tau > 0$ and $K \leq \min(M,N)$. Let us now derive (6.21). Since $S(w) = S(u,v)$ is a fixed point of (6.28) one has

$$S(u,v) = \int_{-\infty}^{\infty} \underline{W}(0,s)\ X(e^{As}[I - P]u, e^{As}Pv)ds \tag{6.42}$$

where $X$ is defined by

$$X(u,v) = G(u,v,S(u,v)) \; .$$

For $u \neq 0$ *and* $v \neq 0$ the infinite integral in (6.42) becomes a finite integral from $T_1$ to $T_2$, where $T_1$ and $T_2$ are given by (6.38). As a result one can bring derivatives with respect to $u$ and $v$ under the integral sign. One then has

$$D_u^p D_v^q S(u,v) = \int_{-\infty}^{\infty} \underline{W}(0,s)(D_u^p D_v^q X)[e^{As}[I - P]]^p [e^{As}P]^q ds \qquad (6.43)$$

provided $0 \leq p + q \leq Q$. For $u \neq 0$ and $v \neq 0$ it follows from (6.11-14) and (6.38) that

$$|e^{As}[I - P]|^p \leq k^p e^{pa\alpha s} = k^p e^{(pa - \mu)\alpha s} e^{(\nu - \sigma)s} \quad , \quad 0 \leq s \leq T_2$$

$$|e^{As}P|^q \leq k^q e^{-qb\beta s} = k^q e^{-(qb - \tau)\beta s} e^{-(\nu-\sigma)s} \quad , \quad T_1 \leq s \leq 0$$

$$|e^{As}[I - P]|^p \leq k^p \quad , \quad s \leq 0$$

$$|e^{As} P|^q \leq k^q \quad , \quad s \geq 0 \; .$$

As a result we see that by (6.11-14) and (6.38) one has

$$|u|^{pa-\mu} \; e^{(pa - \mu)\alpha s} \leq k^{2pa} \left( \frac{3r}{2} \right)^{pa} \; , \quad s \leq T_2$$

$$|v|^{qb-\tau} \; e^{-(qb-\tau)\beta s} \leq k^{qb} \left( \frac{3r}{2} \right)^{qb} \quad , \quad T_1 \leq s \; .$$

Consequently by (6.5) and (6.43) one has

$$|u|^{pa-\mu} \; |v|^{qb-\tau} \; |D_u^p \; D_v^q \; S(u,v)|$$

$$\leq \int_{-\infty}^{\infty} ke^{-\nu|s|} \| D_u^p \; D_v^q \; X \|_\infty \; k^{pa+qb+p+q} \left( \frac{3r}{2} \right)^{pa+qb} e^{(\nu-\sigma)|s|} \; ds$$

$$\leq \frac{2}{\sigma} k^{pa+qb+p+q} \left( \frac{3r}{2} \right)^{pa+qb} \| D_u^p \; D_v^q \; X \|_\infty$$

which proves (6.23). The arguments for (6.21-22) are similar, and we omit these details. This completes the proof of Theorem 2.

The argument for Theorem 1 is straight-forward. When $A$ is stable then, by Lemma 8, the system (6.18) is replaced by a single equation

$$S' = LS + G(v,S)$$

where $x = v + S\langle v\rangle^Q$ , $Q = N + 1$ and $v \in V = X$ . (Here one has $U = \{0\}$ .) The operator $\mathfrak{J}$ in (6.28) then becomes

$$\mathfrak{J}\, S(u) = \int_{-\infty}^{\infty} W(0,s)\ G(v \cdot s\ ,\ S(v \cdot s))ds\ .$$

The rest of the argument now follows the reasoning used above.

An extension of Theorem 1 in the style of Theorem 3 is possible. Let $A$ be stable and let $K \geq 0$ be the largest integer such that $K \leq Q$ and

$$\Lambda^{-}\ (Q - K\rho^{-}) > -\Gamma\ . \tag{6.44}$$

One then has

Theorem 4. Under the assumptions of Theorem 1, Eq. (1.1) admits a $C^K$-linearization where $K$ is given by (6.44).

## VII. Linearization of Diffeomorphisms.

The methods we used above extend directly to the problem of finding sufficient conditions for a smooth linearization in the vicinity of a fixed point of a diffeomorphism. We shall outline the main ideas here.

