AN ISOPERIMETRIC ESTIMATE AND $W^{1,p}$-QUASICONVEXITY IN NONLINEAR ELASTICITY

Stefan Müller
Max-Planck Institute for Mathematics in the Sciences
Inselstr. 22–26
D–04103 Leipzig, Germany

Jeyabal Sivaloganathan
School of Mathematical Sciences
University of Bath
Bath BA2 7AY, UK

Scott J. Spector
Department of Mathematics
Southern Illinois University
Carbondale, IL 62901–4408, USA

December 18, 1997

Abstract.

A class of stored energy densities that includes functions of the form $W(\mathbf{F}) = a|\mathbf{F}|^p + g(\mathbf{F}, \operatorname{adj}\mathbf{F}) + h(\det\mathbf{F})$ with $a > 0$, g and h convex and smooth, and $2 < p < 3$ is considered. The main result shows that for each such W in this class there is a $k > 0$ such that, if a 3 by 3 matrix $\mathbf{F}_0$ satisfies $h'(\det\mathbf{F}_0)|\mathbf{F}_0|^{3-p} \leq k$, then W is $W^{1,p}$-quasiconvex at $\mathbf{F}_0$ on the restricted set of deformations $\mathbf{u}$ that satisfy condition (INV) and $\det\nabla\mathbf{u} > 0$ a.e. (and hence that are one-to-one a.e.). Condition (INV) is (essentially) the requirement that $\mathbf{u}$ be monotone in the sense of Lebesgue and that holes created in one part of the material not be filled by material from other parts. The key ingredient in the proof is an isoperimetric estimate that bounds the integral of the difference of the Jacobians of $\mathbf{F}_0\mathbf{x}$ and $\mathbf{u}$ by the L^p-norm of the difference of their gradients. These results have application to the determination of lower bounds on critical cavitation loads in elastic solids.

1991 *Mathematics Subject Classification.* 73G05, (49K20, 26B10, 73C50).

Key words and phrases. Cavitation, Condition (INV), Distributional Jacobian, Isoperimetric Inequality, Monotonicity in the sense of Lebesgue, Quasiconvexity.

We thank R. D. James for his interest in this paper and especially for his suggestion that we prove a frame indifferent version of our main lemma. We thank the referee for his enthusiasm for this work and in particular for pointing out a similarity between the proof of our main lemma and that of Gehring's Lemma [Ge 73]. This research was initiated while SJS was an EPSRC Visiting Fellow at the University of Bath. It was completed while all three authors were visiting the IMA in Minneapolis. The research of SJS was also supported by the National Science Foundation.

1. Introduction.

We take $2 < p < 3$ and consider a class of stored energy functions that includes

$$W(\mathbf{F}) = a|\mathbf{F}|^p + g(\mathbf{F}, \operatorname{adj}\mathbf{F}) + h(\det \mathbf{F}), \tag{1.1}$$

where $a > 0$, g and h are C^1 and convex, and $\det \mathbf{F}$ is the determinant of the 3 by 3 matrix $\mathbf{F}$, while $\operatorname{adj}\mathbf{F}$ is the adjugate matrix, i.e., the transpose of the cofactor matrix. We show that for each such W there is a constant $k > 0$ such that if

$$h'(\det \mathbf{F}_0)|\mathbf{F}_0|^{3-p} \leq k \tag{1.2}$$

then W is $W^{1,p}$-quasiconvex at $\mathbf{F}_0$ (on a restricted class of deformations), i.e., for every bounded open region $\Omega \subset \mathbb{R}^3$,

$$\int_\Omega W(\mathbf{F}_0)\, d\mathbf{x} \leq \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1.3}$$

for all $\mathbf{u}$ in the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^3)$ that satisfy $\mathbf{u}(\mathbf{x}) = \mathbf{F}_0\mathbf{x}$ on $\partial\Omega$ (in the sense of trace), $\det \nabla\mathbf{u} > 0$ a.e., and whose extension to all of $\mathbb{R}^3$ (as the linear deformation $\mathbf{F}_0\mathbf{x}$) satisfies condition (INV).

Roughly speaking, condition (INV) is the requirement that the deformation $\mathbf{u}$ be monotone in the sense of Lebesgue and that holes created in one part of Ω are not filled by material from another part of Ω. The condition $\det \nabla\mathbf{u} > 0$ a.e. together with condition (INV) prohibits interpenetration of matter, that is, these conditions together imply that $\mathbf{u}$ is one-to-one almost everywhere. If the deformations are not required to satisfy a condition such as (INV) then results of Ball and Murat [BM 84] show that W will not be $W^{1,p}$-quasiconvex at such an $\mathbf{F}_0$. According to [JS 92] this is sometimes due to the ability of the material to interpenetrate matter in order to reduce energy. (There is no apparent energetic penalty to the use of a noninjective deformation in (1.1) and (1.3).)

The heart of our proof is an isoperimetric estimate that bounds the difference of two Jacobians; for every $n \geq 2$ and $p \in (n-1, n)$ there is a constant $\alpha = \alpha(n, p)$ such that for every n by n matrix $\mathbf{F}_0$ with positive determinant and for every bounded open region $\Omega \subset \mathbb{R}^n$

$$\int_\Omega [\det \mathbf{F}_0 - \det \nabla\mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \alpha|\mathbf{F}_0|^{n-p} \int_\Omega |\mathbf{F}_0 - \nabla\mathbf{u}(\mathbf{x})|^p\, d\mathbf{x} \tag{1.4}$$

for all deformations $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ that satisfy $\mathbf{u}(\mathbf{x}) = \mathbf{F}_0\mathbf{x}$ on $\partial\Omega$, $\det \nabla\mathbf{u} > 0$ a.e., and whose extension to all of $\mathbb{R}^n$ (as the linear deformation $\mathbf{F}_0\mathbf{x}$) satisfies condition (INV). If $p \geq n$ this estimate is clear since the left-hand side of (1.4) is zero due to the fact that the Jacobian is a null Lagrangian (see, e.g., [Ba 77]). For $n-1 < p < n$ one can express the left-hand side of (1.4) in terms of the singular part of the distributional Jacobian of $\mathbf{u}$, which is a Radon measure (cf. [Mü 90]) under our hypotheses. This singular measure is then estimated locally via the isoperimetric inequality and a standard covering argument finishes the proof.

An important application of these results is to cavitation problems in elastic solids. Experimental observations on elastomers (see, e.g., [GL 58], [OB 65], or [GP 84]) indicate that, when the material is subjected to tensile loads, a major failure mechanism in such materials is the formation and growth of holes. The fundamental analysis that viewed cavitation as the spontaneous creation of such holes was done by Ball. In [Ba 82] he analyzed the radial problem on the unit ball $B \subset \mathbb{R}^n$ for a class of isotropic, stored energy functions of slow growth ($p < n$). He showed in particular that when $\mathbf{F}$ is of the form λ times the identity matrix there is a critical value λ_{cr} such that for all $\lambda > \lambda_{cr}$ and $p \in [1, n)$ the energy density W is not $W^{1,p}$-quasiconvex at the deformation $\lambda\mathbf{x}$. This failure of $W^{1,p}$-quasiconvexity is due to the existence of a radial equilibrium solution of lower energy that creates a hole at the center of the ball.

Following Ball's work there have been a number of results on cavitation in elastic materials (see the survey [HP 95] and the references therein). Most of this work has concentrated on the radial problem. In regard to the nonradial problem [JS 92] have shown that for $1 \leq p < \infty$ (see [Me 65] for $W^{1,\infty}$-quasiconvexity) any energy minimizing deformation $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ must be $W^{1,p}$-quasiconvex at each point of smoothness of $\mathbf{u}$. An existence theory for minimizers that may create voids has been given by [MS 95]. Not much else is known about the creation of holes by nonradial deformations. The major unanswered question in this area is:

(1) Are the radial solutions obtained by Ball, and many others, in fact global minimizers of the energy?

Since the above question is (essentially[1]) unanswered at present, it is of interest to answer some potentially simpler questions:

(2) Are the radial solutions local minimizers of the energy?

(3) Are the radial solutions minimizers if the class of competing deformations is restricted to those that open a single cavity? What if one requires, in addition, that this cavity be located at the center of the ball?

