Aleksandrov Reflection and Nonlinear Evolution Equations, I: The n-sphere and n-ball*

Bennett Chow Robert Gulliver
School of Mathematics, University of Minnesota

January 5, 1998

1 Introduction

In a brilliant series of papers, A. D. Aleksandrov (1956,1957,1958a,1958b) and Aleksandrov-Volkov (1958) introduced a reflection method based upon the Hopf boundary-point lemma and strong maximum principle. Aleksandrov used his method to show that for a general class of curvature functions, any constant curvature hypersurface embedded in either Euclidean space, hyperbolic space, or a hemisphere of the sphere, is a round sphere of codimension one. J. Serrin (1971), by a beautiful application of the reflection method, proved that solutions to the Poisson equation on a domain with over-determined boundary conditions must be a radial solution on the ball. In a pair of fundamental papers, B. Gidas, W.-M. Ni, and L. Nirenberg (1979,1981) proved symmetry of positive solutions to a class of nonlinear second order spherically symmetric elliptic equations.

In each of the above papers, the proof is based upon Aleksandrov's method of reflecting the solution about a *moving* plane. In this paper, the first in a series, we introduce a new variation of the reflection method. Instead of reflecting a fixed solution about a moving plane, we reflect a one-parameter

*To appear in *Calculus of Variations and Partial Differential Equations*

family of solutions about a *fixed* plane. This method applies to certain non-linear parabolic and elliptic partial differential equations of second order. The major hypothesis on the non-linear partial differential equation is analogous to assuming that at most one eigenvalue is negative, in the linear case. In particular, we shall show that solutions to a class of parabolic equations on the n-sphere and the n-ball with arbitrary initial data remain a bounded distance from symmetry. This holds even for equations where the solution may *blow up in finite time.* In the parabolic case, our method relies only on the weak maximum principle. On the n-sphere, this allows us to consider *degenerate* parabolic equations of the form (see equation (37))

$$u_t = G(\nabla\nabla u + cug, u, t),$$

where G is a Lipschitz continuous function, nonincreasing in the second variable, and $c \leq 1$ (see section 3).

In section 2 we prove that solutions on $S^1 \times [0, T)$ to the equation (see equation (7))

$$u_t = G(u_{xx} + u),$$

where $G : \mathbb{R} \to \mathbb{R}$ is monotone increasing, satisfy the same gradient bound at each time $t < T$ as the initial data. As a consequence, solutions have bounded oscillation over the circle, independent of time. In section 3 we consider parabolic equations on the n-sphere and generalize the results of the previous section. The results we prove are related to estimates for convex hypersurfaces expanding by curvature-dependent normal vector fields, which shall be discussed in Chow-Tsai (1994a,b). In section 4 we consider parabolic equations on the n-ball and prove that solutions have bounded oscillation on (n-1)-spheres. We also obtain a uniform gradient estimate in the spherical directions.

In the second paper of this series (1994a), we shall extend the methods of this paper to, for example, a fully nonlinear degenerate parabolic equation on a Riemannian manifold with one isometric reflection. We will obtain results analogous to (and more general than) Theorem 4.1, implying bounds on the oscillation of a solution on each orbit, which now consists of only two points. We shall also derive estimates on second derivatives for such equations, under an additional hypothesis on the right-hand side. In the third paper of this series (1994b), we shall treat embedded hypersurfaces in $\mathbb{R}^{n+1}$ which flow according to a function of curvatures. The reflection method becomes reflection of $\mathbb{R}^{n+1}$ in hyperplanes, more closely analogous to Aleksandrov (1957). For an expanding hypersurface, our results imply convergence after rescaling to the round sphere, provided the solution exists until the hypersurface expands to infinity. In the convex case, the support function $u : S^n \to \mathbb{R}$ satisfies equation (37) with $c = 1$, and the analogy with the present part becomes an equivalence. In (1994c) we consider the elliptic analogues of the results in this paper. In particular, we prove that solutions to certain elliptic equations on the n-sphere are affine functions. This result is analogous to the parabolic result of section 3 and generalizes Aleksandrov's theorem concerning embedded hypersurfaces with constant curvature in the special case of convex hypersurfaces, described in terms of the Minkowski support function. We will also treat the case of embedded hypersurfaces, yielding a full generalization of Aleksandrov's theorem.

2 The equation $u_t = G(u_{xx} + u)$ on the circle

In this section we consider certain nonlinear parabolic equations on the circle. We shall generalize the results of this section to the n-sphere in the next section. We first consider the case of the circle to illustrate the main ideas with a minimum of technical complexity.

Recall the heat equation on the unit circle. Let $v : S^1 \times \mathbb{R}_+ \to \mathbb{R}$ be the solution to

$$v_t = v_{xx} \qquad \text{in } S^1 \times \mathbb{R}_+ \tag{1}$$

$$v|_{t=0} = v_o, \tag{2}$$

where $v_o : S^1 \to \mathbb{R}$ is a Lipschitz function. Here we consider S^1 as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. It is well known, and can be easily proved using Fourier series, that there exists a constant $C > 0$ depending only on v_o such that

$$\max_{x\in S^1} v(x,t) - \min_{x\in S^1} v(x,t) \le Ce^{-t}, \tag{3}$$

for all $t \ge 0$. If we define $u(x,t) = e^t v(x,t)$, then

$$u_t = u_{xx} + u \qquad \text{in } S^1 \times \mathbb{R}_+ \tag{4}$$

$$u|_{t=0} = u_o, \tag{5}$$

where $u_o = v_o$. Inequality (3) becomes

$$\max_{x\in S^1} u(x,t) - \min_{x\in S^1} u(x,t) \le C, \tag{6}$$

for all $t \ge 0$.