Let $U$ be an open set in a finite dimensional Banach space $X$ with $0 \in U$. Let $F,G:U \to X$ be two $C^M$-diffeomorphisms with $M \geq 2$ and with $F(0) = G(0) = 0$. A $C^N$ conjugation $H$ between $F$ and $G$ near $x = 0$ is a $C^N$-diffeomorphism with $H(0) = 0$ and $F = H^{-1}GH$ near $x = 0$. If $F$ is $C^N$-conjugate to $G$ where $G(y) = Ay$ is linear and $A = DF(0)$, then $F$ is said to admit a $C^N$-linearization near $x = 0$.

For diffeomorphisms we say that a matrix $A$ is hyperbolic if the eigenvalues $\lambda_1,\ldots,\lambda_n$ satisfy $|\lambda_i| \neq 1$ for all $i$. Also $A$ is said to be stable if $|\lambda_i| < 1$ for all $i$. For $m = (m_1,\ldots,m_n)$ we define $\mu(\lambda,m)$ by

$$\mu(\lambda,m) = \lambda(\lambda_1^{m_1} \ldots \lambda_n^{m_n})^{-1}$$

where $\lambda$ is a complex number. We shall say that $A$ satisfies the Sternberg condition of order $N \geq 2$ provided $\mu(\lambda,m) \neq 1$ for all $\lambda \in \sum(A)$ and all $m$ with $2 \leq |m| \leq N$. Finally $A$ satisfies the strong Sternberg condition of order $N \geq 2$ provided $A$ satisfies the Sternberg condition of order $N$ and

$$|\mu(\lambda,m)| \neq 1$$

for all $\lambda \in \sum(A)$ and all $|m| = N$.

Let $A$ be hyperbolic and let $\sum^+(A)$ or $\sum^-(A)$ denote, respectively, the eigenvalues $\lambda \in \sum(A)$ with $|\lambda| > 1$. A is strictly hyperbolic if $A$ is hyperbolic and both $\sum^+(A)$ and $\sum^-(A)$ are nonempty. If $A$ is hyperbolic and $\sum^i(A) \neq \phi$, we define the spectral spread $\rho^i$ by

$$\rho^i = \frac{\max\{|\log|\lambda||: \lambda \in \Sigma^i(A)\}}{\min\{|\log|\lambda||: \lambda \in \Sigma^i(A)\}}$$

where $i = +$ or $-$.

Let Q be a positive integer and let A be hyperbolic. We define the Q-smoothness of A to be the largest integer $K \geq 0$ such that:

(1) $Q - K\rho^- \geq 0$ , if $\sum^+(A) = \phi$ .

(2) $Q - K\rho^+ \geq 0$ , if $\sum^-(A) = \phi$ .

(3) There exist positive integers M , N with $Q = M + N$ , $M - K\rho^+ \geq 0$ , $N - K\rho^- \geq 0$ when A is strictly hyperbolic.

We then have the following results:

Theorem 5. Let $Q \geq 2$ be an integer and assume that $F:U \to X$ is a $C^{3Q}$-diffeomorphism of U with $0 \in U$ , $F(0) = 0$ and $A = DF(0)$ . Let A be nonsingular and strictly hyperbolic, and consider one of the following two assumptions:

(A) Assume that A satisfies the strong Sternberg condition of order Q .

(B) Assume that $D^P F(0) = 0$ for $2 \leq P \leq Q - 1$ and that $|\mu(\lambda,m)| \neq 1$ for all $\lambda \in \sum(A)$ and all $|m| = Q$ .

Under either assumption (A) or (B), F admits a $C^K$-linearization, where K is the Q-smoothness of A .

Theorem 6. If A is nonsingular and stable, then Theorem 5 remains valid when F is of class $C^{2Q}$ .

The proofs of these theorems involve simple modifications of the arguments used in Theorems 1 and 2. The re-scaling by $\varepsilon$ and the truncation methods described in Section III apply to the nonlinear part $F(x) - Ax$. Likewise the Factorization Lemmas 7 and 8 apply here as well. For the hyperbolic case, the change of variables

$$x = u + v + \sum_{j=0}^{Q} S_j \langle u\rangle^{Q-j} \langle v\rangle^{j} \tag{7.1}$$

leads to the family of difference equations

$$S_j(n + 1) = LS_j(n) + G_j\,(w \cdot n\, ,\, S_j(n))\, , \qquad 0 \leq j \leq Q \tag{7.2}$$