[1]The radial minimizer is a global minimizer for an elastic fluid. A special class of constitutive relations of very slow growth ($1 \leq p < n-1$) for which the radial minimizer is not a global minimizer has been given in [JS 91].

(4) Are radial minimizers $W^{1,p}$-quasiconvex at each value of the deformation gradient that they assume?

(5) Is the value of λ_{cr} that is obtained in the radial problem also the critical boundary deformation at which cavitation first occurs or might a nonradial hole open for some $\lambda < \lambda_{cr}$?

At present very little is known concerning these questions although partial results can be found in [Sp 94], [Si 92], and [Si 95]. In particular, for a large class of materials, radial minimizers are local minimizers with respect to small perturbations with support away from the cavity that open no further holes in the material; and, for one particular constitutive relation W, the radial minimizer is indeed the minimizer among deformations that only create a single hole at the center of a ball. Unfortunately, the proof of this last result depends crucially upon the use of the stored energy density

$$W(\mathbf{F}) = a|\mathbf{F}|^2 + b \det \mathbf{F}$$

($a > 0$, $b > 0$), whose radial minimizers may destroy matter by mapping some set of positive measure onto a set of measure zero.

The current paper gives a partial answer to (5) since Theorem 4.1 increases the lower bound for the critical cavitation load over that previously determined in [Sp 94] (see also [St 93] for the purely radial problem). At this point it is unclear whether the results in this paper will also help answer (4) since it has not been determined whether the values of the deformation gradient that are assumed by radial minimizers, which have been computed in the literature, do indeed satisfy (1.2).

We note that $W^{1,p}$-quasiconvexity (especially with $p = \infty$) is a general hypothesis used to obtain the existence of minimizers in the calculus of variations. Results of Morrey [Mo 52] (see also [AF 84]) as well as more recent results of Ball and Murat [BM 84] show that one must require that W be $W^{1,p}$-quasiconvex in order to obtain the sequential weak (weak star, if $p = \infty$) lower semicontinuity of the corresponding total energy

$$E(\mathbf{u}) = \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}$$

on the space $W^{1,p}$. Sequential weak lower semicontinuity is the condition that is used in the direct method of the calculus of variations to obtain existence of minimizers.

Finally, we note that Marcellini [Ma 86] (see also [Ma 89]) has proposed a different definition of the energy in situations where cavitation may occur. He first defines the energy functional for smooth deformations and then considers that functional's lower semicontinuous extension to $W^{1,p}$. Due to this difference the results in this paper do not help determine bounds on a critical cavitation load for his theory.

2. Preliminaries. The Distributional Jacobian and condition (INV).

In the following, D will denote a nonempty, open subset of $\mathbb{R}^n$, $n \geq 2$. By $L^p(D)$ and $W^{1,p}(D)$ we denote the usual Lebesgue and Sobolev spaces, respectively. We use the notation $L^p(D;\mathbb{R}^m)$, etc., for vector-valued maps. A function φ is in $W^{1,p}_{\text{loc}}(D)$ if $\varphi \in W^{1,p}(U)$ for all open sets $U \subset\subset D$. Sobolev functions on manifolds are defined by the use of local charts (see, e.g., [Mo 66]). Henceforth Ω will denote a bounded open set whose boundary, $\partial\Omega$, is (strongly) Lipschitz (see, e.g., [Mo 66, §3.4] or [EG 92, §4.2.1]). We point out that we *do not identify functions that agree almost everywhere.*

The n-dimensional Lebesgue measure will be denoted by $\mathcal{L}^n$ and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. We write

$$B(\mathbf{a}, r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < r\},$$

for the ball of radius r centered at $\mathbf{a} \in \mathbb{R}^n$. For $\mathbf{a} \in D$ we let

$$r_{\mathbf{a}} := \operatorname{dist}(\mathbf{a}, \partial D),$$

i.e., the distance from $\mathbf{a}$ to the boundary of D.

We write Lin for the set of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$ with norm

$$|\mathbf{L}|^2 = \operatorname{trace}(\mathbf{L}^T\mathbf{L}).$$

We denote by $\text{Lin}^>$ those $\mathbf{L} \in \text{Lin}$ with positive determinant. The mapping $\operatorname{adj} : \text{Lin} \to \text{Lin}$ will be the unique continuous function that satisfies

$$\mathbf{L}(\operatorname{adj}\mathbf{L}) = (\det\mathbf{L})\mathbf{Id}$$

for all $\mathbf{L} \in \text{Lin}$, where $\det\mathbf{L}$ is the determinant of $\mathbf{L}$ and $\mathbf{Id} \in \text{Lin}$ is the identity mapping. Thus, with respect to any orthonormal basis, the matrix corresponding to $\operatorname{adj}\mathbf{L}$ is the transpose of the cofactor matrix corresponding to $\mathbf{L}$.

We briefly recall some facts about the Brouwer degree (see, e.g., [Sc 69] or [FG 95] for more details). Suppose that D is bounded and let $\mathbf{u} : \overline{D} \to \mathbb{R}^n$ be a C^∞ map. If $\mathbf{y}_0 \in R^n \backslash \mathbf{u}(\partial D)$ is such that $\det \nabla \mathbf{u}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)$, one defines

$$\deg(\mathbf{u}, D, \mathbf{y}_0) = \sum_{\mathbf{x} \in \mathbf{u}^{-1}(\mathbf{y}_0)} \operatorname{sgn} \det \nabla \mathbf{u}(\mathbf{x}).$$

If φ is a C^∞ function supported in the connected component of $R^n \backslash \mathbf{u}(\partial D)$ that contains $\mathbf{y}_0$, one can show that

$$\int_D (\varphi \circ \mathbf{u}) \det \nabla \mathbf{u} \, d\mathbf{x} = \deg(\mathbf{u}, D, \mathbf{y}_0) \int_{\mathbb{R}^n} \varphi \, d\mathbf{y}. \tag{2.2}$$

Using this formula and approximating by C^∞ functions, one can define $\deg(\mathbf{u}, D, \mathbf{y}_0)$ for any continuous function $\mathbf{u} : \overline{D} \to \mathbb{R}^n$ and any $\mathbf{y}_0 \in \mathbb{R}^n \backslash \mathbf{u}(\partial D)$. Moreover, the degree only depends on $\mathbf{u}|_{\partial D}$. Accordingly, we write $\deg(\mathbf{u}, \partial D, \mathbf{y}_0)$ instead of $\deg(\mathbf{u}, D, \mathbf{y}_0)$.

Indeed, if D has smooth boundary and $\mathbf{u} \in C^\infty(\overline{D}; \mathbb{R}^n)$, then one can use the divergence theorem and (2.2) to express the degree as a boundary integral:

$$\deg(\mathbf{u}, \partial D, \mathbf{y}_0) \int_{\mathbb{R}^n} \operatorname{div} \mathbf{g} \, d\mathbf{y} = \int_{\partial D} (\mathbf{g} \circ \mathbf{u}) \cdot (\operatorname{adj} \nabla \mathbf{u})^T \boldsymbol{\nu} \, d\mathcal{H}^{n-1} \tag{2.3}$$

for any C^∞ function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi = \operatorname{div} \mathbf{g}$ is supported in the connected component of $R^n \backslash \mathbf{u}(\partial D)$ that contains $\mathbf{y}_0$. Here $\boldsymbol{\nu}$ denotes the outward normal to ∂D. Since $(\operatorname{adj} \nabla \mathbf{u})^T \boldsymbol{\nu}$ only depends upon *tangential* derivatives of $\mathbf{u}$, one can use (2.3) to show that, for $p > n - 1$, the degree can be defined on $W^{1,p}(\partial D; \mathbb{R}^n) \cap C^0(\partial D; \mathbb{R}^n)$.