The first application of our version of the Aleksandrov reflection method will be to prove that inequality (6) holds for solutions of nonlinear parabolic equations of the form

$$u_t = G(u_{xx} + u) \qquad \text{in } S^1 \times [0,T) \tag{7}$$

$$u|_{t=0} = u_o, \tag{8}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and $0 < T \le \infty$. Given $u : S^1 \times [0,T) \to \mathbb{R}$, we define

$$\Im(u) = \{u_{xx}(x,t) + u(x,t) | (x,t) \in S^1 \times [0,T)\}. \tag{9}$$

Theorem 2.1 *Let $u \in C^2(S^1 \times [0,T))$ be a solution to equation (7)-(8), where $G|_{\Im(u)}$ is a nondecreasing, uniformly Lipschitz function and u_o is a Lipschitz function. Then there exists a constant $\lambda \ge 0$ depending only on u_o such that*

$$|u(x_1,t) - u(x_2,t)| \le \lambda \left|\sin\left(\frac{x_1 - x_2}{2}\right)\right|, \tag{10}$$

for all $x_1, x_2 \in S^1 = \mathbb{R}/2\pi\mathbf{Z}$ and $t \in [0,T)$. More precisely, the Euclidean Lipschitz constant

$$\max_{x_1 \ne x_2} \frac{u(x_1,t) - u(x_2,t)}{|\sin((x_1 - x_2)/2)|} \tag{11}$$

is a nonincreasing function of $t \in [0,T)$.

Remark. If $f : [0,T) \to \mathbb{R}$ satisfies the ordinary differential equation $\dot{f}(t) = G(f(t))$, then

$$u(x,t) = f(t) + a\cos x + b \sin x, \tag{12}$$

where a and b are constants, is a solution to (7). Since the oscillation of u is constant in time, Theorem 2.1 is sharp.

The proof of Theorem 2.1 depends on the following.

Lemma 2.2 *(i) Given any Lipschitz function $u_o : S^1 \to \mathbb{R}$, there exists $\lambda \in \mathbb{R}$ such that*

$$u_o(\pi - x) + \lambda \cos x \geq u_o(x), \tag{13}$$

for all $-\pi/2 \leq x \leq \pi/2$.

(ii) If $u : S^1 \times [0,T) \to \mathbb{R}$ satisfies the hypotheses of Theorem 2.1, and if also (13) holds for some $\lambda \in \mathbb{R}$, then

$$u(\pi - x, t) + \lambda \cos x \geq u(x,t), \tag{14}$$

for all $-\pi/2 \leq x \leq \pi/2$ and $t \in [0,T)$.

Proof. (i) Define $w_o(x) = u_o(x) - u_o(\pi - x)$. Then w_o is a Lipschitz function on the half-circle $S^1_+ = \{x \in S^1 | -\pi/2 \leq x \leq \pi/2\}$ with $w_o(-\pi/2) = w_o(\pi/2) = 0$. This implies that there exists a $\lambda \in \mathbb{R}$ such that $\lambda \cos x \geq w_o(x)$ for all $x \in S^1_+$. Part (i) follows.

(ii) The idea is that if we reflect the solution u and add to it a large constant multiple of $\cos x$, we then obtain a new solution which by the weak maximum principle is greater than u on the half-circle. Given $\lambda \in \mathbb{R}$ as in part (i), define $u^\lambda(x,t) = u(\pi - x, t) + \lambda \cos x$. Then u^λ is a solution to (7) with initial condition

$$u^\lambda|_{t=0} = u_o^\lambda, \tag{15}$$

where $u_o^\lambda : S^1 \to \mathbb{R}$ is given by

$$u_o^\lambda(x) = u_o(\pi - x) + \lambda \cos x, \qquad x \in S^1. \tag{16}$$

Since both u^λ and u are solutions to (7) and $u_o^\lambda \geq u_o$ on S^1_+, we compute that their difference $u^\lambda - u$ is a solution to the equation

$$\begin{aligned}(u^\lambda - u)_t &= H(u^\lambda_{xx} + u^\lambda, u_{xx} + u)\left((u^\lambda - u)_{xx} + u^\lambda - u\right), &(17)\\ (u^\lambda - u)|_{t=0} &= u_o^\lambda - u_o \geq 0, &(18)\end{aligned}$$

where $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(v,w) = \begin{cases} \dfrac{G(w) - G(v)}{w - v} & \text{if } v \neq w \\ 0 & \text{if } v = w \end{cases}. \tag{19}$$

Since $G|_{\Im}$ is nondecreasing and uniformly Lipschitz, $H|_{\Im\times\Im}$ is a nonnegative bounded function. Moreover, $(u^\lambda_{xx}+u^\lambda, u_{xx}+u) \in \Im\times\Im$. Hence, by the weak maximum principle for parabolic equations of second order, we conclude that $u^\lambda - u \geq 0$ in $S^1_+ \times [0,T)$. Part (ii) follows.

Remark. The weak maximum principle is often stated for strictly parabolic equations; however, the standard proof holds for degenerate parabolic equations. See, for example, the proof in Hamilton (1975), pp.101-2, or the proof of Theorem 3.1 below.

Proof of Theorem 2.1. Given $\theta \in S^1$, define $u_\theta \in C^2(S^1 \times [0,T))$ by

$$u_\theta(y,t) = u(y - \frac{\pi}{2} + \theta, t), \tag{20}$$

for $y \in S^1$, $t \in [0,T)$. Clearly, the rotated function u_θ is a solution to equation (7). Hence, by Lemma 2.2 there exists $\lambda(\theta)$ such that

$$u_\theta(\pi - y, t) + \lambda(\theta)\cos y \geq u_\theta(y,t), \tag{21}$$

for all $y \in [-\pi/2, \pi/2]$, $t \in [0,T)$. Setting $x = y - \pi/2 + \theta$, we obtain for all $\theta \in S^1$

$$u(2\theta - x, t) + \lambda(\theta)\sin(\theta - x) \geq u(x,t), \tag{22}$$

for all $x \in [\theta - \pi, \theta]$, $t \in [0,T)$. Because the circle is compact, we may take $\lambda(\theta)$ independent of θ. Setting $x = x_1$ and $\theta = (x_1+x_2)/2$, we conclude that there exists $\lambda \geq 0$ depending only on u_o, such that

$$u(x_2,t) + \lambda \sin\left(\frac{x_2 - x_1}{2}\right) \geq u(x_1,t), \tag{23}$$

for all $x_1, x_2 \in S^1$, $t \in [0,T)$. Switching x_1 and x_2 in (23) implies inequality (10). The proof of the theorem is complete.