where $w \cdot n$ is the solution of

$$w \cdot (n + 1) = Aw \cdot (n)$$

that satisfies $w \cdot 0 = w$. One seeks a function

$$S = (S_0, \ldots, S_Q) = S(w)$$

as fixed point of the operator

$$\mathfrak{J}\; S(w) = \sum_{n = -\infty}^{\infty} \underline{W}(0,n)\; G(w \cdot n\, ,\, S(w \cdot n)) \tag{7.3}$$

where $G = (G_0, \ldots, G_Q)$ , $\underline{W}(m,n) = \mathrm{diag}(W(m,n))$ ,

$$W(m,n) = \begin{cases} L^m P_0 L^{-n} & , \quad n \leq m \\ L^m[I - P_0] L^{-n} & , \quad m < n , \end{cases}$$

and $P_0$ is the linear projection on $L^s_Q$ with Range(P) being stable manifold of L and Null Space (P) the unstable manifold of L . The theorems in Section V on the existence of smooth solutions extend directly to (7.3). With these observations, the proof of Theorems 5 and 6 follow the corresponding pattern of the differential systems argument.

Remarks 5. As in the case of Theorems 1 and 2, the two theorems for diffeomorphisms admit extensions in the style of Theorems 3 and 4. Define:

$\Gamma = \min \{ |1 - |\gamma(\lambda,m)|| : \gamma(\lambda,m)$ is an eigenvalue of $L\}$

$\Lambda^+ = \min \{ |\lambda| : |\lambda| > 1$ and $\lambda$ is an eigenvalue of $A\}$

$\Lambda^- = \min \{ |\lambda|^{-1} : |\lambda| < 1$ and $\lambda$ is an eigenvalue of $A\}$.

With these values of $\Gamma$, $\Lambda^+$ and $\Lambda^-$ being used in (6.8) and (6.42), Theorems 3 and 4 extend to diffeomorphisms.

6. Theorems 5 and 6 apply directly to the problem of linearization in the vicinity of a periodic solution of an autonomous ordinary differential equation. This problem is reduced to the problem of linearizing a diffeomorphism by studying the Poincare mapping. See Hale (1969) and Hartman (1964) for more details.

7. The problem of linearization in the vicinity of a general compact invariant manifold is more complicated. We will present this theory in a forthcoming paper, Sell (1983b).

## VIII. Dependence on Parameters.

The methods described above extend immediately to the problem of studying the dependence of the linearizing conjugation on a parameter $\theta$, when the coefficients of

$$x' = A(\theta)x + F(x,\theta) \qquad (8.1)$$

depend smoothly on $\theta$, where F is defined for $x \in U$ and $\theta \in \Theta$, and $A(\theta)$ is defined for $\theta \in \Theta$. We can prove the following results:

Theorem 7. Let Q and R be integers where $Q > 2$ and $0 < R < \infty$. Assume that $F(x,\theta)$ is of class $C^{3Q}$ in the x- variable for $x \in U$ with $0 \in U$ and $D_x^P F(0,\theta) = 0$ for $\theta \in \Theta$. Assume further that $A(\theta)$ and $F(x,\theta)$ are of class $C^R$ in the $\theta$-variable. Let $\theta_0 \in \Theta$ be fixed and let $A(\theta_0)$ be strictly hyperbolic, and consider one of the following two assumptions:

(A) $A(\theta_0)$ satisfies the strong Sternberg condition of order Q

(B) $D_x^P F(0,\theta) = 0$ for $0 < P < Q - 1$ and $\operatorname{Re} \gamma(\lambda,m) \neq 0$

for all $\lambda \in \sum(A(\theta))$, all $|m| = Q$, and all $\theta \in \hat{V}$, where $\hat{V}$ is a neighborhood of $\theta_0$.

Under either assumption (A) or (B) there is a neighborhood V of $\theta_0$ and a conjugation

$$x = w + \phi(w,\theta), \qquad \theta \in V$$

between (8.1) and

$$w' = A(\theta)w \qquad (8.2)$$

where $\Phi$ is of class $C^K$ in w and $C^R$ in $\theta$ and K is the Q-smoothness of $A(\theta_0)$.

Theorem 8. If A is stable, then Theorem 7 remains valid when F is of class $C^{2Q}$ in x.

Theorem 9. With the modifications described in Section VII, Theorems 7 and 8 are valid for diffeomorphisms

Proof: We will prove Theorem 7. The modifications required for Theorems 8 and 9 are discussed at the end of Section VI and in Section VII.