Proposition 2.1. (see, e.g.,[VG 76], [Sv 88], [MTY 94]). *Let $p > n-1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary is (strongly) Lipschitz. Suppose that $\overline{\mathbf{u}}$ is the continuous representative of a function in $W^{1,p}(\partial\Omega; \mathbb{R}^n)$. Then $\deg(\overline{\mathbf{u}}, \partial\Omega, \mathbf{y}_0)$ is well-defined, i.e., if $\mathbf{u} \in C^0(\overline{\Omega}; \mathbb{R}^n)$ satisfies $\mathbf{u} = \overline{\mathbf{u}}$ on $\partial\Omega$ then $\deg(\overline{\mathbf{u}}, \partial\Omega, \mathbf{y}_0) = \deg(\mathbf{u}, \partial\Omega, \mathbf{y}_0)$ for every $\mathbf{y}_0 \in R^n \backslash \overline{\mathbf{u}}(\partial\Omega)$.*

Remark. For interesting new developments in degree theory, including a definition for rectifiable currents, approximately differentiable maps, maps with nonintegrable Jacobian, and even maps that are merely in VMO see [GMS 94], [GISS 95], [BN 95], and [BN 96].

Our results will make crucial use of the isoperimetric inequality and the following consequence of the area formula for Sobolev functions on a manifold.

Proposition 2.2. ([MM 73], [Fe 69, Corollary 3.2.20]). *Let Γ be an oriented, smooth, $(n-1)$-dimensional manifold. Suppose that $\mathbf{u} \in W^{1,p}(\Gamma;\mathbb{R}^n) \cap C^0(\Gamma;\mathbb{R}^n)$, with $p > n-1$. Then for any $\mathcal{H}^{n-1}$ measurable $A \subset \Gamma$,*

$$\mathcal{H}^{n-1}(\mathbf{u}(A)) \leq (n-1)^{(1-n)/2} \int_A |D\mathbf{u}|^{n-1} \, d\mathcal{H}^{n-1}.$$

Here $D\mathbf{u}$, the tangential gradient of $\mathbf{u}$, is viewed as a map from the tangent space of Γ to $\mathbb{R}^n$.

Proposition 2.3. Isoperimetric Inequality (see, e.g., [Fe 69 p. 278], [EG 92, p. 190, p. 205]). *For $n \geq 2$ let $\omega = n^{-1}\mathcal{L}^n(B(\mathbf{0},1))^{-1/n}$. Then*

$$\mathcal{L}^n(A)^{\frac{n-1}{n}} \leq \omega \mathcal{H}^{n-1}(\partial^* A)$$

for every bounded measurable set $A \subset \mathbb{R}^n$ of finite perimeter, where $\partial^* A$ denotes the reduced boundary of A.

Let $\mathbf{u} \in W^{1,p}(D;\mathbb{R}^n)$, with $1 \leq p < n$. Since we are interested in pointwise properties of $\mathbf{u}$ as well as restrictions of $\mathbf{u}$ to lower dimensional sets, it is useful to consider a particular representative. We define the **precise representative** $\mathbf{u}^* : D \to \mathbb{R}^n$ by

$$\mathbf{u}^*(\mathbf{x}) = \begin{cases} \lim\limits_{r \to 0^+} \quad \fint_{B(\mathbf{x},r)} \mathbf{u}(\mathbf{z}) \, d\mathbf{z}, & \text{if the limit exists,} \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where $\fint_A$ denotes the average value of the integrand over A, i.e., the integral of the function over A divided by the n-dimensional Lebesgue measure of A. (The definition of $\mathbf{u}^*$ at points where the above limit does not exist is somewhat arbitrary. For a thorough discussion of precise representatives and capacities we refer to [EG 92] or [Zi 89].) The precise representative satisfies many important properties. In particular, if $\mathbf{u} \in W^{1,p}(D;\mathbb{R}^n)$ and $B(\mathbf{a},\rho) \subset D$ then $\mathbf{u}^*|_{\partial B(\mathbf{a},r)} \in W^{1,p}(\partial B(\mathbf{a},r);\mathbb{R}^n)$ for a.e. $r \in (0,\rho)$. Furthermore, if $p > n-1$ then $\mathbf{u}^*|_{\partial B(\mathbf{a},r)} \in C^0(\partial B(\mathbf{a},r);\mathbb{R}^n)$ for such values of r, i.e., $\mathbf{u}^*$ is the continuous representative given by the Sobolev imbedding.

In nonlinear elasticity one is interested in globally invertible maps since, in general, matter cannot interpenetrate itself. We say that $\mathbf{u} \in W^{1,1}(D;\mathbb{R}^n)$ is **invertible almost everywhere** (or equivalently, **one-to-one almost everywhere**) if there is a Lebesgue null set $N \subset D$ such that $\mathbf{u}|_{D\setminus N}$ is injective. We note that invertibility almost everywhere is a property of the equivalence class and not merely of the representative. This notion seems to first appear in [Ba 81] where it occurs in an intermediate step of a proof that, under suitable hypotheses, minimizers for the pure displacement (Dirichlet) problem in nonlinear elasticity are homeomorphisms. Later Ciarlet and Nečas [CN 87] used the area formula to show that invertibility a.e. is preserved under weak convergence in $W^{1,p}(p > n)$. They were thus able to ensure the existence of minimizers for the mixed displacement-traction (Dirichlet-Neumann) problem in the class of almost everywhere invertible maps.

More recently it has also been observed that the notion of invertibility almost everywhere is not as useful in function classes that allow for the formation of cavities. In fact the topological properties of such maps can differ drastically from everywhere invertible maps. The source of the difficulties is that a cavity formed at one point may be filled by material from elsewhere. In order to exclude such behavior the invertibility condition (INV) was introduced in [MS 95].

Let $B(\mathbf{a}, r) \subset D$ and suppose that $\mathbf{u} : \partial B(\mathbf{a}, r) \to \mathbb{R}^n$ is continuous. We define the **topological image** of $B(\mathbf{a}, r)$ under $\mathbf{u}$ by

$$\operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r)) := \{\mathbf{y} \in \mathbb{R}^n \backslash \mathbf{u}(\partial B(\mathbf{a}, r)) : \deg(\mathbf{u}, \partial B(\mathbf{a}, r), \mathbf{y}) \neq 0\}.$$

Thus the topological image of a ball under $\mathbf{u}$ is the topological image of the ball under any continuous function that assumes the same boundary values.

Definition 2.4. We say that $\mathbf{u} : D \to \mathbb{R}^n$ satisfies condition (INV) provided that for every $\mathbf{a} \in D$ there exists an $\mathcal{L}^1$ null set $N_{\mathbf{a}}$ such that, for all $r \in (0, r_{\mathbf{a}}) \backslash N_{\mathbf{a}}$, $\mathbf{u}|_{\partial B(\mathbf{a},r)}$ is continuous,

(i) $\mathbf{u}(\mathbf{x}) \in \operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r)) \cup \mathbf{u}(\partial B(\mathbf{a}, r))$ for $\mathcal{L}^n$ a.e. $\mathbf{x} \in \overline{B(\mathbf{a}, r)}$, and

(ii) $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^n \backslash \operatorname{im}_T(\mathbf{u}, B(\mathbf{a}, r))$ for $\mathcal{L}^n$ a.e. $\mathbf{x} \in D \backslash \overline{B(\mathbf{a}, r)}$.

Here $r_{\mathbf{a}} := \operatorname{dist}(\mathbf{a}, \partial D)$.

Remarks. 1. Condition (i) is equivalent to the monotonicity (in the sense of Lebesgue) of the mapping $\mathbf{u}$. See [VG 76], [Sv 88], and [Ma 94] for related results on monotonicity. Condition (ii) is, essentially, the requirement that holes created in one part of D are not filled by material from other parts of D.

2. An example of a map that satisfies (i) but not (ii) is given in Section 11 of [MS 95], while a map that satisfies (ii) but not (i) is given in Section 5 of [MST 96].

Deformations that satisfy condition (INV) and have nonzero Jacobian are more regular than other elements of the Sobolev spaces $W^{1,p}$, $n - 1 < p < n$. In particular, in [MS 95] it is shown that such deformations are one-to-one a.e. and continuous $\mathcal{H}^{n-p}$ a.e. In addition the following result will be used.