Remark. We only need G to be defined on the set $\Im(u)$.

As an immediate consequence of Theorem 2.1 we have the following gradient estimate for u.

Corollary 2.3 *Suppose* $u : S^1 \times [0,T) \to \mathbb{R}$ *satisfies the hypotheses of Theorem 2.1 and let* $\lambda \geq 0$ *be the constant given in the conclusion of Theorem 2.1. Then*

$$|u_x| \leq \lambda/2 \qquad \text{in } S^1 \times (0,T). \tag{24}$$

Proof. Since $|\sin x| \leq |x|$, Theorem 2.1 implies the Lipschitz estimate

$$\frac{|u(x_1,t) - u(x_2,t)|}{|x_1 - x_2|} \leq \lambda/2, \tag{25}$$

for all $x_1, x_2 \in S^1$, $x_1 \neq x_2$, $t \in [0,T)$. Since $u \in C^1$ for $t > 0$, the corollary follows.

The reflection method may also be used to obtain estimates for certain higher derivatives of the solution u to equation (7). For example, suppose that $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing C^1-diffeomorphism. We define

$$v = G(u_{xx} + u), \tag{26}$$

and compute that

$$v_t = G' \circ G^{-1}(v)(v_{xx} + v). \tag{27}$$

By the weak maximum principle, if $v|_{t=0} \geq c > 0$, then $v \geq c$ for all $t \in [0,T)$. Moreover, we can apply the reflection method to equation (27) under certain hypotheses on G. In fact, we consider equations of the more general form

$$u_t = H(u)(u_{xx} + u) \qquad \text{in } S^1 \times [0,T) \tag{28}$$

$$u|_{t=0} = u_o. \tag{29}$$

Let $\mu = \inf_{S^1 \times [0,T)} u$. Analogous to Theorem 2.1, we have the following.

Theorem 2.4 *Let* $u \in C^2(S^1 \times [0,T))$ *be a solution to equation (28)-(29), where* $H|_{[\mu,\infty)}$ *is a nonnegative, nonincreasing function and* u_o *is a Lipschitz function. Then there exists a constant* $\lambda \geq 0$ *depending only on* u_o *such that*

$$|u(x_1,t) - u(x_2,t)| \leq \lambda \left| \sin\left(\frac{x_1 - x_2}{2}\right)\right|, \tag{30}$$

for all $x_1, x_2 \in S^1$ *and* $t \in [0,T)$.

We omit the proof, which is analogous to that of Theorem 2.1, since in section 3 we shall consider a generalization of both Theorems 2.1 and 2.4 to the n-sphere. In the second paper of this series, we also consider applications to estimates for certain higher derivatives of the solution. In the case of the circle, we have

Corollary 2.5 *Let $u \in C^4(S^1 \times [0,T))$ be a solution to (7)-(8). Suppose $G \in C^1([a_o, a_1])$, $G' > 0$ and G' nonincreasing on $[a_o, a_1]$, where $a_o = \min(u_{xx}+u)$ and $a_1 = \max(u_{xx}+u)$ over $S^1 \times [0,T)$. Also assume u_o has Lipschitz second derivatives on S^1. If v is given by (26), then there is a constant $\lambda \geq 0$, depending only on u_o, such that*

$$|v_x(x,t)| \leq \lambda \tag{31}$$

for all $x \in S^1$ and $t \in [0,T)$.

As a consequence, if $G' \geq \epsilon > 0$, then

$$|u_{xxx}| \leq \lambda/\epsilon \tag{32}$$

and hence

$$\| u_x(\cdot,t) \|_{C^2(S^1)} \leq C, \tag{33}$$

for all $t \in [0,T)$, where C depends only on u_o and ϵ.

See Chow and Tsai (1994) for second-derivative estimates under the hypotheses of Theorem 2.1, and for applications to curves expanding by functions of curvature.

Example. Let $G(x) = x^p$, for $x > 0$, where $p > 0$. By (27), $v = (u_{xx} + u)^p$ satisfies

$$v_t = pv^{1-1/p}(v_{xx} + v). \tag{34}$$

Applying the weak maximum principle to (34) yields: if $(u_o)_{xx} + u_o > 0$, then $u_{xx} + u > 0$ on $S^1 \times [0,T)$. This implies $\Im(u) \subset \mathbb{R}_+$. Since G is an increasing function on $\mathbb{R}_+$, the hypothesis of Theorem 2.1 is satisfied, and hence the oscillation of u remains bounded, even though u may blow up in finite time if $p > 1$. Moreover, if $0 < p \leq 1$, then G' is a nonincreasing function and the hypothesis of Corollary 2.5 is satisfied; thus for all $0 < t < \infty$, v_x remains bounded although u may be unbounded. Compare Urbas (1991).

3 Nonlinear parabolic equations on S^n

In this section we consider generalizations to higher dimensions of the estimates we obtained for solutions to either (7)-(8) or (28)-(29) on the unit circle. Let (S^n, g) denote the unit n-sphere with the standard metric, ∇ the covariant derivative acting on tensors, and $S^2T^*S^n$ the bundle of symmetric covariant 2-tensors on S^n. Let

$$G : S^2T^*S^n \times \mathbb{R} \times [0,T) \to \mathbb{R} \tag{35}$$

be a function invariant under reflection. That is, for any reflection $\rho : S^n \to S^n$, we have

$$G(\rho^*\alpha, v, t) = G(\alpha, v, t), \tag{36}$$

for all $\alpha \in S^2T^*S^n$, $v \in \mathbb{R}$, $t \in [0,T)$.