The first step is to show that statement (A) can be reduced to statement (B). If $A(\theta_0)$ satisfies the strong Sternberg condition of order $Q > 2$, then there is a neighbrhood $\hat{V}$ of $\theta_0$ with the property that $A(\theta)$ satisifes the strong Sternberg condition of order Q for all $\theta \in \hat{V}$. Next we note that the conjugacy $y = H(x)$ described in the Algebraic Reduction Lemma 11 is a $C^R$ - function of $\theta$ for $\theta \in \hat{V}$. One can show this easily in the case that $Q = 2$, which is typical. In this case one seeks a change of variables of the form

$$y = x + S(\theta) \langle x,x\rangle$$

where $S(\theta) \in L_2^s$ for $\theta \in \hat{V}$ and such that the y-equation does not have quadratic terms for $\theta \in \hat{V}$. It turns out that one has $S(\theta) = L(\theta)^{-1} F_2(\theta)$ where

$$F(x,\theta) = F_2(\theta) \langle x,x\rangle + \ldots,$$

$F_2(\theta) \in L_2^s$ and $L(\theta)$ is the linear operator on $L_2^s$ given by

$$L(\theta)S = A(\theta)S - \{S, A(\theta)\}.$$

Since $\gamma(\lambda,m) \neq 0$ for $|m| = 2$ we see that $L(\theta)^{-1}$ exists and is a $C^R$-function of $\theta$. Hence $S(\theta)$, which is the product of $C^R$-functions, is itself a $C^R$-function for $\theta \in \hat{V}$.

The conjugacy between (8.1) and (8.2) then has the form

$$x = w + \Phi(w,\theta) = u + v + \sum_{j=0}^{Q} S_j(w,\theta)\langle u\rangle^{Q-j} \langle v\rangle^{j}, \tag{8.3}$$

see Eq. (6.33). The terms $S = (S_0, \ldots, S_Q)$ are fixed points of the operator (6.28), where G now depends on $\theta \varepsilon \hat{V}$. Since the definition of G involves only derivatives of F with respect to x, we see that G is of class $C^R$ in the θ-variable. Also the Green's function $\underline{W}(0,s)$ depends on the projection $P_0 = P_0(\theta)$, see Eq. (6.5), which is a $C^R$-function of θ since A is a $C^R$-function of θ. Therefore the integrand in (6.28) is a $C^R$-function of θ.

Since the eigenvalues of A vary continuously in θ, there is a neighborhood $V \subseteq \hat{V}$ of $\theta_o$ such that (6.8) is valid for all $\theta \varepsilon V$, where K is the Q-smoothness of $A(\theta_o)$. By restricting V further, if necessary, we can assume that (4.9-10), (6.5), and (6.8-16) are valid uniformly for $\theta \varepsilon V$, where K is the Q-smoothness of $A(\theta_o)$. It then follows from Lemma 10, that the solution $S = S(w,\theta)$ of (6.28) is a $C^R$-function of θ. QED.

## IX. An Illustration in Celestial Mechanics.

An interesting application of our theory can be found in the study of the isosceles three-body problem in celestial mechanics, Moeckel (1983). This problem arises when two of the three bodies have an equal mass $m$, and the motion is confined so that these two bodies remain symmetric about a fixed axis in $R^3$. The motion of the third body is then restricted to this axis of symmetry.

The associated nonlinear system of equations describing this motion becomes a differential system in $R^4$, which we write as follows:

$$\begin{aligned} r' &= vr \cos \phi \\ v' &= U(\phi) \cos \phi - 1/2\, v^2 \cos \phi + 2rh \cos \phi \\ \phi' &= w \\ w' &= U'(\phi) \cos^2 \phi - 1/2\, vw \cos \phi - (2U(\phi) + 2rh - v^2) \sin \phi \cos \phi \end{aligned} \tag{9.1}$$

with the energy relation

$$1/2(v^2 \cos^2\phi + w^2 + \omega^2 r^{-1}) - U(\phi)\cos^2\phi = rh \cos^2\phi , \tag{9.2}$$

where $h < 0$ is the energy, $\omega$ is the angular momentum,

$$U(\phi) = 1/2\, m^{3/2}\, m_3\, [\alpha \sec \phi + 4(1 + 2\alpha \sin^2 \phi)^{-1/2}]$$

and $\alpha = m/m_3$ is the mass ratio. The derivation of these equations, which are real analytic in $R^4$, and the physical description of the $(r,v,\phi,w)$-coordinate system is given in Moeckel (1983). The r-variable is a measure of the "size" of the isosceles configuration.