Proposition 2.5. [MS 95, Lemma 3.5(i) step 2]. *Let* $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(D;\mathbb{R}^n)$ *with* $p > n-1$. *Assume that* $\mathbf{u}^*$ *satisfies condition (INV) and that* $\det \nabla \mathbf{u} \neq 0$ *a.e. Then for every* $\mathbf{a} \in D$ *and almost every* $r \in (0, r_{\mathbf{a}})$ *the set* $\mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, r))$ *has finite perimeter. Moreover, for such* r, *the reduced boundary satisfies*

$$\partial^* \,\mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, r)) \subset \mathbf{u}(\partial B(\mathbf{a}, r)).$$

If $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(D;\mathbb{R}^n)$, with $p > n^2/(n+1)$, then the linear functional $(\mathrm{Det}\,\nabla \mathbf{u}) : C_0^\infty(D) \to \mathbb{R}$ given by

$$(\mathrm{Det}\,\nabla \mathbf{u})(\phi) := -\frac{1}{n}\int_D \nabla\phi \cdot (\mathrm{adj}\,\nabla \mathbf{u})\mathbf{u}\, d\mathbf{x}$$

is a well-defined distribution, which is called the **distributional Jacobian**. If $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(D;\mathbb{R}^n)$, with $p \geq n$ then the identity $\mathrm{Div}(\mathrm{adj}\,\nabla \mathbf{u})^T = \mathbf{0}$ can be used to show that $\mathrm{Det}\,\nabla \mathbf{u}$ is the distribution induced by the function $\det \nabla \mathbf{u}$. (In general this need not be the case and in fact it will not be when cavitation occurs.)

Now suppose that $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(D;\mathbb{R}^n)$, with $p > n-1$. Then the precise representative $\mathbf{u}^*$ is continuous on the sphere $\partial B(\mathbf{a}, r)$ for almost every r and hence $\mathbf{u}^*(\partial B(\mathbf{a}, r))$ is compact for such r. If, in addition, $\mathbf{u}^*$ satisfies condition (INV) then it follows that $\mathbf{u}^* \in L^\infty_{\mathrm{loc}}(D;\mathbb{R}^n)$ and hence that the above functional is once again a well-defined distribution on D. The next result shows that in fact this distribution is a nonnegative Radon measure.

Proposition 2.6. (see [Mü 90], [MS 95, Lemma 8.1]). *Let* $\mathbf{u} \in W^{1,p}_{\mathrm{loc}}(D,\mathbb{R}^n)$ *with* $p > n-1$. *Suppose that* $\det \nabla \mathbf{u} > 0$ *a.e. and that* $\mathbf{u}^*$ *satisfies condition (INV). Then* $\mathrm{Det}\,\nabla \mathbf{u} \geq 0$ *and hence* $\mathrm{Det}\,\nabla \mathbf{u}$ *is a Radon measure. Furthermore,*

$$\mathrm{Det}\,\nabla \mathbf{u} = (\det \nabla \mathbf{u})\mathcal{L}^n + m, \tag{2.4}$$

where m *is singular with respect to Lebesgue measure and for* $\mathcal{L}^1$ *a.e.* $r \in (0, r_{\mathbf{a}})$ *one has*

$$(\mathrm{Det}\,\nabla \mathbf{u})\,(B(\mathbf{a}, r)) = \mathcal{L}^n\big(\mathrm{im}_T(\mathbf{u}, B(\mathbf{a}, r))\big). \tag{2.5}$$

3. Main Lemma.

Let $\mathbf{L} \in \mathrm{Lin}^{>}$ and $p \geq 1$. If $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n)$ satisfies $\mathbf{u} - \mathbf{L}\mathbf{x} \in W_0^{1,p}(\Omega;\mathbb{R}^n)$ then we define its (homogeneous) *extension* $\mathbf{u}^e : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathbf{u}^e(\mathbf{x}) := \begin{cases} \mathbf{u}^*(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{L}\mathbf{x}, & \mathbf{x} \notin \Omega, \end{cases} \tag{3.1}$$

and note that $\mathbf{u}^e \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. For $n-1 < p < n$ define

$$\mathcal{A}_{\mathbf{L},p}(\Omega) := \{\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n) : \mathbf{u}-\mathbf{L}\mathbf{x} \in W_0^{1,p}(\Omega;\mathbb{R}^n), \det \nabla\mathbf{u} > 0 \text{ a.e.}, \mathbf{u}^e \text{ satisfies (INV)}\}.$$

Given such linear boundary values, our main result gives an upper bound for the hole volume created by a deformation that assumes these boundary values.

Main Lemma. *Let $n \geq 2$, $n-1 < p < n$, and $n - 1 \leq q \leq p$. Then there exists a constant $\alpha = \alpha(n, q) > 0$, which is independent of domain, such that*

$$\int_\Omega [\det \mathbf{L} - \det \nabla\mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \alpha |\mathbf{L}|^{n-q} \int_\Omega \Big| |\mathbf{L}| - |\nabla\mathbf{u}(\mathbf{x})| \Big|^q \, d\mathbf{x}$$

and hence

$$\int_\Omega [\det \mathbf{L} - \det \nabla\mathbf{u}(\mathbf{x})]\, d\mathbf{x} \leq \alpha |\mathbf{L}|^{n-q} \int_\Omega |\mathbf{L} - \nabla\mathbf{u}(\mathbf{x})|^q \, d\mathbf{x}$$

for all bounded open sets $\Omega \subset \mathbb{R}^n$, all $\mathbf{L} \in \mathrm{Lin}^{>}$, and all $\mathbf{u} \in \mathcal{A}_{\mathbf{L},p}(\Omega)$.

Remarks. 1. It is clear from (3.4) below that condition (INV) also implies that the left-hand side of the above inequalites is nonnegative.

2. In order to determine bounds upon the critical load at which cavitation first occurs (see the Introduction) it is of interest to obtain numerical bounds on the constant α.

3. Although the requirement that the extension satisfy condition (INV) seems to us to be a bit artificial, we have been unable to prove that it follows from a more natural condition such as:

$$\mathbf{u}(\mathbf{x}) \in \mathbf{L}\overline{\Omega} \quad \text{for a.e. } \mathbf{x} \in \Omega$$

or, for every $\mathbf{a} \in \overline{\Omega}$ and almost every $r > 0$,

$$\mathbf{u} \text{ satisfies (i) and (ii) of (INV) with } B(\mathbf{a}, r) \text{ replaced by } B(\mathbf{a}, r) \cap \Omega. \tag{3.2}$$

The specific technical problem that is encountered is that one does not appear to be able to get information about the degree in a region whose boundary includes part of $\mathbf{L}(\partial\Omega)$. This is due to the possibility that either $\mathbf{u}$ may not be approximately differentiable or the normal component of its approximate derivative may be zero on $\partial\Omega$. If this were not the case then one could use ideas from [MS 95] to show that (3.2) implies that the extension $\mathbf{u}^e$ satisfies condition (INV).

Proof. We first note that the first inequality together with the triangle inequality yield the second inequality. Without loss of generality assume that $\mathbf{0} \in \Omega$. If we replace $\mathbf{u}$ in the inequality by the scaling $\mathbf{u}_\epsilon(\mathbf{x}) = \epsilon\mathbf{u}(\mathbf{x}/\epsilon)$ we find that the inequality is independent of the size of the domain. Thus we may assume that $\Omega \subset B(\mathbf{0}, 1)$. Let $p \in (n-1, n)$, $\mathbf{u} \in \mathcal{A}_{\mathbf{L},p}(\Omega)$, and define $\mathbf{u}^e$ by (3.1). Then by (2.4)

$$\operatorname{Det}\nabla\mathbf{u}^e = (\det\nabla\mathbf{u}^e)\mathcal{L}^n + m, \tag{3.3}$$

where $m \geq 0$ is a Radon measure that is singular with respect to Lebesgue measure. Since $\Omega \subset\subset B(\mathbf{0}, 2)$ the definition of the topological image and (3.1) imply that

$$\operatorname{im}_T(\mathbf{u}^e, B(\mathbf{0}, 2)) = \mathbf{L}B(\mathbf{0}, 2).$$