Remark. Since the group of isometries $O(n+1)$ of S^n is generated by reflections, (36) holds for any isometry ρ of S^n. The action of the isometry group $O(n+1)$ of S^n on $S^2T^*S^n$ identifies the fibers of $S^2T^*S^n$ up to the action of $O(n)$ on each fiber. Given a point $x \in S^n$, we may consider G as a function on $S^2T^*S^n_x \times \mathbb{R} \times [0,T)$. By the invariance of G under isometries, G is invariant under the action of $O(n)$ on $S^2T^*S^n_x$, G depends only on the eigenvalues of α w.r.t. g, and G is independent of x.

We shall consider nonlinear parabolic equations of the form

$$\begin{aligned} u_t &= G(\nabla\nabla u + cug, u, t) \qquad \text{in } S^n \times [0,T) && (37)\\ u|_{t=0} &= u_o, && (38) \end{aligned}$$

where $c \le 1$ is a constant. Equation (37)-(38) is a generalization to higher dimensions of both equations (7)-(8) and (28)-(29). As in the previous section, we assume a monotonicity condition on G. Let

$$\Omega_{\max}(t) = \max_{y \in S^n, |Y|=1} (\nabla\nabla u + cug)(y,t)(Y,Y), \tag{39}$$

and similarly $\Omega_{\min}(t)$. Given $\alpha, \beta \in S^2T^*S^n$, the inequality $\alpha \ge \beta$ means $\alpha - \beta$ is positive semi-definite. Analogous to definition (9), given $\epsilon \in \mathbb{R}_+$, we define

$$\begin{aligned} \Im_\epsilon(u) = \{(\alpha, v, t) \in S^2T^*S^n \times \mathbb{R} \times [0,T) \,|\, \min_{S^n} u(t) \le v \le \max_{S^n} u(t),\\ (\Omega_{\min}(t) - \max\{c, 0\}\epsilon)g \le \alpha \le \Omega_{\max}(t)g\}. \qquad (40) \end{aligned}$$

We shall suppose that for all $(\alpha, v, t), (\beta, w, t) \in \Im_\epsilon(u)$ such that $\alpha \geq \beta$ and $v \leq w$, we have

$$G(\alpha, v, t) \geq G(\beta, w, t). \tag{41}$$

Remark. The condition that G is nondecreasing in the first variable may be taken as a definition of degenerate parabolicity of equation (37).

Theorem 3.1 *Let $u \in C^2(S^n \times [0,T))$ be a solution to (37)-(38), where $c \leq 1$ is a constant and u_o is a Lipschitz function. Suppose that $G|_{\Im_\epsilon(u)}$ satisfies the above conditions (36) and (41), for some $\epsilon > 0$. Moreover, if $c > 0$ assume that $G|_{\Im_\epsilon(u)}$ is uniformly Lipschitz continuous in the first variable, in the direction of g; that is, there exists a constant $C > 0$ such that for all $(\alpha, v, t) \in \Im_\epsilon(u)$ and $a \in \mathbb{R}_+$ with $(\alpha + ag, v, t) \in \Im_\epsilon(u)$,*

$$\frac{G(\alpha + ag, v, t) - G(\alpha, v, t)}{a} \leq C. \tag{42}$$

Then

(i) given any unit vector $\nu \in S^n$, there exists $\lambda(\nu) \in \mathbb{R}_+$ depending only on ν and u_o (e.g., independent of G) such that

$$u(x - 2\langle x, \nu\rangle\nu, t) + \lambda(\nu)\langle x, \nu\rangle \geq u(x, t), \tag{43}$$

for all $(x,t) \in S^n_+ \times [0,T)$, where $S^n_+ = S^n_+(\nu) = \{x \in S^n | \langle x, \nu\rangle \geq 0\}$.

(ii) There exists $\lambda \in \mathbb{R}_+$ depending only on u_o such that

$$|u(x_1, t) - u(x_2, t)| \leq \lambda \sin\left(\frac{dist_{S^n}(x_1, x_2)}{2}\right), \tag{44}$$

for all $x_1, x_2 \in S^n$, $t \in [0,T)$.

(iii) For all $t \in [0,T)$ we have

$$\max_{x \in S^n} u(x,t) - \min_{x \in S^n} u(x,t) \leq \lambda. \tag{45}$$

(iv) For all $x \in S^n$ and $t \in (0,T)$

$$|\nabla u(x,t)| \leq \lambda/2, \tag{46}$$

where in parts (iii) and (iv), λ is the constant given in part (ii).

Remark. Theorem 2.1 is the special case of Theorem 3.1 where $n = 1$, $c = 1$, and G is independent of the second and third variables. Theorem 2.4 is the special case $c = 1$, $n = 1$ with $G(\alpha, v, t) = H(v)\alpha$.

Proof. The proof is analogous to the proof of Theorem 2.1: we reflect the solution u and add to it a large constant multiple of a first eigenfunction of the Laplacian to obtain a supersolution to (37), which by an application of the weak maximum principle is greater than u in a hemisphere.

(i) Given $\nu \in S^n$ and $\lambda \in \mathbb{R}$, define $x^* = x - 2\langle x, \nu\rangle\nu$ and

$$u^\lambda(x,t) = u(x^*, t) + \lambda\langle x, \nu\rangle. \tag{47}$$

Clearly $u^\lambda = u$ on $\partial S^n_+ \times [0,T)$. Let $u^\lambda_o = u^\lambda|_{t=0}$; as in Lemma 2.2 (i), since u_o is Lipschitz continuous, there exists $\lambda(\nu) \in \mathbb{R}_+$ depending only on ν and u_o such that

$$u_o^{\lambda(\nu)}(x) \geq u_o(x), \tag{48}$$

for all $x \in S^n_+$.