With the energy $h$ fixed, the system of equations (9.1) represents a 3-dimensional manifold $M(\omega)$ in $R^4$, which is parameterized by the angular momentum $\omega$. We are especially interested in the manifold $M(0)$, since triple collision can only occur on $M(0)$. The set $(r,v,\phi,w)$ in $M(0)$ with $r = 0$ is a 2-dimensional submanifold; it is the McGehee triple collision manifold.

Let us now examine the fixed points of (9.1). There are six fixed points. First we note that the equation $U'(\phi) = 0$ has three solutions $\{\phi_-,0,\phi_+\}$ where $\phi_- < 0 < \phi_+$. Let $\phi_c$ denote any one of these solutions and define $v_c = (2U(\phi_c))^{1/2} > 0$. Then the six fixed points are $(r,v,\phi,w) = (0,\pm v_c,\phi_c,0)$ where $\phi_c = \phi_-,0,\phi_+$. We will now restrict our attention to the fixed point

$$P = (r,v,\phi,w) = (0,-v_0,0,0) \tag{9.3}$$

where $v_0^2 = 2U(0) = \beta(\alpha + 4)$ and $\beta = m^{3/2}m_3$.

The variational equation of (9.1) at the fixed points (9.3) is given by

$$\begin{pmatrix} \delta r \\ \delta v \\ \delta\phi \\ \delta w \end{pmatrix}' = \begin{pmatrix} -v_0 & 0 & 0 & 0 \\ 2h & +v_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & U''(0) & -v_0/2 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta v \\ \delta\phi \\ \delta w \end{pmatrix} \tag{9.4}$$

The $(4 \times 4)$ matrix $A$ in (9.4) is in block diagonal form, and the eigenvalues $\lambda$ of $A$ are solutions of the equation

$$(\lambda - v_0)(\lambda + v_0)[\lambda(\lambda + v_0/2) - U''(0)] = 0$$

At the fixed point $P$ we obtain the four eigenvalues

$$\lambda_1 = -v_0 \ , \ \lambda_2 = +v_0$$
$$\lambda_3 = -1/4\ v_0 + J \ , \qquad \lambda_4 = -1/4\ v_0 - J$$

where $J = 1/4(v_0^2 + a6U''(0))^{1/2} = 1/4\beta^{1/2}(4 - 55\alpha)^{1/2}$ . On $M(0)$ the meaningful eigenvalues are $\lambda_2$ , $\lambda_3$ , $\lambda_4$ . In this case one has reasonance at the sixth order since

$$\lambda_2 = 2\lambda_2 + 2\lambda_3 + 2\lambda_4 \ , \ \lambda_3 = \lambda_2 + 3\lambda_3 + 2\lambda_4 \ , \ \lambda_4 = \lambda_2 + 2\lambda_3 + 3\lambda_4$$

for all values of the mass ratio $\alpha$ . By a direct, and somewhat tedious calculation, one can show that there are no resonances of order $< 6$ . Hence the eigenvalues $\lambda_2$ , $\lambda_3$ , $\lambda_4$ of (9.4) satisfy the strong Sternberg condition of order 5 . The spectral spreads are given by $\rho^+ = 1$ and

$$\rho^- = \rho = \frac{v_0 + 4J}{v_0 - 4J}$$

for $0 < \alpha < 4/55$, and $\rho^- = 1$ for $\alpha > 4/55$. Theorem 1 then assures us that there is a $C^2$-linearization provided $\rho < 3/2$ , which is equivalent to $\alpha > 3/43$ . Since $\rho$ depends continuously on $\alpha$ , Theorem 3 implies that there is an $\alpha_0 < 3/43$ for which there is a $C^2$-linearization for $\alpha > \alpha_0$ . Furthermore it follows from Theorem 7 that for $3/43 < \alpha < 4/55$ , the $C^2$-conjugacy is a $C^R$-function of $\alpha$ and $\beta$ for every $R$ , $1 < R < \infty$ . The example in Sell (1983a) suggests that there is probably no $C^3$-linearization.

It is instructive to compare these results with the theory developed by Bileckii. First we note that Bileckii (1973, Theorem 4) is applicable to (9.1) on $M(0)$ for all $\alpha > 0$ . For $0 < \alpha < 4/55$ it implies the existence of a $C^1$-linearization for (9.1) on $M(0)$ . Bileckii (1973, Theorem 4) does not apply for $C^2$-linearization since one has, in Bileckii's notation, $e^{\lambda_2} \in I(\Lambda;2)$ for all $\alpha$ , $0 < \alpha < 4/55$.