Thus, if we evaluate $\operatorname{Det}\nabla\mathbf{u}^e$ on the ball $B(\mathbf{0}, 2)$ and make use of (2.5) and (3.3) we find that

$$\begin{aligned}(\det\mathbf{L})\mathcal{L}^n(B(\mathbf{0}, 2)) &= (\operatorname{Det}\nabla\mathbf{u}^e)(B(\mathbf{0}, 2)) \\ &= m(B(\mathbf{0}, 2)) + \int_{B(\mathbf{0},2)} \det\nabla\mathbf{u}^e(\mathbf{x})\,d\mathbf{x} \\ &= m(\overline{\Omega}) + (\det\mathbf{L})\left[\mathcal{L}^n(B(\mathbf{0}, 2)) - \mathcal{L}^n(\Omega)\right] + \int_\Omega \det\nabla\mathbf{u}(\mathbf{x})\,d\mathbf{x},\end{aligned}$$

since $\nabla\mathbf{u}^e = \mathbf{L}$ on $B(\mathbf{0}, 2)\backslash\Omega$ and the support of m is contained in $\overline{\Omega}$. If we rearrange terms we find that

$$\int_\Omega (\det\mathbf{L} - \det\nabla\mathbf{u})\,d\mathbf{x} = m(\overline{\Omega}). \tag{3.4}$$

Next, let $M \subset \overline{\Omega}$ be the support of m. Then there is an $N \subset M$ with $m(N) = 0$ such that

$$\lim_{r\to 0^+} \frac{m\left(\overline{B(\mathbf{a}, r)}\right)}{r^n} = +\infty \quad \text{for every } \mathbf{a} \in M\backslash N, \tag{3.5}$$

since m is singular with respect to Lebesgue measure (see, e.g., [EG 92, Section 1.6]).

Our strategy now will be to use an isoperimetric estimate to bound the local hole volume created at each point of $M\backslash N$ by the deformed surface area enclosing this volume. An integration will then change the surface integral to the L^{n-1}-norm and a suitable covering theorem will yield the desired bound (when $q = n-1$).

Let $\mathbf{a} \in M\backslash N$. By (2.5)

$$(\operatorname{Det} \nabla \mathbf{u}^e)(B(\mathbf{a},t)) = \mathcal{L}^n\left(\operatorname{im}_T(\mathbf{u}^e, B(\mathbf{a},t))\right) \tag{3.6}$$

for a.e. $t > 0$, while Propositions 2.2, 2.3, and 2.5 imply, that for such t,

$$\begin{aligned} \mathcal{L}^n\left(\operatorname{im}_T(\mathbf{u}^e, B(\mathbf{a},t))\right)^{\frac{n-1}{n}} &\le \omega \mathcal{H}^{n-1}\left(\partial^* \operatorname{im}_T(\mathbf{u}^e, B(\mathbf{a},t))\right) \\ &\le \omega \mathcal{H}^{n-1}\left(\mathbf{u}^e(\partial B(\mathbf{a},t))\right) \\ &\le \omega \int_{\partial B(\mathbf{a},t)} |\nabla \mathbf{u}^e|^{n-1}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.7}$$

In view of (3.3) and the nonnegativity of m and $\det \nabla \mathbf{u}^e$ we can combine (3.6) and (3.7) to conclude that (cf. the equation preceding eqn. (28) in [Ge 73])

$$\left[m\left(\overline{B(\mathbf{a},r)}\right)\right]^{\frac{n-1}{n}} \le C \int_{\partial B(\mathbf{a},t)} |\nabla \mathbf{u}^e|^{n-1}\, d\mathcal{H}^{n-1}$$

for almost every $t > r$, where C will now be a generic constant that may vary from line to line.

If we integrate the last inequality with respect to t over the interval $[r, 2r]$ we conclude that

$$\begin{aligned} r\left[m\left(\overline{B(\mathbf{a},r)}\right)\right]^{\frac{n-1}{n}} &\le C \int_{B(\mathbf{a},2r)\backslash B(\mathbf{a},r)} |\nabla \mathbf{u}^e|^{n-1}\, d\mathbf{x} \\ &\le C \int_{B(\mathbf{a},2r)} |\nabla \mathbf{u}^e|^{n-1}\, d\mathbf{x}. \end{aligned} \tag{3.8}$$

Define

$$\left\|\phi\right\|_m := \left(\int_{B(\mathbf{a},2r)} |\phi(\mathbf{x})|^m\, d\mathbf{x}\right)^{\frac{1}{m}}.$$

Then, by the triangle inequality,

$$\begin{aligned} \int_{B(\mathbf{a},2r)} |\nabla \mathbf{u}^e|^{n-1}\, d\mathbf{x} = \left\||\nabla \mathbf{u}^e|\right\|_{n-1}^{n-1} &\le \left[\left\||\nabla \mathbf{u}^e| - |\mathbf{L}|\right\|_{n-1} + \left\||\mathbf{L}|\right\|_{n-1}\right]^{n-1} \\ &\le C\left[\left\||\nabla \mathbf{u}^e| - |\mathbf{L}|\right\|_{n-1}^{n-1} + \left\||\mathbf{L}|\right\|_{n-1}^{n-1}\right] \\ &\le C\left(\int_{B(\mathbf{a},2r)} \left||\nabla \mathbf{u}^e| - |\mathbf{L}|\right|^{n-1} d\mathbf{x} + r^n |\mathbf{L}|^{n-1}\right). \end{aligned} \tag{3.9}$$

Let $q \in [n-1, p]$. Then $|\beta|^{n-1} \leq |\beta|^q + 1$ for every $\beta \in \mathbb{R}$ and hence if we choose $\beta = (|\nabla \mathbf{u}^e| - |\mathbf{L}|)/|\mathbf{L}|$ we find that

$$\Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^{n-1} \leq |\mathbf{L}|^{n-q-1} \Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^q + |\mathbf{L}|^{n-1}. \tag{3.10}$$

Therefore, if we integrate (3.10) over the ball $B(\mathbf{a}, 2r)$ and combine the result with (3.8) and (3.9) we find that

$$\left[\frac{m(\overline{B(\mathbf{a}, r)})}{r^n} \right]^{\frac{n-1}{n}} \leq C \left[|\mathbf{L}|^{n-1-q} \fint_{B(\mathbf{a},2r)} \Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^q \, d\mathbf{x} + |\mathbf{L}|^{n-1} \right]. \tag{3.11}$$

Define

$$\theta_{\mathbf{a}}(r) := |\mathbf{L}|^{n-1-q} \fint_{B(\mathbf{a},r)} \Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^q \, d\mathbf{x}.$$

Then $\theta_{\mathbf{a}} : (0, \infty) \to (0, \infty)$ is a continuous function which, in view of (3.5), (3.11), and the compact support of the integrand, satisfies

$$\lim_{r \to 0^+} \theta_{\mathbf{a}}(2r) = +\infty, \qquad \lim_{r \to +\infty} \theta_{\mathbf{a}}(2r) = 0.$$

Thus

$$\rho_{\mathbf{a}} := \inf\{r > 0 : \theta_{\mathbf{a}}(2r) = |\mathbf{L}|^{n-1}\}$$

is well-defined. Note that, by the continuity of $\theta_{\mathbf{a}}$ and the definition of $\rho_{\mathbf{a}}$

$$\theta_{\mathbf{a}}(\rho_{\mathbf{a}}) > \theta_{\mathbf{a}}(2\rho_{\mathbf{a}}) = |\mathbf{L}|^{n-1}. \tag{3.12}$$

If we now evaluate (3.11) at $r = \rho_{\mathbf{a}}$ and make use of (3.12) and the definition of $\theta_{\mathbf{a}}$ we conclude that

$$\begin{aligned}
\left[\frac{m(\overline{B(\mathbf{a}, \rho_{\mathbf{a}})})}{\rho_{\mathbf{a}}^n} \right]^{\frac{n-1}{n}} &\leq C \left[|\mathbf{L}|^{n-1-q} \fint_{B(\mathbf{a},2\rho_{\mathbf{a}})} \Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^q \, d\mathbf{x} + |\mathbf{L}|^{n-1} \right] \\
&= 2C\theta_{\mathbf{a}}(2\rho_{\mathbf{a}}) \\
&= 2C \Big[\theta_{\mathbf{a}}(2\rho_{\mathbf{a}}) \Big]^{\frac{n-1}{n}} \Big[\theta_{\mathbf{a}}(2\rho_{\mathbf{a}}) \Big]^{\frac{1}{n}} < 2C \Big[\theta_{\mathbf{a}}(\rho_{\mathbf{a}}) \Big]^{\frac{n-1}{n}} \Big[\theta_{\mathbf{a}}(2\rho_{\mathbf{a}}) \Big]^{\frac{1}{n}} \\
&= 2C \left[|\mathbf{L}|^{n-1-q} \fint_{B(\mathbf{a},\rho_{\mathbf{a}})} \Big| |\nabla \mathbf{u}^e| - |\mathbf{L}| \Big|^q \, d\mathbf{x} \right]^{\frac{n-1}{n}} \Big[|\mathbf{L}|^{n-1} \Big]^{\frac{1}{n}}
\end{aligned}$$

and hence

$$m(\overline{B(\mathbf{a},\rho_{\mathbf{a}})}) \le C|\mathbf{L}|^{n-q}\int_{B(\mathbf{a},\rho_{\mathbf{a}})}\Big||\nabla\mathbf{u}^e| - |\mathbf{L}|\Big|^q\,d\mathbf{x}. \tag{3.13}$$