Claim. For $\epsilon > 0$ as in the hypothesis of the theorem, $u^{\lambda(\nu)}$ is a supersolution of (37) in $(S^n_+ \times [0,T)) \cap \{-\epsilon \leq u^{\lambda(\nu)} - u \leq 0\} \cap \{\nabla^2 u^{\lambda(\nu)} \geq \nabla^2 u\}$; that is,

$$u_t^{\lambda(\nu)}(x,t) \geq G\left((\nabla\nabla u^{\lambda(\nu)} + cu^{\lambda(\nu)}g)(x,t), u^{\lambda(\nu)}(x,t), t\right), \tag{49}$$

for all $(x,t) \in S^n_+ \times [0,T)$ such that $-\epsilon \leq u^{\lambda(\nu)}(x,t) - u(x,t) \leq 0$ and $\nabla^2 u^{\lambda(\nu)}(x,t) \geq \nabla^2 u(x,t)$.

Proof of claim. Let $\varphi(x) = \langle x, \nu\rangle$. Then φ is a linear function restricted to S^n. This implies

$$\nabla\nabla\varphi + \varphi g = 0. \tag{50}$$

Define $u^*(x,t) = u(x^*, t)$. Since $G|_{\mathfrak{S}_\epsilon(u)}$ is invariant under reflection (condition (36)), one obtains that u^* is a solution to (37). Therefore

$$\begin{aligned} u_t^{\lambda(\nu)} &= u_t^* = G(\nabla^2 u^* + cu^* g, u^*, t) \qquad (51)\\ &= G(\nabla^2 u^{\lambda(\nu)} + cu^{\lambda(\nu)} g + \lambda(\nu)(1-c)\varphi g, u^{\lambda(\nu)} - \lambda(\nu)\varphi, t), \end{aligned}$$

in $S^n \times [0,T)$. Since $\lambda(\nu)(1-c)\varphi g \geq 0$ and $-\lambda(\nu)\varphi \leq 0$ in S^+, provided we can show that at points (x,t) where $-\epsilon \leq u^{\lambda(\nu)} - u \leq 0$ and $\nabla^2 u^{\lambda(\nu)} \geq \nabla^2 u$

the elements $\omega_1 = (\nabla^2 u^* + cu^* g, u^*, t)$ and $\omega_2 = (\nabla^2 u^{\lambda(\nu)} + cu^{\lambda(\nu)} g, u^{\lambda(\nu)}, t)$ are in $\Im_\epsilon(u)$, we may apply the monotonicity condition (41) to (51) to obtain

$$u_t^{\lambda(\nu)} \geq G(\nabla\nabla u^{\lambda(\nu)} + cu^{\lambda(\nu)} g, u^{\lambda(\nu)}, t), \tag{52}$$

in $(S^n_+ \times [0,T)) \cap \{-\epsilon \leq u^{\lambda(\nu)} - u \leq 0\} \cap \{\nabla^2 u^{\lambda(\nu)} \geq \nabla^2 u\}$, as claimed.

We now show that $\omega_1, \omega_2 \in \Im_\epsilon(u)$. First, since $\lambda(\nu) \geq 0$ and $u^{\lambda(\nu)} \leq u$, we have $\min_{S^n} u(t) \leq u^* \leq u^{\lambda(\nu)} \leq u \leq \max_{S^n} u(t)$. Second, since $\nabla^2 u^{\lambda(\nu)} \geq \nabla^2 u$, we have

$$\Omega_{\max}(t) g \geq \nabla^2 u^* + cu^* g \geq \nabla^2 u^{\lambda(\nu)} + cu^{\lambda(\nu)} g \geq \nabla^2 u + cu^{\lambda(\nu)} g. \tag{53}$$

If $c \leq 0$, then

$$\nabla^2 u + cu^{\lambda(\nu)} g \geq \nabla^2 u + cug \geq \Omega_{\min}(t) g. \tag{54}$$

If $c > 0$, then

$$\nabla^2 u + cu^{\lambda(\nu)} g = \nabla^2 u + cug + c(u^{\lambda(\nu)} - u) g \geq (\Omega_{\min}(t) - c\epsilon) g. \tag{55}$$

In either case, we conclude that $\omega_1, \omega_2 \in \Im_\epsilon(u)$; this proves the claim.

Remark. If $c = 1$, then using $\nabla^2 u^{\lambda(\nu)} + u^{\lambda(\nu)} g = \nabla^2 u^* + u^* g$ one can show that $u^{\lambda(\nu)}$ is a super-solution in $(S^n_+ \times [0,T)) \cap \{u^{\lambda(\nu)} \leq u\}$, even for $\epsilon = 0$ in the hypothesis of the theorem.

Proof of Theorem 3.1, continued. We now apply weak maximum principle-type arguments to the difference of u and $u^{\lambda(\nu)}$. Given $A \in \mathbb{R}_+$, define

$$w(x,t) = e^{-At}(u^{\lambda(\nu)}(x,t) - u(x,t)). \tag{56}$$

Subtracting (37) from (49) we obtain that at points in $S^n_+ \times [0,T)$ where $-\epsilon \leq u^{\lambda(\nu)} - u \leq 0$ and $\nabla^2 u^{\lambda(\nu)} \geq \nabla^2 u$,

$$w_t \geq -Aw + e^{-At}\left(G(\nabla\nabla u^{\lambda(\nu)} + cu^{\lambda(\nu)} g, u^{\lambda(\nu)}, t) - G(\nabla\nabla u + cug, u, t)\right). \tag{57}$$

We prove the theorem by contradiction. Suppose that w is negative somewhere. Since w is continuous, there exists a point $(x_o, t_o) \in \operatorname{int}(S^n_+) \times (0,T)$ such that

$$-\epsilon e^{-At} \leq \min_{S^n_+ \times [0,t_o]} w = w(x_o, t_o) < 0. \tag{58}$$