The estimate in Hartman (1964, p. 257) does not imply a $C^2$-linearization for any value of $\alpha$, $0 < \alpha < 4/55$. If the quantity $\beta$ is sufficiently large, this estimate does show the existence of a $C^1$-linearization for $\alpha$ satisfying $\alpha_1 < \alpha < 4/55$, where $\alpha_1$ is near $4/55$.

In the full 4-dimensional manifold represented by (9.1) one has

$$\lambda_1 = \lambda_1 + \lambda_3 + \lambda_4$$

for all values of the mass ratio $\alpha$. Bileckii's Theorem implies that there is a $C^1$-linearization.

## REFERENCES


1. R. Abraham and J.E. Marsden (1978). Foundations of Mechanics, Second edition. Benjamin/Cummings, Boston.

2. R. Abraham and J. Robbin (1967). Transversal Mappings and Flows. Benjamin/Cummings, Boston.

3. G.R. Belickii (1973). Functional equations and the conjugacy of diffeomorphisms of finite smoothness class. Functional Anal. Appl. 7, 268-277.

4. G.R. Belickii (1978). Equivalence and normal forms of germs of smooth mappings. Russian Math. Surveys 33, 107-177.

5. K.T. Chen (1963). Equivalence and decomposition of vector fields about an elementary critical point. Amer. J. Math. 85, 693-722.

6. W.A. Coppel (1965). Stability and Asymptotic Behavior of Differential Equations. Heath, Boston.

7. J. Dieudonne (1969). Foundations of Modern Analysis. Academic Press, New York.

8. J. Dieudonne (1974). Treatise on Analysis, vol. IV. Academic Press, New York.

9. L.E. Fraenkel (1978). Formulae for high derivatives of composite functions. Math. Proc. Cambridge Phil. Soc. 83, 159-165.

10. G. Glaeser (1958). Etude de quelques algebres Tayloriennes. J. Analyse Math. 11,1-118.

11. D.M. Grobman (1959). Homeomorphisms of systems of differential equations. Dokl. Akad. Nauk SSSR 128, 880-881.

12. D.M. Grobman (1962). Topological classification of the neighborhood of a singular point in n-dimensional space. Mat. Sb. (N.S.) 56 (98), 77-94.

13. J.K. Hale (1969). Ordinary Differential Equations. Wiley, New York.

14. P. Hartman (1960a). On local homeomorphisms of Euclidean spaces. Bol. Soc. Mat. Mexicana 5, 220-241.

15. P. Hartman (1960b). A lemma in the theory of structural stability of differential equations. Proc. Amer. Math. Soc. 11, 610-620.

16. P. Hartman (1963). On the local linearization of differential equations. Proc. Amer. Math. Soc. 14, 568-573.

17. P. Hartman (1964) Ordinary Differential Equations. Wiley, New York.

18. K. Ise and M. Nagumo (1957). On the normal forms of differential equations in the neighborhood of an equilibrium point. Osaka Math. J. 9, 221-234.

19. T. Kato (1976). Perturbation Theory for Linear Operators, Second edition. Springer-Verlag, New York.

20. R. Moeckel (1983). Heteroclinic phenomena in the isosceles three-body problem. (Preprint).

21. E. Nelson (1969). Topics in Dynamics I. Flows. Princeton University Press.

22. K. Palmer (1980). Qualitative behavior of a system of ODE near an equilibrium point. A generalization of the Hartman-Grobman Theorem. Technical Report, Institute fuer Angewandte Mathematik, University of Bonn.

23. H. Poincare (1928). Oeuvres I. Gauthier-Villars, Paris, pp. il-cxxix.

24. C.C. Pugh (19**). Lecture Notes on Dynamical Systems, Univ. of California.

25. C.L. Seigel (1952). Ueber die Normalform analytischer Differential gleichungen in der Naehe einer Gleichgewichtsloesung. Nachr. Akad. Wiss. Goettingen Math. -Phys. Kl. IIa, 21-30.

26. G.R. Sell (1983a). Obstacles to linearizations (to appear).

27. G.R. Sell (1983b). Smooth linearization in the vicinity of a compact invariant manifold (to appear).

28. S. Sternberg (1957). Local contractions and a theorem of Poincare, Amer. J. Math. 79, 809-824.

29. S. Sternberg (1958). On the structure of local homeomorphisms of Euclidean n-space. Amer. J. Math. 80, 623-631.

30. F. Takens (1971). Partially hyperbolic fixed points. Topology 10, 133-147.