Define

$$\mathcal{M} := \{\overline{B(\mathbf{a},\rho_{\mathbf{a}})} : \mathbf{a}\in M\backslash N\}.$$

Note that $\theta_{\mathbf{a}}(r)$ is independent of $\mathbf{a}\in M$ for $r>2$ and hence that

$$\sup\{\rho_{\mathbf{a}} : \mathbf{a}\in M\backslash N\} < \infty.$$

Therefore, by the Besicovitch covering theorem (see, e.g., [EG 92, Theorem 2, p. 30]), there is a constant C_n, which only depends on the dimension n, and families $\mathcal{G}_i\subset\mathcal{M}, i=1,2,3,\dots,C_n$, of pairwise disjoint closed balls, that satisfy

$$M\backslash N \subset \bigcup_{i=1}^{C_n}\ \bigcup_{\overline{B(\mathbf{a},\rho_{\mathbf{a}})}\in\mathcal{G}_i}\overline{B(\mathbf{a},\rho_{\mathbf{a}})}. \tag{3.14}$$

Finally, by (3.13), (3.14), the definitions of M and N, and that fact that the balls in each family $\mathcal{G}_i$ are pairwise disjoint

$$\begin{aligned}
m(\overline{\Omega}) = m(M\backslash N) &\le \sum_{i=1}^{C_n}\ \sum_{\overline{B(\mathbf{a},\rho_{\mathbf{a}})}\in\mathcal{G}_i} m(\overline{B(\mathbf{a},\rho_{\mathbf{a}})})\\
&\le C|\mathbf{L}|^{n-q}\sum_{i=1}^{C_n}\ \sum_{\overline{B(\mathbf{a},\rho_{\mathbf{a}})}\in\mathcal{G}_i}\int_{B(\mathbf{a},\rho_{\mathbf{a}})}\Big||\nabla\mathbf{u}^e|-|\mathbf{L}|\Big|^q\,d\mathbf{x}\\
&\le C|\mathbf{L}|^{n-q}\sum_{i=1}^{C_n}\int_{\Omega}\Big||\nabla\mathbf{u}^e|-|\mathbf{L}|\Big|^q\,d\mathbf{x}\\
&= C_nC|\mathbf{L}|^{n-q}\int_{\Omega}\Big||\nabla\mathbf{u}^e|-|\mathbf{L}|\Big|^q\,d\mathbf{x},
\end{aligned}$$

which together with (3.4) gives the desired result. □

Remark. For $q>n-1$ (but not $q=n-1$) an alternative proof of the above lemma can be obtained by replacing the isoperimetric inequality, the area formula, and the inequality $|\beta|^{n-1}\le|\beta|^q+1$ by the isodiametric inequality and the standard imbedding

$$\sup_{\mathbf{x},\mathbf{z}\in\partial B(\mathbf{a},t)}|\mathbf{u}^*(\mathbf{x})-\mathbf{u}^*(\mathbf{z})|^q \le Ct^{q-n+1}\int_{\partial B(\mathbf{a},t)}|\nabla\mathbf{u}|^q\,d\mathcal{H}^{n-1}.$$

4. $W^{1,p}$-quasiconvexity.

We consider a homogeneous body that, for convenience, will be identified with the bounded region $\overline{\Omega} \subset \mathbb{R}^3$ that it occupies in a fixed homogeneous reference configuration. We assume that the body is hyperelastic with continuous **stored energy** density $W : \mathrm{Lin} \to [0, \infty]$. The quantity $W(\nabla \mathbf{u}(\mathbf{x}))$ gives the energy stored per unit volume in Ω, at any point $\mathbf{x} \in \Omega$ when the body is deformed by a smooth deformation $\mathbf{u}$. Further, we assume that $W(\mathbf{F}) = +\infty$ whenever $\det \mathbf{F} \le 0$.

In particular we are interested in stored-energy functions that satisfy

$$W(\mathbf{F}) = \hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}) + h(\det \mathbf{F}) \tag{4.1}$$

for all $\mathbf{F} \in \mathrm{Lin}^>$, where $h \in C^1((0,\infty),[0,\infty))$ is convex and $\hat{g} : \mathrm{Lin}^> \times \mathrm{Lin}^> \to [0, \infty)$ satisfies the following conditions.

(a) There are constants $c_1 > 0$ and $q \in [2, 3)$ such that for every $\mathbf{K}, \mathbf{M} \in \mathrm{Lin}^>$ there exist $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ such that

$$\hat{g}(\mathbf{N}, \mathbf{P}) \ge \hat{g}(\mathbf{K}, \mathbf{M}) + \mathbf{A} \cdot (\mathbf{N} - \mathbf{K}) + \mathbf{B} \cdot (\mathbf{P} - \mathbf{M}) + c_1 |\mathbf{N} - \mathbf{K}|^q$$

for all $\mathbf{N}, \mathbf{P} \in \mathrm{Lin}^>$.

(b) There are constants $p \in (2, 3)$, $c_2 > 0$, and c_3, with $p \ge q$, such that

$$\hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}) \ge c_2 |\mathbf{F}|^p + c_3.$$

Remarks. 1. Condition (b) ensures that deformations with finite energy belong to a Sobolev space in which condition (INV) makes sense.

2. If $q > 2$ then (a) implies (b) with $p = q$.

3. Condition (a) is slightly stronger than the requirement that the mapping $\hat{g}$ be convex. In particular when $q = 2$ such functions are *uniformly strictly quasiconvex* in the sense of Evans [Ev 86]. A result in [Ev 86] (see the appendix of this paper) shows that (a) is satisfied by

$$\hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}) = a |\mathbf{F}|^q + g(\mathbf{F}, \operatorname{adj} \mathbf{F})$$

where $a > 0$ and g is C^1 and convex. Conditions (a) and (b) are also satisfied (see, e.g., [Ba 77, pp. 229–230]) by certain Ogden [Og 72] materials:

$$\hat{g}(\mathbf{F}, \operatorname{adj} \mathbf{F}) = b |\mathbf{F}|^2 + \sum_{i=1}^{3} \varphi(\lambda_i) + \sum_{i>j} \psi(\lambda_i \lambda_j)$$

where φ and ψ are convex and nondecreasing, $b > 0$, and there is a $p \in (2, 3)$ and a $c > 0$ such that $\varphi(\lambda) \ge c |\lambda|^p$ for all $\lambda > 0$. Here $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of the square root of $\mathbf{F}\mathbf{F}^T$.

Theorem 4.1. *Let the stored energy density W satisfy (4.1), where h is C^1 and convex and $\hat{g}$ satisfies (a) and (b). Then any linear deformation $\mathbf{w}(\mathbf{x}) = \mathbf{L}\mathbf{x}$ that satisfies*

$$h'(\det \mathbf{L})|\mathbf{L}|^{3-q} \leq c_1/\alpha$$

is a global minimizer of the total elastic energy

$$E(\mathbf{u}) := \int_\Omega W(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}$$

in the class $\mathcal{A}_{\mathbf{L},p}(\Omega)$. Here α is the optimal constant from the lemma in section 3.

Remarks. 1. In the terminology of Ball and Murat [BM 84] the function W is $W^{1,p}$-quasiconvex at each such $\mathbf{L}$ (on the restricted class of deformations $\mathcal{A}_{\mathbf{L},p}$). Results of [BM 84] (see also [JS 92]) imply that this result is false if the class of deformations is not restricted.