At (x_o, t_o), we also have $w_t \leq 0$ and $\nabla\nabla w \geq 0$ (i.e., positive semi-definite). This implies

$$-\epsilon \leq u^{\lambda(\nu)}(x_o, t_o) - u(x_o, t_o) < 0 \tag{59}$$

and

$$\nabla\nabla u^{\lambda(\nu)}(x_o, t_o) \geq \nabla\nabla u(x_o, t_o). \tag{60}$$

Hence (57) holds at (x_o, t_o). Applying the inequality $w_t(x_o, t_o) \leq 0$ to (57) yields, at (x_o, t_o),

$$0 > Ae^{At}w \geq G(\nabla\nabla u^{\lambda(\nu)} + cu^{\lambda(\nu)}g, u^{\lambda(\nu)}, t) - G(\nabla\nabla u + cug, u, t). \tag{61}$$

If the hypothesis of the monotonicity condition (41) holds, we may apply it to (61) while using inequalities (59) and (60) to obtain, at (x_o, t_o),

$$0 > Ae^{At}w \geq G(\nabla\nabla u + cu^{\lambda(\nu)}g, u, t) - G(\nabla\nabla u + cug, u, t). \tag{62}$$

Since we have already shown at (x_o, t_o), $\omega_2 = (\nabla\nabla u^{\lambda(\nu)} + cu^{\lambda(\nu)}g, u^{\lambda(\nu)}, t) \in \Im_\epsilon(u)$, we only need to check that at (x_o, t_o), $\omega_3 := (\nabla\nabla u + cu^{\lambda(\nu)}g, u, t) \in \Im_\epsilon(u)$. However, this follows from inequalities (54) and (55).

We now consider inequality (62) in two cases and obtain a contradiction in both cases.

Case 1). $c \leq 0$: Since $cu^{\lambda(\nu)} \geq cu$, by applying the monotonicity condition to (62), we have

$$0 > Ae^{At}w \geq 0, \tag{63}$$

a contradiction.

Case 2). $c > 0$: Applying the Lipschitz hypothesis (42) to (62) implies that there exists a constant $C > 0$ such that, at (x_o, t_o)

$$0 > Ae^{At}w \geq Cc(u^{\lambda(\nu)} - u) = Cce^{At}w. \tag{64}$$

However, we have the freedom to choose A arbitrary large. In particular, if we choose $A > Cc$, then we obtain a contradiction. This completes the proof of part (i).

(ii) Given points $x_1, x_2 \in S^n$ with $x_1 \neq x_2$, let

$$\nu = \frac{x_2 - x_1}{|x_1 - x_2|} \qquad \text{and} \qquad x = x_2. \tag{65}$$

Then $x - 2\langle x, \nu\rangle\nu = x_1$ and by part (i),

$$u(x_1, t) + \lambda\langle x, \nu\rangle \geq u(x_2, t), \tag{66}$$

where $\lambda \in \mathbb{R}_+$ may be taken independent of ν by the compactness of S^n. We have

$$\langle x, \nu\rangle = \frac{1 - \langle x_2, x_1\rangle}{|x_1 - x_2|} = \frac{1}{2}|x_1 - x_2|. \tag{67}$$

Let θ denote the angle formed by x_1 and x_2. Then $\theta = \mathrm{dist}_{S^n}(x_1, x_2)$ is the spherical distance between x_1 and x_2. By elementary trigonometry, we also have $\sin(\theta/2) = |x_1 - x_2|/2$. Therefore

$$\langle x, \nu\rangle = \frac{|x_1 - x_2|}{2} = \sin\left(\frac{\mathrm{dist}_{S^n}(x_1, x_2)}{2}\right), \tag{68}$$

and part (ii) follows from (66), (68), and then switching x_1 and x_2.

(iii) follows from taking x_1 and x_2 to be the points where u attains its maximum and minimum at time t, respectively.

(iv) From (ii) we have the Lipschitz estimate

$$\frac{|u(x_1, t) - u(x_2, t)|}{\mathrm{dist}_{S^n}(x_1, x_2)} \leq \lambda/2. \tag{69}$$

Since $u \in C^1$ for $t > 0$, this implies (iv), and the proof of the theorem is complete.

Remark. In (1994b) we shall consider examples of functions G to which Theorem 3.1 may be applied in the setting of hypersurfaces expanding by curvature-dependent normal vector fields.

4 Nonlinear parabolic equations in the ball

In this section we consider certain nonlinear second-order parabolic equations in the ball in Euclidean n-space. The results we prove are analogous to the results of the previous section for the sphere. However, unlike the sphere, the isometry group of the n-ball does not act transitively, since the orbits are (n-1)-spheres. We shall show that the solutions of the parabolic equations we consider have bounded oscillation on (n-1)-spheres. That is, solutions remain a bounded distance from spherical symmetry. In (1994a) we treat more general spherically symmetric equations with *non-symmetric* boundary data.

Let $B = B_R(0)$ denote the ball of radius R centered at the origin. Let

$$G : \mathbb{R} \times \mathbb{R} \times [0, R] \times [0, T) \to \mathbb{R}, \tag{70}$$

and consider the equation

$$\begin{aligned} u_t &= G(\Delta u + cu, u, |x|, t) && \text{in } B \times [0, T) && (71) \\ u|_{t=0} &= u_o && && (72) \\ u(x,t) &= h(t) && \text{on } \partial B \times [0, T), && (73) \end{aligned}$$

where $h : [0, T) \to \mathbb{R}$, c is a constant, and $u_o \in C^{1,1}(\overline{B})$. We shall assume the following monotonicity condition on G. Let

$$\Omega_{\max}(r, t) = \sup_{S^{n-1}(r) \times \{t\}} (\Delta u + cu), \tag{74}$$

and similarly $\Omega_{\min}(r, t)$. Given $\epsilon \in \mathbb{R}_+$, let

$$\begin{aligned} \Im_\epsilon(u) = \{&(v, w, r, t) \in \mathbb{R} \times \mathbb{R} \times [0, R] \times [0, T) | \\ &\Omega_{\min}(r, t) - \max\{c, 0\}\epsilon \le v \le \Omega_{\max}(r, t), \\ &\min_{S^{n-1}(r)} u(t) \le w \le \max_{S^{n-1}(r)} u(t)\}. \end{aligned} \tag{75}$$