2. Suppose that h satisfies $h'(H) = 0$ for some (unique) $H > 0$. Then the theorem implies that there is an $\epsilon > 0$ such that $\lambda\mathbf{x}$ is a global minimizer of E whenever $\lambda^3 \in (0, H + \epsilon)$. In [Sp 94] it was shown that, for a slightly more general class of energy densities and admissible deformations, the conclusion of the theorem was valid provided $\lambda^3 \in (0, H]$. A physical interpretation of such results is that, for the displacement problem, cavitation can not occur in compression.

Proof of the Theorem. By (a) and the convexity of h

$$\begin{aligned} W(\mathbf{H}) \geq W(\mathbf{L}) &+ \mathbf{A} \cdot [\mathbf{H} - \mathbf{L}] + \mathbf{B} \cdot [\operatorname{adj} \mathbf{H} - \operatorname{adj} \mathbf{L}] \\ &+ h'(\det \mathbf{L})[\det \mathbf{H} - \det \mathbf{L}] + c_1 |\mathbf{H} - \mathbf{L}|^q \end{aligned} \tag{4.2}$$

for every $\mathbf{H} \in \mathrm{Lin}^>$.

Let $\mathbf{u} \in \mathcal{A}_{\mathbf{L},p}$. Then

$$\int_\Omega [\nabla \mathbf{u}(\mathbf{x}) - \mathbf{L}]\, d\mathbf{x} = \mathbf{0}, \tag{4.3}$$

and (see, e.g., [Ba 77, Lemma 3.3a])

$$\int_\Omega [\operatorname{adj} \nabla \mathbf{u}(\mathbf{x}) - \operatorname{adj} \mathbf{L}]\, d\mathbf{x} = \mathbf{0}. \tag{4.4}$$

If we take $\mathbf{H} = \nabla\mathbf{u}(\mathbf{x})$ in (4.2) and integrate over Ω we conclude, with the aid of (4.3) and (4.4), that

$$\begin{aligned}\int_\Omega [W(\nabla\mathbf{u}(\mathbf{x})) - W(\mathbf{L})]\,d\mathbf{x} \ge h'(\det\mathbf{L}) \int_\Omega [\det\nabla\mathbf{u}(\mathbf{x}) - \det\mathbf{L}]\,d\mathbf{x} \\ +c_1 \int_\Omega |\nabla\mathbf{u}(\mathbf{x}) - \mathbf{L}|^q\,d\mathbf{x}.\end{aligned} \tag{4.5}$$

and hence, in view our main lemma, that

$$\int_\Omega [W(\nabla\mathbf{u}(\mathbf{x})) - W(\mathbf{L})]\,d\mathbf{x} \ge [\frac{c_1}{\alpha|\mathbf{L}|^{3-q}} - h'(\det\mathbf{L})] \int_\Omega [\det\mathbf{L} - \det\nabla\mathbf{u}(\mathbf{x})]\,d\mathbf{x},$$

which together with the first remark following the lemma in section 3 yields the desired result. □

Remarks. 1. Supose one replaces (4.1) by the hypothesis $W(\mathbf{F}) = f(\mathbf{F}, \operatorname{adj}\mathbf{F}, \det\mathbf{F})$ where (cf. [Ba 77] and [Ev 86]) f is *p-uniformly strictly polyconvex*, i.e, there is a $p \in (2,3)$ and a constant $c_1 > 0$ such that for every $\mathbf{K}, \mathbf{M} \in \mathrm{Lin}^>$ and $\mu > 0$ there exist $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$ and $\beta \in \mathbb{R}$, which may depend on $\mathbf{K}, \mathbf{M}$ and $\mu > 0$, such that

$$f(\mathbf{N}, \mathbf{P}, \nu) \ge f(\mathbf{K}, \mathbf{M}, \mu) + \mathbf{A}\cdot(\mathbf{N} - \mathbf{K}) + \mathbf{B}\cdot(\mathbf{P} - \mathbf{M}) + \beta(\nu - \mu) + c_1|\mathbf{N} - \mathbf{K}|^p$$

for all $\mathbf{N}, \mathbf{P} \in \mathrm{Lin}^>$ and $\nu > 0$. Then it is clear from the proof that the conclusions of the theorem will remain valid whenever $\beta = \beta(\mathbf{L}, \operatorname{adj}\mathbf{L}, \det\mathbf{L}) \le c_1/(\alpha|\mathbf{L}|^{3-p})$. However, the physics that leads to such an inequality at a particular $\mathbf{L}$ is unclear.

2. Equation (4.5) and our main lemma also imply that

$$E(\mathbf{u}) - E(\mathbf{L}\mathbf{x}) \ge \min\left\{c_1, c_1 - \alpha|\mathbf{L}|^{3-q}h'(\det\mathbf{L})\right\} \int_\Omega |\nabla\mathbf{u}(\mathbf{x}) - \mathbf{L}|^q\,d\mathbf{x}. \tag{4.6}$$

Suppose now that $q > 2$ (so that hypothesis (a) implies hypothesis (b) with $p = q$). Then, whenever $\mathbf{L}$ satisfies $c_1 > \alpha|\mathbf{L}|^{3-q}h'(\det\mathbf{L})$, one can conclude from (4.6) that the mapping $\mathbf{L}\mathbf{x}$ is the *unique* global minimizer of E (among maps in $\mathcal{A}_{\mathbf{L},p}(\Omega)$) and, furthermore, $\mathbf{L}\mathbf{x}$ lies in a potential well. This may have implications for the dynamic stability of such maps.

Appendix.

We here present an alternative proof of a result of Evans [Ev 86, Lemma 8.2] (see also [Zh 91, Lemma 2.15]) since our proof gives a bound on the optimal constant κ.

Proposition A.1. *Let $p \in [2, \infty)$. Then there is a constant $\kappa = \kappa(p) > 0$, which is independent of dimension, such that*

$$|\mathbf{a}|^p \geq |\mathbf{b}|^p + p|\mathbf{b}|^{p-2}\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) + \kappa|\mathbf{a} - \mathbf{b}|^p \tag{A.1}$$

for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Moreover, the largest such κ satisfies $2^{2-p} \leq \kappa \leq p2^{1-p}$.

Proof. For $p = 2$ inequality (A.1) is clear with $\kappa = 1$. We therefore suppose that $p > 2$ and first consider the case when $n = 1$. If $\mathbf{b} = 0$ then (A.1) holds with $\kappa = 1$. By homogeneity we may therefore assume that $|\mathbf{b}| = 1$. Thus letting $t = \operatorname{sgn}(\mathbf{b})(\mathbf{a} - \mathbf{b})$, $|\mathbf{b}| = 1$, and dividing (A.1) by $|t|^p$ we find that the optimal constant κ, which is nonnegative since $t \mapsto |t|^p$ is convex, is given by

$$\kappa = \inf_{\mathbb{R}\setminus\{0\}} \theta, \qquad \theta(t) = \frac{|1+t|^p - 1 - pt}{|t|^p}.$$

Define

$$\psi(s) := \theta(1/s) = |s+1|^p - |s|^p - p|s|^{p-1}\operatorname{sgn}(s).$$

Then $\inf \psi = \inf \theta$ and

$$\psi'(s) = p\left(|s+1|^{p-1}\operatorname{sgn}(s+1) - |s|^{p-1}\operatorname{sgn}(s) - (p-1)|s|^{p-2}\right)$$

is positive on $(0, \infty)$ and negative on $(-\infty, -1)$ since $\sigma \mapsto |\sigma|^{p-1}$ is convex. Therefore ψ attains its infimum at $-\tau \in [-1, 0]$. If $-\tau \in (-1, 0)$ then

$$0 = p^{-1}\psi'(-\tau) = (1-\tau)^{p-1} + \tau^{p-1} - (p-1)\tau^{p-2},$$

and hence

$$\begin{aligned}\kappa = \psi(-\tau) &= (1-\tau)^p - \tau^p + p\tau^{p-1} \\ &= (1-\tau)^p + \tau(1-\tau)^{p-1} - (p-1)\tau^{p-1} + p\tau^{p-1} \\ &= (1-\tau)^{p-1} + \tau^{p-1} \geq 2[2^{1-p}] = 2^{2-p},\end{aligned}$$

where we have used the convexity of $\sigma \mapsto \sigma^{p-1}$. Moreover, $\kappa \leq \psi(-1/2) = p2^{1-p} < 1$. Finally, $\psi(0) = 1$ and $\psi(-1) = p - 1 > 1$. Thus if τ were not in the interior we could conclude that $\kappa = 1$, which is not possible.