For all $(v_1, w_1, r, t), (v_2, w_2, r, t) \in \Im_\epsilon(u)$ with $v_1 \ge v_2$ and $w_1 \le w_2$, we assume that

$$G(v_1, w_1, r, t) \ge G(v_2, w_2, r, t). \tag{76}$$

Let $\lambda_2 = \lambda_2(B)$ denote the second eigenvalue of the Laplacian on B with Dirichlet boundary values. Corresponding to every $\nu \in S^{n-1}$ we have a second eigenfunction for the Laplacian

$$\varphi_2(x) = f(|x|)\langle \frac{x}{|x|}, \nu \rangle, \tag{77}$$

where $f : [0, R] \to \mathbb{R}_+$ is the solution to the ordinary differential equation

$$r^2 f''(r) + (n-1) r f'(r) + \left(\lambda_2 r^2 - (n-1)\right) f(r) = 0, \tag{78}$$

with boundary values

$$f(0) = f(R) = 0. \tag{79}$$

We normalize the second eigenfunction by the condition $\max_{r \in [0,R]} f(r) = 1$ (or equivalently, $\max_B \varphi_2 = 1$).

Remark. Suppose G depends only on the first variable, $G(0) \leq 0$ and $h(t) \leq 0$. If $c \leq \lambda_1(B)$, an application of the weak maximum principle implies that solutions u to equation (71) are less than a constant multiple of the first eigenfunction and hence uniformly bounded for all time. However, when $c > \lambda_1(B)$, solutions to (71) may blow up, even in finite time. We shall assume that $c \leq \lambda_2(B)$.

Theorem 4.1 *Let $u : B \times [0, T) \to \mathbb{R}$ be a solution to (71)-(73) with $u_o \in C^{1,1}(\overline{B})$ and $c \leq \lambda_2(B)$. Suppose that $G|_{\Im_\epsilon(u)}$ satisfies (76) for some $\epsilon > 0$. Moreover, if $c > 0$ assume that $G|_{\Im_\epsilon(u)}$ is uniformly Lipschitz in the first variable. Then*

(i) for every $\nu \in S^{n-1}$, there exists $\lambda(\nu) \in \mathbb{R}_+$ depending only on u_o and ν, such that

$$u(x - 2\langle x, \nu\rangle \nu, t) + \lambda(\nu) f(|x|) \langle \frac{x}{|x|}, \nu \rangle \geq u(x, t), \tag{80}$$

for all $x \in B$ such that $\langle x, \nu \rangle \geq 0$, and $t \in [0, T)$.

(ii) There exists $\lambda \geq 0$ depending only on u_o, such that for every $x_1, x_2 \in B$ with $|x_1| = |x_2| = r \in (0, R]$ and $t \in [0, T)$, we have

$$|u(x_1, t) - u(x_2, t)| \leq \lambda f(r) \sin\left(\frac{dist_{S^{n-1}(r)}(x_1, x_2)}{2r}\right). \tag{81}$$

(iii) For every $r \in [0, R]$, $t \in [0, T)$,

$$\max_{x \in S^{n-1}(r)} u(x, t) - \min_{x \in S^{n-1}(r)} u(x, t) \leq \lambda f(r). \tag{82}$$

(iv) For every $r \in (0, R]$, $x \in S^{n-1}(r)$, and $t \in (0, T)$, we have

$$|\nabla_{S^{n-1}(r)} u(x,t)| \leq \frac{\lambda}{2r} f(r), \tag{83}$$

where the constant λ in parts (iii) and (iv) is the same λ given by part (ii).

Proof. We follow the method of sections 2 and 3. Given $\nu \in S^{n-1}$, recall that the reflected solution is given by $u^*(x,t) = u(x - 2\langle x, \nu\rangle \nu, t)$. For $\lambda \in \mathbb{R}$, define

$$\begin{aligned} u^\lambda(x,t) &= u^*(x,t) + \lambda \varphi_2(x) \\ &= u(x - 2\langle x, \nu\rangle \nu, t) + \lambda f(|x|)\langle x/|x|, \nu\rangle. \end{aligned} \tag{84}$$

Let $B_+ = \{x \in B | \langle x, \nu\rangle \geq 0\}$. We first show the following.

Claim. There exists $\lambda(\nu) \in \mathbb{R}_+$ such that

$$u^{\lambda(\nu)}|_{t=0} \geq u_o \qquad \text{in } B_+. \tag{85}$$

Proof of claim. Let $u_o^* = u^*|_{t=0}$. Since $u_o \in C^{1,1}(\overline{B})$, we also have $u_o^* - u_o \in C^{1,1}(\overline{B}_+)$. Moreover, $u_o^* - u_o = 0$ on ∂B_+. In particular, $\partial/\partial r(u_o^* - u_o) = 0$ when $\langle x, \nu\rangle = 0$. Hence, there exists a constant C such that

$$\left|\frac{\partial u_o^*}{\partial r}(x) - \frac{\partial u_o}{\partial r}(x)\right| = \left|\frac{\partial u_o}{\partial r}(x - 2\langle x, \nu\rangle\nu) - \frac{\partial u_o}{\partial r}(x)\right| \leq \frac{C}{|x|}\langle x, \nu\rangle, \tag{86}$$

for all $x \in B_+$, and where $\partial/\partial r = \sum_{i=1}^n x^i/|x| \cdot \partial/\partial x^i$. Integrating the above inequality along the line segment joining x and $Rx/|x|$, we obtain