Now consider $n > 1$. Once again by homogeneity we may assume that $|\mathbf{b}| = 1$. Let $\mathbf{a} = t\mathbf{e} + \mathbf{b}$ and $\alpha = \mathbf{e} \cdot \mathbf{b}$, where $t \in [0, +\infty)$, $|\mathbf{e}| = 1$, and (consequently) $\alpha \in [-1, 1]$. Then by (A.1) the optimal constant $\kappa \geq 0$ is given by

$$\kappa = \inf_{(0,\infty)\times[-1,1]} \omega, \qquad \omega(t, \alpha) := \frac{[1 + 2\alpha t + t^2]^{p/2} - 1 - \alpha p t}{t^p}.$$

For fixed $t > 0$ we minimize ω on the compact set $-1 \leq \alpha \leq 1$. If the infimum occurs at $\alpha = \pm 1$ then the vectors $\mathbf{a}$ and $\mathbf{b}$ are colinear and hence the problem reduces to the case $n = 1$. Otherwise, we differentiate ω with respect to α and set the result equal to zero to conclude that $\alpha = -t/2$, which necessitates $t \leq 2$. In this case we find that

$$\kappa = \omega(t, -t/2) = \frac{1}{2} p t^{2-p} \leq p 2^{1-p}. \quad \square$$

References.

[AF 84] Acerbi, E. and N. Fusco, Semicontinuity problems in the calculus of variations. Arch. Rational Mech. Anal. **86** (1984), 125–145.

[Ba 77] Ball, J. M., Constitutive inequalities and existence theorems in nonlinear elastostatics, *Nonlinear Analysis and Mechanics*, Vol. I (R. J. Knops, ed.), Pitman, 1977.

[Ba 81] Ball, J. M., Global invertibility of Sobolev functions and the interpenetration of matter. Proc. Roy. Soc. Edinburgh, **88A** (1981), 315–328.

[Ba 82] Ball, J. M., Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Phil. Trans. Roy. Soc. London, A **306** (1982), 557–611.

[BM 84] Ball, J. M. and F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. J. Func. Anal. **58** (1984), 225–253.

[BN 95] Brezis, H. and L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries, Selecta Math. (N. S.) **1** (1995), 197–263.

[BN 96] Brezis, H. and L. Nirenberg, Degree theory and BMO. II. Compact manifolds with boundaries. With an appendix by the authors and Petru Mironescu, Selecta Math. (N. S.) **2** (1996), 309–368.

[CN 87] Ciarlet, P. G. and J. Nečas, Injectivity and self-contact in non-linear elasticity. Arch. Rational Mech. Anal. **97** (1987), 171–188.

[Ev 86] Evans, L. C., Quasiconvexity and partial regularity in the calculus of variations. Arch. Rational Mech. Anal. **95** (1986), 227–252.

[EG 92] Evans, L. C. and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.

[Fe 69] Federer, H., *Geometric Measure Theory*, Springer, 1969.

[FG 95] Fonseca, I. and W. Gangbo, *Degree Theory in Analysis and Applications*, Oxford Press, 1995.

[Ge 73] Gehring, F. W., The L^p-integrability of the partial derivatives of a quasiconformal mapping, Acta Math. **130** (1973), 265–277.

[GMS 94] Giaquinta, M., G. Modica, J. Souček, Remarks on the degree theory, J. Func. Anal. **125** (1994), 172–200.

[GISS 95] Greco, L.,T. Iwaniec, C. Sbordone, and B. Stroffolini, Degree formulas for maps with nonintegrable Jacobian, Top. Meth. Nonlin. Anal. **6** (1995), 81–95.

[GL 58] Gent, A. N. and P. B. Lindley, Internal rupture of bonded rubber cylinders in tension. Proc. R. Soc. Lond. A **249** (1958), 195–205.

[GP 84] Gent, A. N. and B. Park, Failure processes in elastomers at or near a rigid spherical inclusion. J. Mater. Sci. **19** (1984), 1947–1956.

[HP 95] Horgan, C. O. and D. A. Polignone. Cavitation in nonlinearly elastic solids: A review. Appl. Mech. Rev. **48** (1995), 471–485.

[JS 91] James, R. D. and S. J. Spector, The formation of filamentary voids in solids. J. Mech. Phys. Solids **39** (1991), 783–813.

[JS 92] James, R. D. and S. J. Spector, Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter and function spaces for elasticity. Anal. non linéaire **9** (1992), 263–280.

[Ma 86] Marcellini, P., On the definition and the lower semicontinuity of certain quasiconvex integrals. Anal. non linéaire **3** (1986), 391–409.

[Ma 89] Marcellini, P., The stored-energy for some discontinuous deformations in nonlinear elasticity. *Partial Differential Equations and the Calculus of Variations, Vol. II* (F. Colombini, A. Marino, L. Modica, and S. Spagnolo, eds.), Birkhäuser, 1989.

[MM 73] Marcus, M. and V. J. Mizel, Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems. Bull. Am. Math. Soc. **79** (1973), 790–795.

[Ma 94] Manfredi, J. J., Weakly monotone functions. J. Geom. Anal. **4** (1994), 393–402.

[Me 65] Meyers, N. G., Quasi-convexity and lower semi-continuity of multiple variational integrals of any order. Trans. Amer. Math. Soc. **119** (1965), 125–149.

[Mo 52] Morrey, C. B., Quasi-convexity and the lower semicontinuity of multiple integrals. Pacific J. Math. **2** (1952), 25–53.

[Mo 66] Morrey, C. B., *Multiple Integrals in the Calculus of Variations*, Springer, 1966.

[MST 96] Müller, S., S. J. Spector and Q. Tang, Invertibility and a topological property of Sobolev maps. SIAM J. Math. Anal. **27** (1996), 959–976.

[MS 95] Müller, S. and S. J. Spector, An existence theory for nonlinear elasticity that allows for cavitation. Arch. Rational Mech. Anal. **131** (1995), 1–66.

[MTY 94] Müller, S., Q. Tang and B. S. Yan, On a new class of elastic deformations not allowing for cavitation. Anal. non linéaire **11** (1994), 217–243.

[Mü 90] Müller, S., Det = det - A remark on the distributional determinant. C. R. Acad. Sci. Paris, Ser. I, **311** (1990), 13–17.

[OB 65] Oberth, A. E. and R. S. Bruenner, Tear phenomena around solid inclusions in castable elastomers. Trans. Soc. Rheol. **9** (1965), 165–185.

[Og 72] Ogden, R. W., Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubberlike solids. Proc. Roy. Soc. London, A **328** (1972), 567–583.

[Sc 69] Schwartz, J. T., *Nonlinear Functional Analysis*, Gordon and Breach, 1969.

[Si 92] Sivaloganathan, J., Singular minimizers in the calculus of variations: a degenerate form of cavitation. Anal. non linéaire **9** (1992), 657–681.

[Si 95] Sivaloganathan, J., On the stability of cavitating equilibria. Q. Appl. Math. **53** (1995), 301–313.

[Sp 94] Spector, S. J., Linear deformations as global minimizers in nonlinear elasticity. Q. Appl. Math. **52** (1994), 59–64.

[St 93] Stuart, C. A., Estimating the critical radius for radially symmetric cavitation. Q. Appl. Math. **51** (1993), 251–263.

[Sv 88] Šverák, V., Regularity properties of deformations with finite energy. Arch. Rational Mech. Anal. **100** (1988), 105–127.

[VG 76] Vodop'yanov, S. K. and V. M. Gol'dshtein, Quasiconformal mappings and spaces of functions with generalized first derivatives. Siberian Math. J. **17** (1976), 399–411.

[Zh 91] Zhang, K., Energy minimizers in nonlinear elastostatics and the implicit function theorem. Arch. Rational Mech. Anal. **114** (1991), 95–117.

[Zi 89] Ziemer, W. P., *Weakly Differentiable Functions*, Springer, 1989.