$$\begin{aligned} u_o(x) - u_o^*(x) &\leq -\int_{|x|}^R \left(\frac{\partial u_o}{\partial r}(\rho x/|x|) - \frac{\partial u_o^*}{\partial r}(\rho x/|x|)\right) d\rho \\ &\leq \frac{C}{|x|}\langle x, \nu\rangle \int_{|x|}^R d\rho = \frac{C}{|x|}(R - |x|)\langle x, \nu\rangle, \end{aligned} \tag{87}$$

for $x \in B_+$. On the other hand, the Lipschitz bound on u_o implies there exists a constant C' such that

$$u_o(x) - u_o^*(x) \leq C'\langle x, \nu\rangle, \tag{88}$$

in B_+. However, there exists $\lambda(\nu) \in \mathbb{R}_+$ such that

$$\lambda(\nu)\frac{f(|x|)}{|x|} \geq \min\left\{\frac{C}{|x|}(R-|x|), C'\right\}. \tag{89}$$

Combining (87), (88), and (89) implies

$$u_o(x) - u_o^*(x) \leq \lambda(\nu)\frac{f(|x|)}{|x|}\langle x, \nu\rangle = \lambda(\nu)\varphi_2(x), \tag{90}$$

for $x \in B_+$, as claimed.

Proof of Theorem 4.1, continued. Since u^* is a solution to (71), we have

$$\begin{aligned} u_t^{\lambda(\nu)} &= u_t^* = G(\Delta u^* + cu^*, u^*, |x|, t) \qquad (91)\\ &= G(\Delta u^{\lambda(\nu)} + cu^{\lambda(\nu)} + (\lambda_2 - c)\lambda(\nu)\varphi_2, u^{\lambda(\nu)} - \lambda(\nu)\varphi_2, |x|, t). \end{aligned}$$

Since $(\lambda_2 - c)\lambda(\nu)\varphi_2 \geq 0$ and $-\lambda(\nu)\varphi_2 \leq 0$ in B_+, we may apply the monotonicity condition (76) to (91) at points where $-\epsilon \leq u^{\lambda(\nu)} - u \leq 0$ and $\Delta u^{\lambda(\nu)} \geq \Delta u$, to obtain

$$u_t^{\lambda(\nu)} \geq G(\Delta u^{\lambda(\nu)} + cu^{\lambda(\nu)}, u^{\lambda(\nu)}, |x|, t), \tag{92}$$

for all $(x,t) \in B_+ \times [0,T) \cap \{-\epsilon \leq u^{\lambda(\nu)} - u \leq 0\} \cap \{\Delta u^{\lambda(\nu)} \geq \Delta u\}$.

Since u is a solution to (71) and $u^{\lambda(\nu)}$ is a super-solution to (71) in $B_+ \times [0,T) \cap \{-\epsilon \leq u^{\lambda(\nu)} \leq u \leq 0\} \cap \{\Delta u^{\lambda(\nu)} \geq \Delta u\}$, with the boundary inequalities $u^{\lambda(\nu)}|_{\partial B_+ \times [0,T)} = u|_{\partial B_+ \times [0,T)}$ and $u^{\lambda(\nu)}|_{t=0} \geq u_o$, by an application of weak maximum principle-type arguments similar to that in Theorem 3.1, using the monotonicity condition (and when $c > 0$, the uniform Lipschitz condition) on G, we conclude that $u^{\lambda(\nu)}(x,t) \geq u(x,t)$, for all $x \in B_+$ and $t \in [0,T)$. We leave the details, which are analogous to those in the proof of Theorem 3.1, to the reader. This proves part (i). Likewise, we omit the proofs of parts (ii)-(iv), which are analogous to the proofs of Theorem 3.1 (ii)-(iv).

Acknowledgements. We would like to thank Professors Hans Weinberger and John Lowengrub for helpful discussions. B.C. would also like to thank Professor Wei-Ming Ni for his encouragement.

References

Aleksandrov, A. D. (1956) Uniqueness theorems for the surfaces in the large, I, Vestnik Leningrad Univ. 11, no. 19, 5-17; English transl. in Amer. Math. Soc. Transl. 21 (1962) 341-354.

Aleksandrov, A. D. (1957) II, ibid. 12, no. 7, 15-44; 354-388.

Aleksandrov, A. D. (1958a) III, ibid. 13, no. 7, 14-26; 389-403.

Aleksandrov, A. D. (1958b) V, ibid. 13, no. 19, 5-8; 412-416.

Aleksandrov, A. D., Volkov, Ju. A. (1958) IV, ibid. 13, no. 13, 27-34; 403-411.

Chow, B., Gulliver, R. (1994a) Aleksandrov reflection and nonlinear evolution equations, II: manifolds with reflectional symmetry. In preparation.

Chow, B., Gulliver, R. (1994b) Aleksandrov reflection and geometric evolution of embedded hypersurfaces. In preparation.

Chow, B., Gulliver, R. (1994c) A generalization of Aleksandrov's theorem on embedded hypersurfaces with constant curvature. In preparation.

Chow, B., Tsai, D.H. (1994a) Geometric expansion of convex plane curves. Preprint.

Chow, B., Tsai, D.H. (1994b) In preparation.

Gidas, B., Ni, W.-M., Nirenberg, L. (1979) Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68, 209-243.

Gidas, B., Ni, W.-M., Nirenberg, L. (1981) Symmetry of positive solutions of nonlinear equations in R^n, Math. Anal. and Applic., Part A, Advances in Math. Suppl. Studies 7A, ed. L. Nachbin, Academic Press, 369-402.

Hamilton, R. S. (1975) Harmonic maps of manifolds with boundary, Lecture Notes in Math. 471, Springer-Verlag.

Serrin, J. (1971) A symmetry problem in potential theory, Arch. Rat. Mech. 43, 304-318.

Urbas, J.I.E. (1991) An expansion of convex hypersurfaces, J. Diff. Geom. 33, 91-125; (1992) Correction to, ibid. 35, 763-765.